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Oscillations of Hyperbolic Systems with Functional Arguments*

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Abstract

Hyperbolic systems with functional arguments are studied, and sufficient conditions are obtained for every solution of boundary value problems to be weakly oscillatory (that is, at least one of its components is oscillatory) in a cylindrical domain. Robin-type boundary condition is considered. The approach used is to reduce the multi-dimensional oscillation problems to one-dimensional oscillation problems by using some integral means of solutions.

Keywords: Oscillation; hyperbolic systems; functional arguments
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1. Introduction

We are concerned with the oscillation of the system of hyperbolic equations with functional arguments

$$\begin{aligned}&\frac{\partial^2}{\partial t^2}\left(U(x,t)+\sum_{i=1}^{\ell} H_i(t)U(x,\rho_i(t))\right) - A(t)\Delta U(x,t)\\ &\quad -\sum_{i=1}^{K} B_i(t)\Delta U(x,\tau_i(t)) + \sum_{i=1}^{m} P_i(x,t)\varphi_i\big(U(x,\sigma_i(t))\big)\\ &= F(x,t), \quad (x,t)\in\Omega\equiv G\times(0,\infty),\end{aligned} \tag{1}$$

where G is a bounded domain in R^n with piecewise smooth boundary ∂G, Δ is the Laplacian in R^n, and

$$H_i(t) = \left(h_{ijk}(t)\right)_{j,k=1}^M, \quad A(t) = \left(a_{jk}(t)\right)_{j,k=1}^M, \quad B_i(t) = \left(b_{ijk}(t)\right)_{j,k=1}^M,$$
$$P_i(x,t) = \left(p_{ijk}(x,t)\right)_{j,k=1}^M,$$
$$U(x,t) = \left(u_1(x,t),\dots,u_M(x,t)\right)^T,$$
$$U(x,\rho_i(t)) = \left(u_1(x,\rho_i(t)),\dots,u_M(x,\rho_i(t))\right)^T,$$
$$U(x,\tau_i(t)) = \left(u_1(x,\tau_{i1}(t)),\dots,u_M(x,\tau_{iM}(t))\right)^T,$$
$$\varphi_i\left(U(x,\sigma_i(t))\right) = \left(\varphi_{i1}(u_1(x,\sigma_i(t))),\dots,\varphi_{iM}(u_M(x,\sigma_i(t)))\right)^T,$$
$$F(x,t) = \left(f_1(x,t),\dots,f_M(x,t)\right)^T,$$

the superscript T denoting the transpose.

It is easy to see that (1) can be written in the following system:

$$\begin{aligned}\frac{\partial^2}{\partial t^2}\left(u_j(x,t) + \sum_{i=1}^{\ell}\sum_{k=1}^{M} h_{ijk}(t)u_k(x,\rho_i(t))\right) \\ - \sum_{k=1}^{M} a_{jk}(t)\Delta u_k(x,t) - \sum_{i=1}^{K}\sum_{k=1}^{M} b_{ijk}(t)\Delta u_k(x,\tau_{ik}(t)) \\ + \sum_{i=1}^{m}\sum_{k=1}^{M} p_{ijk}(x,t)\varphi_{ik}\left(u_k(x,\sigma_i(t))\right) = f_j(x,t) \quad (j = 1, 2, \dots, M)\end{aligned} \tag{2}$$

for $(x,t) \in \Omega \equiv G \times (0,\infty)$.

The boundary condition to be considered is the following:

$$\alpha(\hat{x})\frac{\partial u_j}{\partial \nu} + \mu(\hat{x})u_j = \alpha(\hat{x})\tilde{\psi}_j + \mu(\hat{x})\psi_j \quad \text{on} \quad \partial G \times (0,\infty) \ (j = 1, 2, \dots, M), \tag{BC}$$

where $\psi_j, \tilde{\psi}_j \in C(\partial G \times (0,\infty); \mathbb{R})$, $\alpha(\hat{x}), \mu(\hat{x}) \in C(\partial G; [0,\infty))$, $\alpha(\hat{x})^2 + \mu(\hat{x})^2 \neq 0$, and ν denotes the unit exterior normal vector to ∂G.

In case $\alpha(\hat{x}) \equiv 0$ on ∂G, then $\mu(\hat{x}) \neq 0$ on ∂G, and hence the boundary condition (BC) reduces to

$$u_j = \psi_j \quad \text{on} \quad \partial G \times (0,\infty) \quad (j = 1, 2, \dots, M).$$

If $\alpha(\hat{x}) = 1$ on ∂G, then (BC) can be written in the form

$$\frac{\partial u_j}{\partial \nu} + \mu(\hat{x})u_j = \tilde{\psi}_j + \mu(\hat{x})\psi_j \quad \text{on} \quad \partial G \times (0,\infty) \quad (j = 1, 2, \dots, M),$$

moreover, if $\mu(\hat{x}) \equiv 0$ on ∂G, then (BC) can be written as

$$\frac{\partial u_j}{\partial \nu} = \tilde{\psi}_j \quad \text{on} \quad \partial G \times (0,\infty) \quad (j = 1, 2, \dots, M).$$

We assume that:

