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Christophe Golé
Smith College, cgole@smith.edu

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CHRISTOPHE GOLÉ*

Abstract. Completely ordered invariant circles are found for the gradiant of the energy flow in the state space, containing the critical sets corresponding to the Birkhoff orbits of all rotation number. In particular, these ghost circles contain the Aubry-Mather sets and map-invariant circles as completely critical sets when these exist. We give a criterion for a sequence of rational ghost circles to converge to a completely critical one.

Key words. twist maps, ghost circles, Aubry-Mather sets

1. Introduction. We continue the study of invariant sets for the gradiant flow attached to twist map and their higher dimensional analogs that are called monotone maps in [Go 1,2]. These are maps of $\mathbf{T}^n \times \mathbf{R}^n$ that are exact symplectic and come equipped with a discrete variational problem, via a generating function (Aubry)([A-L], [B-K], [Go 1,2]). In [Go 1,2], we proved the existence of invariant sets for the energy flow that we called *ghost tori* because their cohomology contains the one of a torus and that in the completely integrable maps, they correspond to the map-invariant tori. This enabled us to prove a generalisation of the Poincaré-Birkhoff theorem, i.e. the existence of a topological lower bound on the the number of periodic orbits of a given rotation vector.

The results in this paper, with the exception of the appendix, only concern twist maps (i.e. $n = 1$). We hope some will generalise to monotone maps. Making use of the fact that, in the twist map case, the gradient flow of the energy is a monotone flow (with respect to the partial order on sequences), we construct, in spaces of sequences of any given rotation number ω, an invariant circle for that *flow* (Theorem 3.5). More precisely, given any Aubry-Mather set (as embedded in the space of sequences as a critical set), we can fill its gaps with monotone orbits. This construction uses a result of Matano [Mto], recently generalised by E.N Dancer and P.Hess [D-H] on monotone flows.

This result answers positively , in the case $n = 1$, to a conjecture we made in [Go 1,2] about the existence of ghost tori of all rotation vector.

These circles , that we call ghost circles, have the remarkable property of being completely ordered with respect to the partial order on sequences, i.e. they lay inside a field of cones in the Banach space of sequences. Projected (diffeomorphically) onto the two first components, which are identified with the annulus, this gives a uniform Lipshitz condition on these circles (Proposition 4.20). In the annulus, ghost circles appear as graphs bridging the gaps between elements in a " saturated" Aubry-Mather set: for example a minimal Aubry-Mather set together with its homoclinic orbits, or, trivially, an invariant circle (for the map) when there is one. They have the property that they intersect their own image

*Department of Mathematics and IMA, University of Minnesota, MN55455

under the map only at critical points. This makes it easy to give a formula for the flux through these circles (see 4.25).

If we also assumes that the energy is a Morse function on all the spaces of p/q sequences, (Property 2.17) the rational ghost circles can be built (using results by

Angenent [An]) to contain the minimum of the energy for the p/q-sequences, as well as the minimax (Theorem 3.6) and their (flow) unstable manifold. The resulting ghost circle are C^1. By a theorem of Robinson [R], 2.17 is shown (appendix) to be a generic condition, in all dimensions.

Both ghost circles and Aubry-Mather sets fall into a larger class of sets that we call σ-Aubry-Mather sets (abreviated σAM). These are subsets of $\mathbb{R}^{\mathbb{Z}}$ that are closed, completely ordered, shift and integer translation invariant. Section 4 is devoted to the study of the σAM sets.

The ghost circles and are connected, *flow invariant* σAM sets which we show must be homeomorphic to circles (Theorem 4.6). The Aubry-Mather sets are in one to one correspondance with the σAM sets that are completely critical for the energy flow. The rotation number is a continuous function on the space of σAM sets (Theorem 4.6) and therefore also on ghost circles.

As an application, we give a criterion for a limit of rational ghost circles to converge to a completely critical one, i.e. corresponding to a map invariant circle (Theorem 5.2). This criterion is very related to the ΔW criterion of Mather and Katok [Ma 3], [K].

We hope that this more detailed study of the dimension two case will give some insight as to what might (or should not) be expected for ghost tori in higher dimensions. Finally, we hope that the flux formula for ghost circles (4.25) together with their good limiting properties could be helpful in giving a sharp estimate for the rate of transport in a region of unstability.

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1. Monotone maps and the energy flow.

In this section we introduce the energy flow in the general context of monotone maps.

We let $\mathbb{T}^n$ be $\mathbb{R}^n/\mathbb{Z}^n$ and consider the space $\mathbb{A}^n := \tilde{\mathbb{T}}^n \times \mathbb{R}^n \cong \mathbb{R}^n \times \mathbb{R}^n$ with (global) coordinates (x,y). The group $\mathbb{Z}^n$ acts as deck transformations by

$$T_m(x,y) = (x+m, y), \quad m \in \mathbb{Z}^n.$$

$\mathbb{A}^n$ is endowed with the canonical symplectic form

$$\Omega = \sum_{k=1}^{n} dx_k \wedge dy_k = d\alpha, \text{ where } \alpha = \sum_{k=1}^{n} y_k dx_k.$$

A diffeomorphism F of $\mathbb{A}^n$ is *symplectic* if

$$F^*\Omega = \Omega, \tag{1.1}$$

and *exact symplectic* if $F^*\alpha - \alpha$ is exact, that is if:

$$F^*\alpha - \alpha = dS, \tag{1.2}$$

where S is a C^2 real valued function on $\mathbb{A}^n$. Note that some authors use "exact symplectic" for maps that are homologous to the identity, i.e., time one maps of (time dependant) Hamiltonian flows (what E.Zehnder now calls hamiltonian maps). Whereas for the case $n = 1$ (twist maps) it is true that the maps we consider can be suspended by a hamiltonian flow ([Mo]), it seems to be an open question whether this holds for the monotone maps we define below.

In the following, we write $F(x,y) = (X,Y)$.

DEFINITION 1.3 (MONOTONE MAPS, GENERATING FUNCTIONS).

A diffeomorphism F of $\mathbb{A}^n$ is called a*monotone map* if:

(1) F is exact symplectic: $F^*\alpha - \alpha = dS, \quad S : \mathbb{A}^n \to \mathbb{R}$

(2) F is a lift: $F \circ T_m = T_m \circ F$

(3) For each x_0,the map $(x_0, y) \to (x_0, X)$ is a diffeomorphism of $\mathbb{R}^n$ (and hence $(x,y) \to (x,X)$ is a diffeomorphism of $\mathbb{R}^{2n}$)

The function S of (1.2) is called the *generating function* of the monotone map, thought of as a function of (x, X).

That $(x, X) \to (x, y)$ is a diffeomorphism implies the nondegeneracy condition:

$$det\partial_1\partial_2 S(x, X) \neq 0 \tag{1.4}$$

which generalizes the so-called twist condition (∂_1 (resp. ∂_2) means the partial derivative with respect to the first (resp. the second) component).

In the (x,X) coordinates, the $\mathbb{Z}^n$-action is given by $T_m(x,X) = (x+m, X+m)$. Because of (2) in the definition, S is a lift of a function on $\mathbb{T}^n \times \mathbb{R}^n$, that is, it satisfies the following periodicity condition:

$$S \circ T_m(x,X) = S(x+m, X+m) = S(x,X), \quad \text{for all } \ m \in \mathbb{Z}^n \tag{1.5}$$

When $n = 1$, we have:

DEFINITION 1.6 (TWIST MAPS).

A diffeomorphism f of $\mathbb{A}$ is called an area preserving monotone twist map, or, in short (in this paper), twist map if it is monotone and if the scalar $-\partial_1\partial_2 S(x,X)$ is strictly positive for all (x,X).

REMARK 1.7. Definition 1.3 is slightly different from the one that we used previously [Go 1,2] where we put condition 1.4 in the definition and found certain extra conditions on S under which the map $(x,X) \rightarrow (x,y)$ is a diffeomorphism. The term monotone was introduced by Herman ([He 2]) in a more general setting : his monotone maps do not necessarily have a generating function. Some authors impose another condition on S, i.e. that $-\partial_1\partial_2 S$ be positive definite. They call these maps symplectic twist maps ([K-M]).

Definition 1.6, on the other hand, is equivalent to the usual one: f is an area preserving map with zero flux which sends any vertical line $x = c$ into a graph over the x-axis satisfying $f(c, +\infty) = (+\infty, +\infty)$ and $f(c, -\infty) = (-\infty, -\infty)$.

The function S generates a monotone F in the following (classical mechanic) sense:

$$\begin{aligned} y &= -\partial_1 S(x,X) \\ Y &= \partial_2 S(x,X) \end{aligned} \tag{1.8}$$

Let $z_k = F^k(z_0) = (x_k, y_k)$. The orbit $\{z_k\}$ is completely determined by the sequence x_k of $(\mathbb{R})^{\mathbb{Z}}$. Indeed, from (1.8), we deduce:

$$y_k = -\partial_1 S(x_k, x_{k+1}) = \partial_2 S(x_{k-1}, x_k) \tag{1.9}$$

This can be written:

$$\partial_1 S(x_k, x_{k+1}) + \partial_2 S(x_{k-1}, x_k) = 0 \tag{1.10}$$

Equation 1.10 can be formally interpreted as:

$$\begin{aligned} &\nabla W(\mathbf{x}) = 0, \quad \text{for} \\ &W(\mathbf{x}) = \sum_{-\infty}^{+\infty} S(x_k, x_{k+1}) \ \text{and} \ \ \mathbf{x} \in (\mathbb{R}^n)^{\mathbb{Z}}. \end{aligned} \tag{1.11}$$

One can think of the above construction as a discrete version of the classical one: the map $(x,X) \to (x,X)$ is the analog to the Legendre transformation ($X-x$ is the discretised velocity) and equation (1.11) is a formulation of the "least action principle".

Of course, W is not well defined, since the sum is not convergent. However, "∇W" is well defined and generates a flow on $(\mathbb{R}^n)^{\mathbb{Z}}$ that we call the energy flow.

More precisely when S is a C^p function, the infinite system of O.D.E's:

$$-(\nabla W(\mathbf{x}))_k = \dot{x}_k = -[\partial_1 S(x_k, x_{k+1}) + \partial_2 S(x_{k-1}, x_k)] \tag{1.12}$$

defines a C^p local flow ζ^t on $(\mathbb{R}^n)^Z$ (with the usual product topology) whose critical points are in one to one correspondance with the orbits of the map. This flow can also be made global by assuming relevant boundary conditions (e.g. that the map be completely integrable outside a bounded strip, or at infinity).

The flow in the case $n=1$ has the striking property that it is order preserving or *monotone* in the space of sequences.

We want to recall a few facts about monotone flows (see [Hi]).

