Chapter

Asymptotic Behavior by Krasnoselskii Fixed Point Theorem for Nonlinear Neutral Differential Equations with Variable Delays

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Abstract

In this paper, we consider a neutral differential equation with two variable delays. We construct new conditions guaranteeing the trivial solution of this neutral differential equation is asymptotic stable. The technique of the proof based on the use of Krasnoselskii's fixed point Theorem. An asymptotic stability theorem with a necessary and sufficient condition is proved. In particular, this paper improves important and interesting works by Jin and Luo. Moreover, as an application, we also exhibit some special cases of the equation, which have been studied extensively in the literature.

Keywords: fixed points theory, stability, neutral differential equations, integral equation, variable delays
AMS Subject Classifications: 34K20, 34K30, 34B40

1. Introduction

For more than one hundred years, Liapunov's direct method has been very effectively used to investigate the stability problems of a wide variety of ordinary, functional, and partial differential, integro-differential equations. The success of Liapunov's direct method depends on finding a suitable Liapunov function or Liapunov functional. Nevertheless, the applications of this method to problems of stability in differential and integro-differential equations with delays have encountered serious difficulties if the delays are unbounded or if the equation has unbounded terms (see [1–3]). Therefore, new methods and techniques are needed to address those difficulties. Recently, Burton and his co-authors have applied fixed point theory to investigate the stability, which shows that some of these difficulties vanish when applying fixed point theory [1–22]. It turns out that the fixed point method is becoming a powerful technique in dealing with stability problems for indeterministic scenes (see for instance [16, 17, 21, 23]).

For example, Luo [16] studied the mean-square asymptotic stability for a class of linear scalar neutral stochastic differential equations by means of Banach's fixed point theory. The author did not use Lyapunov's method; he got interesting results for the stability even when the delay is unbounded. The author also obtained necessary and sufficient conditions for the asymptotic stability. Moreover, it possesses the advantage that it can yield the existence, uniqueness, and stability criteria of the considered system in one step.

Neutral delay differential equations are often used to describe the dynamical systems which not only depend on present and past states but also involve derivatives with delays, (see [24–28]). It has been applied to describe numerous intricate dynamical systems, such as population dynamics [18], mathematical biology [27], heat conduction, and engineering [28], etc.

In particular, qualitative analysis for neutral type equations such as stability and periodicity, oscillation theory, has been an active field of research, both in the deterministic and stochastic cases. We can refer to [6, 7, 13, 15–17, 19–21, 23, 29–31], and the references cited therein.

With this motivation, in this paper, we aim to discuss the boundedness and stability for neutral differential equations with two delays (1). It is worth noting that our research technique is based on Krasnoselskii's fixed point theory. We will give some new conditions to ensure that the zero solution is asymptotically stable. Namely, a necessary and sufficient condition ensuring the asymptotic stability is proved. Our findings generalize and improve some results that can be found in the literature. In our result, the delays can be unbounded and the coefficients in the equations can change their sign. This paper is organized as follows. In Section 1 we present some basic preliminaries and the form of the neutral functional differential equations which will be studied. In Section 2, we present the inversion of the equation and we state Krasnoselskii's fixed point theorem. The boundedness and stability of the neutral differential Eq. (1) are discussed in Section 3 via Krasnoselskii's fixed point theory. Finally, in Section 4 an example is given to illustrate our theory and our method, also to compare our result by using the fixed point theory with the known results by Ardjouni and Djoudi [6].

In this work, we consider the following class of neutral differential equations with variable delays,

$$x'(t) = -a(t)x(t-\tau_1(t)) + c(t)x'(t-\tau_1(t)) + b(t)x^{\sigma}(t-\tau_2(t)), t \geq t_0, \tag{1}$$

denote $x(t) \in \mathbb{R}$ the solution to (1) with the initial condition

$$x(t) = \psi(t) \text{ for } t \in [m(t_0), t_0], \tag{2}$$

where $\psi \in C([m(t_0), t_0], \mathbb{R})$, $\sigma \in (0,1)$ is a quotient with odd positive integer denominator. We assume that $a, b \in C(\mathbb{R}^+, \mathbb{R})$, $c \in C^1(\mathbb{R}^+, \mathbb{R})$ and $\tau_i \in C(\mathbb{R}^+, \mathbb{R}^+)$ satisfy $t - \tau_i(t) \to \infty$ as $t \to \infty$, $i = 1, 2$ and for each $t_0 \geq 0$,

$$m_i(t_0) = \inf\{t - \tau_i(t), t \geq t_0\}, m(t_0) = \min\{m_i(t_0), i = 1, 2\}. \tag{3}$$

Special cases of Eq. (1) have been recently considered and studied under various conditions and with several methods. Particularly, in the case $\sigma = 1/3$, and $c(t) = 0$, in [14] Jin and Luo using the fixed point theorem of Krasnoselskii obtained boundedness and asymptotic stability for the following equation:

$$x'(t) = -a(t)x(t-\tau_1(t)) + b(t)x^{\frac{1}{3}}(t-\tau_2(t)), t \geq 0. \tag{4}$$

More precisely, the following result was established.

Theorem A (Jin and Luo [14]). *Let τ_1 be differentiable and suppose that there exists $\alpha \in (0.1)$, $k_1, k_2 > 0$, and a function $h \in C([m(0), \infty), \mathbb{R}^+)$ such that for $|t_1 - t_2| \leq 1$,*

$$\left| \int_{t_1}^{t_2} |b(u)| du \right| \leq k_1 |t_1 - t_2|, \tag{5}$$

and

$$\left|\int_{t_1}^{t_2} h(u)du\right| \le k_2|t_1 - t_2|, \tag{6}$$

while for $t \ge 0$,

$$\begin{aligned}&\int_{t-\tau_1(t)}^{t} |h(u)|du + \int_0^t e^{-\int_s^t h(u)du}|h(s)|\left(\int_{s-\tau_1(s)}^{s}|h(u)|du\right)ds \\ &+\int_0^t e^{-\int_s^t h(u)du}\left\{\left|h(s-\tau_1(s))\left(1-\tau_1'(s)\right)-a(s)\right| + |b(s)|\right\}ds \le \alpha.\end{aligned} \tag{7}$$

Then there is a solution $x(t, 0, \psi)$ *of* (4) *on* $\mathbb{R}^+$ *with* $|x(t, 0, \psi)| \le 1$.

Notice that when $c(t) = 0$ in the second term on the right-hand side of (1), then (1) reduces to (4). On the other hand, in the case, $\tau_1(t) = \tau_1$, a constant, Eq. (4) reduces to the one in [9]. Therefore, we considered the more general system than in [9, 14].

Very recently, by the same method of Jin and Luo [14], Ardjouni and Djoudi [6] improved the results of Jin and Luo [14] to the generalized nonlinear neutral differential equation with variable delays of the form

$$x'(t) = -a(t)x(t-\tau_1(t)) + c(t)x'(t-\tau_1(t)) + b(t)G(x^\sigma(t-\tau_2(t))), t \ge 0, \tag{8}$$

where $G : \mathbb{R} \to \mathbb{R}$ is locally Lipschitz continuous in x. That is, there is an $L > 0$ so that if $|x|, |y| \le 1$ then

$$|G(x) - G(y)| < |x - y| \text{ and } G(0) = 0.$$

We note that due to the presence of the term $c(t)x'(t - \tau_1(t))$, once the equation is inverted then once will face with the term $\frac{c(t)}{1-\tau_1'(t)}x(t-\tau_1(t))$, (where, $\tau_1'(t) \neq 1$ for $t \ge 0$) which produces a restrictive condition for the stability of (8) (as described in more detail below).