(H_1) $h_{ijk}(t) \in C^2([0,\infty);[0,\infty))$ $(i = 1, 2, \ldots, \ell\,; j, k = 1, 2, \ldots, M)$,
$a_{jk}(t) \in C([0,\infty);[0,\infty))$ $(j, k = 1, 2, \ldots, M)$,
$b_{ijk}(t) \in C([0,\infty);[0,\infty))$ $(i = 1, 2, \ldots, K;\, j, k = 1, 2, \ldots, M)$,
$p_{ijk}(x,t) \in C(\overline{G} \times [0,\infty);[0,\infty))$ $(i = 1, 2, \ldots, m\,; j, k = 1, 2, \ldots, M)$,
$f_j(x,t) \in C(\overline{G} \times [0,\infty);\mathrm{R})$ $(j = 1, 2, \ldots M)$;

(H_2) $\rho_i(t) \in C^2([0,\infty);\mathrm{R})$, $\lim_{t\to\infty} \rho_i(t) = \infty$ $(i = 1, 2, \ldots, \ell)$,
$\tau_{ik}(t) \in C([0,\infty);\mathrm{R})$, $\lim_{t\to\infty} \tau_{ik}(t) = \infty$ $(i = 1, 2, \ldots, K; k = 1, 2, \ldots, M)$,
$\sigma_i(t) \in C([0,\infty);\mathrm{R})$, $\lim_{t\to\infty} \sigma_i(t) = \infty$ $(i = 1, 2, \ldots, m)$;

(H_3) $\varphi_{ik}(\xi) \in C(\mathrm{R};\mathrm{R})$, $\varphi_{ik}(\xi) \ge 0$ for $\xi \ge 0$, $\varphi_{ik}(-\xi) = -\varphi_{ik}(\xi)$ for $\xi > 0$ and $\varphi_{ik}(\xi)$ are convex in $(0,\infty)$ $(i = 1, 2, \ldots, m\,; k = 1, 2, \ldots, M)$;

(H_4) $h_{ikk}(t) - \sum_{\substack{j=1 \\ j\neq k}}^{M} h_{ijk}(t) \ge 0$ $(i = 1, 2, \ldots, \ell\,; k = 1, 2, \ldots, M)$;

(H_5) $a_{kk}(t) - \sum_{\substack{j=1 \\ j\neq k}}^{M} a_{jk}(t) \ge 0$ $(k = 1, 2, \ldots, M)$;

(H_6) $b_{ikk}(t) - \sum_{\substack{j=1 \\ j\neq k}}^{M} b_{ijk}(t) \ge 0$ $(i = 1, 2, \ldots, K; k = 1, 2, \ldots, M)$;

(H_7) $p_{ikk}(x,t) - \sum_{\substack{j=1 \\ j\neq k}}^{M} p_{ijk}(x,t) \ge 0$ $(i = 1, 2, \ldots, m\,; k = 1, 2, \ldots, M)$;

(H_8) $\hat{\varphi}_i(\xi) \equiv \min_{1\le k\le M} \varphi_{ik}(\xi)$ is nondecreasing and convex in $(0,\infty)$ $(i = 1, 2, \ldots, m)$.

Definition 1: By a *solution* of system (2) we mean a vector function $(u_1(x,t), \ldots, u_M(x,t))$ such that $u_j(x,t) \in C^2(\overline{G} \times [t_{-1},\infty);\mathrm{R}) \cap C(\overline{G} \times [\tilde{t}_{-1},\infty);\mathrm{R})$, and $u_j(x,t)$ $(j = 1, 2, \ldots, M)$ satisfy (2) in Ω, where

$$t_{-1} = \min\left\{0, \min_{\substack{1\le i\le K \\ 1\le k\le M}} \left\{\inf_{t\ge 0} \tau_{ik}(t)\right\}, \min_{1\le i\le \ell} \left\{\inf_{t\ge 0} \rho_i(t)\right\}\right\},$$

$$\tilde{t}_{-1} = \min\left\{0, \min_{1\le i\le m} \left\{\inf_{t\ge 0} \sigma_i(t)\right\}\right\}.$$

Definition 2: A solution $(u_1(x,t), \ldots, u_M(x,t))$ of system (2) is said to be *weakly oscillatory* in Ω if at least one of its components is oscillatory in Ω (cf. Ladde, Lakshmikantham and Zhang (1987, Definition 6.2.1)).

In 1984 the oscillations of delay hyperbolic equations have been first investigated by Mishev and Bainov (1984) (cf. Mishev 1989, Mishev and Bainov, 1986). Parhi and Kirane (1994) investigated the oscillatory properties of solutions of coupled hyperbolic equations. Oscillation of hyperbolic systems with deviating arguments was studied by Li (1997), and then oscillation results have been established by several authors, see, e.g., Li (2000), Agarwal, Meng and Li (2002) and the references cited therein. However, all of them pertain to the case where the matrices $H_i(t)$ are the diagonal matrices or $H_i(t) \equiv 0$.

The purpose of this paper is to derive sufficient conditions for every solution of the boundary value problem (2), (BC) to be weakly oscillatory in a cylindrical domain $G \times (0,\infty)$. We note that the matrices $H_i(t)$ are not necessarily the diagonal matrices.

