1-1-1999

Lagrangian Systems on Hyperbolic Manifolds

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Abstract: This paper gives two results that show that the dynamics of a time-periodic Lagrangian system on a hyperbolic manifold are at least as complicated as the geodesic flow of a hyperbolic metric. Given a hyperbolic geodesic in the Poincaré ball, Theorem A asserts that there are minimizers of the lift of the Lagrangian system that are a bounded distance away and have a variety of approximate speeds. Theorem B gives the existence of a collection of compact invariant sets of the Euler-Lagrange flow that are semiconjugate to the geodesic flow of a hyperbolic metric. These results can be viewed as a generalization of the Aubry-Mather theory of twist maps and the Hedlund-Morse-Gromov theory of minimal geodesics on closed surfaces and hyperbolic manifolds.

Section 0: Introduction.

The notion of stability in Dynamical Systems refers to dynamical behavior that persists under perturbation. Stability under small perturbations is perhaps best known, but dynamical persistence under large perturbations (in a restricted class) is often studied and has proved to be quite powerful. Large perturbation theories usually have a strong topological component. This is because behavior that persists under large perturbations must be very fundamental to the system, and the most fundamental aspect of a dynamical system is the topology of the underlying manifold. In applying stability results, one usually begins with a model system whose dynamics are understood and then perturbs it. The stability theorems indicate which dynamics of the model system must be present in the perturbed system. This provides a framework for the investigation the other dynamics present in the perturbed system.

This paper presents stability results for the dynamics of time-periodic Lagrangian (or Hamiltonian) systems for which the configuration manifold carries a hyperbolic metric, *i.e.* a metric of constant negative curvature. In this case the model system is the geodesic flow of a hyperbolic metric. The results generalize and/or are closely related to several theories that contain what may be viewed as stability results, for example, the Aubry-Mather theory of twist maps and the Hedlund-Morse theory of minimal geodesics on closed surfaces. The connection between geodesic flows, Euler-Lagrange flows, and the Aubry-Mather theory has been explored in [B1], [BK], [BP], [Ma1], [Mo] and elsewhere. Our work also builds on a generalization of these theories due to Mather [Ma1], [Ma2] (see also [Mn]). (For a complete survey of the connection of the results here to various other theories, see [BG].)

These theories share the property that the orbits of the dynamical system under consideration correspond to extremals of a variational problem defined in the universal cover of the configuration space. The orbits that correspond to minimums of the variational problem have special properties; they behave approximately like the solutions to the variational problem associated with the model system. This is natural because all orbits of the model problem are minimizers.

Theorem A gives our first way of formalizing the idea that time-periodic Lagrangian systems on hyperbolic manifolds are at least as complicated as the geodesics of a hyperbolic

Stony Brook IMS Preprint #1996/1a
January 1996

metric. Given a hyperbolic geodesic in the Poincaré ball, $\mathbb{H}^n$, the theorem asserts that there are minimizers of the Lagrangian system that are a bounded distance away and have a variety of approximate speeds. Given a path $\gamma : \mathbb{R} \to \mathbb{H}^n$, the notation $\rho(\gamma; a, b)$ means the average displacement in $\mathbb{H}^n$ over the time interval $[a, b]$., *i.e.* the distance from $\gamma(a)$ to $\gamma(b)$ divided by $b - a$.

Theorem A: *Let (M, g) be a closed hyperbolic manifold. Given a Lagrangian L which satisfies Hypothesis 1.0, there are sequences k_i, κ_i, T_i in $\mathbb{R}^+$ depending only on L, with k_i increasing to infinity, such that, for any hyperbolic geodesic $\Gamma \subset \mathbb{H}^n = \tilde{M}$, there are minimizers $\gamma_i : \mathbb{R} \to \tilde{M}$ with $d(\gamma_i, \Gamma) \leq \kappa_i$, $\gamma_i(\pm\infty) = \Gamma(\pm\infty)$, and $k_i \leq \rho(\gamma; c, d) \leq k_{i+1}$ whenever $d - c \geq T_i$.*

The basic idea of the proof is a limit argument that goes back to Morse [M]. Given a hyperbolic geodesic and a speed, we approximates the geodesic by a long minimizing segment with the correct average speed. We then let the approximating segment get longer and longer and pass to the limit. In order to pass to this limit we need some uniform control on the speed and geometry of the minimizing segments. This control comes from showing (Prop 2.1) that minimizing segments are quasi-geodesics in the sense of Gromov. Further, the quasi-geodesic constants depend only on the average speed of the minimizer.

Exact symplectic twist maps on the cotangent bundle of a manifold (defined in §1.3) are in many ways the discrete analogs of the E-L flow. Throughout the paper we indicate how the results for Lagrangian systems can be adapted for the twist map case. In particular, there is a twist map version of Theorem A.

The second main result, Theorem B, focuses on the dynamics generated by the Lagrangian and is a kind of globalization of Theorem A. One way to formulate the fact that a perturbed system is at least as complicated as the model system (*i.e.* the dynamics of the model systems don't go away) is to show that the perturbed system always has a invariant set that carries the dynamics of the minimal model. More precisely, one shows that there is a compact invariant set that is semiconjugate to the minimal model. MacKay and Denvir ([MD]) have recently extended Morse's results to the case with boundary and proved a result giving this semiconjugacy. Also Gromov ([G1]) and others have obtained semiconjugacies in the case of geodesic flows.

Theorem B is a semiconjugacy theorem for time-periodic Lagrangian systems. Given such a Lagrangian, its Euler-Lagrange equations yield a second order time-periodic differential equation on the tangent bundle TM, and thus a vector field on $TM \times S^1$. The solution flow of the vector field (when it exists) is called the E-L flow. The set $\mathcal{M} \subset TM \times S^1$ consists of all the orbits that correspond to minimizing paths in the configuration space.

Theorem B: *Let (M, g) be a closed hyperbolic manifold with geodesic flow g_t. Given a Lagrangian L which satisfies Hypotheses 1.0 with E-L flow ϕ_t, there exists sequences k_i and T_i with k_i increasing to infinity, and a family of compact, ϕ_t-invariant sets $X_i \subset \mathcal{M}$ so that for all i, (X_i, ϕ_t) is semiconjugate to $(T_1 M, g_t)$ and $k_i \leq \rho(\phi_t(x); 0, T) \leq k_{i+1}$, whenever $T \geq T_i$ and $x \in X_i$.*

Note that the geodesic flow of a hyperbolic metric is transitive Anosov and is therefore Bernoulli, has positive entropy, etc (see, *eg.* [HK]). Thus Theorem B implies that the E-L flow is always dynamically very complicated.

In the last subsection we remark on how the semiconjugacy in Theorem B can be used to find ergodic ϕ_t-invariant measures for which an average speed exists almost everywhere. Further, each of these measures "shadow" a unique ergodic g_t-invariant measure.

