Applications and Applied Mathematics: An International Journal (AAM)

Volume 3 | Issue 1 | Article 1

6-2008

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Available at
http://pvamu.edu/aam
Appl. Appl. Math.
ISSN: 1932-9466

Vol. 3, Issue 1 (June 2008) pp. 01 – 17
(Previously, Vol. 3, No. 1)

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An International Journal
(AAM)

Oscillatory Behavior of Second Order Neutral Differential Equations with Positive and Negative Coefficients

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Received September 26, 2007; accepted November 9, 2007

Abstract

Oscillation criteria are obtained for solutions of forced and unforced second order neutral differential equations with positive and negative coefficients. These criteria generalize those of Manojlović, Shoukaku, Tanigawa and Yoshida (2006).

Key Words: Oscillation; second order; positive and negative coefficients

AMS 2000 Mathematical Subject Classifications: 35B05

1. Introduction

In the last few years, there has been an increasing interest in the study of oscillatory behavior of solutions of first order neutral delay differential equations with positive and negative coefficients (see, for example, Chuanxi and Ladas (1990), Farrel, Grove and Ladas (1988), Ruan (1991), Yu (1991). Compared to the first-order differential equations, the study of second-order equations with positive and negative coefficients has received considerably less attention.

In this paper we consider the oscillation of the second order neutral delay differential equations

$$(\mathrm{E}_1) \quad \left[r(t)\left[x(t)+\sum_{i=1}^{l} h_i(t)x(t-\rho_i) \right]' \right]' + \sum_{i=1}^{m} p_i(t)x(t-\delta_i) - \sum_{i=1}^{n} q_i(t)x(t-\sigma_i) = 0, \quad t>0,$$

$$(\mathrm{E}_2) \quad \left[r(t)\left[x(t)-\sum_{i=1}^{l} h_i(t)x(t-\rho_i) \right]' \right]' + \sum_{i=1}^{m} p_i(t)x(t-\delta_i) - \sum_{i=1}^{n} q_i(t)x(t-\sigma_i) = 0, \quad t>0.$$

Sufficient conditions for oscillation of solutions of the equation (E_1) for the case where $q_i(t)=0$ is considered by several authors (see, for example, Grace and Lalli (1987), Tanaka (2004)). Moreover, Parhi and Chand (1999) and Manojlović, Shoukaku, Tanigawa and Yoshida (2006) obtained some oscillatory criteria for equations (E_1) and (E_2) with $r(t)=1$. Namely, sufficient conditions for oscillation of all bounded solutions of equations (E_1) and (E_2) with $r(t)=1$ are given in Parhi and Chand (1999). On the other hand, results established in the paper Manojlović, Shoukaku, Tanigawa and Yoshida (2006) are in fact improvement of results in Parhi and Chand (1999), in the sense that the assumption of boundedness of solutions was removed, i.e. sufficient conditions for oscillation of *all solutions* of equations (E_1) and (E_2) with $r(t)=1$ are given in Manojlović, Shoukaku, Tanigawa and Yoshida (2006).

The purpose of this paper is to derive sufficient conditions for every solution of (E_1) and (E_2) to be oscillatory. It is assumed throughout this paper that:

(H_1) $m \ge n$,
ρ_i $(i=1,\ 2,\dots,l)$, δ_i $(i=1,\ 2,\dots,m)$ and σ_i $(i=1,\ 2,\dots,n)$ are nonnegative constants,
$\delta_i \ge \sigma_i$ $(i=1,\ 2,\dots,n)$;

(H_2) $r(t)\in C([0,\infty);(0,\infty))$,
$h_i(t)\in C([0,\infty);[0,\infty))$ $(i=1,\ 2,\ \dots,l)$,
$p_i(t)\in C([0,\infty);[0,\infty))$ $(i=1,\ 2,\ \dots,m)$,
$q_i(t)\in C([0,\infty);[0,\infty))$ $(i=1,\ 2,\ \dots,n)$;

(H_3) $\int_{T_0}^{\infty} \frac{1}{r(t)}dt=\infty$ for some $T_0>0$;

(H_4) $\begin{cases} q_i(t) \le q_i(t-\sigma_i) & (i=1,\ 2,\dots,n), \\ p_i(t) \ge q_i(t-\delta_i) & (i=1,\ 2,\dots,n); \end{cases}$

(H_5) $p_j(t) - q_j(t-\delta_j) \ge k_j > 0$ for some $j \in \{1,\ 2,\dots,n\}$ and some $k_j > 0$.

Definition 1: By a *solution* of (E_1) or (E_2) we mean a continuous function $x(t)$ which is defined for $t \ge t_0 - T$, and satisfies $\sup\{|x(t)| : t \ge t_1\} > 0$ for all $t_1 \ge t_0$, where

$$T = \max\{\rho_i,\ \delta_j,\ \sigma_k : 1 \le i \le l,\ 1 \le j \le m,\ 1 \le k \le n\}.$$

Definition 2: A nontrivial solution of (E_1) or (E_2) is called *oscillatory* if it has arbitrary large zeros, otherwise, it is called nonoscillatory. The equation is called oscillatory if all its solutions are oscillatory.

In Section 2 we give the sufficient conditions for oscillation of solutions of the equation (E_1), while in the Section 3 we deals with the equation (E_2). Oscillation results for nonhomogeneous cases of (E_1) and (E_2) are given in Section 4.

2. Oscillation of solutions of the equation (E_1)

In this section we obtain the following oscillation criteria for the equation (E_1).