Let X be a Banach space. A *partial order* on X is given by a convex, closed cone V_+ such that, if we denote by $-V_+ = V_-$, we have: $V_- \cap V_+ = \{0\}$. Here, we will also assume V_+ has non empty interior $int\ V_+$. For x, y in X, we define :

$$\begin{aligned} x \leq y \quad &\Leftrightarrow \quad y \in x + V_+ \overset{\text{def}}{=} V_+(x) \\ x < y &\Leftrightarrow x \leq y \text{ and } x \neq y \\ x \ll y \quad &\Leftrightarrow \quad y \in x + int\ V_+ \\ [x,y] &= \{z \in X \quad | \quad x < z < y\} \\ [[x,y]] &= \{z \in X \quad | \quad x \ll z \ll y\} \end{aligned} \tag{1.13}$$

DEFINITION 1.14. A map $A : X \to A$ is called *monotone* if:

$$x < y \Rightarrow Ax < Ay$$

It is *strongly monotone* if:

$$x < y \Rightarrow Ax \ll Ay.$$

A flow ζ^t in X is (strongly) monotone if for all positive t, the map ζ^t is (strongly) monotone.

$\mathbb{R}^{\mathbb{Z}}$ is endowed with the natural partial order on sequences defined by the following: Let $\mathbf{x}$ and $\mathbf{x}'$ be elements of $\mathbb{R}^{\mathbb{Z}}$, with kth terms denoted by x_k and x'_k. Then:

$$V_+ := \{\mathbf{y} \in \mathbb{R}^{\mathbb{Z}} \quad | \quad y_k \geq 0, \forall k \in \mathbb{Z}\}. \tag{1.15}$$

That is :

$$(1.16)\qquad \mathbf{x}\le\mathbf{x}' \Leftrightarrow \quad x_k\le x'_k,\quad \text{all } k,$$

and similarily for $<$ and $\ll$.

We will sometimes restrain ourselves to some finite dimensional subspaces of $\mathbb{R}^{\mathbb{Z}}$ defined by:

$$(1.17)\qquad \begin{aligned} X_{p,q} &:= \{\mathbf{x}\in(\mathbb{R}^n)^{\mathbb{Z}} \mid x_{k+q}=x_k+p\}\\ &\cong \{(x_0,\dots,x_{q-1})\in(\mathbb{R}^n)^q\}\end{aligned}$$

It is easy to see that, the difference $\dot{x}_{k+q}-\dot{x}_k$ for a sequence $\mathbf{x}$ in $X_{p,q}$ must be zero, by periodicity of S. Hence $X_{p,q}$ is invariant under the flow ζ^t.

The order in $X_{p,q}$ is induced by the one on $\mathbb{R}^{\mathbb{Z}}$. The positive cone is just the positive quadrant of $\mathbb{R}^q\cong X_{p,q}$. One has to be carefull however, with the use of $\ll$: in $\mathbb{R}^{\mathbb{Z}}$, *int* V_+ does not contain (strictly) positive sequences tending to 0, whereas in $X_{p,q}$, *int* V_+ contains all (strictly) positive sequences. To remedy this discrepancy, we also use the notation:

$$(1.18)\qquad V_{++}:=\{\mathbf{y}\in\mathbb{R}^{\mathbb{Z}} \mid y_k>0,\forall k\in\mathbb{Z}\},$$

$$\mathbf{x}\prec\mathbf{x}'\Leftrightarrow\mathbf{x}'\in\mathbf{x}+V_{++}$$

and we will say that a map $A:\mathbb{R}^{\mathbb{Z}}\to\mathbb{R}^{\mathbb{Z}}$ is *strictly monotone* when

$$(1.19)\qquad \mathbf{x}<\mathbf{x}'\Rightarrow A\mathbf{x}\prec A\mathbf{x}'.$$

It is clear that in $X_{p,q}$ the notion of strong monotonicity and strict monotonicity are equivalent. We will also use the sets, both in $\mathbb{R}^{\mathbb{Z}}$ and in $X_{p,q}$:

$$(1.21)\qquad \begin{aligned} [\mathbf{x},\mathbf{x}'] &:= \{\mathbf{y}\in\mathbb{R}^{\mathbb{Z}}\ \ (resp.\ X_{p,q}) \mid \mathbf{x}\le\mathbf{y}\le\mathbf{x}'\}\\ [[\mathbf{x},\mathbf{x}']] &:= \{\mathbf{y}\in\mathbb{R}^{\mathbb{Z}}\ \ (resp.\ X_{p,q}) \mid \mathbf{x}\ll\mathbf{y}\ll\mathbf{x}'\}\end{aligned}$$

which are called, respectively a *closed order interval*, an *open order interval.* and

We now state an important lemma, proved by Angenent [An].

LEMMA 1.22. *In the case $n=1$ (twist maps), assuming that the second derivative of the function S is bounded, the flow ζ^t defined by the system (1.12) is strictly monotone with respect to the natural partial order on sequences in $\mathbb{R}^{\mathbb{Z}}$ (and therefore strongly monotone in $X_{p,q}$).*

proof. We let the reader show that if the operator solution of the linearised equation:

$$(1.23)\qquad \dot{u}=-HessW(\mathbf{x}(t))u(t)$$

is strictly positive, then the flow is strictly monotone. One can check that $-HessW(\mathbf{x}(t))$ is an infinite tridiagonal matrix with positive off diagonal terms $-\partial_1\partial_2 S(x_k, x_{k+1})$. The diagonal terms $\partial_1\partial_1 S(x_k, x_{k+1}) + \partial_2\partial_2 S(x_{k-1}, x_k)$ are uniformaly bounded by our assumption. Hence, we can find a positive λ such that:

$$B(t) = -HessW(\mathbf{x}(t)) + \lambda Id$$

is a positive matrix with strictly positive off diagonal terms (λ does not depend on the particular orbit $\mathbf{x}(t)$). If $u(t)$ is solution of the equation 1.23 then $e^{\lambda t}u(t)$ is solution of :

$$\dot{v}(t) = B(t)v(t), \tag{1.24}$$

and hence the strict positivity of the solution operator for 1.24 is equivalent to that of 1.23. Looking at the integral equation:

$$v(t) = v(0) + \int_0^t B(s)v(s)ds, \tag{1.25}$$

one sees that Picard's iteration will give positive solutions for positive vector $v(0)$. This will imply, assuming that $v_k(0) > 0, v_l(0) \geq 0, for l \neq k$:

$$v_{k+1}(t) \geq v_{k+1}(0) + \int_0^t B_{k,k+1}(s)v_k(s)ds > 0 \tag{1.26}$$

The same holding for v_{k-1}. By induction, $v(t) \succ \mathbf{0}$ and the operator solution is strictly positive. □

EXAMPLES 1.27. The completely integrable map

$$f_0(x, y) = (x + y, y)$$

has the generating function $S_0(x, x') = \frac{1}{2}(x' - x)$ and its energy flow is given by the infinite system of O.D.E.'s:

$$\dot{x}_k = -2x_k + x_{k-1} + x_{k+1},$$

i.e. a discretised heat equation.

One can add a "potential" to S_0 : let

$$S(x, x') = \frac{1}{2}(x, x') - g(x).$$

Then S generates $F(x, y) = (x + y + h(x), y + h(x))$ where $h(x) = g'(x)$. The associated flow is given by:

$$\dot{x}_k = -2x_k + x_{k-1} + x_{k+1} + h(x_k).$$

Angenent remarked that this was the discretisation of $x_s = x_{tt} + h(t, x)$ and proved important results for twist maps by developping this analogy. It was in this optic that he looked at ζ^t as a monotone flow.

2. The Poincaré-Birkhoff theorem.

One of the dynamical features on which our work relies is:

DEFINITION 2.1 ((P,Q)-PERIODIC POINTS). Let (p,q) be an element of $\mathbb{Z}^n \times \mathbb{Z}$. (x_0, y_0) in $\mathbb{A}^n$ is a (p,q)-periodic point ((p,q)-point, in short) if, when we denote by $(x_k, y_k) = F^k(x_0, y_0)$, the following holds:

$$x_{k+q} = x_k + p \tag{2.2}$$

The orbit of a (p,q)-point is called a (p,q)-*orbit*.

From now on, we will assume that (p,q) are relatively prime, i.e. that q is prime with at least one of the components of p. To look for such orbits, one restricts the flow ζ^t to the set of periodic sequences defined in 1.13 in the twist map case:

$$\begin{aligned} X_{p,q}(\mathbb{R}^n) = X_{p,q} := &\{\mathbf{x} \in (\mathbb{R}^n)^{\mathbb{Z}} \quad | \quad x_{k+q} = x_k + p\} \\ &\cong \{(x_0, \dots, x_{q-1}) \in (\mathbb{R}^n)^q\} \end{aligned}$$

Because of the periodicity of S, it is easy to see that $X_{p,q}$ is invariant under ζ^t. Moreover ζ^t is in fact the gradiant flow of the well defined function:

$$W_{p,q}(\mathbf{x}) = \sum_{k=0}^{q-1} S(x_k, x_{k+1}) \tag{2.3}$$

with respect to the coordinates $(x_0, \dots, x_{q-1})$ of $X_{p,q}$ (with the convention that $x_q = x_0 + p$).

Hence critical points of ζ^t in $X_{p,q}$ correspond to (p,q)-points.

When $n = 1$, one is interested by orbits that are ordered like those of a rigid rotation by p/q.

DEFINITION 2.4 (WELL ORDERED, FINITE DIMENSIONAL CASE). A sequence $\mathbf{x}$ in $X_{p,q}(\mathbb{R})$ is called well-ordered if and only if

$$x_{k+u} + n \geq x_k \tag{2.5}$$

Where $(u, n) \in \mathbb{Z}^2$ is uniquely defined by the number theoretic condition:

$$\begin{aligned} &up + nq = 1 \\ &0 \leq u < q \end{aligned} \tag{2.6}$$

The set of all well-ordered sequences in $X_{p,q}$ is denoted by $WO_{p,q}$. A (p,q)-orbit (x_k, y_k) is called well-ordered if and only if $\{x_k\}_{k\in\mathbb{Z}}$ is in $WO_{p,q}$.

Geometrically, we are looking at the "extended sequence" of a sequence $\mathbf{x}$ in $X_{p,q}$, i.e. the set $\{x_k + m \quad | \quad (m,k) \in \mathbb{Z}^2\}$ of all integer translates of the terms of $\mathbf{x}$. We are asking that this set be ordered like the extended orbit of a rigid translation by p/q. Indeed, the nearest right hand side neighbor of a point x in the extended orbit of a translation by p/q is given by $x + u(p/q) + n = x + 1/q$. For a more general definition, holding for irrational rotation numbers as well, see 4.9.

THEOREM 2.7 (POINCARÉ-BIRKHOFF ETC...). *Let f be a twist map ($n = 1$). Then f has at least two distinct (p,q)-orbits. Furthermore, these orbits may be chosen to be well-ordered.*

COMMENTS 2.8.