Theorem B (Ardjouni and Djoudi [6])**.** *Let* τ_1 *be twice differentiable and suppose that* $\tau_1'(t) \neq 1$ for all $t \in [m(0), \infty)$ *and suppose that there are constants* $\alpha \in (0.1)$, $k_1, k_2 > 0$, *and a function* $h \in C([m(0), \infty), \mathbb{R}^+)$ *such that for* $|t_1 - t_2| \le 1$,

$$\left|\int_{t_1}^{t_2} |b(u)|du\right| \le k_1|t_1 - t_2|, \tag{9}$$

and

$$\left|\int_{t_1}^{t_2} h(u)du\right| \le k_2|t_1 - t_2|, \tag{10}$$

while for $t \ge 0$,

$$\begin{aligned}&\left|\frac{c(t)}{1-\tau_1'(t)}\right| + \int_{t-\tau_1(t)}^{t}|h(u)|du \\ &+\int_0^t e^{-\int_s^t h(u)du}|h(s)|\left(\int_{s-\tau_1(s)}^{s}|h(u)|du\right)ds \\ &+\int_0^t e^{-\int_s^t h(u)du}\{|h(s-\tau_1(s))(1-\tau'(s))-a(s)-\mu(s)| + L|b(s)|\}ds \le \alpha,\end{aligned} \tag{11}$$

where

$$\mu(t) = \frac{(c(t)h(t) + c'(t))\left(1 - \tau_1'(t)\right) + c(t)\tau_1''(t)}{\left(1 - \tau_1'(t)\right)^2}.$$

Then there is a solution $x(t, 0, \psi)$ *of* (8) *on* $\mathbb{R}^+$ *with* $|x(t, 0, \psi)| \leq 1$.

By letting $c(t) = 0$ and $G(x^\sigma(t - \tau_2(t))) = x^\sigma(t - \tau_2(t))$ in (8), the present authors [14] have studied, the asymptotic stability and the stability by using Krasnoselskii's fixed point theorem, under appropriate conditions, of the Eq. (4) and improved the results claimed in [9]. Consequently, Theorem B improves and generalizes Theorem A. Following the technique of Jin and Luo [14], Ardjouni and Djoudi [6] studied the stability properties of (8). However, the condition (11) in Ardjouni and Djoudi [6] is restrictive. By employing two auxiliary functions p and g for constructing a fixed point mapping argument, the alternative condition (21) in Theorem 3.1 is obtained. Note that the condition

$$\left|\frac{c(t)}{1 - \tau_1'(t)}\right| < \alpha,$$

for some constant $\alpha \in (0, 1)$, is not needed in Theorem 3.1. In the present paper, we also adopt Krasnoselskii's fixed point theory to study the boundedness and stability of (1). A new criteria for asymptotic stability with a necessary and sufficient condition is given. The considered neutral differential equations, the results and assumptions to be given here are different from those that can be found in the literature and complete that one. These are the contributions of this paper to the literature and its novelty and originality. In addition, an example is provided to illustrate the effectiveness and the merits of the results introduced.

2. Inversion of equation

In this section, we use the variation of parameter formula to rewrite the equation as an integral equation suitable for the Krasnoselskii theorem. The technique for constructing a mapping for a fixed point argument comes from an idea in [21]. In our consideration we suppose that:

A1) Let τ_1 be twice differentiable and suppose that $\tau_1'(t) \neq 1$ for all $t \in [m(t_0), \infty[$.

A2) There exists a bounded function $p : [m(t_0), \infty[\to (0, \infty)$ with $p(t) = 1$ for $t \in [m(t_0), t_0]$ such that $p'(t)$ exists for all $t \in [m(t_0), \infty[$.

Let $y(t) = \psi(t)$ on $t \in [m(t_0), t_0]$, and let

$$x(t) = p(t)y(t) \text{ for } t \geq t_0. \tag{12}$$

Make substitution of (12) into (1) to show

$$\begin{aligned} y'(t) = &-\frac{p'(t)}{p(t)}y(t) - \frac{a(t)p(t - \tau_1(t)) - c(t)p'(t - \tau_1(t))}{p(t)}y(t - \tau_1(t)) \\ &+ \frac{c(t)p(t - \tau_1(t))}{p(t)}y'(t - \tau_1(t)) \\ &+ b(t)\frac{p^\sigma(t - \tau_2(t))}{p(t)}y^\sigma(t - \tau_2(t)), t \geq t_0, \end{aligned} \tag{13}$$

then it can be verified that x satisfies (1).

We now re-write Eq. (13) in an equivalent form. To this end, we use the variation of parameter formula and rewrite the equation in an integral from which we derive a Krasnoselskii fixed point theorem. Besides, the integration by parts will be applied.

We need the following lemma in our proof of the main theorem.

Lemma 2.1. *Let $h : [m(t_0), \infty) \to \mathbb{R}^+$ be an arbitrary continuous function and suppose that (A1) and (A2) hold. Then y is a solution of* (13) *if and only if*

$$\begin{aligned} y(t) = &\left(\psi(t_0) - \frac{p(t_0 - \tau_1(t_0))}{p(t_0)} \frac{c(t_0)}{(1 - \tau_1'(t_0))} \psi(t_0 - \tau_1(t_0)) \right. \\ &\left. - \int_{t_0 - \tau_1(t_0)}^{t_0} \left(h(u) - \frac{p'(u)}{p(u)} \right) y(u) du \right) e^{-\int_{t_0}^{t} h(s) ds} \\ &+ \frac{p(t - \tau_1(t))}{p(t)} \frac{c(t)}{1 - \tau_1'(t)} y(t - \tau_1(t)) + \int_{t - \tau_1(t)}^{t} \left(h(u) - \frac{p'(u)}{p(u)} \right) y(u) du \\ &+ \int_{t_0}^{t} e^{-\int_s^t h(u) du} \left\{ -\overline{\mu}(s) + \left(h(s - \tau_1(s)) - \frac{p'(s - \tau_1(s))}{p(s - \tau_1(s))} \right) (1 - \tau_1'(s)) - \overline{\beta}(s) \right\} \\ &\times y(s - \tau_1(s)) ds \\ &- \int_{t_0}^{t} e^{-\int_s^t h(u) du} h(s) \left(\int_{s - \tau_1(s)}^{s} \left(h(u) - \frac{p'(u)}{p(u)} \right) y(u) du \right) ds \\ &+ \int_{t_0}^{t} e^{-\int_s^t h(u) du} b(s) \frac{p^\sigma(s - \tau_2(s))}{p(s)} y^\sigma(s - \tau_2(s)) ds, \end{aligned} \tag{14}$$

where

$$\overline{\mu}(t) = \frac{a(t)p(t - \tau_1(t)) - c(t)p'(t - \tau_1(t))}{p(t)}, C(t) = \frac{c(t)p(t - \tau_1(t))}{p(t)}. \tag{15}$$

and

$$\overline{\beta}(t) = \frac{[C(t)h(t) + C'(t)](1 - \tau_1'(t)) + C(t)\tau_1''(t)}{(1 - \tau_1'(t))^2}. \tag{16}$$

Proof. Let $y(t)$ be a solution of Eq. (13). Rewrite (13) as

$$\begin{aligned} y'(t) + h(t)y(t) = &\left(h(t) - \frac{p'(t)}{p(t)} \right) y(t) - \frac{a(t)p(t - \tau_1(t)) - c(t)p'(t - \tau_1(t))}{p(t)} y(t - \tau_1(t)) \\ &+ \frac{c(t)p(t - \tau_1(t))}{p(t)} y'(t - \tau_1(t)) \\ &+ b(t) \frac{p^\sigma(t - \tau_2(t))}{p(t)} y^\sigma(t - \tau_2(t)), t \geq t_0. \end{aligned} \tag{17}$$