2. Oscillation results

In this section we establish a lemma and two oscillation theorems for the boundary value problem (2), (BC). Two examples are also given in this section to illustrate oscillation results.

$$-\Delta w = \lambda\, w \quad \text{in} \quad G,$$
$$\alpha(\hat{x})\frac{\partial w}{\partial \nu} + \mu(\hat{x})\, w = 0 \quad \text{on} \quad \partial G$$

is nonnegative, and the corresponding eigenfunction $\Phi(x)$ can be chosen so that $\Phi(x) > 0$ in G (see Ye and Li, 1990, Theorem 3.3.22). In case $\mu(\hat{x}) \equiv 0$ on ∂G, then we can choose $\lambda_1 = 0$ and $\Phi(x) = 1$. If $\mu(\hat{x}) \not\equiv 0$ on ∂G, then there exist $\lambda_1 > 0$ and the eigenfunction $\Phi(x) > 0$ in G.

We use the notation :

$$\Gamma_1 = \{\hat{x} \in \partial G\, ;\, \alpha(\hat{x}) = 0\},$$
$$\Gamma_2 = \{\hat{x} \in \partial G\, ;\, \alpha(\hat{x}) \neq 0\}.$$

Lemma: If u_j satisfy the boundary condition (BC) and $\Phi(x)$ is the eigenfunction corresponding to the smallest eigenvalue $\lambda_1 \geq 0$, then we obtain

$$K_\Phi \int_{\partial G} \left[\frac{\partial u_j}{\partial \nu} \Phi(x) - u_j \frac{\partial \Phi(x)}{\partial \nu} \right] dS = \Psi_j(t), \tag{3}$$

where

$$K_\Phi = \left(\int_G \Phi(x)\, dx \right)^{-1},$$
$$\Psi_j(t) = K_\Phi \left(- \int_{\Gamma_1} \psi_j \frac{\partial \Phi(x)}{\partial \nu} dS + \int_{\Gamma_2} \left(\tilde{\psi}_j + \frac{\mu(\hat{x})}{\alpha(\hat{x})} \psi_j \right) \Phi(x)\, dS \right).$$

Proof: It is evident that

$$\int_{\partial G}\left[\frac{\partial u_j}{\partial \nu}\Phi(x)-u_j\frac{\partial \Phi(x)}{\partial \nu}\right]dS=\int_{\Gamma_1}\left[\frac{\partial u_j}{\partial \nu}\Phi(x)-u_j\frac{\partial \Phi(x)}{\partial \nu}\right]dS$$
$$+\int_{\Gamma_2}\left[\frac{\partial u_j}{\partial \nu}\Phi(x)-u_j\frac{\partial \Phi(x)}{\partial \nu}\right]dS. \tag{4}$$

Since $u_j=\psi_j$ on Γ_1 and $\Phi(x)=0$ on Γ_1, we obtain

$$\int_{\Gamma_1}\left[\frac{\partial u_j}{\partial \nu}\Phi(x)-u_j\frac{\partial \Phi(x)}{\partial \nu}\right]dS=-\int_{\Gamma_1}\psi_j\frac{\partial \Phi(x)}{\partial \nu}dS. \tag{5}$$

From the boundary condition (BC) we see that

$$\frac{\partial u_j}{\partial \nu}=\tilde{\psi}_j+\frac{\mu(\hat{x})}{\alpha(\hat{x})}\psi_j-\frac{\mu(\hat{x})}{\alpha(\hat{x})}u_j \quad \text{on} \quad \Gamma_2,$$
$$\frac{\partial \Phi(x)}{\partial \nu}=-\frac{\mu(\hat{x})}{\alpha(\hat{x})}\Phi(x) \quad \text{on} \quad \Gamma_2.$$

Hence, we observe that

$$\int_{\Gamma_2}\left[\frac{\partial u_j}{\partial \nu}\Phi(x)-u_j\frac{\partial \Phi(x)}{\partial \nu}\right]dS=\int_{\Gamma_2}\left(\tilde{\psi}_j+\frac{\mu(\hat{x})}{\alpha(\hat{x})}\psi_j\right)\Phi(x)\,dS. \tag{6}$$

Combining (4)—(6) yields the desired identity (3).

We note that if $\alpha(\hat{x})=0$ on ∂G, then $\Gamma_2=\emptyset$ and

$$K_\Phi\int_{\partial G}\left[\frac{\partial u_j}{\partial \nu}\Phi(x)-u_j\frac{\partial \Phi(x)}{\partial \nu}\right]dS=-K_\Phi\int_{\partial G}\psi_j\frac{\partial \Phi(x)}{\partial \nu}dS.$$

If $\alpha(\hat{x})=1$ $(\hat{x}\in\partial G)$ and $\mu(\hat{x})=0$ $(\hat{x}\in\partial G)$, then $\Gamma_1=\emptyset$ and

$$K_\Phi\int_{\partial G}\left[\frac{\partial u_j}{\partial \nu}\Phi(x)-u_j\frac{\partial \Phi(x)}{\partial \nu}\right]dS=\frac{1}{|G|}\int_{\partial G}\tilde{\psi}_j dS,$$

where $|G|=\int_G dx$ denotes the volume of G.