Section 1: Preliminaries.

In this section we introduce notation and recall some basic results needed in the sequel. For a general discussion of Lagrangian systems see [AM]. For thorough discussions of Lagrangian systems and minimizers the reader is urged to consult [Ma1], [Ma2], and [Mn]. For more details on symplectic twist maps, the reader is referred to [Gl] or [MMS] (*cf* [BK] and [K])

§ **1.1 Lagrangian systems.** The main objects in the Lagrangian formulation of mechanics are a *configuration manifold* M and a real valued function called a *Lagrangian* defined on the tangent bundle TM. The configuration spaces of interest here are closed manifolds M with a fixed Riemannian metric g. The induced norm on the tangent bundle is denoted $\|v\|$. We consider time-periodic systems determined by a C^2-Lagrangian $L : TM \times S^1 \to \mathbb{R}$. The basic variational problem is to find curves $\gamma : [a, b] \to M$ that are extremal for the *action*

$$A(\gamma) = \int_a^b L(\gamma, \dot{\gamma}, t)dt$$

among all absolutely continuous curves $\beta : [a, b] \to M$ that have the same endpoints $\beta(a) = \gamma(a), \beta(b) = \gamma(b)$.

Under appropriate hypothesis (eg. γ is C^1), such a γ satisfies the Euler-Lagrange second order differential equations

$$\frac{d}{dt}\frac{\partial L}{\partial v}(\gamma(t), \dot{\gamma}(t), t) - \frac{\partial L}{\partial x}(\gamma(t), \dot{\gamma}(t), t) = 0.$$

Using local coordinates these equations yield a first order time-periodic differential equation on TM, and thus in the standard way, a vector field on $TM \times S^1$. Since $TM \times S^1$ is not compact it is possible that trajectories of this vector field are not defined for all time in $\mathbb{R}$ and thus do not fit together to give a global flow (i.e. an $\mathbb{R}$-action). When the flow does exist, it is called the *Euler-Lagrange (E-L) flow*.

All Lagrangians in this paper are assumed to satisfy the following hypotheses.

Standing Hypotheses 1.0: *L is a C^2-function $L : TM \times S^1 \to \mathbb{R}$ that satisfies:*

(a) Convexity: $\frac{\partial^2 L}{\partial v^2}$ is positive definite.

(b) Completeness: The Euler-Lagrange flow determined by L exists.

(c) Superquadratic: There exists a $C > 0$ so that $L(x, v, t) \geq C\|v\|^2$.

These assumption are the same as those in Mather [Ma1] and Mañé [Mn] apart from (c), where they only assume $\frac{L(x,v,t)}{\|v\|} \to \infty$ when $\|v\| \to +\infty$. We need the stronger condition in the proof of Lemma 2.2. Note that the addition of a constant to L does not change the E-L flow.

Remark: Condition 1.0(a) is the classical Legendre condition. Thus the E-L flow derived from such an L is conjugate under the Legendre transform to a Hamiltonian flow on $T^*M \times$

S^1. In particular, if L is time *independent*, orbits of the E-L flow are constrained to the Legendre transforms of the constant energy surfaces of the Hamiltonian flow. In the time independent case, the use of this fact along with the Jacobi metric (or finsler) results in a considerable simplification of many the proofs in this paper. The time dependent case is more much interesting and is our focus here.

Example: Mechanical Lagrangians. As pointed out by Mañé, Hypotheses 1.0 are satisfied for mechanical Lagrangians, i.e. those of the form

$$L(x, v, t) = \frac{1}{2} \|v\|_h^2 - V(x, t), \quad V \leq 0$$

where the norm is taken with respect to any Riemannian metric h. (In fact, under some conditions, one may allow the norm to vary with time. See [Mn], page 44).

§**1.2 Minimizers.** Of particular interest in the Lagrangian theory are extremals of the variational problem that minimize in the following sense. If $\tilde{M}$ is a regular covering space of M, L lifts to a real valued function (also called L) defined on $T\tilde{M} \times S^1$. A curve segment $\gamma : [a, b] \to \tilde{M}$ is called a *$\tilde{M}$-minimizing segment* or an *$\tilde{M}$-minimizer* if it minimizes the action among all absolutely continuous curves $\beta : [a, b] \to \tilde{M}$ which have the same endpoints.

A fundamental theorem of Tonelli implies that if L satisfies Hypotheses 1.0, then given $a < b$ and two distinct points $x_a, x_b \in \tilde{M}$ there is always a minimizer γ with $\gamma(a) = x_a$ and $\gamma(b) = x_b$. Moreover such a γ is automatically C^2 and satisfies the Euler-Lagrange equations (this uses the completeness of the E-L flow). Hence its lift, $(\gamma(t), \dot{\gamma}(t), t)$, to $TM \times S^1$ is a solution of the E-L flow. A curve $\gamma : \mathbb{R} \to \tilde{M}$ is called a *minimizer* if $\gamma_{|_{[a,b]}}$ is a minimizer for all $[a, b] \subset \mathbb{R}$. When the domain of definition of a curve is not explicitly given, it is assumed to be $\mathbb{R}$.

Mather [Ma1] and Mañé [Mn] use $\overline{M}$ minimizers where $\overline{M}$ is the universal free Abelian cover. The universal cover (which we denote $\tilde{M}$ from now on) is used here. If γ is an $\tilde{M}$-minimizer, we will simply say it is a minimizer.

Our main task is to get control of the speed and geometry of minimizers. Given a smooth curve $\gamma : [c, d] \to \tilde{M}$ and a segment $[a, b] \subset [c, d]$, the *average displacement* in the cover over the interval $[a, b]$ is measured by

$$\rho(\gamma; a, b) = \frac{d(\gamma(a), \gamma(b))}{b - a}$$

where d is the topological metric on the universal cover constructed from the lift of the given Riemannian metric g. The length of $\gamma_{|_{[a,b]}}$ is denoted $\ell(\gamma; a, b)$, and the *action* over the interval $[a, b]$ is

$$A(\gamma; a, b) = \int_a^b L(\gamma, \dot{\gamma}, t) dt.$$

In all these notations the absence of the last two arguments indicates the quantity is computed for the entire interval of definition, thus $\rho(\gamma) = \rho(\gamma; c, d)$, etc.

Using the fact that L is superquadratic as assumed in 1.0(c), we obtain simple but very useful estimates on the average action of minimizers. The estimates are essentially in

Mañé [Mn] and Mather [Ma1], but the versions given here are slightly more exact as our assumptions on L are slightly stronger. The proof follows [Mn], Theorem 3.3.