Multiply both sides of (17) the previous equality by $e^{\int_{t_0}^t h(s)ds}$ and then integrate from t_0 to t, we have

$$\begin{aligned} y(t) &= \psi(t_0)e^{-\int_{t_0}^t h(s)ds} + \int_{t_0}^t \left(h(s) - \frac{p'(s)}{p(s)}\right)e^{-\int_s^t h(u)du}y(s)ds \\ &\quad -\int_{t_0}^t e^{-\int_s^t h(u)du}\frac{a(s)p(s-\tau_1(s)) - c(s)p'(s-\tau_1(s))}{p(s)}y(s-\tau_1(s))ds \\ &\quad +\int_{t_0}^t e^{-\int_s^t h(u)du}\frac{c(s)p(s-\tau_1(s))}{p(s)}y'(s-\tau_1(s))ds \\ &\quad +\int_{t_0}^t e^{-\int_s^t h(u)du}b(s)\frac{p^\sigma(s-\tau_2(s))}{p(s)}y^\sigma(s-\tau_2(s))ds. \end{aligned} \tag{18}$$

Performing an integration by parts, we can conclude, for $t \geq t_0$,

$$\begin{aligned} y(t) &= \psi(t_0)e^{-\int_{t_0}^t h(s)ds} + \int_{t_0}^t e^{-\int_s^t h(u)du}d\left(\int_{s-\tau_1(s)}^s \left(h(u) - \frac{p'(u)}{p(u)}\right)y(u)du\right) \\ &\quad +\int_{t_0}^t e^{-\int_s^t h(u)du}\left(h(s-\tau_1(s)) - \frac{p'(s-\tau_1(s))}{p(s-\tau_1(s))}\right) \times (1-\tau_1'(s))y(s-\tau_1(s))ds \\ &\quad -\int_{t_0}^t e^{-\int_s^t h(u)du}\frac{a(s)p(s-\tau_1(s)) - c(s)p'(s-\tau_1(s))}{p(s)}y(s-\tau_1(s))ds \\ &\quad +\int_{t_0}^t e^{-\int_s^t h(u)du}\frac{c(s)p(s-\tau_1(s))}{p(s)}y'(s-\tau_1(s))ds \\ &\quad +\int_{t_0}^t e^{-\int_s^t h(u)du}b(s)\frac{p^\sigma(s-\tau_2(s))}{p(s)}y^\sigma(s-\tau_2(s))ds. \end{aligned}$$

Thus,

$$\begin{aligned} y(t) &= \left(\psi(t_0) - \frac{p(t_0-\tau_1(t_0))}{p(t_0)}\frac{c(t_0)}{(1-\tau_1'(t_0))}\psi(t_0-\tau_1(t_0))\right. \\ &\quad \left.-\int_{t_0-\tau_1(t_0)}^{t_0}\left(h(u) - \frac{p'(u)}{p(u)}\right)y(u)du\right)e^{-\int_{t_0}^t h(s)ds} \\ &\quad +\frac{p(t-\tau_1(t))}{p(t)}\frac{c(t)}{1-\tau_1'(t)}y(t-\tau_1(t)) + \int_{t-\tau_1(t)}^t\left(h(u) - \frac{p'(u)}{p(u)}\right)y(u)du \\ &\quad +\int_{t_0}^t e^{-\int_s^t h(u)du}\left\{-\overline{\mu}(s) + \left(h(s-\tau_1(s)) - \frac{p'(s-\tau_1(s))}{p(s-\tau_1(s))}\right)(1-\tau_1'(s)) - \overline{\beta}(s)\right\} \\ &\quad \times y(s-\tau_1(s))ds \\ &\quad -\int_{t_0}^t e^{-\int_s^t h(u)du}h(s)\left(\int_{s-\tau_1(s)}^s\left(h(u) - \frac{p'(u)}{p(u)}\right)y(u)du\right)ds \\ &\quad +\int_{t_0}^t e^{-\int_s^t h(u)du}b(s)\frac{p^\sigma(s-\tau_2(s))}{p(s)}y^\sigma(s-\tau_2(s))ds, \end{aligned}$$

where $\overline{\mu}(s)$ and $\overline{\beta}(s)$ are defined in (15) and (16), respectively. The proof is complete.

Below we state Krasnoselskii's fixed point theorem which will enable us to establish a stability result of the trivial solution of (1) For more details on Krasnoselskii's captivating theorem, we refer to smart [20] or [3].

Theorem 2.1. (see, [Kranoselskii's fixed point theorem, [20]]). *Suppose that* $(X, \|.\|)$ *is a Banach space and* $\mathcal{M}$ *is a bounded, convex, and closed subset of* X. Suppose further that there exist, two operators, $\mathcal{A}, \mathcal{B} \to \mathcal{M}$ *into* X *such that:*

i. $\mathcal{A}x + \mathcal{B}y \in \mathcal{M}$ *for all* $x, y \in \mathcal{M}$;

ii. $\mathcal{A}$ *is completely continuous*;

iii. $\mathcal{B}$ *is a contraction mapping.*

Then $\mathcal{A} + \mathcal{B}$ *has a fixed point in* $\mathcal{M}$.

3. Stability by Krasnoselskii fixed point theorem

From the existence theory, which can be found in Hale [26] or Burton [3], we conclude that for each $(t_0, \psi) \in \mathbb{R}^+ \times C([m(t_0), t_0], \mathbb{R})$, a solution of (1) through (t_0, ψ) is a continuous function $x : [m(t_0), t_0 + \rho) \to \mathbb{R}$ for some positive constant $\rho > 0$ such that x satisfies (1) on $[t_0, t_0 + \rho)$ and $x = \psi$ on $[m(t_0), t_0]$. We denote such a solution by $x(t) = x(t, t_0, \psi)$. We define $\|\psi\| = \max\{|\psi(t)| : m(t_0) \le t \le t_0\}$.

As we mentioned previously, our results in this section extend and improve the work in [14] by considering more general classes of neutral differential equations presented by (1). Our main results in this part can be applied to the case when

$$\left|\frac{c(t)}{1 - \tau_1'(t)}\right| \ge 1,$$

which improve [14]. In other words, we will establish and prove a necessary and sufficient condition ensuring the boundedness of solutions and the asymptotic stability of the zero solution to Eq. (1). However, the mathematical analysis used in this research to construct the mapping to employ Krasnoselskii's fixed point theorem is different from that of [14].

The results of this work are news and they extend and improve previously known results. To the best of our knowledge from the literature, there are few authors who have used the fixed point theorem to prove the existence of a solution and the stability of trivial equilibrium of several special cases of (1) all at once [9, 14].

Let us know to recall the definitions of stability that will be used in the next section. For stability definitions, we refer to [3].

Definition 3.1. The zero solution of (1) is said to be:

i. stable, if for any $\varepsilon > 0$ and $t_0 \ge 0,$ there exists a $\delta = \delta(\varepsilon, t_0) > 0$ such that $\psi \in C([m(t_0), t_0], \mathbb{R})$ and $\|\psi\| < \delta$ imply $|x(t, t_0, \psi)| < \varepsilon$ for $t \ge t_0$.

ii. asymptotically stable, if the zero solution is stable and for any $\varepsilon > 0$ and $t_0 \ge 0,$ there exists a $\delta = \delta(\varepsilon, t_0) > 0$ such that $\psi \in C([m(t_0), t_0], \mathbb{R})$ and $\|\psi\| < \delta$ imply $|x(t, t_0, \psi)| \to 0$ as $t \to \infty$..

Now, we can state our main result.