We use the notation :

$$F_j(t) = K_\Phi \int_G f_j(x,t)\Phi(x)\,dx \quad (j=1,2,\ldots,M),$$
$$[\Theta(t)]_\pm = \max\{\pm\Theta(t),0\},$$
$$\Gamma = \left\{\left((-1)^{\alpha_1},(-1)^{\alpha_2},\ldots,(-1)^{\alpha_M}\right); \alpha_j = 0,1\ (j=1,2,\ldots,M)\right\}.$$

We note that $\#\Gamma = 2^M$ and $-\gamma \in \Gamma$ for $\gamma \in \Gamma$, and hence $\Gamma = \{\pm\gamma ; \gamma \in \tilde{\Gamma}\}$ for some $\tilde{\Gamma} \subset \Gamma$ with $\#\tilde{\Gamma} = 2^{M-1}$. For example, we let $M = 2$. Then we observe that

$$\Gamma = \{(1,1),(1,-1),(-1,1),(-1,-1)\}$$

and

$$\Gamma = \{\pm(1,1),\pm(1,-1)\} = \{\pm\gamma ; \gamma \in \tilde{\Gamma}\},$$

where

$$\tilde{\Gamma} = \{(1,1),(1,-1)\}.$$

Theorem 1: Assume that the hypotheses (H_1)–(H_8) hold. If the following conditions are satisfied :

(H_9) $\rho_i(t) \le t \quad (i=1,2,\ldots,\ell)$;

(H_{10}) $\sum_{i=1}^{\ell}\sum_{j=1}^{M} \tilde{h}_{ij}(t) \le 1$ on $[t_0,\infty)$ for some $t_0 > 0$; and

(H_{11}) there exist functions $\Theta_\gamma(t) \in C^2([t_0,\infty);\mathbb{R})\ (\gamma \in \Gamma)$ such that $\Theta_\gamma(t)$ is oscillatry at $t=\infty$ and $\Theta''_\gamma = \gamma \cdot \left(G_j(t)\right)_{j=1}^{M}$ ($\cdot$ denotes the scalar product);

if for some $j_0 \in \{1,2,\ldots,m\}$ and for any $c > 0$

$$\int_{t_0}^{\infty} \hat{p}_{j_0}(t)\,\hat{\varphi}_{j_0}\left(\left[\left(1-\sum_{i=1}^{\ell}\sum_{j=1}^{M}\tilde{h}_{ij}(\sigma_{j_0}(t))\right)c + \hat{\Theta}_\gamma(\sigma_{j_0}(t))\right]_+\right)dt = \infty, \tag{7}$$

then every solution $(u_1(x,t),\ldots,u_M(x,t))$ of the boundary value problem (2), (BC) is weakly oscillatory in Ω, where

$$\tilde{h}_{ij}(t) = \max_{1\le k\le M} h_{ijk}(t),$$
$$G_j(t) = \frac{1}{M}\left(F_j(t) + \sum_{k=1}^{M} a_{jk}(t)\Psi_k(t) + \sum_{i=1}^{K}\sum_{k=1}^{M} b_{ijk}(t)\Psi_k(\tau_{ik}(t))\right),$$
$$\hat{p}_i(t) = \min_{x\in G}\left\{\min_{1\le k\le M}\left(p_{ikk}(x,t) - \sum_{\substack{j=1\\ j\ne k}}^{M} p_{ijk}(x,t)\right)\right\},$$
$$\hat{\Theta}_\gamma(t) = \Theta_\gamma(t) - \sum_{i=1}^{\ell}\sum_{j=1}^{M}\tilde{h}_{ij}(t)\Theta_\gamma(\rho_i(t)).$$

Proof: Suppose that there exists a solution $(u_1(x,t),\ldots,u_M(x,t))$ of the problem (2), (BC) which is not weakly oscillatory in Ω. Then, each component $u_j(x,t)$ is nonoscillatory in Ω.

We easily see that there is a number $t_1 \ge t_0$ such that $|u_j(x,t)| > 0$ for $x \in G$, $t \ge t_1$ $(j = 1, 2, \ldots, M)$. Letting

$$w_j(x,t) = \delta_j u_j(x,t),$$

where $\delta_j = \operatorname{sgn} u_j(x,t)$, we see that $w_j(x,t) = |u_j(x,t)| > 0$ in $G \times [t_1, \infty)$. There exists a number $t_2 \ge t_1$ such that $w_j(x,t) > 0$, $w_j(x, \rho_i(t)) > 0$, $w_j(x, \tau_{ij}(t)) > 0$, $w_j(x, \sigma_i(t)) > 0$ in $G \times [t_2, \infty)$. Proceeding as in the proof of Theoem 1 of Shoukaku and Yoshida (2005), we observe that the following identity holds :

$$\frac{d^2}{dt^2}\left(V(t) + \frac{1}{M}\sum_{i=1}^{\ell}\sum_{j,k=1}^{M}\delta_j\delta_k h_{ijk}(t) W_k(\rho_i(t)) \right) + \sum_{i=1}^{m}\hat{p}_i(t)\hat{\varphi}_i\big(V(\sigma_i(t))\big) \le \sum_{j=1}^{M}\delta_j G_j(t), \quad t \ge t_2, \tag{8}$$

where

$$V(t) = \frac{\sum_{k=1}^{M} W_k(t)}{M}.$$

There is a $\gamma \in \Gamma$ such that $\sum_{j=1}^{M}\delta_j G_j(t) = \gamma \cdot \big(G_j(t)\big)_{j=1}^{M}$. Setting

$$Y(t) = V(t) + \frac{1}{M}\sum_{i=1}^{\ell}\sum_{j,k=1}^{M}\delta_j\delta_k h_{ijk}(t) W_k(\rho_i(t)) - \Theta_\gamma(t),$$

we see that

$$Y''(t) \le -\sum_{i=1}^{m}\hat{p}_i(t)\hat{\varphi}_i\big(V(\sigma_i(t))\big) \le 0, \quad t \ge t_2. \tag{9}$$