Theorem 1: Assume that

$$(\mathrm{H}_6) \qquad 0 \le \sum_{i=1}^{l} h_i(t) \le h, \qquad h = const.$$

The equation (E_1) is oscillatory if

$$\sum_{i=1}^{n} \int_0^\infty \frac{1}{r(s)} \int_{s-\delta_i}^{s-\sigma_i} q_i(\xi)d\xi\, ds \le 1. \tag{1}$$

Proof: Suppose that $x(t)$ is a nonoscillatory solution of (E_1). Without any loss of generality, we assume that $x(t) > 0$ for $t \ge t_0$, where t_0 is some positive number. We set

$$z(t) = x(t) + \sum_{i=1}^{l} h_i(t)x(t-\rho_i) - \sum_{i=1}^{n} \int_{t_0}^{t} \frac{1}{r(s)} \int_{s-\delta_i}^{s-\sigma_i} q_i(\xi)x(\xi)d\xi\, ds \tag{2}$$

for $t \ge t_0 + T$, then

$$z'(t) = \left[x(t) + \sum_{i=1}^{l} h_i(t)x(t-\rho_i) \right]' - \frac{1}{r(t)} \sum_{i=1}^{n} \int_{t-\delta_i}^{t-\sigma_i} q_i(s)x(s)ds.$$

Multiplying the above equation by $r(t)$ and differentiating both sides, we have

$$\begin{aligned}
(r(t)z'(t))' &= \left[r(t) \left[x(t) + \sum_{i=1}^{l} h_i(t)x(t-\rho_i) \right]' \right]' \\
&\qquad - \sum_{i=1}^{n} q_i(t-\sigma_i)x(t-\sigma_i) + \sum_{i=1}^{n} q_i(t-\delta_i)x(t-\delta_i) \\
&= \left\{ -\sum_{i=1}^{m} p_i(t)x(t-\delta_i) + \sum_{i=1}^{n} q_i(t)x(t-\sigma_i) \right\} \\
&\qquad - \sum_{i=1}^{n} q_i(t-\sigma_i)x(t-\sigma_i) + \sum_{i=1}^{n} q_i(t-\delta_i)x(t-\delta_i) \\
&= -\sum_{i=1}^{n} \{ q_i(t-\sigma_i) - q_i(t) \} x(t-\sigma_i) \\
&\qquad - \sum_{i=1}^{m} p_i(t)x(t-\delta_i) + \sum_{i=1}^{n} q_i(t-\delta_i)x(t-\delta_i) \\
&\le -\sum_{i=1}^{m} p_i(t)x(t-\delta_i) + \sum_{i=1}^{n} q_i(t-\delta_i)x(t-\delta_i) \\
&\le -\sum_{i=1}^{n} \{ p_i(t) - q_i(t-\delta_i) \} x(t-\delta_i), \qquad t \ge t_0 + T.
\end{aligned}$$

This leads to the following inequality for some $j \in \{1, 2, \ldots, n\}$ and some $k_j > 0$, that

$$(r(t)z'(t))' \le -k_j x(t-\delta_j) \le 0, \quad t \ge t_0 + T, \tag{3}$$

that is, $r(t)z'(t)$ is nonincreasing. Then, we conclude that $z'(t) \ge 0$ or $z'(t) < 0$, $t \ge t_1$ for some $t_1 \ge t_0 + T$. We discuss the following two possible cases:

Case 1. $z'(t) < 0$ for all $t \ge t_1$. Integrating (3) over $[t_1, t]$ yields

$$r(t)z'(t) \le r(t_1)z'(t_1) < 0, \quad t \ge t_1.$$

Multiplying the above inequality by $\frac{1}{r(t)}$ and integrating over $[t_1, t]$, we obtain

$$z(t) \le z(t_1) + r(t_1)z'(t_1) \int_{t_1}^{t} \frac{1}{r(s)} ds, \quad t \ge t_1,$$

and we see from (H_3) that $\lim_{t\to\infty} z(t) = -\infty$. We claim that $x(t)$ is bounded from above. If this is not the case, then there exists a number $t_2 \geq t_1$ such that

$$z(t_2) < 0 \quad \text{and} \quad \max_{t_1 \leq t \leq t_2} x(t) = x(t_2). \tag{4}$$

Then, we have

$$\begin{aligned} 0 > \ z(t_2) \quad &= x(t_2) + \sum_{i=1}^{l} h_i(t_2) x(t_2 - \rho_i) - \sum_{i=1}^{n} \int_{t_0}^{t_2} \frac{1}{r(s)} \int_{s-\delta_i}^{s-\sigma_i} q_i(\xi) x(\xi) d\xi\, ds \\ &\geq \left\{ 1 - \sum_{i=1}^{n} \int_{t_0}^{t_2} \frac{1}{r(s)} \int_{s-\delta_i}^{s-\sigma_i} q_i(\xi) d\xi\, ds \right\} x(t_2) \\ &\geq \left\{ 1 - \sum_{i=1}^{n} \int_{0}^{\infty} \frac{1}{r(s)} \int_{s-\delta_i}^{s-\sigma_i} q_i(\xi) d\xi\, ds \right\} x(t_2) \geq 0, \end{aligned}$$

which is a contradiction, so that $x(t)$ is bounded from above. Hence for every $L > 0$ there exists a $t_3 \geq t_2$ such that $x(t) \leq L$ for all $t \geq t_3$. We then have

$$\begin{aligned} z(t) &\geq -L \sum_{i=1}^{n} \int_{t_0}^{t} \frac{1}{r(s)} \int_{s-\delta_i}^{s-\sigma_i} q_i(\xi) d\xi\, ds \\ &\geq -L \sum_{i=1}^{n} \int_{0}^{\infty} \frac{1}{r(s)} \int_{s-\delta_i}^{s-\sigma_i} q_i(\xi) d\xi\, ds \geq -L > -\infty, \qquad t \geq t_3. \end{aligned}$$

This contradicts the fact that $\lim_{t \to \infty} z(t) = -\infty$.

