On a-ary Subdivision for Curve Design III. 2m-Point and (2m + 1)-Point Interpolatory Schemes

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III. $2m$-Point and $(2m + 1)$-Point
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Abstract

In this paper, we investigate both the $2m$-point a-ary for any $a \geq 2$ and $(2m + 1)$-point a-ary for any odd $a \geq 3$ interpolatory subdivision schemes for curve design. These schemes include the extended family of the classical 4- and 6-point interpolatory a-ary schemes and the family of the 3- and 5-point a-ary interpolatory schemes, both having been established in our previous papers (Lian [9]) and (Lian [10]).

Keywords: Subdivision; Curve design; Stationary; Refinable functions; a-ary

MSC (2000) #: 14H50; 17A42; 65D17; 68U07

1.   Introduction

THIS is a continuation of both (Lian [9]) and (Lian [10]). The former was for extending the classical 4- and 6-point binary interpolatory subdivision schemes for curve design in (Dyn, et al. [5]) and (Weissman [13]) to a-ary interpolatory schemes for any $a \geq 3$, and the latter was for the 3- and 5-point a-ary interpolatory schemes for any odd $a \geq 3$.

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Recall that a function $\phi \in L^2(\mathbb{R})$ is said to be *refinable* with *dilation factor* $a \in \mathbb{Z}_+$, which is ≥ 2, if ϕ satisfies the *two-scale equation*

$$\phi(x) = \sum_{k \in \mathbb{Z}} p_k \phi(ax - k), \tag{1}$$

with $\{p_k\}_{k \in \mathbb{Z}} \in \ell^2(\mathbb{R})$ being ϕ's *two-scale sequence*. The simplest setting of a is 2. (cf., e.g., Chui [2] and Daubechies [4]).

A refinable ϕ is called a *scaling function*, if, in addition, the family $\{\phi(\cdot - k) : k \in \mathbb{Z}\}$ generates a *Riesz basis* of $V_0^\phi \subset L^2(\mathbb{R})$, meaning there are constants $0 < A \leq B < \infty$ such that

$$A \left(\sum_{n \in \mathbb{Z}} |a_n|^2 \right)^2 \leq \left\| \sum_{n \in \mathbb{Z}} a_n \phi(\cdot - n) \right\|_2^2 \leq B \left(\sum_{n \in \mathbb{Z}} |a_n|^2 \right)^2, \quad \forall \{a_n\}_{n \in \mathbb{Z}} \in \ell^2(\mathbb{R}), \tag{2}$$

where V_0^ϕ is the L^2-closure of all linear combinations of the integer translates of ϕ, namely,

$$V_0^\phi = \text{Clos}_{L^2} \text{ span} \{\phi(\cdot - k) : k \in \mathbb{Z}\}.$$

By taking the Fourier transforms of (1) both sides,

$$\widehat{\phi}(\omega) = P\left(e^{-i\omega/a}\right) \widehat{\phi}\left(\frac{\omega}{a}\right), \tag{3}$$

$$P(z) = \frac{1}{a} \sum_{k \in \mathbb{Z}} p_k z^k, \tag{4}$$

where P is called the *two-scale symbol* of ϕ. The two-scale sequence $\{p_k\}_{k \in \mathbb{Z}}$ is normally *finitely supported*, meaning it has finitely many nonzero entries. Such a scaling function is said to have *polynomial preservation* of order $d + 1$, denoted by $\phi \in \mathbb{PP}_d$, if

$$\pi_d \subset \text{ span} V_0^\phi, \tag{5}$$

where π_d is the collection of all polynomials of degree $\leq d$. It is well-known that (5) is equivalent to the two-scale symbol P in (4) with the factor $(1 + z + \cdots + z^{a-1})^{d+1}$.