Theorem 3.1. *Suppose that assumptions (A1) and (A2) hold, and that there are constants* $\alpha \in (0,1)$, $k_1, k_2 > 0$, *and an arbitrary continuous function* $h \in C([m(t_0), \infty), \mathbb{R}^+)$ *such that for* $|t_1 - t_2| \le 1$,

$$\left| \int_{t_1}^{t_2} \left| b(u) \frac{p^\sigma(u - \tau_2(u))}{p(u)} \right| du \right| \le k_1 |t_1 - t_2|, \tag{19}$$

and

$$\left| \int_{t_1}^{t_2} h(u) du \right| \le k_2 |t_1 - t_2|, \tag{20}$$

while for $t \ge t_0$

$$\begin{aligned} &\left| \frac{p(t - \tau_1(t))}{p(t)} \frac{c(t)}{1 - \tau_1'(t)} \right| + \int_{t-\tau_1(t)}^{t} \left| h(u) - \frac{p'(u)}{p(u)} \right| du \\ &+ \int_{t_0}^{t} e^{-\int_s^t h(u) du} \left\{ \left| -\overline{\mu}(s) + \left(h(s - \tau_1(s)) - \frac{p'(s - \tau_1(s))}{p(s - \tau_1(s))} \right) (1 - \tau_1'(s)) - \overline{\beta}(s) \right| \right\} ds \\ &+ \int_{t_0}^{t} e^{-\int_s^t h(u) du} |h(s)| \left(\int_{s-\tau_1(s)}^{s} \left| h(u) - \frac{p'(u)}{p(u)} \right| du \right) ds \\ &+ \int_{t_0}^{t} e^{-\int_s^t h(u) du} |b(s)| \left| \frac{p^\sigma(s - \tau_2(s))}{p(s)} \right| ds < \alpha, \end{aligned} \tag{21}$$

where $\overline{\mu}(s)$ *and* $\overline{\beta}(s)$ *are defined in* (15) *and* (16), *respectively. If* ψ *is a given continuous initial function which is sufficiently small, then there is a solution* $x(t, t_0, \psi)$ *of* (1) *on* $\mathbb{R}^+$ *with* $|x(t, t_0, \psi)| \le 1$.

We are now ready to prove Theorem 3.1.

Proof. We start with some preparation:

Let $\left(X, |.|_g\right)$ be the Banach space of continuous $\varphi : [m(t_0), \infty) \to \mathbb{R}$ with

$$|\varphi|_g := \sup_{t \ge m(t_0)} |\varphi(t)/g(t)| < \infty.$$

For each $t_0 \ge 0$ and $\psi \in C([m(t_0), t_0], \mathbb{R})$ fixed, we define X_ψ as the following space

$$X_\psi = \{\varphi \in X : |\varphi(t)| \le 1 \ \text{for} t \in [m(t_0), \infty) \text{and} \varphi(t) = \psi(t) \text{if} t \in [m(t_0), t_0]\}.$$

It is easy to check that X_ψ is a complete metric space with metric induced by the norm $|.|_g$.

We note that to apply Krasnoselskii's fixed point theorem we need to construct two mappings; one is contraction and the other is compact. Therefore, we use (14) to define the operator $H : X_\psi \to X_\psi$ by

$$(H\varphi)(t) := (\mathcal{A}\varphi)(t) + (\mathcal{B}\varphi)(t),$$

where $\mathcal{A}, \mathcal{B} : X_\psi \to X_\psi$ are given by

$$(\mathcal{A}\varphi)(t) := \int_{t_0}^{t} e^{-\int_s^t h(u) du} b(s) \frac{p^\sigma(s - \tau_2(s))}{p(s)} \varphi^\sigma(s - \tau_2(s)) ds, \tag{22}$$

and

$$
\begin{aligned}
(\mathcal{B}\varphi)(t) := &\left(\psi(t_0) - \frac{p(t_0-\tau_1(t_0))}{p(t_0)} \frac{c(t_0)}{(1-\tau_1'(t_0))} \psi(t_0-\tau_1(t_0))\right.\\
&\left. - \int_{t_0-\tau_1(t_0)}^{t_0} \left(h(u) - \frac{p'(u)}{p(u)}\right)\varphi(u)du\right) e^{-\int_{t_0}^{t} h(s)ds}\\
&+ \frac{p(t-\tau_1(t))}{p(t)} \frac{c(t)}{1-\tau_1'(t)} \varphi(t-\tau_1(t)) + \int_{t-\tau_1(t)}^{t} \left(h(u) - \frac{p'(u)}{p(u)}\right)\varphi(u)du\\
&+ \int_{t_0}^{t} e^{-\int_s^t h(u)du} \left\{-\overline{\mu}(s) + \left(h(s-\tau_1(s)) - \frac{p'(s-\tau_1(s))}{p(s-\tau_1(s))}\right)(1-\tau_1'(s)) - \overline{\beta}(s)\right\}\\
&\times \varphi(s-\tau_1(s))ds\\
&- \int_{t_0}^{t} e^{-\int_s^t h(u)du} h(s) \left(\int_{s-\tau_1(s)}^{s} \left(h(u) - \frac{p'(u)}{p(u)}\right)\varphi(u)du\right)ds.
\end{aligned}
\tag{23}
$$

If we are able to prove that H possesses a fixed point φ on the set X_ψ, then $y(t,t_0,\psi) = \varphi(t)$ for $t \geq t_0$, $y(t,t_0,\psi) = \psi(t)$ on $[m(t_0), t_0]$, $y(t,t_0,\psi)$ satisfies (13) when its derivative exists and $|y(t,t_0,\psi)| < 1$ for $t \geq t_0$. That $\mathcal{A}$ maps X_ψ into itself can be deduced from condition (21).

For $\alpha \in (0,1)$, we choose $\delta > 0$ such that

$$
\begin{aligned}
&\left(1 + \left|\frac{p(t_0-\tau_1(t_0))}{p(t_0)} \frac{c(t_0)}{(1-\tau_1'(t_0))}\right|\right.\\
&\qquad \left. + \int_{t_0-\tau_1(t_0)}^{t_0} \left|h(u) - \frac{p'(u)}{p(u)}\right| du\right) e^{-\int_{t_0}^{t} h(s)ds} \delta + \alpha \leq 1.
\end{aligned}
\tag{24}
$$

Let $\psi : [m(t_0), t_0] \to \mathbb{R}$ be a given continuous initial function with $\|\psi\| < \delta$. Let $g : [m(t_0), \infty) \to [1, \infty)$ be any strictly increasing and continuous function with $g(m(t_0)) = 1$, $g(s) \to \infty$ as $s \to \infty$, such that

$$
\begin{aligned}
&\left|\frac{p(t-\tau_1(t))}{p(t)} \frac{c(t)}{1-\tau_1'(t)}\right| + \int_{t-\tau_1(t)}^{t} \left|h(u) - \frac{p'(u)}{p(u)}\right| g(u)/g(t)du\\
&+ \int_{t_0}^{t} e^{-\int_s^t h(u)du} |h(s)| \left(\int_{s-\tau_1(s)}^{s} \left|h(u) - \frac{p'(u)}{p(u)}\right| g(u)/g(t)du\right)ds\\
&+ \int_{t_0}^{t} e^{-\int_s^t h(u)du} \left|-\overline{\mu}(s) + \left(h(s-\tau_1(s)) - \frac{p'(s-\tau_1(s))}{p(s-\tau_1(s))}\right)(1-\tau_1'(s)) - \overline{\beta}(s)\right|\\
&\times g(s-\tau_1(s))/g(t)ds < \alpha.
\end{aligned}
\tag{25}
$$

Now we split the rest of our proof into three steps.