Therefore, $Y(t) > 0$ or $Y(t) \le 0$ on $[t_3, \infty)$ for some $t_3 \ge t_2$. If $Y(t) \le 0$ on $[t_3, \infty)$, then

$$V(t) + \frac{1}{M}\sum_{i=1}^{\ell}\sum_{j,k=1}^{M}\delta_j\delta_k h_{ijk}(t) W_k(\rho_i(t)) \le \Theta_\gamma(t), \quad t \ge t_3,$$

and therefore

$$V(t) + \frac{1}{M}\sum_{i=1}^{\ell}\sum_{k=1}^{M}\left(h_{ikk}(t) - \sum_{\substack{j=1 \\ j \ne k}}^{M} h_{ijk}(t) \right) W_k(\rho_i(t)) \le \Theta_\gamma(t), \quad t \ge t_3. \tag{10}$$

The left hand side of (10) is positive in view of the hypothesis (H_4), whereas the right hand side of (10) is oscillatory at $t = \infty$. This is a contradiction. Hence, we conclude that $Y(t) > 0$ on $[t_3, \infty)$. Since $Y''(t) \le 0$, $Y(t) > 0$ on $[t_3, \infty)$, we obtain $Y'(t) \ge 0$ on $[t_4, \infty)$ for some $t_4 \ge t_3$. Hence, $Y(t) \ge Y(t_4)$
for $t \ge t_4$. In view of the fact that $V(t) \le Y(t) + \Theta_\gamma(t)$ and $Y(t)$ is nondecreasing, we obtain

$$\begin{aligned} V(t) &= Y(t) - \frac{1}{M}\sum_{i=1}^{\ell}\sum_{j,k=1}^{M}\delta_j\delta_k h_{ijk}(t)W_k(\rho_i(t)) + \Theta_\gamma(t) \\ &\geq Y(t) - \sum_{i=1}^{\ell}\sum_{j=1}^{M}\tilde{h}_{ij}(t)V(\rho_i(t)) + \Theta_\gamma(t) \\ &\geq Y(t) - \sum_{i=1}^{\ell}\sum_{j=1}^{M}\tilde{h}_{ij}(t)\big(Y(\rho_i(t)) + \Theta_\gamma(\rho_i(t))\big) + \Theta_\gamma(t) \\ &\geq \left(1 - \sum_{i=1}^{\ell}\sum_{j=1}^{M}\tilde{h}_{ij}(t)\right)Y(t) + \hat{\Theta}_\gamma(t), \quad t \geq t_4. \end{aligned} \tag{11}$$

Since $Y(t) \geq Y(t_4)$ and $V(t) > 0$ for $t \geq t_4$, from (11) we have

$$V(t) \geq \left[\left(1 - \sum_{i=1}^{\ell}\sum_{j=1}^{M}\tilde{h}_{ij}(t)\right)Y(t_4) + \hat{\Theta}_\gamma(t)\right]_+, \quad t \geq t_4$$

and therefore

$$V(\sigma_{j_0}(t)) \geq \left[\left(1 - \sum_{i=1}^{\ell}\sum_{j=1}^{M}\tilde{h}_{ij}(\sigma_{j_0}(t))\right)Y(t_4) + \hat{\Theta}_\gamma(\sigma_{j_0}(t))\right]_+, \ t \geq T \tag{12}$$

for some $T \geq t_4$. Since $\hat{\varphi}_{j_0}(t)$ is nondecreasing, from (9) and (12) we obtain

$$\begin{aligned} &\hat{p}_{j_0}(t)\hat{\varphi}_{j_0}\left(\left[\left(1 - \sum_{i=1}^{\ell}\sum_{j=1}^{M}\tilde{h}_{ij}(\sigma_{j_0}(t))\right)Y(t_4) + \hat{\Theta}_\gamma(\sigma_{j_0}(t))\right]_+\right) \\ &\leq -Y''(t), \quad t \geq T. \end{aligned} \tag{13}$$

Integrating (13) over $[T,t]$ yields

$$\begin{aligned} &\int_T^t \hat{p}_{j_0}(s)\hat{\varphi}_{j_0}\left(\left[\left(1 - \sum_{i=1}^{\ell}\sum_{j=1}^{M}\tilde{h}_{ij}(\sigma_{j_0}(s))\right)Y(t_4) + \hat{\Theta}_\gamma(\sigma_{j_0}(s))\right]_+\right)ds \\ &\leq -Y'(t) + Y'(T) \leq Y'(T), \quad t \geq T. \end{aligned}$$

This contradicts the hypothesis (7) and completes the proof.

Theorem 2: Assume that the hypotheses (H_1)$-$(H_8) hold. Every solution $(u_1(x,t),\ldots,u_M(x,t))$ of the boundary value problem (2), (BC) is weakly oscillatory in Ω if for any $\gamma \in \Gamma$

$$\liminf_{t\to\infty}\int_T^t\left(1-\frac{s}{t}\right)\left(\gamma\cdot\left(G_j(s)\right)_{j=1}^M\right)ds = -\infty$$

for all large T.