Lemma 1.1: *Given a Lagrangian* L *satisfying Hypothesis 1.0, let* $C_K^{max} = \frac{1}{K}\sup\{L(x,v,t) : \|v\| \leq K\}$ *and* $C_K^{min} = \frac{CK}{4}$*, where* C *is the constant in 1.0(c). If* γ *is a minimizer and* $\rho(\gamma; a, b) = K$*, then*

$$C_K^{min} K \leq \frac{1}{b-a} A(\gamma; a, b) \leq C_K^{max} K.$$

Proof: If $\Gamma : [a,b] \to \tilde{M}$ with $\gamma(a) = \Gamma(a)$ and $\gamma(b) = \Gamma(b)$ is a minimizing *geodesic* segment with respect to the given metric g, then $\left\|\dot{\Gamma}\right\| = \rho(\Gamma; a, b) = \rho(\gamma; a, b) = K$. Thus,

$$A(\gamma) \leq A(\Gamma) \leq \int_a^b K C_K^{max}\, dt = K C_K^{max}(b-a),$$

yielding the upper bound.

For the lower bound, first note that

$$\begin{aligned} K(b-a) = d(\gamma(a), \gamma(b)) &\leq \int_a^b \|\dot{\gamma}\|\ dt = \int_{\|\dot{\gamma}\| > \frac{K}{2}} \|\dot{\gamma}\|\ dt + \int_{\|\dot{\gamma}\| \leq \frac{K}{2}} \|\dot{\gamma}\|\ dt \\ &\leq \int_{\|\dot{\gamma}\| > \frac{K}{2}} \|\dot{\gamma}\|\ dt + \frac{K}{2}(b-a), \end{aligned}$$

and so

$$\int_{\|\dot{\gamma}\| > \frac{K}{2}} \|\dot{\gamma}\|\, dt \geq \frac{K}{2}(b-a).$$

Thus using the Cauchy-Schwartz inequality and 1.0(c),

$$\begin{aligned} A(\gamma) &\geq \int_{\|\dot{\gamma}\| > \frac{K}{2}} L(\gamma, \dot{\gamma}, t)\ dt \geq C \int_{\|\dot{\gamma}\| > \frac{K}{2}} \|\dot{\gamma}\|^2\ dt \\ &\geq \frac{C}{b-a}\left(\int_{\|\dot{\gamma}\| > \frac{K}{2}} \|\dot{\gamma}\|\ dt\right)^2 \geq \frac{CK^2}{4}(b-a). \end{aligned}$$

□

Remark: Note that $K \mapsto C_K^{min}$ is a continuous function that increases monotonically to infinity, while $K \mapsto C_K^{max}$ is continuous and grows to infinity (since $C_K^{max} \geq CK$). Note also that KC_K^{max} is monotone nondecreasing. These facts will be used frequently in the sequel without further mention.

Example: Mechanical Lagrangians. Consider again the mechanical Lagrangian

$$L(x, v, t) = \frac{1}{2}\|v\|_h^2 - V(x, t)$$

where $\| \ \|_h$ comes from a Riemannian metric, and we let $\max V = 0$ and $\min V := V_{min}$. If B_1 and B_2 are the positive constants such that

$$B_1 \|v\|^2 \leq \|v\|_h^2 \leq B_2 \|v\|^2, \tag{1.1}$$

where $\|\cdot\|$ is the norm coming from the fixed reference metric g (*eg.* of constant curvature), one readily computes that

$$C_K^{min} = \frac{B_1 K}{8}, \quad C_K^{max} = \frac{1}{2} B_2 K - \frac{V_{min}}{K}$$

(see [BG] for slightly better estimates in mechanical case).

§**1.3 Exact symplectic twist maps.** An *exact symplectic twist map* F is a map from a subset U of the cotangent bundle of a manifold N (which we allow to be noncompact) into U, which comes equipped with a *generating function* $S : N \times N \to \mathbb{R}$ that satisfies

$$F^*(p\,dx) - p\,dx = P\,dX - p\,dx = dS(x, X), \tag{1.2}$$

where (X, P) are the coordinates of $F(x, p)$ (this can also be written in a coordinate free manner).

Because the one-form $P\,dX - p\,dx$ in (1.2) is exact, one says that F is *exact*. Note that taking the exterior differential of (1.2) yields $dP \wedge dX = dp \wedge dx$, and so any exact F is also symplectic, *i.e.* it preserves the standard symplectic form. The fact that S is expressed using the coordinates (x, X) instead of (x, p) is the *twist condition*. Given S, one can retrieve the map (at least implicitly) from $p = -\frac{\partial S}{\partial x} and P = \frac{\partial S}{\partial X}$. This can be done globally (*i.e.* $U = T^*N$) only when N is diffeomorphic to a fiber of T^*N, for example when N is the covering space of the n-torus or of a manifold of constant negative curvature.

The variational problem for Lagrangian systems translates into a discrete variational problem for twist maps: the role of curves in the continuous setting is taken by sequences of points ("integer time curves"), and the action of a finite sequence $\mathbf{x} = \{x_n, \ldots, x_m\}$ is given by $W(\mathbf{x}) = \sum_n^{m-1} S(x_k, x_{k+1})$. This corresponds closely to the continuous setting when the exact symplectic twist map F is the time-one map of an E-L flow. In this case, $S(x, X) = \int_0^1 L(x, \dot{x}, t)dt$, where $x(t)$ is the minimizer with endpoints x and X.

In direct correspondence to Lagrangian systems, critical points of W (with fixed time and configuration endpoints) correspond to orbits of F (this is closely related to the method of broken geodesics in Riemannian geometry). Action minimizers are sequences that minimize W over any of their subsegments. The natural growth condition on the generating function

$$S(x, X) \geq C\, dist^2(x, X),$$

implies the analog of Tonelli's theorem: minimizers always exist between any two points over any given (integer) interval of time. Moreover, there is an exact analog of Lemma 1.1: the average action of minimizers is bounded below and above by functions of the average displacement. The proof is virtually identical to the continuous time case, replacing geodesics with orbits of the time-one of the geodesic flow.

Example: Generalized standard maps. Let M be $\mathbf{T}^n$ or a closed hyperbolic manifold, and let $N = \tilde{M}$ be the universal cover $\mathbf{R}^n$ or $\mathbb{H}^n$, respectively. On the covering space, define the *generalized standard map* using its generating function $\tilde{M} \times \tilde{M} \to \mathbb{R}$,

$$S(x, X) = \frac{1}{2} dist^2(x, X) + V(x)$$

where the distance $dist$ is induced by the Euclidean metric in $\mathbf{R}^n$, or the hyperbolic metric on $\mathbb{H}^n$, and $V(x)$ is $\pi_1(M)$-equivariant, *i.e.* it descends to a function on M. A short argument shows that one can use the relation (1.2) to solve for (X, P) in terms of (x, p) and thus obtain an exact symplectic twist map on $T^*\tilde{M}$ that, in turn, induces a map on T^*M (also called a twist map). For more general examples, *cf* [Gl].