The first statement of the theorem is a version of the well known Poincaré-Birkhoff theorem which states the existence of at least two (p,q)-orbits for a twist map. Birkhoff's general versions apply for maps that only satisfy a boundry twist condition ([], and Franks assumes even some less stringent recurrence hypothesis ([10]). See also [G-H] where Poincaré's original (unfinished) argument is carried through. In our thesis ([Go 1,2]), we prove an analogue of this theorem in the context of monotone maps. The loss of the monotonicity of the flow ζ^t as one passes to higher dimension is compensated by a boundary condition (at infinity) on the map.

Since we will be using some of the ideas it contains, we include a proof of the theorem in the case of twist maps. This proof is essentially due to Aubry and Katok ([A-L], [K]): it yields automatically the second statement of the theorem, which is itself crucial when one wants to take limits of (p,q)-orbits in order to find quasiperiodic ones. We refer the reader to ([Ha],[B],[LeC]) for recent generalisations of this statement. We recast the proof in the context of monotone flows and Conley index theory.

Proof of Theorem 2.7. We show that, thanks to the strict monotonicity of ζ^t in the space $X_{p,q}$, $WO_{p,q}$ is an attractor block for that flow. We can rewrite

$$(2.9) \qquad WO_{p,q} = \{\mathbf{x} \in X_{p,q} \quad |\sigma_{pq}\mathbf{x} \gg \mathbf{x}\}, \quad \text{where } \{\sigma_{pq}\mathbf{x}\}_k \stackrel{\text{def}}{=} x_{k+u} + n$$

(translation to nearest right neighbor). But, since the energy flow is strictly monotone:

$$(2.10) \qquad \sigma_{pq}\mathbf{x} > \mathbf{x} \Rightarrow \zeta^t(\sigma_{pq}\mathbf{x}) \gg \zeta^t(\mathbf{x}), \quad \text{all } t > 0$$

which means that the compact set $WO_{p,q}$ is mapped into its interior by $\mathbf{x}$. That in itself is enough to prove the existence of a (minimum) critical point in $WO_{p,q}$, since the flow is gradiant. To find more than one critical point, we use the following fact:

LEMMA 2.11. $\underline{WO}_{p,q} := WO_{p,q}/\mathbb{Z} \cong \mathbb{S}^1 \times \Delta_{q-1}$
where Δ_{q-1} represents the unit simplex of dimension $q-1$

Proof. We can rewrite:

$$WO_{p,q} = \{(x_0, \beta_1, \dots, \beta_q) \mid \sum_{k=1}^{q} \beta_k = 1 \quad and\ \beta_k > 0, \quad all\ k\} \tag{2.12}$$

Where the correspondance with definition 2.4 is given by $\beta_k = x_{k+u} + n - x_k$. The $\mathbb{Z}$ -action in these coordinate is given by:

$$T_m(x_0, \beta_1, \dots, \beta_q) = (x_0 + m, \beta_1, \dots, \beta_q) \tag{2.13}$$

□

To finish the proof of the theorem, one can use Conley-Zehnder Morse theory [C-Z, 2] for the isolating block $\underline{WO}_{p,q}$ and the gradiant flow induced on $\underline{WO}_{p,q}$ by the $\mathbb{Z}$-invariant gradiant flow ζ^t. Assume that $W_{p,q}$ is a Morse function (We will see in proposition 2.16 that this is a generic property in the space of monotone maps). Then the generalised Morse inequalities of Conley and Zehnder imply the existence of two critical points of *different* indices in $WO_{p,q}$. Indeed, the cohomology Conley index:

$$H^*(\underline{WO}_{p,q}, (\underline{WO}_{p,q})^-) \cong H^*(\underline{WO}_{p,q}, \emptyset) \cong H^*(\mathbb{S}^1) \tag{2.14}$$

and the morse inequalities imply that the cohomology indices of the critical points must contribute for both $H^0(\mathbb{S}^1)$ and $H^1(\mathbb{S}^1)$. But a nondegenerate critical point can only contribute to one at the time. Therefore, there must be two critical points of different indices and hence two distinct orbits. For the general case, which we only outline here (see [Go 1,2]), one has to take another quotient, namely by σ_{pq} and show that the cuplength of the invariant set contained in $\underline{WO}_{p,q}/\sigma_{pq}$ is still the one of the circle, that is 2. By an argument of Conley and Zehnder, there must be two critical points. But critical points in $\underline{WO}_{p,q}/\sigma_{pq}$ correspond to *orbits* of the map. □

An alternate proof to find more than one orbit is the following: Let $\mathbf{x}$ be the minimum of $W_{p,q}$ in $WO_{p,q}$. Then $\sigma_{pq}\mathbf{x}$ is also a minimum and $[\mathbf{x}, \sigma_{pq}\mathbf{x}]$ is positively invariant under ζ^t. A minimax argument by Mather shows that this interval must contain another critical point. We will prove and use a non degenerate version of this fact in lemma 3.7. In the above proof, we wanted to outline how the topology of $\mathbb{A}$ intervenes in the variational space.

REMARK 2.15 (MONOTONE MAPS AND GHOST TORI). The main problem to generalise this argument in higher dimensions is that there is not (as yet) any analog of $WO_{p,q}$ in $X_{p,q}(\mathbb{R}^n)$. However, in [Go 1,2] we used a boundary condition to produce an isolating block homeomorphic to the product of a torus with a disk, yielding the existence of as many critical points as the torus would have (i.e. $n+1$ and 2^n when nondegenerate). Furthermore, we showed that, when the monotone map F considered can be continued, through monotone maps, to a completely integrable F_0, the invariant set in the isolating block retains the cohomology of the one for F_0. Since there is a natural diffeomorphism between the latter and the F_0-invariant torus of rotation vector p/q, we called these invariant sets *"Ghost tori"*: they are, in the variational space the (Conley-Floer) continuations of the F_0-invariant tori of rational rotation vector. This holds, in particular, in the twist map case, where the ghost circles lay in $WO_{p,q}$. We conjectured ([Go 1,2]) that the irrational tori would also continue as invariant sets. This is what theorem 3.5 proves for the twist map case, among other things. In this paper, we will ask more of our ghost circles than just being a ζ^t- invariant set with the cohomology of a circle: they will be homeomorphic to a circle and be completely ordered (see definition 3.3).

We end this section with a statement that we will use in the rest of the paper.

PROPOSITION 2.16. *The property:*

(2.17) *"For all (p,q) in* $\mathbb{Z}^n \times \mathbb{Z}$ *the function* $W_{p,q}$ *is Morse."*

is a generic property in the set of monotone maps of $\mathbb{A}^n$.

This property is equivalent to the nondegeneracy of all (p,q)-points for the corresponding monotone map F. For a proof of the proposition, we refer the reader to the Appendix.

3. Existence of ghost circles.

The proof of theorem 2.7 suggested that the invariant set for the flow ζ^t in $\underline{WO}_{p,q}$ attached to a twist map f, might contain a circle. In this section, we construct such a circle, made of critical points and connections (Theorem 3.6). In fact we find flow invariant circle of all rotation numbers (Theorem 3.5). The idea of this construction comes from a theorem of Matano ([Mto]) on monotone flows. His theorem , which requires the flow to be strongly monotone, only fits the flow in $X_{p,q}$ (see 1.17). However, Dancer and Hess recently proved a more general version of this theorem which fits our flow in $\mathbb{R}^{\mathbb{Z}}$. We rephrase this theorem in our context:

THEOREM 3.1. ([Mto], theorem 8, [D-H],proposition 1). *Let* $\mathbf{x} < \mathbf{x}'$ *be two critical points of* ζ^t *in the space* $\mathbb{R}^{\mathbb{Z}}$. *Suppose there exists no critical point in* $[[\mathbf{x}, \mathbf{x}']]$. *Then there exists a monotone entire orbit connecting* $\mathbf{x}$ *and* $\mathbf{x}'$.

Entire means defined for all time, whereas *monotone* means monotone in time, with respect to the partial order. The hypothesis required by Dancer and Hess' result are that ζ^t be a monotone semiflow (and not necessarily strongly monotone as in [Mtno]) defined for all positive time and order compact on any ordered Banach space (i.e. the time t image of a closed order interval is compact).

In the following, we denote by:

$$\sigma : \mathbb{R}^{\mathbb{Z}} \to \mathbb{R}^{\mathbb{Z}}$$
$$\{\sigma \mathbf{x}\}_k = x_{k+1} \tag{3.2}$$

the usual shift on sequences. We will make use of the following concepts:

Definitions 3.3 (σ -Aubry-Mather sets, Ghost circles).

(1) A *σ-Aubry-Mather* set (or σAM set) is a closed, completely ordered, $\mathbb{Z}$-invariant, σ-invariant subset of $\mathbb{R}^{\mathbb{Z}}$.
(2) An Aubry-Mather set is a *completely critical* σAM set. We will say that an Aubry-Mather set is *saturated* if it is not strictly contained inside another one.
(3) A *Ghost circle* is a ζ^t-invariant, connected σAM set.

Comments 3.4.

(1) The notion of σAM set is a natural generalisation of the one of Aubry-Mather set (for the connection between our definition of Aubry-Mather sets and the classical one, see (2)) . In a sense, σAM sets are all the potential Aubry-Mather sets for all possible twist maps. σAM sets will be extensively studied in section 4. It will in particular be shown that they are homeomorphic to closed invariant sets of a circle homeomorphism (theorem 4.6).
(2) We remind the reader that the usual definition of Aubry-Mather sets is that of a closed, f-invariant, T-invariant subset E of A such that $\pi_x|_E$ is injective and $f|_E$ preserves the order in the x coordinate (see [K1] or [Che] where the latter also assumes E to be minimal, in the sense that it contains a dense orbit)

   To see that these two definitions are equivalent, it suffices to show that the maps $\mathbf{x} \in \mathbb{R}^{\mathbb{Z}} \to (x_0, x_1) \to (x_0, y_0)$ induces an order preserving homeomorphism between a set given by our definition and one given by the usual one. That comes from the fact that the Aubry-Mather sets in our definition are in fact *strictly ordered*: Let $\mathbf{x} < \mathbf{y}$ be two points in a critical σAM set. By strict monotonicity of the flow $\zeta^t \mathbf{x} \prec \zeta^t \mathbf{y}$ for any $t > 0$. But $\mathbf{x}$ and $\mathbf{y}$ are critical. Hence $\mathbf{x} \prec \mathbf{y}$.
(3) As noted in Remark 2.15, ghost circles will be part of the ζ^t-invariant set in $\mathbb{R}^{\mathbb{Z}}$ that continues from the *completely critical* ghost circles of the completely integrable map f_0 (that there is a Conley continuation could be made precise). It is natural to ask for their "trace" , as one perturbs f_0, to be only invariant for the flow. The depth of the KAM theorem is that, under certain conditions, these completely critical circles remain completely critical. We will see that ghost circles *are* circles in $\mathbb{R}^{\mathbb{Z}}/\mathbb{Z}$ (theorem 4.6) which project to graphs in the annulus. For this and other properties of ghost circles see proposition 4.20.