Proof: Suppose that there exists a solution $(u_1(x,t),\ldots,u_M(x,t))$ of the problem (2), (BC) which is not weakly oscillatory in Ω. Arguing as in the proof of Theorem 1, we observe that

(8) holds, and hence

$$\frac{d^2}{dt^2}\left(V(t)+\frac{1}{M}\sum_{i=1}^{\ell}\sum_{j,k=1}^{M}\delta_j\delta_k h_{ijk}(t)W_k(\rho_i(t))\right)\le\sum_{j=1}^{M}\delta_j G_j(t),\quad t\ge t_2 \qquad (14)$$

for some $t_2 > 0$. Letting

$$\tilde{V}(t)=V(t)+\frac{1}{M}\sum_{i=1}^{\ell}\sum_{j,k=1}^{M}\delta_j\delta_k h_{ijk}(t)W_k(\rho_i(t)),$$

we note that $\tilde{V}(t)>0 \quad (t>t_3)$ for some $t_3 \ge t_2$ (cf. the proof of Theorem 1). Integrating (14) twice over $[t_3,t]$, we obtain

$$\frac{\tilde{V}(t)}{t}\le\frac{c_1}{t}+c_2\left(1-\frac{t_3}{t}\right)+\int_{t_3}^{t}\left(1-\frac{s}{t}\right)\left(\sum_{j=1}^{M}\delta_j G_j(s)\right)ds$$

for some constants c_1 and c_2. The left hand side of the above inequality is positive, whereas the right hand side is not bounded from below by the hypothesis. This is a contradiction.

Remark 1: We find that $\Gamma=\{\pm\gamma\,;\gamma\in\tilde{\Gamma}\}$ for some $\tilde{\Gamma}\subset\Gamma$ with $\#\tilde{\Gamma}=2^{M-1}$. Hence, Theorem 1 holds true if the hypothesis (H_{11}) and the condition (7) are replaced by

($\tilde{H}_{11}$) there exist functions $\Theta_\gamma(t)\in C^2([t_0,\infty);\mathbb{R})(\gamma\in\tilde{\Gamma})$ such that $\Theta_\gamma(t)$ is oscillatory at $t=\infty$ and $\Theta''_\gamma(t)=\gamma\cdot\left(G_j(t)\right)_{j=1}^{M}$ ($\cdot$ denotes the scalar and product)

$$\int_{t_0}^{\infty}\hat{p}_{j_0}(t)\,\hat{\varphi}_{j_0}\left(\left[\left(1-\sum_{i=1}^{\ell}\sum_{j=1}^{M}\tilde{h}_{ij}(\sigma_{j_0}(t))\right)c\pm\hat{\Theta}_\gamma(\sigma_{j_0}(t))\right]_+\right)dt=\infty,$$

respectively.

Remark 2: Under the same hypotheses of Theorem 2, the conclusion of Theorem 2 holds true if for any $\gamma\in\tilde{\Gamma}$

$$\liminf_{t\to\infty}\int_T^t\left(1-\frac{s}{t}\right)\left(\gamma\cdot\left(G_j(s)\right)_{j=1}^{M}\right)ds=-\infty,$$
$$\limsup_{t\to\infty}\int_T^t\left(1-\frac{s}{t}\right)\left(\gamma\cdot\left(G_j(s)\right)_{j=1}^{M}\right)ds=\infty$$

for all large T.

Example 1: We consider the hyperbolic system

$$\begin{cases} \dfrac{\partial^2}{\partial t^2}\left(u_1(x,t) + \dfrac{1}{2}u_1(x,t-2\pi) + \dfrac{1}{4}u_2(x,t-2\pi) \right) \\ \quad -3\dfrac{\partial^2 u_1}{\partial x^2}(x,t) - \dfrac{1}{8}e^{2\pi}\dfrac{\partial^2 u_2}{\partial x^2}(x,t) - \dfrac{\partial^2 u_1}{\partial x^2}(x,t-\pi) - \dfrac{1}{8}\dfrac{\partial^2 u_2}{\partial x^2}(x,t-2\pi) \\ \quad + \dfrac{1}{2}u_1(x,t-\pi) + \dfrac{1}{2}e^{\pi}u_2(x,t-\pi) = e^{2\pi}(\sin x)e^{-t}, \\ \dfrac{\partial^2}{\partial t^2}\left(u_2(x,t) + \dfrac{1}{4}u_1(x,t-2\pi) + \dfrac{1}{4}u_2(x,t-2\pi) \right) \\ \quad -2\dfrac{\partial^2 u_1}{\partial x^2}(x,t) - e^{2\pi}\dfrac{\partial^2 u_2}{\partial x^2}(x,t) - \dfrac{1}{2}\dfrac{\partial^2 u_1}{\partial x^2}(x,t-\pi) - \dfrac{1}{4}\dfrac{\partial^2 u_2}{\partial x^2}(x,t-2\pi) \\ \quad + \dfrac{1}{4}u_1(x,t-\pi) + \dfrac{3}{2}e^{\pi}u_2(x,t-\pi) = (\sin x)\sin t + \left(1+3e^{2\pi}\right)(\sin x)e^{-t}, \\ (x,t) \in (0,\pi)\times(0,\infty) \end{cases} \tag{15}$$