Remark: In certain cases the twist map theory overlaps with the continuous theory. If a twist map f of $T^*\mathbf{T}^n$ has a generating function that is super quadratic in $\|X - x\|$, the mixed partial $\partial_{12}S$ is symmetric, and for some $a > 0$ satisfies the convexity condition

$$< \partial_{12}S(x, X).v, v > \leq -a \|v\|^2$$

uniformly in (x, X), then F is the time-one map of an E-L flow derived from a one-periodic Lagrangian that is superquadratic in the velocity. Moser [Mo] gives the proof in the case $n = 1$. Bialy and Polterovitch remark in [BP] that Moser's proof goes through in the case $n > 1$. This is not quite so, but they subsequently obtained a different proof (personal communication). Note that the generating function for the generalized standard map satisfies these hypothesis.

§**1.4 Hyperbolic Geometry.** We recall some basic facts about hyperbolic geometry and manifolds. For more information see *eg.* [BKS]. A closed manifold M is called *hyperbolic* if there is a Riemannian metric g on M that has curvature identically equal to -1. The universal cover $\tilde{M}$ of a n-dimensional hyperbolic manifold is homeomorphic to $\mathbb{R}^n$. We identify $\tilde{M}$ with the n-dimensional Poincaré Disk $\mathbb{H}^n$, and so the group of covering transformations can be identified with a discrete subgroup, isomorphic to $\pi_1(M)$, of the set of isometrics of $\mathbb{H}^n$. This group action has a fundamental domain with compact closure and under the quotient by the action the metric on $\mathbb{H}^n$ descends to the hyperbolic metric on M. As a consequence of the Mostow Rigidity Theorem, a closed hyperbolic manifold of dimension three or greater carries a unique hyperbolic metric. If M is a surface, it carries many hyperbolic metrics, but their geodesic flows are all orbit equivalent ([G1])

The sphere at infinity, S_∞, is the usual Euclidean sphere that is the boundary of $\mathbb{H}^n$ in the *Euclidean* topology. The geodesics in $\mathbb{H}^n$ are semi-circles that are orthogonal to S_∞. In this paper these hyperbolic geodesics will always be oriented and parameterized by arclength. For a geodesic Γ, the notations $\Gamma(\infty)$ and $\Gamma(-\infty)$ refer to the limit points of Γ on S_∞ in the forward and backward directions, respectively. More generally, for $\gamma : \mathbb{R} \to \mathbb{H}^n$, the notations $\gamma(\infty)$ and $\gamma(-\infty)$ refer to the limit points *in the Euclidean topology* of γ in forward and backward time if these limits exist and are contained in S_∞. Implicit in this notation is the fact that γ has no nontrivial limit points in $\mathbb{H}^n$, *i.e.* $\gamma : \mathbb{R} \to \mathbb{H}^n$ is proper.

For a pair of points $x, y \in \mathbb{H}^n$, $d(x, y)$ denotes their distance in the hyperbolic metric. The notion of a quasi-geodesic, central to Gromov 's work on hyperbolic groups is also of central importance here. Given $\lambda > 1$ and $\epsilon > 0$, a curve $\gamma : \mathbb{R} \to \mathbb{H}^n$ or a curve segment $\gamma : [a, b] \to \mathbb{H}^n$ is called a (λ, ϵ)-quasi-geodesic if

$$\lambda^{-1}(d - c) - \epsilon \leq d(\gamma(c), \gamma(d)) \leq \lambda(d - c) + \epsilon$$

for all $[c, d]$ in the domain of γ.

The next theorem, usually called "Stability of quasi-geodesics", gives the most important property of quasi-geodesics. It is true in the broader context of what are usually called δ-hyperbolic spaces, but we just state the result needed here. Given two closed subsets $X, Y \subset \mathbb{H}^n$, $d(X, Y)$ denotes their Hausdorff distance as induced by the hyperbolic metric. For a proof and more information see [GH], [CDP] or [G2].

Theorem 1.2: *Given $\lambda > 1$ and $\epsilon > 0$, there exists a $\kappa > 0$ so that whenever γ is a (λ, ϵ)-quasi-geodesic segment in $\mathbb{H}^n$ and Γ_0 is the geodesic segment connecting the endpoints of γ, then $d(\gamma, \Gamma_0) < \kappa$. If γ is a (λ, ϵ)-quasi-geodesic, then $\gamma(\infty)$ and $\gamma(-\infty)$ exist and further, if Γ is the geodesic connecting $\gamma(\infty)$ and $\gamma(-\infty)$, then $d(\gamma, \Gamma) < \kappa$.*

Remark: Although the notion of quasi-geodesic makes sense on any Riemannian manifold, or even metric space, it only yields the strong consequence as in the last theorem for hyperbolic manifolds or, more generally, δ-hyperbolic spaces. One can easily construct counter examples to the theorem in Euclidean space. These counter-examples contain the seeds of failure for the analog of Theorem A on the 3-torus. See Section 3.3 in [BG] for more details.

Section 2: Minimizers and quasi-geodesics.

The main purpose of this section is to prove Theorem A.

§**2.1 Minimizing segments are quasi-geodesics.** Throughout this section we fix a Lagrangian L that satisfies the Hypotheses 1.0. The first proposition gives uniform upper and lower bounds on the local average displacement of minimizing segments with a given total average displacement in the cover. Part (a), due to Mather (see the proof of Proposition 4 in [Ma1]), gives an absolute upper bound of the velocity. Mather considers minimizers in the universal free Abelian cover, but his proof works without change in the universal cover. Part (b) says that points on minimizers cannot go too slow for too long. The main ingredient in the proof of (b) is Lemma 2.2. Its proof was inspired by Mather's proof of (a). It uses an argument of curve shortening type. One assumes that a minimizer does not have the desired property, and this allows one can to construct another curve that has lesser action, yielding a contradiction. Part (c) is a consequence of (a) and (b).

Proposition 2.1: *Let $\gamma : [a, b] \to \tilde{M}$ be a minimizing segment with average displacement $\rho(\gamma; a, b) = K$ and $b - a \in \mathbb{N}$.*

(a) (Mather) There is a $K'' > K$ depending only on L and K such that for all $t \in [a, b]$, $\|\dot{\gamma}(t)\| \leq K''$.

(b) There exists $K_0 > 0$ such that for all $K > K_0$ there is a $k'' > 0$ and an $N_0 \in \mathbb{N}$ depending only on K and L so that for any interval $[c, d] \subset [a, b]$ with $d - c \geq N_0$, one has $\rho(\gamma; c, d) \geq k''$.

(c) With K and K_0 as in (b), there are constants $\lambda > 1$ and $\epsilon > 0$ depending only on K and L so that γ is a (λ, ϵ)-quasi-geodesic segment.