<u>Case 2.</u> $z'(t) \geq 0$ for $t \geq t_1$. Then, by integrating (3) over $[t_1, t]$, we obtain

$$\infty > r(t_1) z'(t_1) \geq -r(t) z'(t) + r(t_1) z'(t_1) \geq k_j \int_{t_1}^{t} x(s - \delta_j) ds,$$

and therefore $x(t) \in L^1([t_1, \infty))$. Thus, from the condition (H_6), we have

$$X(t) = x(t) + \sum_{i=1}^{l} h_i(t) x(t - \rho_i) \in L^1([t_1, \infty)). \tag{5}$$

Moreover, it is clear that for $t \geq t_1$

$$X'(t) = \left[x(t) + \sum_{i=1}^{l} h_i(t) x(t - \rho_i) \right]' = z'(t) + \sum_{i=1}^{n} \frac{1}{r(t)} \int_{t-\delta_i}^{t-\sigma_i} q_i(s) x(s) ds \geq 0,$$

which implies that $X(t)$ is nondecreasing. Therefore, $X(t) \geq X(t_1)$, $t \geq t_1$, which yields that $X(t) \notin L^1([t_1, \infty))$. This contradicts the fact that (5) holds. The proof is completed.

Example 1: We consider the equation

$$\left[e^{-t}\left[x(t)+\frac{1}{2}x(t-2\pi)\right]'\right]' + \left(2e^{t+2\pi}+\frac{3}{2}e^{-t}\right)x(t-2\pi)$$
$$+2e^{t+2\pi}x(t-\pi)+\frac{3}{2}e^{-t}x\left(t-\frac{3}{2}\pi\right)-e^{-2t-4\pi}x(t-\pi)-e^{-2t-4\pi}x(t)=0, \quad t>0. \tag{6}$$

Here we have

$$r(t)=e^{-t},\ l=1,\ h_1(t)=\frac{1}{2},\ \rho_1=2\pi$$
$$m=3,\ p_1(t)=2e^{t+2\pi}+\frac{3}{2}e^{-t},\ p_2(t)=2e^{t+2\pi},\ p_3(t)=\frac{3}{2}e^{-t},$$
$$\delta_1=2\pi,\ \delta_2=\pi,\ \delta_3=\frac{3}{2}\pi,$$
$$n=2,\ q_1(t)=q_2(t)=e^{-2t-4\pi},\ \sigma_1=\pi,\ \sigma_2=0,$$

so that, for $t>0$, it is clear that

$$q_1(t)=e^{-2t-4\pi}\le e^{-2t-2\pi}=q_1(t-\sigma_1)=q_1(t-\pi),$$
$$q_2(t)=e^{-2t-4\pi}=q_2(t-\sigma_2),$$
$$p_1(t)-q_1(t-\delta_1)=2e^{t+2\pi}+\frac{3}{2}e^{-t}-e^{-2t}\ge 2e^{2\pi}-1\equiv k_1>0,$$
$$p_2(t)-q_2(t-\delta_2)=2e^{t+2\pi}-e^{-2t-2\pi}\ge 2e^{2\pi}-e^{-2\pi}>0$$

and

$$\int_0^\infty e^s\int_{s-2\pi}^{s-\pi}e^{-2\xi-4\pi}d\xi\,ds+\int_0^\infty e^s\int_{s-\pi}^{s}e^{-2\xi-4\pi}d\xi\,ds=\frac{1}{2}\left(1-e^{-4\pi}\right)<1.$$

Therefore, Theorem 1 implies that every solution $x(t)$ of the equation (6) oscillates. Indeed, $x(t)=\sin t$ is an oscillatory solution of this equation.

Example 2: We consider the equation

$$\left[e^{-t}\left[x(t)+\frac{e^{2\pi}}{2}x(t-2\pi)\right]'\right]' + \left(\frac{3}{2}e^{-t+2\pi}+e^{-2t-2\pi}+\pi\right)x(t-2\pi)$$
$$+\pi e^{-\pi}x(t-\pi)+\frac{3}{2}e^{-t+\frac{\pi}{2}}x\left(t-\frac{\pi}{2}\right)-e^{-2t-4\pi}x(t)=0, \quad t>0. \tag{7}$$

Here we have

$$
\begin{aligned}
&r(t)=e^{-t},\ l=1,\ h_1(t)=\frac{e^{2\pi}}{2},\ \rho_1=2\pi,\\
&m=3,\ p_1(t)=\frac{3}{2}e^{-t+2\pi}+e^{-2t-2\pi}+\pi,\ \delta_1=2\pi,\ p_2(t)=\pi\, e^{-\pi},\ \delta_2=\pi,\\
&\qquad p_3(t)=\frac{3}{2}e^{-t+\frac{\pi}{2}},\ \delta_3=\frac{\pi}{2},\\
&n=1,\ q_1(t)=e^{-2t-4\pi},\ \sigma_1=0,
\end{aligned}
$$

so that, for $t>0$, a straightforward verification shows that

$$
\begin{aligned}
&q_1(t)=q_1(t-\sigma_1),\\
&p_1(t)-q_1(t-\delta_1)=\left(\frac{3}{2}e^{-t+2\pi}+e^{-2t-2\pi}+\pi\right)-e^{-2t}\geq \pi-1\equiv k_1>0
\end{aligned}
$$

and

$$
\int_0^\infty e^s\int_{s-2\pi}^s e^{-2\xi-4\pi}d\xi\, ds=\frac{1}{2}\left(1-e^{-4\pi}\right)<1.
$$

Therefore, Theorem 1 implies that every solution $x(t)$ of the equation (7) oscillates. Indeed, $x(t)=e^t\cos t$ is an oscillatory solution of this equation.

3. Oscillation of solutions of the equation (E_2)

Now, we turn to the oscillation theorem for the equation (E_2).

Theorem 2: Assume that

$$
(\mathrm{H}_7)\qquad h_i(t)\leq h_i\quad (i=1,2,\ldots,l),
$$

where h_i are nonnegative constants such that $\sum_{i=1}^{l}h_i<1$. If

$$
\sum_{i=1}^{l}h_i+\sum_{i=1}^{n}\int_0^\infty \frac{1}{r(s)}\int_{s-\delta_i}^{s-\sigma_i}q_i(\xi)d\xi\, ds<1, \tag{8}
$$

then every solution of (E_2) oscillates or satisfies $\lim_{t\to\infty}x(t)=0$.