with the boundary condition

$$u_j(0,t) = u_j(\pi,t) = 0, \quad t > 0 \quad (j = 1, 2). \tag{16}$$

Here $G = (0,\pi)$, $n = 1$, $M = 2$, $\ell = K = m = 1$, $h_{111}(t) = 1/2$, $h_{112}(t) = h_{121}(t) = h_{122}(t) = 1/4$, $\rho_1(t) = t - 2\pi$, $a_{11}(t) = 3$, $a_{12}(t) = (1/8)e^{2\pi}$, $a_{21}(t) = 2$, $a_{22}(t) = e^{2\pi}$, $b_{111}(t) = 1$, $b_{112}(t) = 1/8$, $b_{121}(t) = 1/2$, $b_{122}(t) = 1/4$, $\tau_{11}(t) = t - \pi$, $\tau_{12}(t) = t - 2\pi$, $p_{111}(x,t) = 1/2$, $p_{112}(x,t) = (1/2)e^{\pi}$, $p_{121}(x,t) = 1/4$, $p_{122}(t) = (3/2)e^{\pi}$, $\sigma_1(t) = t - \pi$, $\varphi_{11}(\xi) = \varphi_{12}(\xi) = \xi$, $f_1(x,t) = e^{2\pi}(\sin x)e^{-t}$, $f_2(x,t) = (\sin x)\sin t + \left(1+3e^{2\pi}\right)(\sin x)e^{-t}$, $\psi_j = 0$ $(j = 1, 2)$, $\alpha(\hat{x}) = 0$ and $\Gamma_2 = \emptyset$. We observe that

$$\tilde{h}_{11}(t) = \frac{1}{2}, \quad \tilde{h}_{12}(t) = \frac{1}{4}, \quad \hat{p}_1(t) = \frac{1}{4}.$$

It is easy to see that $\lambda_1 = 1$, $\Phi(x) = \sin x$, $\Psi_j(t) = 0\,(j = 1, 2)$, $G_1(t) = (1/2)F_1(t) = (\pi/8)e^{2\pi}e^{-t}$ and $G_2(t) = (1/2)F_2(t) = (\pi/8)\sin t + (\pi/8)\left(1+3e^{2\pi}\right)e^{-t}$. Since we can choose

$$\Theta_{(1,1)}(t) = \frac{\pi}{8}\left(-\sin t + \left(1+4e^{2\pi}\right)e^{-t}\right),$$
$$\Theta_{(1,-1)}(t) = \frac{\pi}{8}\left(\sin t - \left(1+2e^{2\pi}\right)e^{-t}\right),$$

we obtain

$$\hat{\Theta}_{(1,1)}(t) = \frac{\pi}{8}\left(-\frac{1}{4}\sin t + \left(1+4e^{2\pi}\right)\left(1 - \frac{3}{4}e^{2\pi}\right)e^{-t} \right),$$
$$\hat{\Theta}_{(1,-1)}(t) = \frac{\pi}{8}\left(\frac{1}{4}\sin t - \left(1+2e^{2\pi}\right)\left(1 - \frac{3}{4}e^{2\pi}\right)e^{-t} \right),$$

and therefore

$$\int_{t_0}^{\infty}\frac{1}{4}\left[\left(1-\frac{3}{4}\right)c\pm\frac{\pi}{8}\left(-\frac{1}{4}\sin(t-\pi)+\left(1+4e^{2\pi}\right)\left(1-\frac{3}{4}e^{2\pi}\right)e^{-(t-\pi)}\right)\right]_+ dt$$
$$\geq\frac{\pi}{32}\int_{t_0}^{\infty}\left[\frac{1}{4}\sin t+\left(1+4e^{2\pi}\right)\left(1-\frac{3}{4}e^{2\pi}\right)e^{\pi}e^{-t}\right]_{\pm} dt$$
$$=\infty$$

and

$$\int_{t_0}^{\infty}\frac{1}{4}\left[\left(1-\frac{3}{4}\right)c\pm\frac{\pi}{8}\left(\frac{1}{4}\sin(t-\pi)-\left(1+2e^{2\pi}\right)\left(1-\frac{3}{4}e^{2\pi}\right)e^{-(t-\pi)}\right)\right]_+ dt$$
$$\geq\frac{\pi}{32}\int_{t_0}^{\infty}\left[-\frac{1}{4}\sin t-\left(1+2e^{2\pi}\right)\left(1-\frac{3}{4}e^{2\pi}\right)e^{\pi}e^{-t}\right]_{\pm} dt$$
$$=\infty$$

for any $c>0$. Hence, it follows from Theorem 1 and Remark 1 that every solution $\left(u_1(x,t),u_2(x,t)\right)$ of the problem (15), (16) is weakly oscillatory in $(0,\pi)\times(0,\infty)$. One such solution is

$$U(x,t)=\begin{pmatrix}(\sin x)\sin t\\ (\sin x)e^{-t}\end{pmatrix}.$$

Example 2: We consider the hyperbolic system

$$\begin{cases}\dfrac{\partial^2}{\partial t^2}\left(u_1(x,t)+\dfrac{1}{2}u_1(x,t-\pi)+\dfrac{1}{2}u_2(x,t-\pi)\right)\\ \quad-\dfrac{\partial^2 u_1}{\partial x^2}(x,t)-\dfrac{\partial^2 u_2}{\partial x^2}(x,t)-e^{(3/2)\pi}\dfrac{\partial^2 u_1}{\partial x^2}(x,t+\dfrac{\pi}{2})-\dfrac{\partial^2 u_2}{\partial x^2}(x,t+\pi)\\ \quad+\dfrac{3}{2}u_1(x,t-\pi)+\dfrac{1}{2}u_2(x,t-\pi)=\left(2+3e^{\pi}\right)(\cos x)e^{-t},\\ \dfrac{\partial^2}{\partial t^2}\left(u_2(x,t)+\dfrac{1}{3}u_1(x,t-\pi)+\dfrac{1}{2}u_2(x,t-\pi)\right)\\ \quad-\dfrac{\partial^2 u_1}{\partial x^2}(x,t)-\dfrac{\partial^2 u_2}{\partial x^2}(x,t)-\dfrac{2}{3}e^{(3/2)\pi}\dfrac{\partial^2 u_1}{\partial x^2}(x,t+\dfrac{\pi}{2})-\dfrac{3}{2}\dfrac{\partial^2 u_2}{\partial x^2}(x,t+\pi)\\ \quad+u_1(x,t-\pi)+u_2(x,t-\pi)=-2(\cos x)\sin t+\left(2e^{\pi}+1\right)(\cos x)e^{-t},\\ (x,t)\in(0,\frac{\pi}{2})\times(0,\infty)\end{cases} \quad (17)$$