Remark: This proposition is true on any compact Riemannian manifold for Lagrangian systems satisfying Hypothesis 1.0. *If the manifold is hyperbolic*, Theorem 1.2 implies that a minimizer γ stays at a uniform distance from a geodesic Γ. However, because Theorem 1.2 is generally not true on non-hyperbolic manifolds, one cannot obtain a version of Theorem A in that case. As noted after Theorem 1.2, this provides a heuristic explanation for why straightforward generalizations of the Aubry-Mather theory fail on the 3-torus. In the Hedlund metric, one can construct a sequence of minimizing segments, each of which are quasi-geodesics by Proposition 2.1, but whose distance to any geodesic grows to infinity (see [BG], Section 4.2).

§**2.2 The main technical lemma.** The main step in proving Proposition 2.1 is a technical lemma that deals with a special case of Proposition 2.1(b). It gives a lower bound on the average displacement when the subinterval has a specified integer length.

Lemma 2.2: *There exists $K_0 > 0$ such that for all $K > K_0$ there is a K' with $0 < K' < K$ and an $N_0 \in \mathbb{N}$ so that whenever $\gamma : [a,b] \to \tilde{M}$ is a minimizing segment with $b - a \in \mathbb{N}$ and $\rho(\gamma; a, b) = K$, then for any interval $[c,d] \subset [a,b]$ with $d - c = N_0$, and $b - d \in \mathbb{N}$, one has $\rho(\gamma; c, d) \geq K'$.*

Proof: Since C_K^{max} is a continuous function of $K \geq 0$ that is bounded below by $\frac{CK}{4}$, $m := \min_{K \geq 0} C_K^{\max}$ is achieved at some finite K. If we let $K_0 = 28m/C$ (with C as in 1.0(c)), then for any $K > K_0$ we can find a K' with $0 < K' < K$ and

$$0 < 7C_{2K'}^{max} < CK/4 = C_K^{min}. \tag{2.1}$$

Now let N_0 be the positive even integer with

$$\frac{K}{K'} \leq \frac{N_0}{2} < \frac{K}{K'} + 1. \tag{2.2}$$

Given $K > K_0$ and K' and N_0 as in the last paragraph, assume γ, a, b, c, d satisfy the hypothesis of the lemma, but $\rho(\gamma; c, d) < K'$. We will construct a curve $\gamma^* : [a,b] \to \tilde{M}$ of lesser action than γ, yielding a contradiction.

First note that we can find $[a', b'] \subset [a,b]$ such that $b' - a' = 1$, $a' - a \in \mathbb{N}$, and $\rho(\gamma; a', b') > K$. Indeed, since $\rho(\gamma; c, d) < K'$, either $\rho(\gamma; a, c) > K$ or $\rho(\gamma; d, b) > K$ (the lost speed must be made up for somewhere). Say $\rho(\gamma; a, c) > K$, the other case is similar. Subdivide $[a, c]$ into intervals of length 1 (recalling that $b - a \in \mathbb{N}$). Clearly, one of these intervals, which we call $[a', b']$, will satisfy the above conditions. Note in particular that $d(\gamma(a'), \gamma(b')) > K$.

We now construct $\gamma^* : [a,b] \to \tilde{M}$. Let $b'' \in [a', b']$ be such that $d(\gamma(a'), \gamma(b'')) = K$, and let $n = \frac{N_0}{2}$. The curve γ^* is defined on sub-intervals of $[a,b]$ as follows:

On:	$[a, a']$,	$\gamma^*(t) = \gamma(t)$
	$[a', b'' + n]$,	γ^* is a minimizing segment with $\gamma^*(a') = \gamma(a')$, $\gamma^*(b'' + n) = \gamma^*(b'')$
	$[b'' + n, c + n]$,	$\gamma^*(t) = \gamma(t - n)$
	$[c + n, d]$,	γ^* is a minimizing segment with $\gamma^*(c + n) = \gamma(c)$, $\gamma^*(d) = \gamma(d)$
	$[d, b]$,	$\gamma^*(t) = \gamma(t)$.

From the time periodicity of L, it follows that

$$A(\gamma) - A(\gamma^*) = A(\gamma; a', b'') + A(\gamma; c, d) - A(\gamma^*; a', b'' + n) - A(\gamma^*; c+n, d)$$
$$\stackrel{\text{def}}{=} A_1 + A_2 - A_1^* - A_2^*.$$

We will show that this difference is positive by deriving estimates for A_1, A_2, A_1^*, and A_2^*.

To estimate A_1, note that

$$\frac{d(\gamma(a'), \gamma(b''))}{b'' - a'} = \frac{K}{b'' - a'} \stackrel{\text{def}}{=} \tilde{K} \geq K.$$

Thus, using Lemma 1.1,

$$A_1 = A(\gamma; a', b'') \geq C_{\tilde{K}}^{min} \tilde{K}(b'' - a') \geq C_K^{min} K.$$

All we need about A_2 is that $A_2 \geq 0$.

Now for A_1^*, note that

$$\frac{d(\gamma^*(a'), \gamma^*(b'' + n))}{b'' + n - a'} = \frac{d(\gamma(a'), \gamma(b''))}{n + b'' - a'} = \frac{K}{n + b'' - a'} < \frac{K}{n} \leq K',$$

and hence using Lemma 1.1 and (2.2),

$$A_1^* = A(\gamma^*; a', b'' + n) \leq C_{K'}^{max} K'(b'' + n - a') \leq 3K C_{2K'}^{max}.$$

Finally, to estimate A_2^*, we observe that

$$\frac{d(\gamma(c+n), \gamma(d))}{d - (c+n)} = \frac{d(\gamma(c), \gamma(d))}{n} = \frac{d(\gamma(c), \gamma(d))}{d - c} \cdot \frac{d - c}{n} < 2K',$$

and so using (2.2),

$$A_2^* = A(\gamma^*; c+n, d) \leq C_{2K'}^{max} 2K'(d - c - n) = 2C_{2K'}^{max} K' n \leq 4C_{2K'}^{max} K.$$