Proof: Suppose that $x(t)$ is a nonoscillatory solution of (E_2) such that $x(t)>0$ for $t\geq t_0$, where t_0 is some positive number. We denote by

$$w(t) = x(t) - \sum_{i=1}^{l} h_i(t)x(t-\rho_i) - \sum_{i=1}^{n} \int_{t_0}^{t} \frac{1}{r(s)} \int_{s-\delta_i}^{s-\sigma_i} q_i(\xi)x(\xi)d\xi\, ds. \tag{9}$$

Then as in the proof of Theorem 1, from the equation (E_2) we obtain

$$(r(t)w'(t))' \leq -k_j x(t-\delta_j) \leq 0, \quad t \geq t_0 + T \tag{10}$$

for some $j \in \{1, 2, \ldots, n\}$. Therefore, $r(t)w'(t)$ is nonincreasing, and hence $w'(t) < 0$ or $w'(t) \geq 0$, $t \geq t_1$ for some $t_1 \geq t_0 + T$.

<u>Case 1.</u> $w'(t) < 0$ for $t \geq t_1$. Then, as in the proof of Theorem 1, taking into account the assumption (H_3), we have that $\lim_{t\to\infty} w(t) = -\infty$. We claim that $x(t)$ is bounded from above. If it is not the case, there exists a number $t_2 \geq t_1$ such that (4) holds, so that we come to the following contradiction

$$\begin{aligned} 0 > \quad w(t_2) &= x(t_2) - \sum_{i=1}^{l} h_i(t_2)x(t_2-\rho_i) - \sum_{i=1}^{n} \int_{t_0}^{t_2} \frac{1}{r(s)} \int_{s-\delta_i}^{s-\sigma_i} q_i(\xi)x(\xi)d\xi\, ds \\ &\geq \left\{1 - \sum_{i=1}^{l} h_i - \sum_{i=1}^{n} \int_{t_0}^{t_2} \frac{1}{r(s)} \int_{s-\delta_i}^{s-\sigma_i} q_i(\xi)d\xi\, ds\right\} x(t_2) \\ &\geq \left\{1 - \sum_{i=1}^{l} h_i - \sum_{i=1}^{n} \int_{0}^{\infty} \frac{1}{r(s)} \int_{s-\delta_i}^{s-\sigma_i} q_i(\xi)d\xi\, ds\right\} x(t_2) \geq 0. \end{aligned}$$

Therefore, $x(t)$ must be bounded from above. Consequently, for every $L > 0$, there exists some $t_3 \geq t_1$ such that $x(t) \leq L$ for $t \geq t_3$. It follows from (9) that

$$\begin{aligned} w(t) &\geq -L\left\{\sum_{i=1}^{l} h_i(t) + \sum_{i=1}^{n} \int_{t_0}^{t} \frac{1}{r(s)} \int_{s-\delta_i}^{s-\sigma_i} q_i(\xi)d\xi\, ds\right\} \\ &\geq -L\left\{\sum_{i=1}^{l} h_i(t) + \sum_{i=1}^{n} \int_{0}^{\infty} \frac{1}{r(s)} \int_{s-\delta_i}^{s-\sigma_i} q_i(\xi)d\xi\, ds\right\} \geq -L > -\infty, \quad t \geq t_3, \end{aligned}$$

which contradicts the fact that $\lim_{t\to\infty} w(t) = -\infty$.

<u>Case 2.</u> $w'(t) \geq 0$ for $t \geq t_1$. Integration of (10) over $[t_1, t]$ yields that $x(t) \in L^1([t_1, \infty))$. From (9), it follows that

$$\left[x(t) - \sum_{i=1}^{l} h_i(t)x(t-\rho_i)\right]' = w'(t) + \sum_{i=1}^{n} \frac{1}{r(t)} \int_{t-\delta_i}^{t-\sigma_i} q_i(s)x(s)ds \geq 0,$$

so that, $X(t) = x(t) - \sum_{i=1}^{l} h_i(t)x(t-\rho_i)$ is a nondecreasing function. If we let

$$\lim_{t\to\infty} X(t) = \lim_{t\to\infty}\left\{x(t) - \sum_{i=1}^{l} h_i(t)x(t-\rho_i)\right\} = \mu,$$

taking into account the fact that $x(t) \in L^1([t_1,\infty))$, we conclude that $\mu \neq \infty$, and therefore $\mu \in (-\infty,\infty)$.

(i) If $0 < \mu < \infty$, then for any ε with $0 < \varepsilon < \mu$, there exists a number $t_4 \geq t_1$ such that

$$X(t) > \mu - \varepsilon$$

But, this implies that $x(t) \notin L^1([t_4,\infty))$, which is a contradiction.

(ii) If $-\infty < \mu < 0$, then for any ε with $0 < \varepsilon < -\mu$, there exists a number $t_5 \geq t_1$ such that

$$X(t) < \mu + \varepsilon, \quad t \geq t_5.$$

Hence we have

$$\sum_{i=1}^{l} x(t-\rho_i) > -(\mu+\varepsilon), \quad t \geq t_5,$$

which again contradicts the fact that $x(t) \in L^1([t_5,\infty))$.