A subdivision scheme corresponding to such a scaling function is given by

$$\lambda_{ak+\ell}^{(n+1)} = \sum_j p_{aj+\ell} \lambda_{k-j}^{(n)}, \quad \ell = 0, \ldots, a - 1; \ n \in \mathbb{Z}_+, \tag{6}$$

where $\lambda_k^{(0)}$, $k \in \mathbb{Z}$, are the initial control points. When $a = 2$, it is called a binary scheme, and when $a = 3$, it is called a ternary scheme. For the generic a, it is simply called an a-ary scheme.

The main objective of this note is to extend both results in (Lian [9]) and (Lian [10]) to $2m$-point a-ary (for any $a \geq 2$) and $(2m + 1)$-point a-ary (for any odd $a \geq 3$) interpolatory subdivision schemes, respectively. Our results are stated in Section 2 as Theorem 1 and Theorem 2, with their proofs given in Section 3. Another "short" way of proving Theorems 1 & 2 is given in Section 4. Two explicit examples, as direct consequences of our elegant formulations, are demonstrated in Section 5.

2.   Main Results

Let $^a\phi_{2m}$ and $^a\phi_{2m+1}$ be the scaling functions with dilation factor a, which correspond the $2m$- and $(2m+1)$-point interpolatory subdivision schemes for curve design. The smoothness of $^a\phi_{2m}$ and $^a\phi_{2m+1}$ for all dilation factors $a \geq 2$ is important for the convergence of their corresponding interpolatory subdivision schemes. We do not go to details here. However, for some previous studies of scaling functions their smoothness with different dilation factors $a \geq 3$, the readers are referred to, e.g., (Chui & Lian [3]), (Han [6]), (Belogay & Wang [1]), (Shui, Bao, & Zhang [12]), (Peng & Wang [11]).

For $^a\phi_{2m}$, we have the following:

Theorem 1. The scaling function $^a\phi_{2m} \in \mathbb{PP}_{2m}$ with the smallest support, is determined by the two-scale symbol $^aP_{2m}$ of the form

$$
\begin{aligned}
^aP_{2m}(z) &= \frac{1}{a}\sum_{k\in\mathbb{Z}}\left(_{2m}^{a}p_k\right)z^k \\
&= z^{1-am}\left(\frac{1}{a}\frac{1-z^a}{1-z}\right)^{2m}S_{2m-2}(z),
\end{aligned}
\tag{7}
$$

where $_{2m}^{a}p_k = 0$ for $|k| > ma - 1$, and

$$
\begin{aligned}
_{2m}^{a}p_{-k} = _{2m}^{a}p_k &= \frac{1}{(m-\ell-1)!\,(m+\ell)!\,a^{2m-1}}\prod_{\xi=1}^{m-\ell-1}(\xi a + k)\prod_{\eta=1}^{m+\ell}(\eta a - k), \\
k &= \ell a, \ell a + 1, \ldots, (\ell+1)a - 1; \quad \ell = 0, \ldots, m-1,
\end{aligned}
\tag{8}
$$

and S_{2m-2} is a polynomial of exact degree $2m-2$, satisfying $S_{2m-2}(1) = 1$ and $S_{2m-2}(z) = z^{2m-2}S_{2m-2}(1/z)$.

It then follows from (6) and (8) that the $2m$-point a-ary interpolatory subdivision scheme is

$$
\begin{aligned}
\lambda_{ak}^{(n+1)} &= \lambda_k^{(n)}, \\
\lambda_{ak+\ell}^{(n+1)} &= \sum_{j=-m}^{m-1}{}_{2m}^{a}p_{-aj+\ell}\lambda_{k+j}^{(n)}, \quad \ell = 1, \ldots, a-1; \; n \in \mathbb{Z}_+,
\end{aligned}
\tag{9}
$$

for any dilation factor $a \geq 2$. For $^a\phi_{2m+1} \in \mathbb{PP}_{2m+1}$, we have the following.