Now, if K and K' satisfy (2.1), clearly

$$A(\gamma) - A(\gamma^*) = A_1 + A_2 - A_1^* - A_2^* \geq K(C_K^{min} - 7C_{2K'}^{max}) > 0,$$

a contradiction to the assumption that γ is a minimizer. $\square$

§**2.3 Proof of Proposition 2.1.** As noted above, part (a) is proved in [Ma1]. For part (b), first note that Lemma 1.2 implies that whenever $d - c = N_0$ and $b - d \in \mathbb{N}$, we have

$$\ell(\gamma; c, d) \geq d(\gamma(c), \gamma(d)) \geq K'(d - c). \tag{2.3}$$

We now show that slightly weaker is true for more general $[c, d]$, specifically, that for any $[c, d] \subset [a, b]$ with $d - c > N_0$,

$$\ell(\gamma; c, d) \geq \frac{K'}{2}(d - c). \tag{2.4}$$

Let c' be the smallest integer translate of a that is bigger than c (i.e. $c' = a + \lfloor c - a \rfloor + 1$). Divide $[c', d]$ into m intervals $[c_i, c_{i+1}]$ of length N_0, plus possibly one interval of lesser length r (i.e. $d - c' = mN_0 + r, r < N_0$). If we let $d' = c' + mN_0 = d - r$, then

$$\frac{\ell(\gamma; c, d)}{d - c} \geq \frac{\ell(\gamma; c', d')}{d - c}$$

and

$$\frac{\ell(\gamma; c', d')}{d - c} = \frac{1}{d - c} \sum_{0}^{m-1} \ell(\gamma; c_i, c_{i+1}) \geq \frac{N_0}{d - c} \sum_{0}^{m-1} \frac{d(\gamma(c_i), \gamma(c_{i+1}))}{N_0} \geq \frac{mN_0}{d - c} K',$$

the last inequality coming from (2.3). But

$$\frac{mN_0}{d - c} K' = \frac{mN_0}{mN_0 + r + (c - c')} K' > \frac{mN_0}{(m + 1)N_0 + 1} K' \geq \frac{mN_0}{(m + 1)N_0} K'.$$

Since $\frac{m}{m+1}$ increases with m and $m \geq 1$, we have

$$\frac{\ell(\gamma; c, d)}{d - c} \geq \frac{1}{2} K',$$

which proves (2.4).

Now using the Cauchy-Schwartz inequality and what we have just proved,

$$\begin{aligned} \frac{1}{d - c} \int_c^d L(\gamma, \dot{\gamma}, t) dt &\geq \frac{C}{d - c} \int_c^d \|\dot{\gamma}\|^2 \, dt \geq \frac{C}{(d - c)^2} \left(\int_c^d \|\dot{\gamma}\| \, dt \right)^2 \\ &= \frac{C}{(d - c)^2} \left(\ell(\gamma; c, d) \right)^2 \geq \frac{C(K')^2}{4} \end{aligned}$$

On the other hand, part (a) shows that $\rho(\gamma; a, b) = K$, implies $\rho(\gamma; c, d) \leq K''$ for any $[c, d] \subset [a, b]$ and so by Lemma 1.1,

$$\frac{1}{d - c} \int_c^d L(\gamma, \dot{\gamma}, t) dt \leq C_{K''}^{max} \rho(\gamma; c, d) = \frac{C_{K''}^{max} d(\gamma(c), \gamma(d))}{d - c}.$$

Part (b) then follows by letting

$$k'' = \frac{C(K')^2}{4C_{K''}^{max}}.$$

Part (c) follows from (a) and (b) by letting $\lambda = \max\{K'', 1/k'', 1\}$ and $\epsilon = N_0/\lambda$. $\square$

§**2.4 Proof of Theorem A.** Given a path $\gamma : \mathbb{R} \to \tilde{M}$, we denote by $D\gamma = (\gamma, \dot{\gamma})$ its differential, which is a path in $T\tilde{M}$.

Fix an oriented geodesic with a given parameterization by arclength $\Gamma : \mathbb{R} \to \mathbb{H}^n$ and a $K > K_0$ with K_0 as in Proposition 2.1. Let $\gamma_N : [-N, N] \to \mathbb{H}^n$ be a minimizing segment with $\gamma_N(-N) = \Gamma(-KN)$ and $\gamma_N(N) = \Gamma(KN)$, and thus $\rho(\gamma_N; -N, N) = K$. By

Proposition 2.1(c), γ_N is a (λ, ϵ)-quasi-geodesic and so by Theorem 1.1, $d(\gamma_N, \Gamma([-KN, KN])) < \kappa$, with λ, ϵ and κ depending on K and not on N.

Fix a fundamental domain of the manifold M in the universal cover $\mathbb{H}^n$ and call it S. We may assume that S has compact closure and that Γ intersects S. If S_κ denotes the closure of $\{x : d(x, S) \leq \kappa\}$, then γ_N intersects S_κ for all N. Thus we may find a t_N with $\gamma_N(t_N) \in S_\kappa$.

Now let $z_N = D\gamma_N(t_N)$ and let $\tau_N \in S^1$ be the residue of t_N mod 1. Then using the $K'' = K''(K)$ from Proposition 2.1(a), we have that for all N, (z_N, τ_N) is contained in the compact space $\{(x, v, t) \in T\tilde{M} \times S^1 : x \in S_\kappa \text{ and } \|v\| \leq K''\}$. Thus there exits (z, τ) and a subsequence N_i, with $(z_{N_i}, \tau_{N_i}) \to (z, \tau)$ in $T\tilde{M} \times S^1$.

Let $y = (z, \tau)$, and let $y(t) = (\gamma(t), \dot{\gamma}(t), t)$ be the trajectory of the lift of the E-L flow with initial conditions y. If k'', N_0, λ and ϵ are as in Proposition 2.1, then the continuity of solutions of differential equations with respect to initial conditions implies that $\gamma(t)$ satisfies $k'' \leq \rho(\gamma; c, d) \leq K''$ whenever $d - c \geq N_0$, since each γ_N satisfies this inequality. Further, γ is a (λ, ϵ)-quasi-geodesic. Thus by Theorem 1.1, there is an oriented geodesic Γ_1 with $d(\gamma, \Gamma_1) < \kappa$ and $\gamma(\pm\infty) = \Gamma_1(\pm\infty)$. Finally, it is clear that $\Gamma_1 = \Gamma$ since the γ_N converge to γ pointwise and all the γ_N and γ are quasi-geodesics with the same constants.

Doing this construction for each K yields a family of minimizers. This family clearly contains the sequence required for the theorem. $\square$

Remarks:

Note that in contrast to the Aubry-Mather theory, Theorem A does not yield all speeds in every direction (in the Aubry-Mather case there is just one direction). This is almost certainly not just artifact of the proof as there are autonomous mechanical systems on the two-torus that have gaps in the speed spectrum (see Section 6 in [BG]). Nonetheless, one would expect that Theorem A could be improved.

If Γ is a closed geodesic, then the shadowing minimizer does not necessarily have to be a closed orbit. By minimizing the action in the space of loops of integer period in each free homotopy class, one find periodic orbits of all free homotopy types (see [Bn] and Proposition 7.1 in [Mn]) However, these periodic orbits may not be minimizers in the sense used here.

The proof of analog of Theorem A for symplectic twist maps is virtually identical to that of the continuous case. By using the integers to define a discrete time, one has an analogous notion of quasi-geodesic for which the analog of Theorem 1.2 holds. Thus one obtains the Theorem using the twist map version of Proposition 2.1.