(iii) If $\mu = 0$, then we claim that $x(t)$ is bounded from above. If this is not the case, then there exists a sequence $\{t_{\tilde{n}}\}_{\tilde{n}=1}^{\infty}$ such that

$$\lim_{\tilde{n}\to\infty} t_{\tilde{n}} = \infty, \quad \max_{t_1 \leq t \leq t_{\tilde{n}}} x(t) = x(t_{\tilde{n}}), \quad \lim_{\tilde{n}\to\infty} x(t_{\tilde{n}}) = \infty.$$

Then, we see that

$$X(t_{\tilde{n}}) = x(t_{\tilde{n}}) - \sum_{i=1}^{l} h_i(t_{\tilde{n}})x(t_{\tilde{n}} - \rho_i) \geq \left[1 - \sum_{i=1}^{l} h_i\right] x(t_{\tilde{n}}),$$

and taking the limit as $\tilde{n} \to \infty$, we are led to a contradiction in view of the facts that $\lim_{\tilde{n}\to\infty} X(t_{\tilde{n}}) = \mu = 0$ and $\lim_{\tilde{n}\to\infty} x(t_{\tilde{n}}) = \infty$. Hence, $x(t)$ is bounded from above. Using the assumption (H_7), we have that

$$x(t) - \sum_{i=1}^{l} h_i(t)x(t-\rho_i) \geq x(t) - \sum_{i=1}^{l} h_i \cdot x(t-\rho_i).$$

Taking the upper limit of the above inequality as $t \to \infty$, we obtain

$$
\begin{aligned}
0 &\ge \limsup_{t\to\infty}\left[x(t)-\sum_{i=1}^{l}h_i\cdot x(t-\rho_i)\right]\\
&\ge \limsup_{t\to\infty}x(t)+\liminf_{t\to\infty}\left[-\sum_{i=1}^{l}h_i\cdot x(t-\rho_i)\right]\\
&\ge \limsup_{t\to\infty}x(t)-\sum_{i=1}^{l}h_i\cdot\limsup_{t\to\infty}x(t-\rho_i)\\
&\ge \left[1-\sum_{i=1}^{l}h_i\right]\limsup_{t\to\infty}x(t).
\end{aligned}
$$

Therefore, we observe that $\lim_{t\to\infty}x(t)=0$. This completes the proof of the theorem.

Example 3: We consider the equation

$$
\begin{aligned}
&\left[\frac{1}{t}\left[x(t)-e^{2\pi-7}x(t-2\pi)\right]'\right]' + \left(1-e^{-7}\right)\left(\frac{1}{t^2}+1\right)e^{\frac{3}{2}\pi}x\left(t-\frac{3}{2}\pi\right)\\
&\quad+\left(1-e^{-7}\right)\left(\frac{2}{t}+1\right)e^{\frac{\pi}{2}}x\left(t-\frac{\pi}{2}\right)+\left(1-e^{-7}\right)\frac{e^{2\pi}}{t^2}x(t-2\pi)\\
&\quad-\frac{1}{2}e^{-t-2\pi}x(t-\pi)-\frac{1}{2}e^{-t-3\pi}x(t)=0,\quad t>0.
\end{aligned}
\tag{11}
$$

Here, it is easily checked that

$$
\begin{aligned}
&r(t)=\frac{1}{t},\ l=1,\ h_1(t)=e^{2\pi-7},\ \rho_1=2\pi,\\
&m=3,\ p_1(t)=\left(1-e^{-7}\right)\left(\frac{1}{t^2}+1\right)e^{\frac{3}{2}\pi},\ \delta_1=\frac{3}{2}\pi,\ p_2(t)=\left(1-e^{-7}\right)\left(\frac{2}{t}+1\right)e^{\frac{\pi}{2}},\ \delta_2=\frac{\pi}{2},\\
&\qquad p_3(t)=\left(1-e^{-7}\right)\frac{e^{2\pi}}{t^2},\ \delta_3=2\pi,\\
&n=2,\ q_1(t)=\frac{1}{2}e^{-t-2\pi},q_2(t)=\frac{1}{2}e^{-t-3\pi},\ \sigma_1=\pi,\ \sigma_2=0.
\end{aligned}
$$

Thus, the conditions (H_4) and (H_5) for the coefficients of the equation (11) are fulfilled, since

$$
\begin{aligned}
&q_1(t)=\frac{1}{2}e^{-t-2\pi}<\frac{1}{2}e^{-t-\pi}=q_1(t-\sigma_1),\\
&q_2(t)=\frac{1}{2}e^{-t-3\pi}=q_2(t-\sigma_2)=q_2(t),
\end{aligned}
$$

$$p_1(t)-q_1(t-\delta_1)=\left(1-e^{-7}\right)\left(\frac{1}{t^2}+1\right)e^{\frac{3}{2}\pi}-\frac{1}{2}e^{-t-\frac{\pi}{2}}\geq\left(1-e^{-7}\right)e^{\frac{3}{2}\pi}-\frac{1}{2}e^{-\frac{\pi}{2}}>0,$$

$$p_2(t)-q_2(t-\delta_2)=\left(1-e^{-7}\right)\left(\frac{2}{t}+1\right)e^{\frac{\pi}{2}}-\frac{1}{2}e^{-t-\frac{5}{2}\pi}\geq\left(1-e^{-7}\right)e^{\frac{\pi}{2}}-\frac{1}{2}e^{-\frac{5}{2}\pi}\equiv k_1>0.$$

Moreover, the condition (8) of Theorem 2 is also verified by

$$e^{2\pi-7}+\frac{1}{2}\int_0^\infty s\int_{s-\frac{3}{2}\pi}^{s-\pi}e^{-\xi-2\pi}d\xi\,ds+\frac{1}{2}\int_0^\infty s\int_{s-\frac{\pi}{2}}^{s}e^{-\xi-3\pi}d\xi\,ds$$

$$=e^{2\pi-7}+\frac{1}{2}\left(e^{-\frac{\pi}{2}}-e^{-\pi}\right)+\frac{1}{2}\left(e^{-\frac{5}{2}\pi}-e^{-3\pi}\right)<1.$$

So, Theorem 2 implies that every solution $x(t)$ of eq. (11) is oscillatory or tends to zero limits as $t\to\infty$. In fact, $x(t)=e^t\cos t$ is the oscillatory solution of this equation.