Theorem 2. The scaling function $^a\phi_{2m+1} \in \mathbb{PP}_{2m+1}$ with the smallest support, is determined from the two-scale symbol $^aP_{2m+1}$ of the form

$$
\begin{aligned}
^aP_{2m+1}(z) &= \frac{1}{a}\sum_{k\in\mathbb{Z}}\left(_{2m+1}^{a}p_k\right)z^k \\
&= z^{-ma-(a-1)/2}\left(\frac{1}{a}\frac{1-z^a}{1-z}\right)^{2m+1}S_{2m}(z),
\end{aligned}
\tag{10}
$$

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where $a \in \mathbb{Z}_+$ is odd, $a \geq 3$, $_{2m+1}^{a}p_k = 0$ for $|k| > ma + (a-1)/2$, and

$$_{2m+1}^{a}p_{-k} = {}_{2m+1}^{a}p_k = \frac{1}{(m-\ell)!\,(m+\ell)!\,a^{2m}} \prod_{\xi=1}^{m-\ell}(\xi a + k) \prod_{\eta=1}^{m+\ell}(\eta a - k),$$

$$k = 0, \ldots, (a-1)/2, \quad \text{when} \quad \ell = 0;$$
$$k = a\ell - (a-1)/2, \ldots, a\ell + (a-1)/2, \quad \text{when} \quad \ell = 1, \ldots, m, \tag{11}$$

and S_{2m} is a polynomial of exact degree $2m$, satisfying $S_{2m}(1) = 1$ and $S_{2m}(z) = z^{2m}\,S_{2m}(1/z)$.

Correspondingly, it follows from (6) and (11) that the $(2m+1)$-point a-ary interpolatory subdivision scheme is

$$\lambda_{ak}^{(n+1)} = \lambda_k^{(n)},$$
$$\lambda_{ak+\ell}^{(n+1)} = \sum_{j=-m}^{m} \left({}_{2m+1}^{a}p_{-aj+\ell}\right) \lambda_{k+j}^{(n)}, \quad |\ell| = 1, \ldots, (a-1)/2;\ n \in \mathbb{Z}_+. \tag{12}$$

3. Proofs of Main Results

Proof of Theorem 1.

Analogous to the proof in (Lian [9]), an a-ary $2m$-point scheme needs at most $2ma$ weights, i.e., the two-scale sequence $\{_{2m}^{a}p_j\}_{k\in\mathbb{Z}}$ of $^a\phi_{2m}$ has at most $2ma$ consecutive nontrivial entries. For $\mathbb{PP}_{2m}$, the two-scale symbol $^aP_{2m}$ must have the form

$$^aP_{2m}(z) = z^{1-am} \left(\frac{1}{a}\frac{1-z^a}{1-z}\right)^{2m} S_{2m-2}(z)$$

for some polynomial $S_{2m-2} \in \pi_{2m-2}$ of exact degree $2m-2$ satisfying $S_{2m-2}(1) = 1$ and $S_{2m-2}(z) = z^{2m-2}\,S_{2m-2}(1/z)$. It suffices to show that S_{2m-2} is uniquely determined under the interpolatory condition

$$\sum_{\ell=0}^{a-1} {}^aP_{2m}(w_\ell z) = 1, \quad |z| = 1, \tag{13}$$

where $\{w_\ell\}_{\ell=0}^{a-1}$ are the a distinct roots of $z^a = 1$, namely,

$$w_\ell = \exp\left(\frac{-2\ell\pi i}{a}\right), \quad \ell = 0, \ldots, a-1.$$

First, with $S_{2m-2}(z) = \sum_{k=0}^{2m-2} s_k z^k$, we define

$$\sum_{j=0}^{\infty} g_j z^j = S_{2m-2}(z)\frac{1}{(1-z)^{2m}} = \sum_{k=0}^{2m-2} s_k z^k \sum_{\ell=0}^{\infty} \binom{2m+\ell-1}{2m-1} z^\ell,$$

so that

$$g_j = \sum_{k=0}^{j} \binom{2m+j-k-1}{j-k} s_k, \quad j \in \mathbb{Z}_+. \tag{14}$$