Section 3: Semiconjugacy with the geodesic flow

The previous section was concerned with minimizing curves in the universal cover and their relation to hyperbolic geodesics. In this section we consider the dynamical implications of those results and focus on the E-L flow on $T\tilde{M} \times S^1$. We prove Theorem B by uniformizing the results of the previous section and obtain semiconjugacies from subsets of the set of minimizers to the geodesic flow of the fixed hyperbolic metric.

§**3.1 Definitions.** The E-L flow ϕ_t on $TM \times S^1$ lifts to a flow $\tilde{\phi}_t$ on $T\tilde{M} \times S^1$. There are two projections that are of importance here. The projection from the covering space to the base is $\pi : T\tilde{M} \times S^1 \to TM \times S^1$. The projections from bundles to the bases are both denoted p and are $p : T\tilde{M} \times S^1 \to \tilde{M}$ and $p : TM \times S^1 \to M$.

Take an orbit $\tilde{\phi}_t(z)$ of the lift of the E-L flow and let $\gamma(t) = p(\tilde{\phi}_t(z))$. The notions that we used in the previous section to describe curves γ will also be applied to orbits. Thus if γ is a minimizer or a (λ, ϵ)-quasi-geodesic, then this same label is attached to the orbit. The subset $\mathcal{M}$ of $TM \times S^1$ or $T\tilde{M} \times S^1$ denotes the set of orbits that are minimizers. We define $\rho(\tilde{\phi}_t(z); T_1, T_2) = \rho(\gamma; T_1, T_2)$, $\omega(z) = \gamma(\infty)$, and $\alpha(z) = \gamma(-\infty)$ (when the latter two exist).

The E-L equations are turned into a vector field using local coordinates in which the second coordinate is the velocity. Thus if π_2 is projection on the second component (*i.e.* $z = (x, v, \tau) \in TM \times S^1$, and $\pi_2(z) = v$), then for an orbit $\phi_t(z)$, $\pi_2(\phi_t(z)) = d(p(\phi_t(z)))/dt$.

The space of all the geodesics in $\mathbb{H}^n$ is denoted $\mathcal{G}$. This space will always be given the topology of Hausdorff convergence on compact subsets. With this topology $\mathcal{G}$ is homeomorphic to $S_\infty \times S_\infty - \{\text{diagonal}\}$ where a geodesic Γ is identified with the pair $(\Gamma(-\infty), \Gamma(\infty))$. Recall that we have fixed a hyperbolic metric g on M. Its geodesic flow is defined on T_1M (and is in fact the restriction to the invariant set T_1M of the E-L flow of the Lagrangian $L = \|v\|_g^2$). The geodesic flow g_t lifts to a flow $\tilde{g}_t$ on $T_1\mathbb{H}^n$.

Two flows (X, ϕ_t) and (Y, ψ_t) are said to be ***semiconjugate*** (or sometimes orbit semi-equivalent) if there is a continuous surjection $f : X \to Y$ that takes orbits of ϕ_t to those of ψ_t preserving the direction of the flow, but not necessarily the time parameterization. Note that f is locally injective when restricted to an orbit of ϕ_t, but f may take many orbits of ϕ_t to the same orbit of ψ_t.

Given a point in $\mathbb{H}^n$ and a geodesic, (hyperbolic) orthogonal projection sends the point to a point on the geodesic. To get a image point in the unit tangent bundle we define $\Sigma : \mathcal{G} \times \mathbb{H}^n \to T_1\mathbb{H}^n$ via $\Sigma(\Gamma, z) = (x, v)$ where x is the orthogonal projection of z onto Γ and v is the unit vector tangent to Γ at x.

§**3.2 Proof of Theorem B.** Given the Lagrangian L, find K_0 as in Proposition 2.1. Now fix $K > K_0$ and let $\lambda, \epsilon, k'', K''$, and N_0 (all depending on K) be as in that proposition. Define the set $Q_K \subset T\tilde{M} \times S^1$ as the set of z that satisfy

(1) The orbit $\tilde{\phi}_t(z)$ is a minimizer and a (λ, ϵ)-quasi-geodesic.

(2) $k'' \le \rho(\tilde{\phi}_t(z); T_1, T_1 + T)$ for all T_1, whenever $T \ge N_0$.

(3) $\left\|\pi_2(\tilde{\phi}_t(z))\right\| \le K''$, for all $t \in \mathbb{R}$.

Note that Q_K is $\tilde{\phi}_t$ invariant and closed and $\pi(Q_K) \subset TM \times S^1$ is compact. Since each orbit in Q_K is a (λ, ϵ)-quasi-geodesic, by Theorem 1.2 there is a constant κ and for each $z \in Q_K$ a unique geodesic denoted Γ_z with $\Gamma_z(\infty) = \omega(z)$, $\Gamma_z(-\infty) = \alpha(z)$, and $d(\Gamma_z, p(\tilde{\phi}_t(z))) \le \kappa$. Thus $z_i \to z$ in Q_K implies that $\Gamma_{z_i} \to \Gamma_z$, and so the map $Q \to \mathcal{G}$ given by $z \mapsto \Gamma_z$ is continuous.

This implies that the "projection" $\sigma : Q_K \to T_1\tilde{M}$ defined by $\sigma(z) = \Sigma(\Gamma_z, p(z))$ is also continuous. In addition, by construction, σ takes orbits of $\tilde{\phi}_t$ to those of the lift of the geodesic flow $\tilde{g}_t$. Also, σ is equivariant, *i.e.* it descends to a map $\pi(Q_K) \to T_1M$. Further, by Theorem A, σ is onto. Unfortunately, σ does not preserve the direction of time as it is perhaps not locally injective when restricted to the orbits of ϕ_t. This is remedied using an averaging technique due to Fuller [F].

Fix a parameterization by arclength for each geodesics in $\mathbb{H}^n$. We will use the parameterization to add and subtract elements on the geodesics. Given $z \in Q_K$ and $t \in \mathbb{R}$, let

$a(z,t) = \sigma(\tilde{\phi}_t(z)) - \sigma(z)$, or equivalently, $a(z,t)$ is the unique $s \in \mathbb{R}$ with $\tilde{g}_s(\sigma(z)) = \sigma(\tilde{\phi}_t(z))$. Note that a is an additive cocycle for $\tilde{\phi}_t$, *i.e.* $a(z, t_1 + t_2) = a(z, t_1) + a(\tilde{\phi}_{t_1}(z), t_2)$, for all t_1, t_2. Given $\alpha_1 > 0$ define

$$\bar{\sigma}_{\alpha_1}(z) = \sigma(z) + \frac{1}{\alpha_1} \int_0^{\alpha_1} a(z,t)\, dt.$$

Equivalently, $\bar{\sigma}_{\alpha_1}(z) = \tilde{g}_s(\sigma(z))$, where $s = \frac{1}{\alpha_1} \int_0^{\alpha_1} a(z,t)\, dt$. Informally, $\bar{\sigma}_{\alpha_1}(z)$ is the average value of σ over the orbit segment $\tilde{\phi}_{[0,\alpha_1]}(z)$.