with the boundary condition

$$\begin{cases}-\dfrac{\partial u_1}{\partial x}(0,t)=0, & \dfrac{\partial u_1}{\partial x}\left(\dfrac{\pi}{2},t\right)=-e^{-t},\\ -\dfrac{\partial u_2}{\partial x}(0,t)=0, & \dfrac{\partial u_2}{\partial x}\left(\dfrac{\pi}{2},t\right)=-\sin t, \quad t>0.\end{cases} \quad (18)$$

Here $G=(0,(\pi/2))$, $n=1$, $M=2$, $\ell=K=m=1$, $h_{111}(t)=h_{112}(t)=1/2$, $h_{121}(t)=1/3$, $h_{122}(t)=1/2$, $\rho_1(t)=t-\pi$, $a_{11}(t)=a_{12}(t)=a_{21}(t)=a_{22}(t)=1$, $b_{111}(t)=e^{(3/2)\pi}$,

$b_{112}(t)=1, b_{121}(t)=(2/3)e^{(3/2)\pi}, b_{122}(t)=3/2, \tau_{11}(t)=t+(\pi/2), \tau_{12}(t)=t+\pi$, $p_{111}(x,t)=3/2$, $p_{112}(x,t)=1/2$, $p_{121}(x,t)=p_{122}(x,t)=1$, $\sigma_1(t)=t-\pi$, $\varphi_{11}(\xi)=\varphi_{12}(\xi)=\xi$, $f_1(x,t)=\left(2+3e^{\pi}\right)(\cos x)e^{-t}$, $f_2(x,t)=-2(\cos x)\sin t+\left(2e^{\pi}+1\right)(\cos x)e^{-t}$, $\alpha(\hat{x})=1$, $\mu(\hat{x})=0$ and $\Gamma_1=\emptyset$. We find that

$$\tilde{h}_{11}(t)=\tilde{h}_{12}(t)=\frac{1}{2}, \quad \hat{p}_1(t)=\frac{1}{2}.$$

It is easy to check that $\lambda_1=0$, $\Phi(x)=1$ and that

$$G_1(t)=\frac{1}{2\pi}\left(2+4e^{\pi}\right)e^{-t},$$
$$G_2(t)=\frac{1}{2\pi}\left(-3\sin t+\frac{8}{3}e^{\pi}e^{-t}\right).$$

Choosing

$$\Theta_{(1,1)}(t)=\frac{1}{2\pi}\left(3\sin t+\left(\frac{20}{3}e^{\pi}+2\right)e^{-t}\right),$$
$$\Theta_{(1,-1)}(t)=\frac{1}{2\pi}\left(-3\sin t+\left(\frac{4}{3}e^{\pi}+2\right)e^{-t}\right),$$

we see that

$$\hat{\Theta}_{(1,1)}(t)=\frac{1}{2\pi}\left(6\sin t+\left(\frac{20}{3}e^{\pi}+2\right)\left(1-e^{\pi}\right)e^{-t}\right),$$
$$\hat{\Theta}_{(1,-1)}(t)=\frac{1}{2\pi}\left(-6\sin t+\left(\frac{4}{3}e^{\pi}+2\right)\left(1-e^{\pi}\right)e^{-t}\right).$$

Since

$$\begin{aligned}&\int_{t_0}^{\infty}\frac{1}{2}\left[\pm\hat{\Theta}_{(1,1)}(t-\pi)\right]_+ dt\\ &=\frac{1}{4\pi}\int_{t_0}^{\infty}\left[-6\sin t+\left(\frac{20}{3}e^{\pi}+2\right)\left(e^{\pi}-e^{2\pi}\right)e^{-t}\right]_{\pm} dt\\ &=\infty\end{aligned}$$

and

$$\begin{aligned}&\int_{t_0}^{\infty}\frac{1}{2}\left[\pm\hat{\Theta}_{(1,-1)}(t-\pi)\right]_+ dt\\ &=\frac{1}{4\pi}\int_{t_0}^{\infty}\left[6\sin t+\left(\frac{4}{3}e^{\pi}+2\right)\left(e^{\pi}-e^{2\pi}\right)e^{-t}\right]_{\pm} dt\\ &=\infty,\end{aligned}$$

Theorem 1 and Remark 1 imply that every solution $\left(u_1(x,t),u_2(x,t)\right)$ of (17), (18) is weakly oscillatory in $\left(0,\frac{\pi}{2}\right)\times(0,\infty)$. In fact,

$$U(x,t)=\begin{pmatrix}(\cos x)e^{-t}\\(\cos x)\sin t\end{pmatrix}$$

is such a solution.

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