Second, by defining $g_j = 0$ for $j < 0$,

$$^a P_{2m}(z) = a^{-2m} z^{1-am} (1-z^a)^{2m} \left[S_{2m-2}(z) \frac{1}{(1-z)^{2m}} \right]$$

$$= a^{-2m} z^{1-am} \sum_{k=0}^{2m} (-1)^k \binom{2m}{k} z^{ak} \sum_{\ell \in \mathbb{Z}} g_\ell z^\ell$$

$$= a^{-2m} z^{1-am} \sum_{\ell=0}^{a-1} z^\ell \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} (-1)^k \binom{2m}{k} g_{a(j-k)+\ell} z^{aj}$$

$$= a^{-2m} \sum_{\ell=0}^{a-1} z^\ell \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} (-1)^k \binom{2m}{k} g_{a(j-k)+\ell} z^{aj-am+1}.$$

Third, the interpolatory condition (13) now leads to

$$a^{1-2m} \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} (-1)^k \binom{2m}{k} g_{a(j+m-k)-1} z^j = 1,$$

which is equivalent to

$$\sum_{k=0}^{j+m-1} (-1)^k \binom{2m}{k} g_{a(j+m-k)-1} = a^{2m-1} \delta_{j,0}, \quad j = -m+1, \cdots, m-1. \tag{15}$$

Fourth, solving the lower triangular system (15), we have

$$g_{aj-1} = \binom{m+j-1}{2m-1} a^{2m-1}, \quad j = 1, \cdots, 2m-1, \tag{16}$$

and the coefficients $s_0, \ldots, s_{2m-2}$ of S_{2m-2} can now be evaluated from (14) and (16), namely,

$$\sum_{k=0}^{2m-2} \binom{2m+aj-k-2}{2m-1} s_k = \binom{m+j-1}{2m-1} a^{2m-1}, \quad j = 1, \cdots, 2m-1. \tag{17}$$

Solving (17) for $s_0, \ldots, s_{2m-2}$ and computing $\{g_j\}_{j \in \mathbb{Z}_+}$ by (14), we have

$$g_\ell = \frac{\ell+1}{(2m-1)!} \prod_{k=1}^{m-1} \left((\ell+1)^2 - k^2 a^2 \right), \quad \ell \in \mathbb{Z}_+. \tag{18}$$

Finally, by using (7), we arrive at the values of $_{2m}^{a} p_k$ in (8). This completes the proof of Theorem 1. $\square$

Proof of Theorem 2.

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Denote by ${}^{a}P_{2m+1}$ the two-scale symbol of ${}^{a}\phi_{2m+1}$. Completely analogous to the proof of Theorem 1, both ${}^{a}P_{2m+1} \in \mathbb{PP}_{2m+1}$ and ${}^{a}\phi_{2m+1}$ having symmetric two-scale sequence $\{{}^{a}_{2m+1}p_k\}_{k=-ma-(a-1)/2}^{ma+(a-1)/2}$ lead to

$$
{}^{a}P_{2m+1}(z) = z^{-ma-(a-1)/2} \left(\frac{1}{a} \frac{1-z^a}{1-z} \right)^{2m+1} S_{2m}(z)
$$

for a to-be-determined polynomial S_{2m} of exact degree $2m$, satisfying $S_{2m}(1) = 1$ and $S_{2m}(z) = z^{2m} S_{2m}(1/z)$, where $a \geq 3$ is a positive odd integer. Again, by writing $S_{2m}(z) = \sum_{k=0}^{2m} s_k z^k$, introducing

$$
\sum_{j=0}^{\infty} g_j z^j = S_{2m}(z) \frac{1}{(1-z)^{2m+1}},
$$

$$
g_j = \sum_{k=0}^{j} \binom{2m+j-k}{j-k} s_k, \quad j \in \mathbb{Z}_+. \tag{19}
$$

and defining $g_j = 0$ for $j < 0$, we have

$$
{}^{a}P_{2m+1}(z) = a^{-2m-1} \sum_{\ell=-(a-1)/2}^{(a-1)/2} z^{\ell} \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} (-1)^k \binom{2m+1}{k} g_{a(j-k)+\ell+(a-1)/2} z^{aj-am}.
$$