4. Oscillation of solutions of equations (E_1) and (E_2) with forcing terms

In this section we consider equations (E_1) and (E_2) with forcing term

$$(\mathrm{E}_3)\quad \left[r(t)\left[x(t)+\sum_{i=1}^{l}h_i(t)x(t-\rho_i)\right]'\right]'+\sum_{i=1}^{m}p_i(t)x(t-\delta_i)-\sum_{i=1}^{n}q_i(t)x(t-\sigma_i)=f(t),t>0,$$

$$(\mathrm{E}_4)\quad \left[r(t)\left[x(t)-\sum_{i=1}^{l}h_i(t)x(t-\rho_i)\right]'\right]'+\sum_{i=1}^{m}p_i(t)x(t-\delta_i)-\sum_{i=1}^{n}q_i(t)x(t-\sigma_i)=f(t),t>0,$$

where

$f(t)\in C([0,\infty);\mathbf{R})$.

Theorem 3: Assume that (H_6) holds and that

$$(\mathrm{H}_8)\quad \begin{cases}\text{there exists a function } F(t)\in C^1([0,\infty);\mathbf{R}) \text{ such that}\\ r(t)F'(t)\in C^1([0,\infty);\mathbf{R})\\ [r(t)F'(t)]'=f(t)\\ \lim\limits_{t\to\infty}F(t)=0.\end{cases}$$

Every solution of the equation (E_3) is oscillatory or satisfies $\lim\limits_{t\to\infty}x(t)=0$, if the condition (1) is satisfied.

Proof: Suppose that $x(t)$ is a nonoscillatory solution of (E_3) such that $x(t) > 0$ for $t \geq t_0$, where t_0 is some positive number. If we choose

$$Z(t) = z(t) - F(t), \tag{12}$$

where $z(t)$ is defined by (2), then we obtain from the eq. (E_3)

$$(r(t)Z'(t))' \leq -k_j x(t-\delta_j) \leq 0, \qquad t \geq t_0 + T \tag{13}$$

for some $j \in \{1, 2, \ldots, n\}$. We claim that $Z'(t)$ is eventually nonnegative function. If we suppose on the contrary that $Z'(t) < 0$, $t \geq t_1$ for some $t_1 \geq t_0 + T$, then using (H_3) we have that $\lim_{t\to\infty} Z(t) = -\infty$. First, we prove that $x(t)$ is bounded from above. As a matter of fact, if $x(t)$ is unbounded from above, there exists a sequence $\{t_{\hat{n}}\}_{\hat{n}=1}^{\infty}$ satisfying

$$\lim_{\hat{n}\to\infty} t_{\hat{n}} = \infty, \quad \lim_{\hat{n}\to\infty} Z(t_{\hat{n}}) = -\infty, \quad \lim_{\hat{n}\to\infty} F(t_{\hat{n}}) = 0,$$
$$\max_{t_1 \leq t \leq t_{\hat{n}}} x(t) = x(t_{\hat{n}}), \quad \lim_{\hat{n}\to\infty} x(t_{\hat{n}}) = \infty.$$

Then, we have

$$\begin{aligned} Z(t_{\hat{n}}) &= x(t_{\hat{n}}) + \sum_{i=1}^{l} h_i(t_{\hat{n}}) x(t_{\hat{n}} - \rho_i) - \sum_{i=1}^{n} \int_{t_0}^{t_{\hat{n}}} \frac{1}{r(s)} \int_{s-\delta_i}^{s-\sigma_i} q_i(\xi) x(\xi) d\xi\, ds - F(t_{\hat{n}}) \\ &\geq \left\{1 - \sum_{i=1}^{n} \int_{t_0}^{t_{\hat{n}}} \frac{1}{r(s)} \int_{s-\delta_i}^{s-\sigma_i} q_i(\xi) d\xi\, ds \right\} x(t_{\hat{n}}) - F(t_{\hat{n}}) \\ &\geq \left\{1 - \sum_{i=1}^{n} \int_{0}^{\infty} \frac{1}{r(s)} \int_{s-\delta_i}^{s-\sigma_i} q_i(\xi) d\xi\, ds \right\} x(t_{\hat{n}}) - F(t_{\hat{n}}) \end{aligned}$$

and taking the limit as $\hat{n} \to \infty$, leads to the contradiction

$$\lim_{\hat{n}\to\infty} Z(t_{\hat{n}}) \geq \left\{1 - \sum_{i=1}^{n} \int_{0}^{\infty} \frac{1}{r(s)} \int_{s-\delta_i}^{s-\sigma_i} q_i(\xi) d\xi\, ds \right\} \lim_{\hat{n}\to\infty} x(t_{\hat{n}}) = \infty.$$

Therefore, $x(t)$ is bounded from above, so that for arbitrary constant $L > 0$, there exists a number $t_2 \geq t_1$ such that $x(t) \leq L$ for $t \geq t_2$. Hence, from (12) we have

$$Z(t) \geq -L \sum_{i=1}^{n} \int_{0}^{t} \frac{1}{r(s)} \int_{s-\delta_i}^{s-\sigma_i} q_i(\xi) d\xi\, ds - F(t), \quad t \geq t_2,$$

which according to the assumption (1), yields the following contradiction

$$\lim_{t\to\infty} Z(t) \geq -L\sum_{i=1}^{n}\int_0^\infty \frac{1}{r(s)}\int_{s-\delta_i}^{s-\sigma_i} q_i(\xi)d\xi\, ds \geq -L$$

to the fact that $\lim_{t\to\infty} Z(t) = -\infty$. Accordingly, $Z'(t)$ is eventually nonnegative i.e. $Z'(t) \geq 0$ for $t \geq t_1$. Now, we denote by

$$X(t) = x(t) + \sum_{i=1}^{l} h_i(t)x(t-\rho_i), \quad Y(t) = X(t) - F(t).$$

From (13), we have that $x(t) \in L^1([t_1,\infty))$ and consequently $X(t) \in L^1([t_1,\infty))$. From (12), we obtain

$$Y'(t) = Z'(t) + \sum_{i=1}^{n} \frac{1}{r(t)}\int_{t-\delta_i}^{t-\sigma_i} q_i(s)x(s)ds \geq 0,$$

so that $Y(t)$ is a nondecreasing function. Therefore, using the hypothesis (H_8), we have

$$\lim_{t\to\infty} Y(t) = \lim_{t\to\infty} X(t) = \mu \in [0,\infty).$$

If $0 < \mu < \infty$, then there exists a number $t_3 \geq t_1$ such that

$$X(t) > \mu - \varepsilon, \quad t \geq t_3$$

for arbitrary $\varepsilon \in (0,\mu)$. Hence, $X(t) \notin L^1([t_3,\infty))$, which is a contradiction. If $\mu = 0$, then since $x(t) \leq x(t) + \sum_{i=1}^{l} h_i(t)x(t-\rho_i)$, $t \geq t_1$, we find that $\lim_{t\to\infty} x(t) = 0$. This completes the proof.