Theorem 3.5 (Existence of ghost circles). *Let f be any twist map of $\mathbb{S}^1 \times \mathbb{R}$. Then for all ω in $\mathbb{R}$ there is a ghost circle G_ω with intrinseque rotation number $R(G_\omega) = \omega$.*

Furthermore, any Aubry-Mather set can be embedded in a ghost circle. In particular, one can find a ghost circle passing through a minimal Aubry-Mather set and an orbit homoclinic to it.

We will see (theorem 4.6) that σ acts on a ghost circle as the lift of a circle homeomorphism, hence the notion of rotation number.

Proof. We will see that the second assertion of the theorem easily implies the first: if we find a ghost circle through an Aubry-Mather set of rotation number $\rho = \omega$ then this ghost circle has to have rotation number ω, by lemma 4.11.

Take a *saturated* Aubry-Mather set E (seen as a completely critical σAM set), i.e. one such that :

$$\mathbf{x} < \mathbf{y} \in E \text{ and } (\mathbf{x}, \mathbf{y}) \cap E = \emptyset \Rightarrow \text{ there are no critical points in } (\mathbf{x}, \mathbf{y}).$$

Let $\rho(E) = \omega$.

By theorem 4.6 (see also remark 3.4), E is homeomorphic to a closed set on the line. Consider a complementary interval with endpoints $\mathbf{x}, \mathbf{y}$. Since E is saturated, the order interval $(\mathbf{x}, \mathbf{y})$ does not contain any critical points. We will show that there must be an orbit connecting $\mathbf{x}$ and $\mathbf{y}$.

The interval $[\mathbf{x}, \mathbf{y}]$ is positively invariant under the flow, by monotonicity. Since it is compact (a Hilbert cube in the product topology), solutions there are defined for all positive time. Obviously, the flow is also order compact since closed order intervals are compact. We can therefore use Dancer and Hess' Proposition 1 (see theorem 3.1) to find an entire, monotone, connecting orbit between $\mathbf{x}$ and $\mathbf{y}$. Thus, we fill up all the gaps of E by monotone orbits, taking care to fill the gaps $[\sigma^m T^k \mathbf{x}, \sigma^m T^k \mathbf{y}]$ with the corresponding images of the orbit through $[\mathbf{x}, \mathbf{y}]$. The set obtained is a ghost circle. □

Theorem 3.6 (C^1 rational ghost circles). *Suppose that the function $W_{p,q}$ attached to a twist map is Morse (property 2.17). Then:*

1. *There exists a ghost circle in $WO_{p,q}$ containing all absolute minima of $W_{p,q}$ and made of unstable manifolds of mountain pass points between them. These unstable manifolds join C^1 at the minima.*
2. *Alternatively, one can construct a ghost circle containing the extended orbits of an absolute minimum and its minimax. This ghost circle is also made out of unstable manifolds of moutain passes.*
3. *In both cases, these ghost circles project diffeomorphically in the annulus to C^1 circles which are graphs over the x-axis and such that their images by f are also graphs. Intersections of such a circle and its image only occur at the periodic points.*

proof. Let M_{pq} be the set of minima of $W_{p,q}$ in the invariant set $WO_{p,q}$. By a lemma of Aubry (see [A-L],[Che]), M_{pq} is completely ordered. In fact it is a σAM set. (see definition 3.3). Consider all the critical points of index 0 that are ordered with respect to M_{pq}. By adding such points, and their σ and T translates, we can create new σAM sets containing M_{pq}. We (partial) order this set of σAM sets by inclusion and take a maximum one, say $\tilde{M}_{pq}$.

Chose one $\mathbf{x} \in \tilde{M}_{pq}$. Since $W_{p,q}$ is Morse,

$$[\mathbf{x}, \sigma_{pq}\mathbf{x}] \cap \tilde{M}_{pq} = 0(\mathbf{x}^0 \dots \mathbf{x}^k) \text{ with } x^0 = \mathbf{x}, \mathbf{x}^k = \sigma_{pq}\mathbf{x} \text{ and } l < j \Rightarrow \mathbf{x}^l < \mathbf{x}^j$$

Obviously, we can not hope for connecting orbits between the points of $\tilde{M}_{pq}$. However, we can go from one to the other through mountain passes. To prove this, we need:

LEMMA 3.7 (MOUNTAIN PASS FOR MONOTONE GRADIANT FLOWS). *Let $\mathbf{x} < \mathbf{x}'$ be two local minima of a Morse function $f : \mathbb{R}^n \to \mathbb{R}$, where $\mathbb{R}^n$ is endowed with the usual partial order. Suppose that the gradiant flow of f is monotone. Then $[\mathbf{x}, \mathbf{x}']$ contains a critical point of index 1.*

Proof of Lemma 3.7. We make use of a variant of Ambrosetti and Rabinowitz's Mountain pass lemma due to Chang ([Cha], Lemma 7.1). There, one considers a function f satisfying the Palais-Smale condition on a Banach space X. In our case, we will consider the open domain of $\mathbb{R}^n$ defined by:

$$X = \operatorname{int}(B_\epsilon(\mathbf{x}) \cup [\mathbf{x}, \mathbf{x}'] \cup B_\epsilon(\mathbf{x}')), \tag{3.8}$$

where $B_\epsilon(\mathbf{x})$ means a ball of radius ϵ small enough so that it contains only $\mathbf{x}$ as critical point. We use the fact that, rounding off its corner, X is diffeomorphic to $\mathbb{R}^n$. Indeed, one can add small neiborhoods to the vertices of the set $[\mathbf{x}, \mathbf{x}']$ so as to make X a star shaped region with smooth retraction on a ball. Then one can apply lemma 17, Ch11 of [S]. To prove that f satisfies the Palais-Smale condition in our situation, it is enough to show that there are no critical points on the boundary $\partial\overline{X}$. For this we note that, if $\mathbf{x}$ and $\mathbf{x}'$ are any two critical points, then:

$$\zeta^t([\mathbf{x}, \mathbf{x}'] - \{\mathbf{x}, \mathbf{x}'\}) \subset [[\mathbf{x}, \mathbf{x}']], \tag{3.9}$$

which is due to the strict monotonicity of the flow. On the other hand, the B_ϵ's do not contain any critical points on their boundaries either. Hence $\partial\overline{X}$ does not contain any critical points, and neither does our smoothed out version of it.

Let:

$$F = \{\gamma \in C([0,1], X) \quad | \quad \gamma(0) = \mathbf{x}, \gamma(1) = \mathbf{x}'\}$$

and:

$$c = \inf_{\gamma \in F} \sup_{\mathbf{y} \in \gamma} f(\mathbf{y}) \tag{3.10}$$

The mountain pass lemma then implies that c is a critical value and moreover, that the (isolated) critical points corresponding to this value are of index 1 (The hessian has 1 negative eigenvalue). This finishes the proof of Lemma 3.7. □

Next, we join the moutain passes to the minima. In the proof of his theorem 1, Angenent [An] proves that, if a rest point $\mathbf{y}$ for ζ^t has a positive eigenvalue, then there are two orbits $\mathbf{y}_{\pm}(t)$ (fast unstable manifolds) such that:

$$\mathbf{y}_{-}(t) < \mathbf{y} < \mathbf{y}_{+}(t), \quad \text{for all } t,$$

$$\mathbf{y}_{\pm}(t) \to \mathbf{y} \quad \text{as } t \to -\infty, \tag{3.11}$$

and

$$\mathbf{y}'_{-}(t) < 0 < \mathbf{y}'_{+}(t).$$

Note that one can actually say more, i.e.:

$$\mathbf{y}'_{-}(t) \ll 0 \ll \mathbf{y}'_{+}(t). \tag{3.12}$$

Indeed, this is true for t close to $-\infty$, since then the normalised $\mathbf{y}'_{+}$ tends to the normalised, strictly positive eigenvector that Angenent ([An], lemma 3.4) finds in the positive cone (and likewise for $\mathbf{y}'_{-}$ which is strictly negative). But the differential of the flow is strictly positive (see lemma 1.22), that is, it sends the positive (resp. negative) cone inside its interior, which proves 3.12.

Moreover, denoting by

$$\mathbf{a} = \lim_{t\to\infty} \mathbf{y}_{-}(t) \text{ and } \mathbf{b} = \lim_{t\to\infty} \mathbf{y}_{+}(t) \tag{3.13}$$

Angenent proves that $\mathbf{a}$ and $\mathbf{b}$ are critical points of index 0. Note that, given our sign convention, Angenent's flow ϕ^t and our ζ^t are the same (his $h = -$ our S). In our situation, these orbits can be taken to be the unstable manifolds $\mathbf{y}^k_{\pm}$ of the restpoint $\mathbf{y}^k$ of index 1 that exists in $[[\mathbf{x}^k, \mathbf{x}^{k+1}]]$, thanks to Lemma 3.7. Because of the strict monotonicity of ζ^t, it is easy to see that $\mathbf{y}^k_{\pm}$ are contained in $[[\mathbf{x}^k, \mathbf{x}^{k+1}]]$. Hence, since $\tilde{M}_{pq}$ is maximum,

$$\mathbf{a}^k \stackrel{\text{def}}{=} \lim_{t\to\infty} \mathbf{y}^k_{-}(t) = \mathbf{x}^k, \tag{3.14}$$

otherwise we could add $\mathbf{a}^k$ and all its σ and T translates to $\tilde{M}_{pq}$ and get a bigger σAM set. We proceed in the same fashion to the right of $\mathbf{y}^k$ to connect it to $\mathbf{x}^{k+1}$.

Finally we show that these unstable manifolds meet C^1 at the critical points. This is trivial at the mountain pass points, since we are taking their unstable manifolds. To show that they meet tangentially at the minima, we again use the fact (see [An], Lemma 3.4) that the positive-negative cone V only contains a one dimensional eigenspace at a critical point. Because Angenent proves in the same lemma that the corresponding eigenvalue is simple and is the biggest, a monotone trajectory of the flow that approaches a minimum has to do so tangentially to this unique, one dimensional eigenspace contained in V: The differential of the flow along this orbit induces a map on the unit tangent sphere that contracts the positive orthant.

Hence, the trajectory coming from the right in our construction is continued C^1 (as manifolds) by the one coming from the left. This finishes the proof of (1).

To prove (2), note that lemma 3.7 insures that the minimax related to a minimum $\mathbf{x}$ as defined by Mather is a mountain pass point, i.e. of index 1. Call this point $\mathbf{y}$ and take as before a maximum collection $\{\mathbf{x} = \mathbf{x}^0 \ll \mathbf{x}^1 \ll, \dots, \ll \mathbf{x}^n \ll \mathbf{y}\}$ where $\mathbf{x}^n = \mathbf{a}$ as defined in 3.14 and each $\mathbf{x}^k$ is a critical point of index 0. Join these points as before and repeat the argument to close the circle to the right of $\mathbf{y}$.