Now since for every $z \in Q_K$ we have that $\omega(z) = \Gamma_z(\infty)$, it follows that for each z there is an α_z so that $a(z, \alpha_z) > 0$. Since $\pi(Q_K)$ is compact, we may find an α with $a(z,\alpha) > 0$ for all $z \in Q_K$. Let $\bar{\sigma} = \bar{\sigma}_\alpha$. Now $\bar{\sigma}$ is clearly continuous, equivariant, onto and takes orbits to orbits. We will show that it is injective on orbits of $\tilde{\phi}_t$ by showing that for any $\beta > 0$ and $z \in Q_K$, $\bar{\sigma}(\tilde{\phi}_\beta(z)) - \bar{\sigma}(z) > 0$.

$$\begin{aligned}
\bar{\sigma}(\tilde{\phi}_\beta(z)) - \bar{\sigma}(z) &= \sigma(\tilde{\phi}_\beta(z)) - \sigma(z) + \frac{1}{\alpha}\left(\int_0^\alpha (a(z,\beta+t) - a(z,\beta)) - \int_0^\alpha a(z,t)\right) \\
&= \frac{1}{\alpha}\left(\int_0^\alpha a(z,\beta+t) - \int_0^\alpha a(z,t)\right) \\
&= \frac{1}{\alpha}\left(\int_\beta^{\alpha+\beta} a(z,t) - \int_0^\alpha a(z,t)\right) \\
&= \frac{1}{\alpha}\left(\int_\alpha^{\alpha+\beta} a(z,t) - \int_0^\beta a(z,t)\right) \\
&= \frac{1}{\alpha}\int_0^\beta a(z,t+\alpha) - a(z,t) \\
&= \frac{1}{\alpha}\int_0^\beta a(\tilde{\phi}_t(z),\alpha) \\
&> 0,
\end{aligned}$$

where all integrals are with respect to t, and in the first and sixth equalities we used the cocycle equation for a.

Thus for each $K > K_0$ we have a $\pi(Q_K)$ with $(\pi(Q_K), \phi_t)$ semiconjugate to $(T_1 M, g_t)$. The set of all such $\pi(Q_K)$ clearly contains a sequence X_i as needed for the Theorem. $\square$

Remark: A different perspective can be gained on Theorems A and B by considering the simple case $L(x, \dot{x}, t) = \|\dot{x}\|^2 - V(x,t)$, where $\|\cdot\|$ comes from the hyperbolic metric. In this case, for large velocities (high up in the tangent bundle), the time-one map Φ of the E-L flow (considered as a map from $TM \to TM$) may be thought of as a small perturbation of the time-one map G of the full geodesic flow of the hyperbolic metric. In TM above each periodic geodesic, G has a normally hyperbolic annulus. Restricted to this annulus, G is a twist map. Under small perturbation high up in the bundle, one expects this annulus to persist and Φ restricted to it will also be a twist map. Using Aubry-Mather theory on these various annuli and taking limits one sees that the dynamics of F near infinity to reflect those of G.

There are a number of technical problems with making these arguments precise, but almost certainly these can be overcome. However, we prefer the techniques used here as they are-self contained, more general and allow for fairly explicit estimates on where the persistence occurs.

§**3.3 Semiconjugacies, time changes and ergodic measures.** Theorems A and B give the existence of a large collection of minimizers, but the theorems are somewhat unsatisfactory because they do not yield minimizers for which an asymptotic speed necessarily exits. To obtain results of this type, either one needs a great deal of control over the minimizers (for twist maps of the annulus this comes from the low dimensionality), or else one uses Ergodic Theory. In this subsection we briefly consider the latter.

The first ingredient, contained in Theorem B, is the semiconjugacy $\bar{\sigma}$ from a compact ϕ_t-invariant set X_i onto (T_1M, g_t). To simplify the exposition, let us fix an X_i and call it X, denote the semiconjugacy by f, and change the name of time on X to s, so the flow restricted to X is denoted ϕ_s. A well known construction using cocycles (*eg.* see [Pa] or [HK]) allows us to perform a time change on the flow ϕ_s to obtain a new flow $\hat{\phi}_t$ such that f is a time-preserving semiconjugacy, *i.e.* $f\hat{\phi}_t = g_t f$ for all t. Further, to each ergodic ϕ_s-invariant measure μ there corresponds an ergodic $\hat{\phi}_t$-invariant measure $\hat{\mu}$ with μ and $\hat{\mu}$ mutually absolutely continuous.

Since f is continuous, for each ergodic g_t-invariant η, the set $f_*^{-1}(\eta)$ is a nonempty compact convex set in the weak topology on measures. The extreme points of this set are ergodic $\hat{\phi}_t$-invariant measures. Thus to each ergodic g_t-invariant measure η there is at least one ergodic ϕ_t-invariant μ with $f_*(\hat{\mu}) = \eta$. We say that η ***corresponds*** to μ.

The second ingredient is a way of measuring the progress of orbits in the universal cover. Given $z \in TM \times S^1$ and $t \in \mathbb{R}$, pick a lift $\tilde{z} \in \tilde{M}$ and let $D(z,t) = d(p(\tilde{\phi}_t(\tilde{z})), p(\tilde{z}))$. Note that the definition of D is independent of the choice of lift $\tilde{z}$ and that $D(z,t) = \rho(z; 0, t)$, with ρ as defined above. Now let

$$D^*(z) = \lim_{t\to\infty} \frac{D(z,t)}{t}$$

if the limit exists. The triangle inequality for the metric d implies that D satisfies $D(z, t+s) \leq D(z,t) + D(\phi_t(x), s)$ for all z, s and t. Thus D is a subadditive cocycle for ϕ_t, and so by Kingman's subadditive ergodic theorem (see, for example, [Po]) we have that D^* exists and is constant almost everywhere with respect to an ergodic ϕ_t-invariant measure μ.

Recalling the original situation from Theorem B, we now see that for each i and for each ergodic g_t-invariant measure η there corresponds a ϕ_t-invariant measure μ supported on X_i such that $D^*(z)$ exists and is constant for μ almost every point z. Further, using the speed bounds from the theorem, we have $k_i \leq D^*(z) \leq k_{i+1}$.

Given the ergodic measure μ, the pair $(D^*(\mu), \eta)$, where η corresponds to μ, can be interpreted as giving the length and direction of a kind of rotation vector. One can see from the definition that the correspondence between ergodic measures invariant under ϕ_t and g_t means vaguely that the dynamical behavior of one "shadows" the other. What is lacking is a precise meaning of this correspondence which doesn't require *a priori* knowledge of a semiconjugacy. This lack is remedied in a subsequent paper ([Bd]).

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