Example 4: Consider the equation

$$\begin{aligned}
&\left[e^{-t}\left[x(t)+\frac{e^{\pi}}{2}x(t-\pi)\right]'\right]' + \left(\frac{1}{2}e^{-t+\frac{\pi}{2}} + e^{\frac{3}{2}\pi}\right)x\left(t-\frac{\pi}{2}\right) + e^{\frac{5}{2}\pi}x\left(t-\frac{3}{2}\pi\right) \\
&\quad + \left(\frac{1}{2}e^{-t+2\pi} + e^{-2t+\pi}\right)x(t-2\pi) - e^{-2t-\pi}x(t) \qquad (14)\\
&= \left(e^{\frac{5}{2}\pi} + e^{\frac{11}{2}\pi}\right)e^{-2t} + \left(6 + \frac{1}{2}e^{\frac{3}{2}\pi} + 3e^{3\pi} + \frac{1}{2}e^{6\pi}\right)e^{-3t} + \left(e^{5\pi} - e^{-\pi}\right)e^{-4t}, \qquad t > 0.
\end{aligned}$$

Here we have

$$r(t)=e^{-t},\ l=1,\ h_1(t)=\frac{e^{\pi}}{2},\ \rho_1=\pi,$$
$$m=3,\ p_1(t)=\frac{1}{2}e^{-t+\frac{\pi}{2}}+e^{\frac{3}{2}\pi},\ \delta_1=\frac{\pi}{2},\ p_2(t)=e^{\frac{5}{2}\pi},\ \delta_2=\frac{3}{2}\pi,$$
$$p_3(t)=\frac{1}{2}e^{-t+2\pi}+e^{-2t+\pi},\ \delta_3=2\pi,$$
$$n=1,\ q_1(t)=e^{-2t-\pi},\ \sigma_1=0,$$
$$f(t)=\left(e^{\frac{5}{2}\pi}+e^{\frac{11}{2}\pi}\right)e^{-2t}+\left(6+\frac{1}{2}e^{\frac{3}{2}\pi}+3e^{3\pi}+\frac{1}{2}e^{6\pi}\right)e^{-3t}+\left(e^{5\pi}-e^{-\pi}\right)e^{-4t}.$$

Since

$$q_1(t)-q_1(t-\sigma_1)=0,$$
$$p_1(t)-q_1(t-\delta_1)=\frac{1}{2}e^{-t+\frac{\pi}{2}}+e^{\frac{3}{2}\pi}-e^{-2t}\geq e^{\frac{3}{2}\pi}-1\equiv k_1\geq 0,$$

we have that conditions (H_4) and (H_5) are satisfied. Moreover, there exists a function

$$F(t)=\frac{1}{2}\left(e^{\frac{5}{2}\pi}+e^{\frac{11}{2}\pi}\right)e^{-t}+\frac{1}{6}\left(6+\frac{1}{2}e^{\frac{3}{2}\pi}+3e^{3\pi}+\frac{1}{2}e^{6\pi}\right)e^{-3t}+\frac{1}{12}\left(e^{5\pi}-e^{-\pi}\right)e^{-4t}$$

satisfying (H_8). The condition (1) is also fulfilled, since we get

$$\int_0^{\infty}e^{s}\int_{s-\frac{\pi}{2}}^{s}e^{-2\xi-\pi}d\xi\, ds=\frac{1}{2}\left(1-e^{-\pi}\right)<1.$$

Accordingly, by Theorem 3, it follows that every solution of the equation (14) is oscillatory. In fact, $x(t)=e^{-2t}+e^{t}\sin t$ is such a solution.

Theorem 4: Assume that (H_7) and (H_8) hold. If the condition (8) holds, then every solution of the equation (E_4) is oscillatory or satisfies $\lim_{t\to\infty}x(t)=0$.

Proof: Suppose that $x(t)$ is a nonoscillatory solution of (E_4) such that $x(t)>0$, $t\geq t_0$, where t_0 is some positive number. Let we denote with

$$W(t)=w(t)-F(t),$$

where $w(t)$ is defined by (9). Then, we see immediately that

$$(r(t)W'(t))'\leq -k_j x(t-\delta_j)\leq 0,\quad t\geq t_0+T \tag{15}$$

for some $j\in\{1,2,\ldots,n\}$. Therefore, we have the following two cases:

Case 1. $W'(t) < 0$, $t \ge t_1$ for some $t_1 \ge t_0 + T$ which implies that $\lim_{t\to\infty} W(t) = -\infty$. On the other hand, $x(t)$ must be bounded from above. Otherwise, there exists a sequence $\{t_{\breve{n}}\}_{\breve{n}=1}^{\infty}$ satisfying

$$\lim_{\breve{n}\to\infty} t_{\breve{n}} = \infty, \quad \lim_{\breve{n}\to\infty} W(t_{\breve{n}}) = -\infty, \quad \lim_{\breve{n}\to\infty} F(t_{\breve{n}}) = 0,$$
$$\max_{t_1 \le t \le t_{\breve{n}}} x(t) = x(t_{\breve{n}}), \quad \lim_{\breve{n}\to\infty} x(t_{\breve{n}}) = \infty.$$