In both cases, we can chose to parametrise the curve obtained in such a way that its tangent vector $\frac{d}{ds}\mathbf{x}(s)$ is strictly inside the positive cone. Consider the map:

$$\pi : \mathbb{R}^{\mathbb{Z}} \to \mathbb{A} \qquad (3.15)$$
$$\pi(\mathbf{x}) = (x_0, y(x_0, x_1))$$

where $y(x_0, x_1)$ is given thanks to 1.3 (3). The tangent vector to the projected curve, i.e. $(\frac{dx_0}{ds}, \frac{dy(x_0,x_1)}{ds})$, has a strictly positive first component and hence the projected curve C is a graph. To see that its image is also graph, we look in the (x_0, x_1) coordinates of the annulus . (see 1.3 (3)). There the map f has differential (see [Ma 2],[McK-M]):

$$\begin{pmatrix} 0 & 1 \\ -\frac{b_0}{b_1} & -\frac{a_1}{b_1} \end{pmatrix} \qquad (3.16)$$

where a_1 and b_0, b_1 are given by second derivatives of S. The point being that a vector with positive components is sent into one with positive first component. Hence in the (x, y) coordinates the image of the tangent vector to our curve C always has strictly positive x component and the image of C is also a graph.

We need to show that $C \cap f(C)$ only contains the periodic points corresponding to the $\mathbf{x}^k, \mathbf{y}^k$'s. Writting $f(x, y) = (X, Y)$ as in 1.3, we only need to check that the sign of :

$$\begin{aligned} Y(x_{k-1}(t), x_k(t)) - y(x_k(t), x_{k+1}(t)) &= \partial_2 S(x_{k-1}(t), x_k(t)) + \partial_1 S(x_k(t), x_{k+1}(t)) \\ &= \frac{dx_k(t)}{dt} \end{aligned} \qquad (3.17)$$

is constant along any segment of our ghost circle which does not contain a critical point. But that is the case, since by 3.12, the connections are strictly monotone and the term 3.17 is just the kth component of the tangent to one of these connections. This finishes the proof of theorem 3.6. □

COMMENTS 3.17. The above construction may fail to be unique at two steps: First in the choice of $\tilde{M}_{pq}$. The unicity of $\tilde{M}_{pq}$ would mean that all orbits corresponding to critical points of index 0 that do not link with the minima do not link amoung themselves (orbits are said to link when their suspension does, see [Ha]). Second, in the choice of the mountain pass points $\mathbf{y}_k$. There the non-unicity seems to imply, through more careful analysis of the morse theory in the set X considered in lemma 3.7, the existence of critical points of higher index. The non unicity might be related to the discontinuity of the $\Delta W(\omega)$ of Mather at the rational ω's. Note that theorem 3.5 and 3.6 together imply that the ghost circle in $W_{p,q}$ won't be unique whenever there is a critical point of index greater than 1: theorem 3.5 gives a ghost circle passing through such a point, whereas theorem 3.6 gives one containing critical points of index less than or equal to one.

4. Properties of σ-Aubry-Mather sets and ghost circles.

In this section, we show that the set ΣAM of σAM sets is a closed set for the Hausdorff topology and that the rotation number is a well defined, continuous function on that set. These results may be seen as a generalisation of Katok's proposition 3 ([K]) on Mather sets. We could have used this to find ghost circles of all rotation number. These ghost circles can be made to contain the minimum energy state (in the sense of Aubry) for each given rotation number.

We remind the reader that σAM sets are closed, completely ordered, $\mathbb{Z}$ and σ invariant subsets of $\mathbb{R}^{\mathbb{Z}}$ (see definition 3.3). In this section, denote by T the unit translation in $\mathbb{R}^{\mathbb{Z}}$, i.e. the map given by the $\mathbb{Z}$-action.

We will proceed in the following way: first, we show that ΣAM is closed (corollary 4.3). Second, we show that any σAM set is included in a connected one (proposition 4.4). Third, the quotient by T of the connected σAM sets are homeomorphic to a circle (proposition 4.4). That makes σ restricted to such a set conjugate to the lift of a circle homeomorphism. The continuity of the rotation number on ΣAM derives then from its continuity on the space of lifts of circle homeomorphisms. As a bonus, we get that any σAM set is the lift of a minimal set for a circle homeomorphism: a full line, a Cantor set or a finite set.

Some of what we will prove will hold for general ordered Banach spaces.

Let X be a Banach space, whose order is given by a positive cone V_+ (i.e., V_+ is convex, closed, has non empty interior and $V_+ \cap -V_+ = \{0\}$: see 1.13). A set E is completely ordered iff

$$x, y \in E \Rightarrow y \in V(x),$$

where $V(x)$ denotes the closed set $(V_+ \cup V_-) + x$. Denote by H the set of closed sets in X. For $A \in H$, define:

$$U_\epsilon(A) = \cup_{a \in A} B_\epsilon(a)$$

We give H a (bounded) Hausdorff metric:

$$D(A,B) = min\,(glb\{\epsilon \in \mathbb{R} \quad | \quad A \subset U_\epsilon(B) \text{ and } B \subset U_\epsilon(A)\} \quad , \quad 1 \quad \} \tag{4.1}$$

Then H is complete for this metric ([Mu]). The following proposition is inspired by Katok's ([K], Prop.3):

PROPOSITION 4.2. *Suppose a sequence $\{E_k\}_{k\in\mathbb{Z}^+}$ of closed, completely ordered subsets of X converges to the closed subset E. Then E is also completely ordered.*

Proof. Let x, y be distinct elements of E. Pick two sequences x_k, y_k with x_k and y_k in E_k and $x_k \to x$, $\quad y_k \to y$. Since E_k is completely ordered, $y_k \in V(x_k)$. Consider the sequence:

$$\overline{y}_k = y_k + x - x_k$$

Then $\overline{y}_k$ is in $V(x)$. Since $V(x)$ is closed, $\lim_{k\to\infty} \overline{y}_k \in V(x)$, that is, $y \in V(x)$. This proves that E is completely ordered. □

COROLLARY 4.3. *The set of σAM set, the set of ghost circles, and the set of Aubry-Mather sets are all closed for the Hausdorff topology.*

Proof. We only need to show that the limit of a sequence of σAM sets is invariant under both T and σ. But this derives automatically from the continuity of these maps. As for ghost circles and Aubry-Mather sets, just note that flow invariant, flow critical, and connected are three epithets conserved by Hausdorff limits. □

PROPOSITION 4.4. *Suppose that the order $<$ on the Banach space X satisfies:*

(1) $0 < x < y \Rightarrow \|x\| < \|y\|$

(2) $[x, y]$ *is compact*

Then, any completely ordered, closed subset E of X can be embedded in a connected one. Furthermore, if E is connected, then, for any x, y in E, the set $[x, y] \cap E$ is homeomorphic to a closed interval of $\mathbb{R}$, endpoints corresponding to endpoints.

Proof. Let E be a completely ordered subset of X . We construct a map δ from E to $\mathbb{R}$ which is continuous and is strictly increasing. Pick a point x_0 of E. Define δ by:

$$\delta(x) = \begin{cases} \|x - x_0\| & x > x_0 \\ -\|x - x_0\| & x \leq x_0 \end{cases} \tag{4.5}$$

This makes sense since E is completely ordered. δ is obviously continuous. And the condition (1) says that it is strictly increasing, hence one to one. Let x, y be any two points of E. Then $E \cap [x, y]$ is compact and so is its image under δ (which is homeomorphic to it). The set $[\delta(x), \delta(y)] \setminus \{\delta(E \cap [x, y])\}$ is a union of disjoint open intervals of the form $(\delta(a), \delta(b))$ in $\mathbb{R}$, with a and b in E. Because δ is increasing, $[a, b] \cap E = \{a, b\}$. Now, join a and b by a straight line segment. Since $b - a > 0$ the points on this segment are ordered with respect to all other points of E, as well as with respect to one another. In this fashion, we close all the "gaps" in E and make it connected.

To prove the second assumption of the proposition, assume E is connected. Then $d(E \cap [x, y])$ must also be connected: if not, suppose $[x, y] = A \underset{\neq}{\cup} B$ where A and B are closed in $[x, y]$ (and hence in E). There are two possibilities: either both x and y belong to the same set, say A or else $x \in A$, $y \in B$. In the first case, We would have:

$$E = ((V_-(x) \cap E) \cup A \cup (V_+(y) \cap E)) \underset{\neq}{\cup} B$$

Since all these sets are closed, we get a contradiction. The other case is similar. Hence we proved that $\delta[x, y] = [\delta(x), \delta(y)]$. □

The following theorem gives a complete picture of what σAM sets look like. It is a corollary of proposition 4.4.

THEOREM 4.6.

(1) *If E is a connected σAM set, then it is homeomorphic to $\mathbb{R}$ and $E/T = E/\mathbb{Z}$ is homeomorphic to a circle. σ induces a circle homeomorphism on E/T.*

(2) *Any σAM set can be embedded in a connected σAM set and therefore is homeomorphic to a closed invariant set of a circle homeomorphism.*

(3) *The rotation number $R(E)$ of $\sigma\mid_E$ is a continuous function of E in ΣAM.*

Proof. The condition (1) of proposition 4.4 are satisfied for $\mathbb{R}^{\mathbb{Z}}$ when we give it the norm (product topology):

$$\|\mathbf{x}\| = \frac{1}{3} \sum_{-\infty}^{+\infty} \frac{1}{2^{|k|}} |x_k| \tag{4.7}$$

The second condition is satisfied because of Tychonoff's theorem: the space $[\mathbf{x}, \mathbf{y}] = \{\mathbf{z} \in \mathbb{R}^{\mathbb{Z}} \mid x_k \leq z_k \leq y_k\}$ is a product of intervals of $\mathbb{R}$ and, possibly, points.

We now have to make sure that, if E is σ and T invariant, so is its connected extension. But if $[a, b]$ is a gap interval, so are $T[a, b]$ and $\sigma[a, b]$, by invariance of E under these maps. Since σ and T are linear, they send the straight line segment that we used to bridge the gaps in proposition 4.4 onto one another.

As for the second statement of the Theorem, note that (calling $\mathbf{x}^0$ a base point for E)

$$\delta(\mathbf{x}^0 + \mathbf{1}) = \|(\mathbf{1} + \mathbf{x}^0) - \mathbf{x}^0\| = 1, \tag{4.8}$$

where $\mathbf{1}$ is the sequence of constant term 1, and hence, since E is T invariant, it covers any finite interval of $\mathbb{R}$ under the map δ. Hence E is homeomorphic to $\mathbb{R}$.

Remember that E is T and σ invariant. Because of 4.8, the action of T on E is conjugated, via the homeomorphism δ to the usual unit translation on $\mathbb{R}$. Hence δ induces a homeomorphism $E/T \cong \mathbb{R}/\mathbb{Z}$. That σ induces a circle homeomorphism on E/T comes from the fact that T and σ commute.