First step: We now show that $\varphi, \phi \in X_\psi$ implies that $\mathcal{A}\varphi + \mathcal{B}\phi \in X_\psi$. Now, let $\|.\|$ be the supremum norm on $[m(t_0), \infty)$ of $\varphi \in X_\psi$ if φ is bounded. Note that if $\varphi, \phi \in X_\psi$ then

$$|(\mathcal{A}\varphi)(t) + (\mathcal{B}\phi)(t)| \leq$$

$$\|\psi\|\left(1+\left|\frac{p(t_0-\tau_1(t_0))}{p(t_0)}\right|\left|\frac{c(t_0)}{(1-\tau_1'(t_0))}\right|+\int_{t_0-\tau_1(t_0)}^{t_0}\left|h(u)-\frac{p'(u)}{p(u)}\right|du\right)e^{-\int_{t_0}^{t}h(s)ds}$$

$$+\|\phi\|\left|\frac{p(t-\tau_1(t))}{p(t)}\frac{c(t)}{1-\tau_1'(t)}\right|$$

$$+\|\phi\|\int_{t-\tau_1(t)}^{t}\left|h(u)-\frac{p'(u)}{p(u)}\right|du$$

$$+\|\phi\|\int_{t_0}^{t}e^{-\int_s^t h(u)du}\left\{\left|-\overline{\mu}(s)+\left(h(s-\tau_1(s))-\frac{p'(s-\tau_1(s))}{p(s-\tau_1(s))}\right)(1-\tau_1'(s))-\overline{\beta}(s)\right|\right\}ds$$

$$+\|\phi\|\int_{t_0}^{t}e^{-\int_s^t h(u)du}|h(s)|\left(\int_{s-\tau_1(s)}^{s}\left|h(u)-\frac{p'(u)}{p(u)}\right|du\right)ds$$

$$+\|\varphi^\sigma\|\int_{t_0}^{t}e^{-\int_s^t h(u)du}|b(s)|\left|\frac{p^\sigma(s-\tau_2(s))}{p(s)}\right|ds$$

$$\leq\left(1+\left|\frac{p(t_0-\tau_1(t_0))}{p(t_0)}\right|\left|\frac{c(t_0)}{(1-\tau_1'(t_0))}\right|\right.$$

$$\left.+\int_{t_0-\tau_1(t_0)}^{t_0}\left|h(u)-\frac{p'(u)}{p(u)}\right|du\right)e^{-\int_{t_0}^{t}h(s)ds}\delta+\alpha\leq 1.$$

By applying (24), we see that $|(\mathcal{A}\varphi)(t)+(\mathcal{B}\phi)(t)|\leq 1$ for $t\in[m(t_0),\infty)$.

We see that also $\mathcal{B}$ maps X_ψ into itself by letting $\varphi=0$ in the preceding sum.

Second step: Next, we will show that $\mathcal{A}X_\psi$ is equicontinuous and $\mathcal{A}$ is continuous. We first show that $\mathcal{A}X_\psi$ is equicontinuous. If $\varphi\in X_\psi$ and if $0\leq t_1<t_2$ with $t_2-t_1<1$, then

$$|(\mathcal{A}\varphi)(t_2)-(\mathcal{A}\varphi)(t_1)|=\left|\int_{t_0}^{t_2}e^{-\int_s^{t_2}h(u)du}b(s)\frac{p^\sigma(s-\tau_2(s))}{p(s)}ds\right.$$

$$\left.-\int_{t_0}^{t_1}b(s)\frac{p^\sigma(s-\tau_2(s))}{p(s)}e^{-\int_s^{t_1}h(u)du}ds\right|$$

$$\leq\left|\int_{t_1}^{t_2}b(s)\frac{p^\sigma(s-\tau_2(s))}{p(s)}e^{-\int_s^{t_2}h(u)du}ds\right|$$

$$+\left|\int_{t_0}^{t_1}\left(e^{-\int_s^{t_2}h(u)du}-e^{-\int_s^{t_1}h(u)du}\right)b(s)\frac{p^\sigma(s-\tau_2(s))}{p(s)}ds\right|$$

$$\leq\int_{t_1}^{t_2}e^{-\int_s^{t_2}h(u)du}d\left(\int_{t_1}^{s}\left|b(v)\frac{p^\sigma(v-\tau_2(v))}{p(v)}\right|dv\right)$$

$$+\left|e^{-\int_{t_1}^{t_2}h(u)du}-1\right|\int_{t_0}^{t_1}e^{-\int_s^{t_1}h(u)du}\left|b(s)\frac{p^\sigma(s-\tau_2(s))}{p(s)}\right|ds$$

$$\leq\int_{t_1}^{t_2}\left|b(u)\frac{p^\sigma(u-\tau_2(u))}{p(u)}\right|du\left(1+\int_{t_1}^{t_2}h(u)e^{-\int_s^{t_2}h(u)du}ds\right)+\alpha\left|e^{-\int_{t_1}^{t_2}h(u)du}-1\right|$$

$$\leq 2\int_{t_1}^{t_2}\left|b(u)\frac{p^\sigma(u-\tau_2(u))}{p(u)}\right|du + \alpha\left|\int_{t_1}^{t_2} h(u)du\right| \leq (2k_1+\alpha k_2)|t_2-t_1|,$$

by (19)–(21). In the space $\left(X, |.|_g\right)$, the set $\mathcal{A}X_\psi$ is uniformly bounded and equicontinuous. Hence by Ascoli-Arzela theorem $\mathcal{A}X_\psi$ resides in a compact set.

Next, we need to show that $\mathcal{A}$ is continuous. Let $\varepsilon > 0$ be given and let $\varphi, \phi \in X_\psi$. Now y^σ, is uniformly continuous on $[-1, +1]$ so for a fixed $T > 0$ with $4/g(T) < \varepsilon$ there is an $\eta > 0$ such that $|y_1 - y_2| < \eta g(T)$ implies $|y_1^\sigma - y_2^\sigma| < \varepsilon/2$. Thus for $|\varphi(t) - \phi(t)| < \eta g(t)$ and for $t > T$ we have

$$\begin{aligned}
&|(\mathcal{A}\varphi)(t) - (\mathcal{A}\phi)(t)|/g(t)\\
&= (1/g(t))\int_{t_0}^{t} e^{-\int_s^t h(u)du}\left|b(s)\frac{p^\sigma(s-\tau_2(s))}{p(s)}\right||\varphi^\sigma(s-\tau_2(s)) - \phi^\sigma(s-\tau_2(s))|ds\\
&\leq (1/g(t))\left\{\int_{t_0}^{T} e^{-\int_s^t h(u)du}\left|b(s)\frac{p^\sigma(s-\tau_2(s))}{p(s)}\right||\varphi^\sigma(s-\tau_2(s)) - \phi^\sigma(s-\tau_2(s))|ds\right.\\
&\left.+2\int_{T}^{t} e^{-\int_s^t h(u)du}\left|b(s)\frac{p^\sigma(s-\tau_2(s))}{p(s)}\right|ds\right\}\\
&\leq \{(\alpha\varepsilon/2g(t)) + (2\alpha/g(T))\} \leq \alpha\varepsilon.
\end{aligned}$$