The interpolatory property yields

$$
\sum_{k \in \mathbb{Z}} (-1)^k \binom{2m+1}{k} g_{a(j+m-k)+(a-1)/2} = a^{2m} \delta_{j,0}, \quad j = -m, \cdots, m-1,
$$

which leads to

$$
g_{aj+(a-1)/2} = \binom{m+j}{2m} a^{2m}, \quad j = 0, \cdots, 2m. \tag{20}
$$

Solving the linear system

$$
\sum_{k=0}^{2m} \binom{2m+aj-k}{2m} s_k = \binom{m+j}{2m} a^{2m}, \quad j = 0, \cdots, 2m, \tag{21}
$$

for $s_0, \ldots, s_{2m}$ and computing $\{g_j\}_{j \in \mathbb{Z}_+}$ for general j by (19), we have

$$
g_\ell = \sum_{k=0}^{\ell} \binom{2m+\ell-k}{\ell-k} s_k = \frac{1}{2^{2m}(2m)!} \prod_{k=1}^{m-1} \left((2\ell+1)^2 - (2k-1)^2 a^2 \right), \quad \ell \in \mathbb{Z}_+. \tag{22}
$$

Finally, $\{{}^{a}_{2m+1}p_k\}_{k=-ma-(a-1)/2}^{ma+(a-1)/2}$ is obtained as listed in (11). $\square$

4.   Another "Short" Way of Proving Theorem 1 & Theorem 2

The proofs of our two theorems in Section 3 were direct and clear-cut. We show in this section another "short" or maybe not so-short way, by using the Taylor expansions of the expression

$\left(\dfrac{a(1-z)}{1-z^a} \right)^m$. For $a = 2$, see (Daubechies [4]). For $a = 3$, see (Chui & Lian [3]). For generic a and multiwavelet setting, see (Lian [8]).

First, for a subdivision scheme generated from a scaling function ϕ to be symmetric, the two-scale symbol P of ϕ has to be reciprocal. For this purpose, $P(z)$ needs to be a function of $z + z^{-1}$. For this reason, the factor $\left(\dfrac{a(1-z)}{1-z^a} \right)^m$, a requirement for $\phi \in \mathbb{PP}_m$, has to be a function of $z + z^{-1}$. This is equivalent to the positive integer $(a-1)m$ has to be even. Hence, if m is even then the dilation factor (or sampling rate) a can be any positive integer ≥ 2. However, if m is odd, $a - 1$ has to be even. That is why for a $(2m+1)$-point subdivision scheme, a has to be odd. To get ready for the new short proofs of the two theorems, we establish the following:

Lemma 1. For any positive even integer a,

$$\frac{1}{z^{a-1}} \left(\frac{1 + z + \cdots + z^{a-1}}{a} \right)^2 = 1 + \frac{(z-1)^2}{a^2 z} \left[\binom{a+1}{3} \right.$$
$$\left. + \sum_{k=0}^{a-3} \binom{k+3}{3} (z^{a-2-k} + z^{-a+2+k}) \right]. \tag{23}$$

For any positive odd integer a,

$$\frac{1}{z^{(a-1)/2}} \frac{1 + z + \cdots + z^{a-1}}{a} = 1 + \frac{(z-1)^2}{az} \left[\binom{(a+1)/2}{2} \right.$$
$$\left. + \sum_{k=0}^{(a-5)/2} \binom{k+2}{2} (z^{(a-3)/2-k} + z^{-(a-3)/2+k}) \right]. \tag{24}$$

The proof of Lemma is straightforward. We are now in the position of the new proofs of Theorems 1 & 2.

Another "Short" Proof of Theorem 1.