To show that R is continuous on ΣAM, it is enough to consider the connected ones. We want to show that, given a sequence $E_k \to E$ of connected σAM sets,

$$\lim_{k\to\infty} R(E_k) = R(E) \tag{4.9}$$

Take a base point $\mathbf{x}^k$ in each E_k, with $\mathbf{x}_k \to \mathbf{x}, \quad \mathbf{x} \in E$. Denote by δ_k the "distance" homeomorphism defined as in proposition 4.4, on each E_k. Each $h_k = \delta_k \circ \sigma \circ \delta_k^{-1}$ is a lift of a circle homeomorphism on $\mathbb{R}/\mathbb{Z}$. Because of the continuity of the distance and of σ, $\lim_{k\to\infty} h_k = h$ (in the uniform topology) where h is the homeomorphism conjugated to $\sigma\mid_E$. The continuity of R derives from the continuity of the rotation number on the space of lifts of circle homeomorphisms. □

Remember that the rotation number of a sequence $\mathbf{x}$ in $\mathbb{R}^{\mathbb{Z}}$ is defined by:

$$\rho(\mathbf{x}) = \lim_{k\to\pm\infty} \frac{x_k}{k} \tag{4.10}$$

Note that another way of saying that $\mathbf{x}$ is ordered is: the closure $\overline{eo(\mathbf{x})}$ of its extended orbit is a σAM set.

LEMMA 4.11. *Let E be a σAM set and $X_{p,q}$ any element in E. Then $\rho(\mathbf{x})$ exists and $R(E) = \rho(\mathbf{x}))$ for all $\mathbf{x}$ in E. Moreover, if $R(E) = \omega$, then*

$$E \subset Y_\omega \stackrel{\text{def}}{=} \{\mathbf{x} \in \mathbb{R}^{\mathbb{Z}} \quad | \quad \sup_{k\in\mathbb{Z}} |x_k - k\omega| < \infty\} \tag{4.11}$$

The space Y_ω is in fact invariant under the flow ζ^t ([An],[Go 1,2]) and made out of sequences of rotation number ω, although *not necessarily ordered like orbits of a rigid rotation.* The definition of Y_ω obviously generalises to higher dimensions. In [Go 1,2], we conjectured the existence of a ghost torus in each of the Y_ω's. Theorem 3.5 together with the theorem of Aubry-Mather and lemma 4.11 establish this conjecture in the twist map case. Note that the following proof actually gives more information on the nature of the elements of a σAM set: if we denote the *extended orbit* of $\mathbf{x} \in \mathbb{R}^{\mathbb{Z}}$ by

$$eo(\mathbf{x}) \stackrel{\text{def}}{=} \{T^m \sigma^k \mathbf{x}\}_{(m,k)\in\mathbb{Z}^2}, \tag{4.12}$$

then, because a σAM set E is completely ordered, a fortiori $eo(\mathbf{x})$ is completely ordered for all $\mathbf{x}$ in E. In this sence, a natural generalisation of $WO_{p,q}$ for an irrational rotation number ω is the set of $\mathbf{x}$ whose extended orbit is ordered and live in Y_ω.

Proof. Since the map σ on E is conjugated to a rotation of angle ω, we have:

$$T^n\sigma^m\mathbf{x} \leq T^{n'}\sigma^{m'}\mathbf{x} \Leftrightarrow m\omega + n <\leq m'\omega + n'. \tag{4.13}$$

Taking the 0^{th} element of these sequences, we get:

$$m\omega + n \leq m'\omega + n' \Rightarrow x_m + n \leq x_{n'} + n'$$

(Equality can appear in the last term even if it does not in the first). In particular,

$$n \leq m\omega \leq n+1 \Rightarrow n + x_0 \leq x_m \leq n + x_0 + 1, \tag{4.14}$$

which implies that $lim_{m\to\infty} x_m/m = \omega$ and also that

$$|x_m - m\omega| \leq |x_0| + 1.$$

□

The next theorem tells us that we can indeed take limits of ghost circles.

THEOREM 4.15. *Let (S_{ω_k}) be a sequence of σAM with rotation number $\omega_k \to \omega$. Then (S_{ω_k}) has a subsequence converging to a σAM S_ω of rotation number ω. In particular, if (S_{ω_k}) is a sequence of ghost circles (resp. Aubry-Mather sets), S_ω is a ghost circle (resp Aubry-Mather set) of rotation number ω*

Proof. We first show that the sequence (G_{ω_k}) projects to one which is included in a compact subset of $\mathbb{R}^{\mathbb{Z}}/\mathbb{Z}$. Consider the linear map:

$$\begin{aligned} &\Psi : \mathbb{R}^{\mathbb{Z}} \to \mathbb{R}^{\mathbb{Z}} \\ &\Psi(\dots, x_{-1}, x_0, x_1, \dots) = (x_0, \dots, a_{-1}, a_0, a_1, \dots), \quad \text{where } a_k = x_k - x_{k-1} \end{aligned} \tag{4.16}$$

Then it is not hard to see that Ψ is an isomorphism of the Banach space $\mathbb{R}^{\mathbb{Z}}$, i.e. a linear change of coordinates.

In the $(x_0, \mathbf{a})$ coordinates, 4.14 becomes:

$$|a_j| < |\omega_k| + 2(|x_0| + 1) \tag{4.18}$$

When we mod out by T in the $(x_0, \mathbf{a})$ coordinate, we do not change the $\mathbf{a}$ coordinate and may think of $\mathbb{R}^{\mathbb{Z}}/\mathbb{Z} \cong \mathbb{R}/\mathbb{Z} \times \mathbb{R}^{\mathbb{Z}}$. Hence we can assume that $0 \leq x_0 \leq 1$. Call $\underline{S}_{\omega_k}$

the projection of S_{ω_k} in the quotient space $\mathbb{R}^{\mathbb{Z}}/\mathbb{Z}$. Since the sequence ω_k converges, it is bounded, say $\alpha_0 \leq \omega \leq \alpha_1$. Thus, for any $k \in \mathbb{Z}$ and $(x_0, \mathbf{a}) \in \underline{S}_{\omega_k}$, we have:

$$\underline{S}_{\omega_k} \subset Z_{[\alpha_0,\alpha_1]} := \{(x_0, \mathbf{a}) \in \mathbb{R}^{\mathbb{Z}}/\mathbb{Z} \quad | \quad |a_j| < \sup\{|\alpha_0|, |\alpha_1\} + 4\}\} \tag{4.19}$$

The set $Z_{[\alpha_0,\alpha_1]}$ is homeomorphic to the product of $\mathbb{S}^1$ with a Hilbert cube, and hence compact.

Then the set of compact subsets of $Z_{[\alpha_0,\alpha_1]}$ is itself compact for the Hausdorff topology.

The set of projection of σAM sets belonging to $Z_{[\alpha_0,\alpha_1]}$ is thus compact, intersection of a closed set with a compact one. Thus the sequence $\{\underline{S}_{\omega_k}\}_{k\in\mathbb{Z}}$ has a converging subsequence, that we denote in the same way and which must converge to the projection of a σAM set S_ω of rotation number $R(S_\omega) = \omega$, by theorem 4.6. since the limit of connected (resp. ζ^t-invariant) sets must be connected (resp. ζ^t invariant), the set S_ω is a σAM set. □

To understand some of the meaning the ghost circles might have in the "real" space $\mathbb{A}$, we have the following proposition:

PROPOSITION 4.20 (PROPERTIES OF GHOST CIRCLES). *Let* $\pi_{01} : \mathbb{R}^{\mathbb{Z}} \to \mathbb{R}^2$ *and* $\pi_x : \mathbb{A} \to \mathbb{R}$ *denote the projections:*

$$\begin{aligned} &\pi_{01}(\mathbf{x}) = (x_0, x_1), \qquad \textit{and } \pi_x(x, y) = x \\ &\psi(x, X) = (x, y), \end{aligned} \tag{4.21}$$

the change of coordinates defined in 1.3.(3)

(1) *Let G be a ghost circle. Then $\pi_{01}(G)$ forms a lipshitz graph over the (first) diagonal in R^2.*
(2) *G is strictly ordered and thus $\pi_x \circ \psi \circ \pi_{01} |_G$ is an order preserving homeomorphism (diffeomorphism if G is differentiable). In particular $G' = \psi \circ \pi_{01}(G)$ is a graph over the x axis.*
(3) *$f(G')$ is also a graph and*

$$f(G') \cap G' = \psi \circ \pi_{01}\{\mathbf{x} \in G \quad | \quad \nabla W(\mathbf{x}) = 0\}$$

By order preserving, we mean with respect to the partial order in $\mathbb{R}^{\mathbb{Z}}$ and the order in $\mathbb{R}$.

Proof. The argument in 3.16 proves the last statement. We only prove the key property that ghost circles are strictly ordered. The other statements are left to the reader. Let G be a ghost circle. In 3.4 (2), we showed that if $\mathbf{x} < \mathbf{y}$ are critical, then in fact $\mathbf{x} \prec \mathbf{y}$. For $\mathbf{x} < \mathbf{y} \in G$ we have two other cases to consider: $\mathbf{x}$ is critical and $\mathbf{y}$ is not, or neither are critical. Suppose $\mathbf{x}$ is critical, but that $\mathbf{y}$ is not. Since G is homeomorphic to $\mathbb{R}$ (theorem

4.6) the flow is a parametrisation of a neiborhood of $\mathbf{y}$ in G, hence it is monotone with respect to the order in G. Suppose $\zeta^{-t}\mathbf{y} < \mathbf{y}$ for $t > 0$. Let $\|\mathbf{x} - \mathbf{y}\| = \alpha$ where the norm was defined in the proof of theorem 4.6. By proposition 4.4 this norm induces a homeomorphism between the set $[\mathbf{x}, \mathbf{y}] \cap G$ and an interval of length α in $\mathbb{R}$. Since $\|\mathbf{y} - \zeta^{-t}\| < \alpha/2$ for small enough t we have

$$\mathbf{x} < \zeta^{-t}\mathbf{y}$$

and, applying the strict monotone flow: $\mathbf{x} \prec \mathbf{y}$. If on the other hand $\zeta^{t}\mathbf{y} < \mathbf{y}$, by strict monotonicity we must in fact have: $\mathbf{x} \prec \zeta^{t}\mathbf{y} < \mathbf{y}$ and thus $\mathbf{x} \prec \mathbf{y}$. Finally, the case when neither $\mathbf{x}$ nor $\mathbf{y}$ are critical reduces to the situation $\mathbf{y} = \zeta^{t}\mathbf{x}$ for some positive t and say, $\mathbf{x} < \mathbf{y}$. Then $\zeta^{t-\epsilon}\mathbf{x} < \zeta^{-\epsilon}\mathbf{y}$ which implies $\mathbf{x} \prec \mathbf{y}$. □

As noticed in 3.13, the ghost circle G_{pq} is not necessarily unique. For irrational ω's, we have:

LEMMA 4.22. *When G_ω, ω irrational, is completely critical and transitive for σ (i.e. its projection is transitive for f), then it is the unique ghost circle with rotation number ω containing critical points.*

Proof. Let G be a ghost circle of rotation number ω containing a critical point $\mathbf{x}$. The closure of the extended orbit of $\mathbf{x}$ forms an Aubry-Mather set contained in G. But this set must also be contained in G_ω ([He 1], [Ma 4]), and hence be G_ω itself. □

COMMENTS 4.23 FLUX FOR GHOST CIRCLES ETC.. 1) At this point, we are unable to rule out the case of a completely noncritical ghost circle. We think that it is unlikely to occur. Also, the restriction to transitive ghost circles might not be necessary in lemma 4.22. On the other hand, one can not expect unicity when the circle is not critical: Mather [Ma 4] showed that, when there is no invariant circle of irrational rotation number ω, there are uncountably many distinct minimal Aubry-Mather sets of rotation number ω.