Third step: Finally, we show that $\mathcal{B}$ is a contraction with respect to the norm $|.|_g$ with constant α. Let $\mathcal{B}$ be defined by (23). Then for $\phi_1, \phi_2 \in X_\psi$ we have

$$\begin{aligned}
&|(\mathcal{B}\phi_1)(t) - (\mathcal{B}\phi_2)(t)|/g(t) \leq \left|\frac{p(t-\tau_1(t))}{p(t)}\frac{c(t)}{1-\tau_1'(t)}\right||\phi_1(t-\tau_1(t)) - \phi_2(t-\tau_1(t))|/g(t)\\
&+\int_{t-\tau_1(t)}^{t}\left|h(u) - \frac{p'(u)}{p(u)}\right||\phi_1(u) - \phi_2(u)|/g(t)du\\
&+\int_{t_0}^{t} e^{-\int_s^t h(u)du}\left|-\overline{\mu}(s) + \left(h(s-\tau_1(s)) - \frac{p'(s-\tau_1(s))}{p(s-\tau_1(s))}\right)(1-\tau_1'(s)) - \overline{\beta}(s)\right|\\
&\times|\phi_1(s-\tau_1(s)) - \phi_2(s-\tau_1(s))|/g(t)ds\\
&+\int_{t_0}^{t} e^{-\int_s^t h(u)du}|h(s)|\left(\int_{s-\tau_1(s)}^{s}\left|h(u) - \frac{p'(u)}{p(u)}\right||\phi_1(u) - \phi_2(u)|/g(t)du\right)ds\\
&\leq |\phi_1 - \phi_2|_g\left\{\left|\frac{p(t-\tau_1(t))}{p(t)}\frac{c(t)}{1-\tau_1'(t)}\right|\right.\\
&+\int_{t-\tau_1(t)}^{t}\left|h(u) - \frac{p'(u)}{p(u)}\right|g(u)/g(t)du\\
&+\int_{t_0}^{t} e^{-\int_s^t h(u)du}|h(s)|\left(\int_{s-\tau_1(s)}^{s}\left|h(u) - \frac{p'(u)}{p(u)}\right|g(u)/g(t)du\right)ds\\
&+\int_{t_0}^{t} e^{-\int_s^t h(u)du}\\
&\left.\times\left|-\overline{\mu}(s) + \left(h(s-\tau_1(s)) - \frac{p'(s-\tau_1(s))}{p(s-\tau_1(s))}\right)(1-\tau_1'(s)) - \overline{\beta}(s)\right|g(s-\tau_1(s))/g(t)ds\right\}\\
&\leq \alpha|\phi_1 - \phi_2|_g, \text{ by (22).}
\end{aligned}$$

Since $\alpha \in (0,1)$, we can conclude that $\mathcal{B}$ is a contraction on $\left(X_\psi, |.|_g\right)$.

The conditions of Krasnoselskii's theorem are satisfied with $\mathcal{M} = X_\psi$. Hence, we deduce that $H : X_\psi \to X_\psi$ has a fixed point $y(t)$, which is a solution of (13) with $y(s) = \psi(s)$ on $s \in m(t_0), t_0]$ and $|y(t, t_0, \psi)| \leq 1$ for $t \in [m(t_0), \infty)$. Since there exists a bounded function $p : [m(t_0), \infty[\to (0, \infty)$ with $p(t) = 1$ for $t \in [m(t_0), t_0]$, by hypotheses (12) and from the above arguments we deduce that there exists a solution x of (1) with $x = \psi$ on $[m(t_0), t_0]$ satisfies $|x(t, t_0, \psi)| \leq 1$ for all $t \in [m(t_0), \infty)$. The proof is complete.

Letting $\sigma = 1/3$, and $c(t) = 0$ in Theorem 3.1. Then we have the following corollary.

Corollary 3.1. *Let* (19) *and* (20) *hold, and* (21) *be replaced by*

$$\int_{t-\tau_1(t)}^{t} \left| h(u) - \frac{p'(u)}{p(u)} \right| du$$

$$+ \int_{t_0}^{t} e^{-\int_s^t h(u)du} \left\{ \left| \left(h(s - \tau_1(s)) - \frac{p'(s - \tau_1(s))}{p(s - \tau_1(s))} \right) (1 - \tau_1'(s)) - a(s) \frac{p(s - \tau(s))}{p(s)} \right| \right\} ds$$

$$+ \int_{t_0}^{t} e^{-\int_s^t h(u)du} |h(s)| \left(\int_{s-\tau_1(s)}^{s} \left| h(u) - \frac{p'(u)}{p(u)} \right| du \right) ds \leq \alpha. \tag{26}$$

Then there is a solution $x(t, t_0, \psi)$ *of* (4) *on* $\mathbb{R}^+$ *with* $|x(t, t_0, \psi)| \leq 1$.

Remark 3.2: When $p(t) = 1$, then Corollary 3.1 reduces to Theorem A, which was recently stated in Jin and Luo [14]. Therefore, the paper (Jin and Luo [14]) is a particular case of ours.

For the next Theorem, we manipulate function spaces defined on infinite t -intervals. So, for compactness, we need an extension of the Arzelà-Ascoli theorem. This extension is taken from ([3], Theorem 1.2.2 p. 20).

Theorem 3.2. *Let* (19)–(21) *hold and assume that*

$$\int_{t_0}^{t} e^{-\int_s^t h(u)du} \left| b(s) \frac{p^\sigma(s - \tau_2(s))}{p(s)} \right| ds \to 0 \text{ as } t \to \infty, \tag{27}$$

and

$$\liminf_{t \to \infty} \int_{t_0}^{t} h(s)ds > -\infty. \tag{28}$$

If ψ *is given continuous initial function which is sufficiently small, then* (1) *has a solution* $x(t, t_0, \psi) \to 0$ *as* $t \to \infty$ *if and only if*

$$\int_{t_0}^{t} h(s)ds \to \infty \text{ as } t \to \infty. \tag{29}$$

Proof. We set

$$K = \sup_{t \geq t_0} \left\{ e^{-\int_{t_0}^{t} h(s)ds} \right\}, \tag{30}$$

by (28), K is well defined. Suppose that (29) holds.

Since p is bounded, it remains to prove that the zero solution of (1) is asymptotically stable.

All of the calculations in the proof of Theorem 3.1 hold with $g(t) = 1$ when $|.|_g$ is replaced by the supremum norm $\|.\|$.

For

$$\varphi \in X_\psi,$$
$$|(\mathcal{A}\varphi)(t)| \le \int_{t_0}^{t} e^{-\int_s^t h(u)du} \left| b(s) \frac{p^\sigma(s - \tau_2(s))}{p(s)} \right| ds =: q(t), \tag{31}$$

where $q(t) \to 0$ as $t \to \infty$ by (27).

Add to X_ψ the condition that $\varphi \in X_\psi$ implies that $\varphi(t) \to 0$ as $t \to \infty$. We can see that for $\varphi \in X_\psi$ then $(\mathcal{A}\varphi)(t) \to 0$ as $t \to \infty$ by (31), and $(\mathcal{B}\varphi)(t) \to 0$ as $t \to \infty$ by (29).

Since $\mathcal{A}X_\psi$ has been shown to be equicontinuous, $\mathcal{A}$ maps X_ψ into a compact subset of X_ψ. By Krasnoselskii's theorem, there is $y \in X_\psi$ with $\mathcal{A}y + \mathcal{B}y = y$. As $y \in X_\psi, y(t, t_0, \psi) \to 0$ as $t \to \infty$. By condition (12), it is very easy to show that there exists a solution $x \in X_\psi$ of (1) with $x(t, t_0, \psi) \to 0$ as $t \to \infty$.

Conversely, we suppose that (29) fails. From (28) there exists a sequence $\{t_n\}$ with $t_n \to \infty$ as $n \to \infty$ such that $\lim_{n\to\infty} \int_{t_0}^{t_n} h(u)du = \xi$ for some $\xi \in \mathbb{R}^+$. We may also choose a positive constant J satisfying

$$-J \le \int_{t_0}^{t_n} h(u)du \le +J,$$

for all $n \ge 1$. To simplify the expression, we define

$$\omega(s) := \left| -\overline{\mu}(s) + \left(h(s - \tau_1(s)) - \frac{p'(s - \tau_1(s))}{p(s - \tau_1(s))} \right) (1 - \tau_1'(s)) - \overline{\beta}(s) \right|$$
$$+ |h(s)| \int_{s-\tau_1(s)}^{s} \left| h(u) - \frac{p'(u)}{p(u)} \right| du + \left| b(s) \frac{p^\sigma(s - \tau_2(s))}{p(s)} \right|,$$

for all $s \ge 0$. By (21), we have

$$\int_{t_0}^{t_n} e^{-\int_s^{t_n} h(u)du} \omega(s) ds \le \alpha.$$

This yields

$$\int_{t_0}^{t_n} e^{\int_0^s h(u)du} \omega(s) ds \le \alpha e^{\int_0^{t_n} h(u)du} \le e^J.$$

The sequence $\left\{ \int_{t_0}^{t_n} e^{\int_0^s h(u)du} \omega(s) ds \right\}$ is bounded, hence there exists a convergent subsequence. Without loss of generality, we can assume that

$$\lim_{n\to\infty} \int_{t_0}^{t_n} e^{\int_0^s h(u)du} \omega(s) ds = \theta,$$

for some $\theta \in \mathbb{R}^+$. Let m be an integer such that

$$\int_{t_m}^{t_n} e^{\int_0^s h(u)du}\omega(s)ds \le \frac{\delta_0}{4K},$$

for all $n \ge m$, where $\delta_0 > 0$ satisfies $2\delta_0 Ke^J + \alpha \le 1$.