When the polynomial preservation is of even order, we can write

$$^aP_{2m}(z) = \frac{1}{a} \sum_{k \in \mathbb{Z}} \binom{a}{2m}p_k z^k = z^{-(a-1)m-n} \left(\frac{1}{a} \frac{1 - z^a}{1 - z} \right)^{2m} S_{2n}(z), \tag{25}$$

for some reciprocal polynomial S_{2n}. Then the interpolatory condition (13) yields

$$z^{-n} S_{2n}(z) = \left[\left(\frac{a}{1 + z + \cdots + z^{a-1}} \right)^2 z^{a-1} \right]^m$$
$$- \sum_{\ell=1}^{a-1} \left(\frac{1-z}{1 - w_\ell z} \right)^{2m} w_\ell^{-(a-1)m-n} S_{2n}(w_\ell z). \tag{26}$$

It follows from Lemma 1 that for either even a or odd a, the first term in (26) is a function of $\dfrac{(1-z)^2}{z}$ and the last $a - 1$ terms in (26) have the factor $\left(\dfrac{(1-z)^2}{z} \right)^m$. Hence, the minimum

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degree of n is $m-1$, and $z^{-(m-1)}S_{2m-2}(z)$ is the mth order Taylor polynomial of

$$\left(\frac{a}{1+z+\cdots+z^{a-1}}\right)^{2m}z^{(a-1)m} \tag{27}$$

in terms of the variable $t = -\dfrac{(1-z)^2}{z}$. More precisely, by using Lemma 1, when a is even,

$$\left(\frac{a}{1+z+\cdots+z^{a-1}}\right)^{2m}z^{(a-1)m} = \left[\left(\frac{a}{1+z+\cdots+z^{a-1}}\right)^2 z^{a-1}\right]^m$$

$$= \frac{1}{\left(1+\dfrac{(z-1)^2}{a^2 z}\left[\binom{a+1}{3}+\displaystyle\sum_{k=0}^{a-3}\binom{k+3}{3}\left(z^{a-2-k}+z^{-a+2+k}\right)\right]\right)^m}, \tag{28}$$

and when a is odd,

$$\left(\frac{a}{1+z+\cdots+z^{a-1}}\right)^{2m}z^{(a-1)m} = \left[\frac{az^{(a-1)/2}}{1+z+\cdots+z^{a-1}}\right]^{2m}$$

$$= \frac{1}{\left(1+\dfrac{(z-1)^2}{az}\left[\binom{(a+1)/2}{2}+\displaystyle\sum_{k=0}^{(a-5)/2}\binom{k+2}{2}\left(z^{(a-3)/2-k}+z^{-(a-3)/2+k}\right)\right]\right)^{2m}}. \tag{29}$$

In other words, the exact expressions of ${}^aP_{2m}(z)$ are formed from either the Taylor polynomials of (28) for even a or the Taylor polynomials of (29) for odd a. This completes the short proof of Theorem 1. $\qquad\square$

Another "Short" Proof of Theorem 2.

When the polynomial preservation is of odd order, the dilation factor a has to be odd as well, and we can write

$${}^aP_{2m+1}(z) = \frac{1}{a}\sum_{k\in\mathbb{Z}}\left({}_{2m+1}^{\quad a}p_k\right)z^k$$

$$= z^{-(2m+1)(a-1)/2-n}\left(\frac{1+z+\cdots+z^{a-1}}{a}\right)^{2m+1}S_{2n}(z), \tag{30}$$

for some reciprocal polynomial S_{2n}. Again, the interpolatory condition (13) leads to

$$z^{-n}S_{2n}(z) = \left[\frac{az^{(a-1)/2}}{1+z+\cdots+z^{a-1}}\right]^{2m+1}$$

$$\qquad - \sum_{\ell=1}^{a-1}\left(\frac{1-z}{1-w_\ell z}\right)^{2m}w_\ell^{-(a-1)(2m+1)/2-n}S_{2n}(w_\ell z). \tag{31}$$