Since we have that

$$\begin{aligned} W(t_{\breve{n}}) &= x(t_{\breve{n}}) - \sum_{i=1}^{l} h_i(t_{\breve{n}}) x(t_{\breve{n}} - \rho_i) - \sum_{i=1}^{n} \int_{t_0}^{t_{\breve{n}}} \frac{1}{r(s)} \int_{s-\delta_i}^{s-\sigma_i} q_i(\xi) x(\xi) d\xi \, ds - F(t_{\breve{n}}) \\ &\ge \left\{ 1 - \sum_{i=1}^{l} h_i - \sum_{i=1}^{n} \int_{t_0}^{t_{\breve{n}}} \frac{1}{r(s)} \int_{s-\delta_i}^{s-\sigma_i} q_i(\xi) d\xi \, ds \right\} x(t_{\breve{n}}) - F(t_{\breve{n}}) \\ &\ge \left\{ 1 - \sum_{i=1}^{l} h_i - \sum_{i=1}^{n} \int_{0}^{\infty} \frac{1}{r(s)} \int_{s-\delta_i}^{s-\sigma_i} q_i(\xi) d\xi \, ds \right\} x(t_{\breve{n}}) - F(t_{\breve{n}}), \end{aligned}$$

letting $\breve{n} \to \infty$ we obtain

$$\lim_{\breve{n}\to\infty} W(t_{\breve{n}}) \ge \left\{ 1 - \sum_{i=1}^{l} h_i - \sum_{i=1}^{n} \int_{0}^{\infty} \frac{1}{r(s)} \int_{s-\delta_i}^{s-\sigma_i} q_i(\xi) d\xi \, ds \right\} \lim_{\breve{n}\to\infty} x(t_{\breve{n}}) > 0,$$

which is the contradiction. Therefore, $x(t)$ is bounded from above, so that for every $L > 0$ there exists a number $t_2 \ge t_1$ such that $x(t) \le L$ for $t \ge t_2$. Then

$$W(t) \ge -L \sum_{i=1}^{l} h_i - L \sum_{i=1}^{n} \int_{0}^{t} \frac{1}{r(s)} \int_{s-\delta_i}^{s-\sigma_i} q_i(\xi) \, d\xi \, ds - F(t), \quad t \ge t_2,$$

and letting $t \to \infty$, we get

$$\lim_{t\to\infty} W(t) \ge -L \left\{ \sum_{i=1}^{l} h_i + \sum_{i=1}^{n} \int_{0}^{\infty} \frac{1}{r(s)} \int_{s-\delta_i}^{s-\sigma_i} q_i(\xi) \, d\xi \, ds \right\} \ge -L,$$

which contradicts the fact that $\lim_{t\to\infty} W(t) = -\infty$.

Case 2. $W'(t) \ge 0$ for $t \ge t_1$. From (15) we obtain $x(t) \in L^1([t_1, \infty))$. We see that

$$\left[x(t) - \sum_{i=1}^{l} h_i(t) x(t - \rho_i) - F(t) \right]' = W'(t) + \sum_{i=1}^{n} \frac{1}{r(t)} \int_{t-\delta_i}^{t-\sigma_i} q_i(s) x(s) ds \ge 0, \quad t \ge t_1,$$

so that

$$\lim_{t\to\infty}\left\{x(t)-\sum_{i=1}^{l}h_i(t)x(t-\rho_i)-F(t)\right\}=\lim_{t\to\infty}\left\{x(t)-\sum_{i=1}^{l}h_i(t)x(t-\rho_i)\right\}=\mu,$$

where $\mu\in(-\infty,\infty]$. The rest of the proof is similar to the proof Theorem 2, and so, we are led to the contradiction in the cases when $\mu\neq 0$, while in the case of $\mu=0$ we conclude that $\lim_{t\to\infty}x(t)=0$. Therefore, the proof is completed.

Example 5: Consider the equation

$$\left[e^{-t}\left[x(t)-\frac{1}{2e^2}x(t-1)\right]'\right]'+\left(1+e^{-2t-5}\right)x(t-1)-e^{-2t-3}x(t)=3e^{-3t}+e^{-2t+2},\quad t>0. \tag{16}$$

Here we have

$$\begin{aligned}&l=m=n=1,\ r(t)=e^{-t},\ h_1(t)=\frac{1}{2e^2},\ \rho_1=1,\\&p_1(t)=1+e^{-2t-5},\ \delta_1=1,\ q_1(t)=e^{-2t-3},\ \sigma_1=0,\\&f(t)=3e^{-3t}+e^{-2t+2}\end{aligned}$$

so that, for $t\geq 0$, it is obvious that

$$\begin{aligned}&q_1(t)-q_1(t-\sigma_1)=0,\\&p_1(t)-q_1(t-\delta_1)=\left(1+e^{-2t-5}\right)-e^{-2t-1}\geq 1-e^{-1}\equiv k_1>0.\end{aligned}$$

Moreover, there exists a function

$$F(t)=\frac{e^{-2t}}{2}\left(e^{t+2}+1\right)$$

satisfying (H_8), which can be easily verified. The condition (8) is also satisfied, since we have that

$$\int_0^\infty e^s\int_{s-1}^{s}e^{-2\xi-3}d\xi\,ds=\frac{1}{2e}\left(1-e^{-2}\right)<1-h_1=1-\frac{1}{2e^2}.$$

Therefore, Theorem 4 implies that every solution of the equation (16) is oscillatory or satisfies $\lim_{t\to\infty}x(t)=0$. In fact, $x(t)=e^{-2t}$ is a solution of the equation (16) which tends to zero as $t\to\infty$.

5. Conclusion

In this paper, we studied the oscillations of second order neutral differential equations with positive and negative coefficients. We derived sufficient conditions for every solution of (E_1) or (E_2) to be oscillatory. Our results generalize those of Manojlović, Shoukaku, Tanigawa and Yoshida (2006).

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