2) The argument in 3.17 actually gives a way to compute the*geometric flux* of f through the projection G' of a ghost circle G. By geometric flux, we mean the area above G' which is below $f(G')$. Note that this notion is well defined for G', since G' and $f(G')$ are graphs. We use property (iii) in proposition 4.20 in what follows. Consider the set of connections between critical points in G and E the quotient of this set by σ and T, i.e. the set of extended orbits of connections (see 4.12), denoted by $\underline{\mathbf{x}}(t)$, where $\mathbf{x}(t)$ is a representant.

$$\begin{aligned} Flux(G') &= \frac{1}{2} \sum_{\underline{\mathbf{x}}(t) \in E} \sum_{k \in \mathbb{Z}} \left| \int_{-\infty}^{+\infty} Y(x_{k-1}(t), x_k(t)) - y(x_k(t), x_{k+1}(t)) dt \right| \\ &= \frac{1}{2} \sum_{\underline{\mathbf{x}}(t) \in E} \sum_{k \in \mathbb{Z}} \left| \int_{-\infty}^{+\infty} \partial_2 S(x_{k-1}(t), x_k(t)) + \partial_1 S(x_k(t), x_{k+1}(t)) dt \right| \qquad (4.24) \\ &= \frac{1}{2} \sum_{\underline{\mathbf{x}}(t) \in E} \sum_{k \in \mathbb{Z}} \left| \int_{-\infty}^{+\infty} \frac{d}{dt} \mathbf{x}_k(t) dt \right| \end{aligned}$$

The flow being strictly monotone, the integrand in the last part is of constant sign along a given $\underline{\mathbf{x}}(t)$. Letting $\lim_{t\to\pm\infty}\mathbf{x}(t)=\mathbf{x}^{\pm}$ one gets:

$$\text{(4.25)}\qquad Flux(G')=\frac{1}{2}\sum_{\underline{\mathbf{x}}(t)\in E}\left|\sum_{k\in\mathbb{Z}}S(x_k^+,x_{k+1}^+)-S(x_k^-,x_{k+1}^-)\right|$$

Because of their geometric interpretation, the above sums converge. One can get rid of the factor 1/2 and the absolute value by chosing only the elements of E that give us positive terms. One can see the flux as the global variation of the energy W on G (eventhough W is not necessarily well defined there). When there is a direct connection $\mathbf{x}(t)$ between a min and a minimax, the sum corresponding to $\mathbf{x}(t)$ is the ΔW of Mather [Ma 2].

T he set G_ω may also contain critical points that are not minima (when they exist): the limits of mountain pass points. We will not explore this issue in this paper. For related results, see [Ma 2], [K 2]

5. A criterion of non existence of invariant circles.

In this section, we use the ghost circles to give a proof of a criterion related to Mather's ΔW criterion for the nonexistence of invariant circles for a twist map f [Ma 2].

The crucial tool that we will be using is the following:

DEFINITION 5.1. Let G_{pq} be a ghost circle in $X_{p,q}$ for a given twist map f. Then:

$$\text{(5.1)}\qquad \Delta W(G_{pq})=\max_{\mathbf{x},\mathbf{x}'\in G_{pq}}|W_{p,q}(\mathbf{x})-W_{p,q}(\mathbf{x}')|$$

It is clear that $\Delta W(G_{pq})$ is attained at critical points of $W_{p,q}$ in G_{pq}. If G_{pq} is a ghost circle such as the ones we constructed in theorem 3.6, the points $\mathbf{x}$ and $\mathbf{x}'$ will be a minimum and a mountain pass point. Note however that this mountain pass point needs not be the same as the minimax defined by Mather. For one thing, as we noted in 3.13, the ghost circle constructed in 3.6 needs not be unique. Hence the above definition might or might not coincide with the $\Delta W_{p,q}$ of Mather. What we have, for a ghost circle G_{pq} passing through a minimum, is:

$$\Delta W_{p,q}\le\Delta W(G_{pq})$$

We consider a converging sequence of ghost circles $\{G_{p_kq_k}\}_{k\in\mathbb{Z}}$ with

$$G_{p_kq_k}\to G_\omega \text{ and } p_k/q_k\to\omega,$$

where ω is irrational. If f does not satisfy the generic condition (2.17), we assume that each $G_{p_kq_k}$ is itself a limit of ghost circles for maps of the generic kind.

Remember that we find an f-invariant circle each time we find a completely critical ghost circle (see 4.15).

THEOREM 5.2.

(1) *If* $\lim_{k\to\infty} \Delta W(G_{p_k q_k}) = 0$ *then* G_ω *is a completely critical ghost circle (and hence projects to an* f*-invariant circle in* $\mathbb{A}$*).*

(2) *Conversally, if* G_ω *is completely critical then any sequence of rational ghost circles* $G_{p_k q_k}$ *with* $G_{p_k q_k} \to G_\omega$ *will satisfy:* $\lim_{k\to\infty} \Delta W(G_{p_k q_k}) = 0$

COMMENTS: THE CRITERION.

As a consequence of lemma 4.22 and theorem 5.2, we deduce that if G_ω *has a dense orbit then any sequence* $G_{p_k q_k} \to G_\omega$ *has* $\Delta W(G_{p_k q_k}) \to 0$*. Another way to say this is the following criterion:*

If $G_{p_k q_k}$ *is a converging sequence of ghost circles and if* $\Delta W(G_{p_k q_k}) \not\to 0$*, then there is no transitive invariant circles of rotation number* $\omega = \lim p_k/q_k$*.*

We could refine this criterion by erasing "transitive" in the above criterion if we could prove the unicity of G_ω *when it is a completely critical set, not necessarily transitive.*

Proof. We start by making a remark on the norms we will be using. Remember that the product topology on $\mathbb{R}^{\mathbb{Z}}$ can be given the norm:

$$\|\mathbf{x}\| = \frac{1}{3}\sum_{-\infty}^{+\infty} \frac{1}{2^{|k|}}|x_k|$$

which is normalised so that the sequence $\mathbf{1}$ of constant term 1 has norm 1. On the other hand we can give a sequence the sup norm:

$$\|\mathbf{x}\|_\infty = \sup_{j\in\mathbb{Z}} |x_j|$$

which might be infinite. Because of our normalisation, we have:

$$\|\mathbf{x}\| \leq \|\mathbf{x}\|_\infty \tag{5.3}$$

with equality holding on constant sequences.

Next define:

$$Q(\mathbf{x}) = \partial_1 S(x_0, x_1) + \partial_2 S(x_{-1}, x_0), \tag{5.4}$$

that is , according to the notation used in 1.12:

$$Q(\mathbf{x}) = (\nabla W(\mathbf{x}))_0.$$

Thus we can rewrite:

$$(\nabla W(\mathbf{x}))_j = Q(\sigma^j \mathbf{x})$$

From 5.3, we have:

$$\sup_{j\in\mathbb{Z}} |Q(\sigma^j(\mathbf{x})| = \|\nabla W(\mathbf{x})\|_\infty \geq \|\nabla W(\mathbf{x})\| \tag{5.4}$$

It turns out that, restricted to G_ω, $\|\nabla W(\mathbf{x})\|_\infty$ is well defined. But we won't need this fact here. Note that both Q and ∇W are continuous for the product topology. In fact, since we are working with a converging sequence we may assume that everything that we do takes place in the compact set $Z_{[\alpha_0,\alpha_1]}$ introduced in 4.19. From this assumption, it is not hard to see that ∇W is in fact lipshitz (of constant K, say), since it is C^1.

To proceed with the proof of the theorem, suppose by contradiction that:

$$\lim_{k\to\infty} \Delta W(G_{p_k q_k}) = 0, \text{ but } \nabla W\mid_{G_\omega} \not\equiv 0. \tag{5.5}$$

That is, assume G_ω is not completely critical. That means that we can find $\mathbf{x}$ in G_ω such that:

$$\|\nabla W(\mathbf{x})\| = 3\alpha > 0$$

and hence, by 5.4 there is a j such that:

$$|Q(\sigma^j\mathbf{x})| \geq 2\alpha$$

To simplify the notation we may assume that $j = 0$ and thus:

$$|Q(\mathbf{x})| \geq 2\alpha. \tag{5.6}$$

By continuity of Q, we can find a ball (product metric) $B_\epsilon(\mathbf{x})$ around $\mathbf{x}$ such that:

$$\mathbf{y} \in B_\epsilon(\mathbf{x}) \Rightarrow |Q(\mathbf{y})| \geq \alpha \tag{5.7}$$

Let $\mathbf{x}^k \in G_{p_k q_k}$ be such that $\mathbf{x}^k \to \mathbf{x}$. For $k > k_0$ large enough, we have:

$$B_{\epsilon/2}(\mathbf{x}^k) \subset B_\epsilon(\mathbf{x}) \tag{5.8}$$

From the fact that $G_{p_k q_k}$ is a connected σAM set (see proof of proposition 4.4), one deduces that:

$$B_{\epsilon/2}(\mathbf{x}^k) \cap G_{p_k q_k} = [\overline{\mathbf{a}}^k, \mathbf{a}^k] \cap G_{p_k q_k} \tag{5.9}$$

where both $\overline{\mathbf{a}}^k$ and $\mathbf{a}^k$ are elements of $G_{p_k q_k}$. Since the above set is connected and can not contain any critical points, we can follow the flow ζ^t to get from $\mathbf{x}^k$ to $\mathbf{a}^k$, say in positive time s, remaining in this set the whole time. Thus, the following holds:

$$\begin{aligned}\left|W_{p_k q_k}(\mathbf{x}^k) - W_{p_k q_k}(\mathbf{a}^k)\right| &= \left|\int_0^s \nabla W_{p_k q_k}(\zeta^t \mathbf{x}^k).\frac{d(\zeta^t \mathbf{x}^k)}{dt} dt\right| \\ &= \left|\int_0^s \sum_{j=0}^{q_k} \left[Q\left(\sigma^j \zeta^t(x^k)\right)\right]^2 dt\right| \\ &\geq \int_0^s (Q(\zeta^t \mathbf{x}^k))^2 dt \\ &\geq s\alpha^2\end{aligned} \tag{5.10}$$

But s must satisfy:

$$\sup_{\mathbf{z}\in B_\epsilon(\mathbf{X})} \|\nabla W(\mathbf{z})\|.s \geq \|\zeta^s(\mathbf{x}^k) - \mathbf{x}^k\| = \|\mathbf{a}^k - \mathbf{x}^k\| = \epsilon/2 \tag{5.11}$$

where the sup is well defined since we are working in $Z_{[\alpha_0,\alpha_1]}$. In fact, it can be taken to be the lipshitz constant K of ∇W. Hence:

$$s \geq K^{-1}\epsilon/2 \tag{5.12}$$

where K is independant of k. Using the assumption 5.7 we get:

$$\Delta W(G_{p_k q_k}) \geq \left|W_{p_k q_k}(\mathbf{x}^k) - W_{p_k q_k}(\mathbf{a}^k)\right| \geq s\alpha^2 \geq K^{-1}\alpha^2\epsilon/2 \tag{5.13}$$

The last term is positive, independant of k and thus, since ϵ is arbitrary, we get a contradiction to our assumption 5.5. This proves part (1) of the theorem.