We now consider the solution $y(t) = y(t, t_m, \psi)$ of (1) with $\psi(t_m) = \delta_0$ and $|\psi(s)| \le \delta_0$ for $s \le t_m$. We may choose ψ so that $|y(t)| \le 1$ for $t \ge t_m$ and

$$\begin{aligned}\psi(t_m) &- \frac{p(t_m - \tau_1(t_m))}{p(t_m)}\frac{c(t_m)}{(1-\tau_1'(t_m))}\psi(t_m - \tau_1(t_m))\\ &- \int_{t_m-\tau_1(t_m)}^{t_m}\left(h(u) - \frac{p'(u)}{p(u)}\right)z(u)du \ge \frac{1}{2}\delta_0.\end{aligned}$$

In follows from (22) and (23) with $y(t) = (\mathcal{A}y)(t) + (\mathcal{B}y)(t)$ that for $n \ge m$

$$\begin{aligned}&\left|y(t_n) - \frac{p(t_n - \tau_1(t_n))}{p(t_n)}\frac{c(t_n)}{(1-\tau_1'(t_n))}y(t_n - \tau_1(t_n)) - \int_{t_n-\tau_1(t_n)}^{t_n}\left(h(s) - \frac{p'(s)}{p(s)}\right)y(s)ds\right|\\ &\ge \frac{1}{2}\delta_0 e^{-\int_{t_m}^{t_n}h(u)du} - \int_{t_m}^{t_n} e^{-\int_s^{t_n}h(u)du}\omega(s)ds\\ &= e^{-\int_{t_m}^{t_n}h(u)du}\left(\frac{1}{2}\delta_0 - e^{-\int_0^{t_m}h(u)du}\int_{t_m}^{t_n} e^{\int_0^s h(u)du}\omega(s)ds\right)\\ &\ge e^{-\int_{t_m}^{t_n}h(u)du}\left(\frac{1}{2}\delta_0 - K\int_{t_m}^{t_n} e^{\int_0^s h(u)du}\omega(s)ds\right)\\ &\ge \frac{1}{4}\delta_0 e^{-\int_{t_m}^{t_n}h(u)du} \ge \frac{1}{4}\delta_0 e^{-2J} > 0.\end{aligned} \tag{32}$$

On the other hand, if the zero solution of (13) $y(t) = y(t, t_m, \psi) \to 0$ as $t \to \infty$, since $t_n - \tau_i(t_n) \to \infty$ as $t \to \infty$, $i = 1, 2,$ and (21) holds, we have

$$y(t_n) - \frac{p(t_n - \tau_1(t_n))}{p(t_n)}\frac{c(t_n)}{(1-\tau_1'(t_n))}y(t_n - \tau_1(t_n)) - \int_{t_n-\tau_1(t_n)}^{t_n}\left(h(s) - \frac{p'(s)}{p(s)}\right)y(s)ds \to 0$$

as $t \to \infty$, which contradicts (32). Hence condition (29) is necessary for the asymptotic stability of the zero solution of (13), and hence the zero solution of (1) is asymptotically stable. The proof is complete.

For the special case $c(t) = 0$ and $\sigma = \frac{1}{3}$, we can get.

Corollary 3.2. *Let* (19), (20) *and* (27) *hold and* (21) *be replaced by*

$$\begin{aligned}&\int_{t-\tau_1(t)}^{t}\left|h(u) - \frac{p'(u)}{p(u)}\right|du\\ &+ \int_{t_0}^{t} e^{-\int_s^t h(u)du}\left\{\left|\left(h(s-\tau_1(s)) - \frac{p'(s-\tau_1(s))}{p(s-\tau_1(s))}\right)(1-\tau_1'(s)) - a(s)\frac{p(s-\tau(s))}{p(s)}\right|\right\}ds\\ &+ \int_{t_0}^{t} e^{-\int_s^t h(u)du}|h(s)|\left(\int_{s-\tau_1(s)}^{s}\left|h(u) - \frac{p'(u)}{p(u)}\right|du\right)ds \le \alpha.\end{aligned}$$

Then the zero solution $x(t, t_0, \psi)$ of (4) *with a small continuous function $\psi(t)$ is asymptotically stable if only if*

$$\int_{t_0}^{t} h(s)ds \to \infty \text{ as } t \to \infty.$$

Remark 3.3. The method in this paper can be applied to more general nonlinear neutral differential equations than Eq. (1).

Remark 3.4. Theorem 3.1 is still true if condition (21) is satisfied for $t \geq t_\rho$ with some $t_\rho \in \mathbb{R}^+$.

4. Example

In this section, we now give an example to show the applicability of Theorem 3.1.

Example. Let us consider the following neutral differential equation of first order with two variable delays, which is a special case of (1):

$$x'(t) = -a(t)x(t-\tau_1(t)) + \ln\left(\frac{0.95t+1}{4(t+1)}\right)x'(t-\tau_1(t)) + \frac{0.6(0.95t+1)^{\frac{1}{3}}}{(t+1)^2}x^{\frac{1}{3}}(t-\tau_2(t)), \tag{33}$$

for $t \geq 0$ where $\tau_2(t) = 0.5t,\ \tau_1(t) = 0.05t,$ and $a(t)$ satisfies

$$\left|-\overline{\mu}(t) + \left(h(t-\tau_1(t)) - \frac{p'(t-\tau_1(t))}{p(t-\tau_1(t))}\right)(1-\tau_1'(t)) - \overline{\beta}(t)\right| \leq \frac{0.03}{t+1},$$

where $\overline{\mu}(t)$ and $\overline{\beta}(t)$ are defined in (15) and (16), respectively. Choosing $h(t) = \frac{1.5}{t+1}$ and $p(t) = \frac{1}{t+1}$. By straightforward computations, we can check that condition (21) in Theorem 3.1 holds true. As $t \to \infty,$ we have

$$\begin{aligned}
&\left|\frac{p(t-\tau_1(t))}{p(t)}\frac{c(t)}{1-\tau_1'(t)}\right| \leq \left|\frac{1}{4\times 0.95}\right| \leq 0.263,\\
&\int_{t-\tau_1(t)}^{t}\left|h(u) - \frac{p'(u)}{p(u)}\right|du \leq 0.026,\\
&\int_0^t e^{-\int_s^t h(u)du}|h(s)|\left(\int_{s-\tau_1(s)}^{s}\left|h(u) - \frac{p'(u)}{p(u)}\right|du\right)ds \leq 0.026,\\
&\int_0^t e^{-\int_s^t h(u)du}\left|-\overline{\mu}(s) + \left(h(s-\tau_1(s)) - \frac{p'(s-\tau_1(s))}{p(s-\tau_1(s))}\right)(1-\tau_1'(s)) - \overline{\beta}(s)\right|ds\\
&\leq \int_0^t e^{-\int_s^t \frac{1.5}{u+1}du}\frac{0.3}{s+1}ds \leq 0.2,
\end{aligned}$$

and

$\int_0^t e^{-\int_s^t h(u)du}|b(s)|\left|\frac{p^\sigma(s-\tau_2(s))}{p(s)}\right|ds \leq 0.4,$ and since $\int_0^t h(s)ds \to \infty$ as $t \to \infty,\ p(t) \leq 1$. Let $\alpha = 0.263 + 0.026 + 0.026 + 0.2 + 0.4$. It is easy to see that all the conditions of Theorem 3.1 hold for $\alpha \simeq 0.915 < 1$. Thus, Theorem 3.1 implies that the zero solution of (33) is asymptotic stable.