Therefore, $n \geq m$, and $z^{-m}S_{2m}(z)$ is the $(m+1)^{\text{st}}$ order Taylor polynomial of

$$\left(\frac{az^{(a-1)/2}}{1+z+\cdots+z^{a-1}}\right)^{2m+1} \tag{32}$$

in terms of the variable $t = -\dfrac{(1-z)^2}{z}$. This completes the short proof of Theorem 2. $\qquad$ $\square$

For convenience, we fix

$$t = 2 - z - z^{-1} = -\frac{(1-z)^2}{z} \tag{33}$$

in the sequel, so that the Taylor expansions are in terms of t. For example, it is easy to see, by using t in (33), that for $a = 2, 3, 4$, and 5, the expressions we need for Taylor expansions in t are

$$\left(\frac{2}{1+z}\right)^2 z = \frac{1}{1 - \dfrac{t}{4}}; \tag{34}$$

$$\frac{3z}{1+z+z^2} = \frac{1}{1 - \dfrac{t}{3}}; \tag{35}$$

$$\left(\frac{4}{1+z+z^2+z^3}\right)^2 z^3 = \frac{1}{1 - \dfrac{t}{16}(20 - 8t + t^2)}; \tag{36}$$

$$\frac{5z^2}{1+z+z^2+z^3+z^4} = \frac{1}{1 - \dfrac{t}{5}(5 - t)}. \tag{37}$$

5. Demonstration by Two Explicit Examples

We demonstrate our elegant formulations in this section by giving two explicit examples as direct consequences of Theorems 1 & 2.

Example 1. **8-point binary interpolatory:** $m = 4, a = 2$.

It follows from (18), (17), and (8) that

$$\{g_0, \cdots, g_6\} = \left\{ -\frac{5}{16}, 0, \frac{9}{16}, 0, -\frac{33}{16}, 0, \frac{429}{16} \right\};$$

$$\{s_0, \cdots, s_6\} = \left\{ -\frac{5}{16}, \frac{5}{2}, -\frac{131}{16}, 13, -\frac{131}{16}, \frac{5}{2}, -\frac{5}{16} \right\};$$

$$\{{}_8^2 p_{-7}, {}_8^2 p_{-5}, \cdots, {}_8^2 p_7\} = \left\{ -\frac{5}{2048}, \frac{49}{2048}, -\frac{245}{2048}, \frac{1225}{2048}, \frac{1225}{2048}, -\frac{245}{2048}, \frac{49}{2048}, -\frac{5}{2048} \right\},$$

so that, by using (9), the 8-point binary interpolatory scheme is given by

$$\lambda_{2k}^{(n+1)} = \lambda_k^{(n)},$$

$$\lambda_{2k+1}^{(n+1)} = \sum_{j=-4}^{3} {}_8^2 p_{-2j+1} \lambda_{k+j}^{(n)} \tag{38}$$

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$$= \frac{1}{2048}\left[-5\left(\lambda_{k-4}^{(n)} + \lambda_{k+4}^{(n)}\right) + 49\left(\lambda_{k-3}^{(n)} + \lambda_{k+3}^{(n)}\right)\right.$$

$$\left. -245\left(\lambda_{k-2}^{(n)} + \lambda_{k+2}^{(n)}\right) + 1225\left(\lambda_{k-1}^{(n)} + \lambda_{k+1}^{(n)}\right)\right], \quad n \in \mathbb{Z}_+. \quad (39)$$

The subdivision scheme in (38)–(39) can also be established by (26), with $m = 4$ and $a = 2$, or by using the cubic Taylor polynomial expansion of (34), with t in (33). More explicitly, it follows from

$$z^{-3}S_6(z) = \sum_{k=0}^{3}\binom{4+k-1}{k}\left(\frac{t}{4}\right)^k = \sum_{k=0}^{3}\binom{3+k}{k}\left(\frac{1}{2} - \frac{1}{2}\frac{z+z^{-1}}{2}\right)^k \quad (40)$$

that

$$^2P_8(z) = \left[\frac{1}{z^4}\left(\frac{1+z}{2}\right)^8\right]z^{-3}S_6(z)$$

$$= \left[\frac{1}{z^4}\left(\frac{1+z}{2}\right)^8\right]\sum_{k=0}^{3}\binom{3+k}{k}\left(\frac{1}{2} - \frac{1}{2}\frac{z+z^{-1}}{2}\right)^k, \quad (41)$$

which also yields the subdivision scheme in (38)–(39).