To prove part (2) of the theorem, we assume that G_ω is completely critical and take any sequence $G_{p_k q_k} \subset X_{p_k q_k}$ of ghost circles with $G_{p_k q_k} \to G_\omega$. We will also assume, for the moment, that the map$f$ satisfies the generic assumption 2.17. Since G_ω is completely critical:

$$Q\mid_{G_\omega} \equiv 0 \tag{5.14}$$

and thus, for any ϵ there is an α such that:

$$D(G_\omega, E) < \alpha \Rightarrow \|Q\mid_E\| < \epsilon, \tag{5.15}$$

by continuity of Q. Here E is any compact set in $Z_{[\alpha_0,\alpha_1]}$, and D is the Hausdorff metric. To prove (2) in the theorem , it therefore suffices to show that:

$$\|Q\mid_{G_{p_k q_k}}\| < \epsilon \Rightarrow \Delta W(G_{p_k q_k}) < 3\epsilon \tag{5.16}$$

Take $\mathbf{x}^k > \mathbf{y}^k \in G_{p_k q_k}$ such that:

$$\Delta W(G_{p_k q_k}) = \left|W_{p_k q_k}(\mathbf{x}^k) - W_{p_k q_k}(\mathbf{y}^k)\right| . \tag{5.17}$$

Such a pair exists since $G_{p_k q_k}/\mathbb{Z}$ is compact.

Since $W_{p_k q_k}$ is constant on $eo(\mathbf{x}^k)$ (see 4.12) for the definition of $eo(\mathbf{x})$, we may assume that $\mathbf{x}^k$ is the closest to the the right of $\mathbf{y}^k$ in $eo(\mathbf{x}^k)$ and hence:

$$T^m \sigma^j \left([\mathbf{x}^k, \mathbf{y}^k]\right) \cap [\mathbf{x}^k, \mathbf{y}^k] = \varnothing \text{ for } 0 \leq j < q_k \text{ and } m \in \mathbb{Z} \tag{5.18}$$

Note that, because of this assumption on $\mathbf{x}^k$, we have $\mathbf{x}^k \in [\mathbf{y}^k, T\mathbf{y}^k]$.

Let π_{012} denote the projection $\mathbb{R}^{\mathbb{Z}} \to \mathbb{R}^3$:

$$\pi_{012}(\mathbf{x}) = (x_0, x_1, x_2) \tag{5.19}$$

We will use the projection of $G_{p_k q_k}$ by π_{012} to get an estimate on $\Delta W(G_{p_k q_k})$. For this purpose, we need to assume that $G_{p_k q_k}$ is at least piecewise differentiable, which is the case when $W_{p_k q_k}$ is Morse. Also remember that $\pi_{012} \mid_{G_{p_k q_k}}$ is a diffeomorphism in this case (see Theorem 3.6).

Let $\gamma(s)$ be an arclength parametrisation of $\pi_{012}(G_{p_k q_k})$. The curve γ has the property that $\gamma + (1,1,1) = \gamma$. Let:

$$\pi_{012}(\sigma^j \mathbf{y}^k) = \gamma(s_{2j}), \quad \pi_{012}(\sigma^j \mathbf{x}^k) = \gamma(s_{2j+1}) \tag{5.20}$$

Finally denote by $r(\gamma(s)) = Q\left(\pi_{012}^{-1}(\gamma(s))\right)$. With this notation, we have the following estimate:

$$\left|W_{p_k q_k}(\mathbf{x}^k) - W_{p_k q_k}(\mathbf{y}^k)\right| \leq \sum_{j=0}^{q_k-1} \left| \int_{s_{2k}}^{s_{2k+1}} r(\gamma(s)).\gamma'(s) ds \right| \tag{5.21}$$

But the length of the segment of γ between $\gamma(0)$ and $\gamma(0)+1$ is bounded above by 3, since this segment is a monotone curve included in the cube $[\gamma(0), \gamma(0)+1]$ (partial order on $\mathbb{R}^3$). Because this segment includes only one integer translate of the segment of γ between $\gamma(s_{2j})$ and $\gamma(s_{2j+1})$ for $0 \leq j < q_k$ (see 5.18), we have:

$$\sum_{j=1}^{q_k} (s_{2k+1} - s_{2k}) \leq 3 \tag{5.22}$$

On the other hand, we have $|r \mid_\gamma| = |Q \mid_{G_{p_k q_k}}| < \epsilon$ and of course $|\gamma'(s)| = 1$. Thus:

$$\left|W_{p_k q_k}(\mathbf{x}^k) - W_{p_k q_k}(\mathbf{y}^k)\right| \leq 3\epsilon \tag{5.23}$$

which finishes the proof of (2) in the case where $W_{p_k q_k}$ is Morse. If it is not, we can still conclude by taking a sequence $G^l_{p_k q_k}$ indexed on $\mathbb{Z}^2$ of ghost circles for maps $f_{k,l} \to f$ as k or $l \to \infty$, with the assumption that each $f_{k,l}$ has the generic property (2.17) and such that $\lim_{l\to\infty} G^l_{p_k q_k} = G_{p_k q_k}$. □

QUESTION: CONTINUITY OF ΔW. :Mather [Ma 2] extended his ΔW, as a function of the rotation number, to the irrationals. He showed that this function was continuous at the irrationals but not at the rationals. Can we extend similarly our ΔW to all ghost circles and show its continuity on this set? Intuitively, the discontinuity observed by Mather should come from the non unicity of the rational ghost circles.

Appendix: genericity result for Monotone maps.

In this appendix, we give a proof of Proposition 2.16 on the genericity of the assumption:

(2.17) *"For all (p,q) in* $\mathbb{Z}^n \times \mathbb{Z}$ *the function* $W_{p,q}$ *is Morse."*.

The proof will be made of the two following lemmata.

LEMMA A.1. : *The property 2.17 is equivalent to :*

(A.1)

"For all (p,q) in $\mathbb{Z}^n \times \mathbb{Z}$ *the (p,q)-periodic points of the map f are nondegenerate"*

Here we are using the following definition:

DEFINITION A.2. A (p,q) periodic point $\mathbf{z}$ is said to be nondegenerate if:

$$\det(D(f^q)_{\mathbf{z}} - I) \neq 0 \tag{A.2}$$

LEMMA A.3. *The property A.1 is satisfied by a residual set in the space of C^1 monotone maps.*

We proved lemma A.1 in [Go 1,2], (lemma 2.17). We reproduce the proof here. As for the proof of lemma A.3, it is a simple modification of Robinson's theorem ([R], theorem 1Bi), which he proves in the general context of symplectic maps.

Proof of lemma A.1. Since (p,q) points are in one to one correspondance to critical points of $W_{p,q}$, what we need to show is $Df^q_{\mathbf{z}}$ has eigenvalue 1 if and only if $HessW_{p,q}(\mathbf{x})$ has eigenvalue 0, where $\mathbf{x}$ is the critical point corresponding to $\mathbf{z}$.

Let $\{\lambda_i, \frac{1}{\lambda_i}\}_{i=1}^n$ represent the complete set of eigenvalues (multipliers) of $Df^q_{\mathbf{z}}$. Define:

$$\begin{aligned} \rho_i &= \lambda_i + \frac{1}{\lambda_i} \\ R_i &= \frac{1}{4}(2 - \rho_i) \end{aligned} \tag{A.4}$$

since $\rho_i = 2$ when $\lambda_i = 1$, $\mathbf{z}$ is nondegenerate if and only if $R_i = 0$ (see definition A.2). Let:

$$\mathbf{b}_k = \mathbf{b}(f^k(\mathbf{z})) \tag{A.5}$$

where

$$\mathbf{b}(\mathbf{z}) = \left(\frac{\partial X}{\partial y}\right)_{\mathbf{z}} \tag{A.6}$$

when $z = (x, y)$ and $(X, Y) = f(x, y)$.

Using implicit differentiation on $\partial_1 S(x, X(x, y)) = -y$, one gets:

$$\mathbf{b}(\mathbf{z}) = -[\partial_1 \partial_2 S(\mathbf{z})]^{-1} \tag{A.7}$$

with this notation, the equation (29) of Kook and Meiss [K-M] becomes

$$\prod_{i=1}^{n} R_i = (-\frac{1}{4})^n (\det \nabla^2 W_{p,q}(\mathbf{x})) \prod_{k=1}^{q} \det \mathbf{b}_k \tag{A.8}$$

(our $\mathbf{b}_k$ = their $\mathbf{b}_t^{-1}$). Since we know that $\det \mathbf{b}_k \neq 0$ (see 1.4), the lemma is proved. □

Proof of lemma A.3. Robinson [R], in his theorem 1Bi, proves that the set of C^k symplectic maps with nondegenerate periodic points is residual in the space of all C^k symplectic maps. He proceeds by induction on the period q of the points(*). We want to adapt his proof to the space M^1 of C^1 monotone maps of $\mathbb{A}^n$. First note that M^1 is an open set in the space of C^1 exact symplectic maps: the condition (1.3.3) in our definition of monotone maps is an open one. The only thing that we have to check, therefore, is that the perturbations that Robinson uses to kill degeneracy transform exact symplectic maps into exact symplectic maps. But this is not hard to check: each of these perturbations is given by composing the original map f with the time one map of the hamiltonian flow associated to a bump function in a small neibourghood of a given periodic point. Taking as representants of the generators of $H_1(\mathbb{T}^n \times \mathbb{R}^n)$ closed curves $c_1, \ldots, c_n$ that do not pass through the support of the bump, we see that:

$$\int_{c_i} \tilde{f}^*(ydx) - ydx = \int_{c_i} f^*(ydx) - ydx = \int_{c_i} dS = 0, \tag{A.9}$$

where $\tilde{f}$ denotes the perturbed f. By DeRham's theorem, this is equivalent to the one form $\tilde{f}^*(ydx) - ydx$ being 0 as a cohomological class, that is :

$$\tilde{f}^*(ydx) - ydx = d\tilde{S} \tag{A.10}$$

for a "0-form", or function $\tilde{S}$. □

(*)C.Robinson actually deals with higher order resonnances as well, i.e, roots of unity in the spectrum of $Df^q_{\mathbf{z}}$.

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