However, for the asymptotic stable of the zero solution of (33), the corresponding conditions used by the fixed point theory in Ardjouni and Djoudi [6] are

$$\lim \left|\frac{c(t)}{1-\tau_1'(t)}\right| = \lim \left|\frac{1}{0.95}\ln\left(\frac{0.95t+1}{4(t+1)}\right)\right| = 1.513 \text{as} t \to \infty.$$

This implies that condition (11) does not hold. So it is clear that the reduction of the conservatism by our method is quite significant when compared to Ardjouni and Djoudi [6].

Remark 4.1. It is an open problem whether the zero solution of (1) is uniform asymptotically stable, perseverance, and so on.

5. Conclusion

This work is a new attempt at applying the fixed point theory to the stability analysis of neutral differential equations with variable delays, several special cases of which have been studied in [9, 14]. Some of the results, like Theorem B, is mainly dependent on the constraint

$$\left|\frac{c(t)}{1-\tau_1'(t)}\right| < 1.$$

But in many environments, the constraint does not hold. So by employing two auxiliary continuous functions g and p to define an appropriate mapping, and present new criteria for asymptotic stability of Eq. (1) which makes stability conditions more feasible and the results in [14] are improved and generalized. From what has been discussed above, we see that Krasnoselskii's fixed point theorem is effective for not only the investigation of the existence of solution but also for the boundedness and the stability analysis of trivial equilibrium. We introduce an example to verify the applicability of the results established. In the future, we will continue to explore the application of other kinds of fixed point theorems to the stability research of fractional neutral systems with variable delays.

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References

[1] T. A. Burton, Stability by fixed point theory or Liapunov's theory: A comparison, Fixed Point Theory 4 (2003), 15–32.

[2] T. A. Burton, Liapunov functionals, fixed points, and stability by Krasnoselskii's theorem, Nonlinear Studies 9 (2001), 181–190.

[3] T. A. Burton, Stability by fixed point theory for functional differential equations, Dover Publications, New York, (2006).

[4] Ardjouni A, Djoudi A. Fixed points and stability in linear neutral differential equations with variable delays. Nonlinear Analysis. 2011;**74**:2062-2070

[5] A. Ardjouni, A. Djoudi; Stability by Krasnoselskii fixed point theorem for neutral differential equations with variable delays, Acta Universitatis Apulensis Mathematica, (2012), no. 32, 17–30.

[6] A. Ardjouni, A. Djoudi; Fixed points and stability in neutral nonlinear differential equations with variable delays, Opuscula Mathematica 32 (2012), no. 1, 5–19.

[7] M. Ad var, H. C. Koyuncuoğlu and Y. N. Raffoul, Existence of periodic solutions in shifts $\delta\pm$ for neutral nonlinear dynamic systems on time scales, Appl. Math. Comput. 242 (2014), 328–339.

[8] S. Althubiti, H.A. Makhzoum, Y. N. Raffoul, Periodic solution and stability in nonlinear neutral system with infinite delay, Applied Mathematical Sciences, Vol. 7, (2013), no. 136, 6749–6764.

[9] T. A. Burton, T. Furumochi, Asymptotic behavior of solutions of functional differential equations by fixed point theorems, Dynam. Systems Appl. 11 (2002), 499–519.

[10] T. A. Burton, T. Furumochi, Krasnoselskii's fixed point theorem and stability, Nonlinear Anal. 49 (2002), 445–454.

[11] G. Chen, D. Li, O. Gaans, S. Lunel, Stability of nonlinear neutral delay differential equations with variable delays, Electronic Journal of Differential Equations, Vol. 2017 (2017), no. 118, 1–14.

[12] Y. M. Dib, M. R. Maroun, Y. N. Raffoul, Periodicity a stability in neutral nonlinear differential equations with functional delay. Electronic Journal of Differential Equations, Vol. 2005 (2005), no. 142, 1072–6691.

[13] M. N. Islam and Y. N. Raffoul, Periodic solutions of neutral nonlinear system of differentiale quations with functional delay, J. Math. Anal. Appl, 331 (2007), 1175–1186.

[14] C. H. Jin, J.W. Luo, Stability in functional differential equations established using fixed point theory, Nonlinear Anal. 68 (2008), 3307–3315.

[15] E. R. Kaufman and Y. N. Raffoul, Periodic solutions for a neutral nonlinear dynamical equation on a time scale, J. Math. Anal. Appl, 319 (2006), 315–325.

[16] J. W. Luo, Fixed points and stability of neutral stochastic delay differential equations, J. Math. Anal. Appl., 334 (2007), 431–440.

[17] J. W. Luo and T. Taniguchi, Fixed points and stability of stochastic neutral partial differential equations with infinite delays, Stoch. Anal. Appl., 27 (2009), 1163–1173.

[18] B. Lisena, Global attractivity in nonautonomous logistic equations with delay. Nonlinear Analysis: RWA, 9 (2008), 53–63.

[19] Y. N. Raffoul, Existence of positive periodic solutions in neutral nonlinear equations with functional delay. Rocky Mountain J. Math. 42 (2012), no. 6, 1983–1993.

[20] D. Zhao, New criteria for stability of neutral differential equations with variable delays by fixed points method, Zhao Advances in Difference Equations (2011), 2011:48.

[21] D. Zhao, The boundedness and stability of neutral delay differential equations with damped stochastic perturbations via fixed point theory, Applied Mathematics and Computation 219 (2013), 6792–6803.

[22] B. Zhang, Fixed points and stability in differential equations with variable delays, Nonlinear Anal. 63 (2005) e233–e242.

[23] Y. Guo, X. Chao, W. Jun, Stability analysis of neutral stochastic delay di ¤erential equations by a generalization of Banach.s contraction principle. International Journal of Control, 90 (2017), no. 8, 1555-1560.

[24] R. K. Brayton, Bifurcation of periodic solutions in a nonlinear difference-differential equation of neutral type, Quart. Appl. Math. 24 (1966), 215–224.

[25] J. K. Hale and S.M.V. Lunel , Introduction to functional differential equations, Ser. Applied Mathematical Sciences. New York: Springer-Verlag, Vol. 99, (1993).

[26] J. K. Hale and K.R. Meyer, A class of functional equations of neutral type, Mem. Am. Math. Soc. 76(1967), 1–65.

[27] Y. Kuang, Delay Differential Equations with Applications in Population Dynamics, Academic Press, San Diego, (1993).

[28] V. B. Kolmanovskii, A.D. Myshkis, Applied Theory of Functional Differential Equations, Kluwer Academic, Dordrecht, 1992.

[29] M. Liu, I. Dassios, and F. Milano, "On the stability analysis of systems of neutral delay differential equations, " Circuits, Systems, and Signal Processing, Vol. 38, (2019), Issue 4, 1639–1653.

[30] Y. N. Raffoul, Stability in neutral nonlinear differential equations with functional delays using fixed-point theory, Math. Comput. Modelling 40 (2004), 691–700.

[31] D. R. Smart, Fixed points theorems. Cambridge University Press, Cambridge, (1980).