Example 2. 7-point ternary interpolatory: $m = a = 3$.

Again, it follows from (22), (21), and (11) that

$$\{g_0, \cdots, g_6\} = \left\{-\frac{28}{9}, 0, \frac{35}{9}, \frac{44}{9}, 0, -\frac{91}{9}, -\frac{151}{9}\right\};$$

$$\{s_0, \cdots, s_6\} = \left\{-\frac{28}{9}, \frac{196}{9}, -\frac{553}{9}, \frac{779}{9}, -\frac{553}{9}, \frac{196}{9}, -\frac{28}{9}\right\};$$

$$\{^3_7p_{-10}, {}^3_7p_{-9}, \cdots, {}^3_7p_{10}\} = \left\{-\frac{28}{6561}, 0, \frac{35}{6561}, \frac{80}{2187}, 0, -\frac{112}{2187},\right.$$

$$-\frac{350}{2187}, 0, \frac{700}{2187}, \frac{5600}{6561}, 1, \frac{5600}{6561}, \frac{700}{2187}, 0, -\frac{350}{2187},$$

$$\left.-\frac{112}{2187}, 0, \frac{80}{2187}, \frac{35}{6561}, 0, -\frac{28}{6561}\right\}.$$

Hence, it follows from (12) that the 7-point ternary interpolatory scheme is given by

$$\lambda_{3k-1}^{(n+1)} = \sum_{j=-3}^{3}{}^3_7p_{-3j-1}\lambda_{k+j}^{(n)}$$

$$= \frac{1}{6561}\left[35\,\lambda_{k-3}^{(n)} - 336\,\lambda_{k-2}^{(n)} + 2100\,\lambda_{k-1}^{(n)} + 5600\,\lambda_k^{(n)}\right.$$

$$\left. -1050\,\lambda_{k+1}^{(n)} + 240\,\lambda_{k+2}^{(n)} - 28\,\lambda_{k+3}^{(n)}\right], \quad (42)$$

$$\lambda_{3k}^{(n+1)} = \lambda_k^{(n)}, \quad (43)$$

$$\lambda_{3k+1}^{(n+1)} = \sum_{j=-3}^{3}{}^3_7p_{-3j+1}\lambda_{k+j}^{(n)}$$

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$$= \frac{1}{6561} \Bigg[- 28\, \lambda_{k-3}^{(n)} + 240\, \lambda_{k-2}^{(n)} - 1050\, \lambda_{k-1}^{(n)} + 5600\, \lambda_{k}^{(n)}$$

$$+ 2100\, \lambda_{k+1}^{(n)} - 336\, \lambda_{k+2}^{(n)} + 35\, \lambda_{k+3}^{(n)} \Bigg], \quad n \in \mathbb{Z}_+. \tag{44}$$

Similar to (40)–(41) in Example 1, (42)–(44) can also be established by applying (30), with $m = a = 3$, or, with t in (33), by using the cubic Taylor polynomial expansion of (35). More explicitly, it follows from

$$z^{-3} S_6(z) = \sum_{k=0}^{3} \binom{7+k-1}{k} \left(\frac{t}{3}\right)^k = \sum_{k=0}^{3} \binom{6+k}{k} \left(\frac{2-z-z^{-1}}{3}\right)^k$$

that

$$^3P_7(z) = \left(\frac{1+z+z^2}{3z}\right)^7 z^{-3} S_6(z)$$

$$= \left(\frac{1+z+z^2}{3z}\right)^7 \sum_{k=0}^{3} \binom{6+k}{k} \left(\frac{2-z-z^{-1}}{3}\right)^k,$$

which also generates the subdivision scheme in (42)–(44).

Acknowledgment

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