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Design of model predictive control for constrained Markov jump linear systems with multiplicative noises and online portfolio selection

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Summary

In this paper, we consider model predictive control for a class of constrained discrete-time Markov jump linear systems with multiplicative noises. A generalized performance criterion is composed of a weighted sum of a linear combination of the (a) expected value of quadratic forms of state and control vectors, (b) quadratic forms of the expected value of the state vector, and (c) the linear component of the expected value of the state vector. The goal of the present paper is to design optimal control strategies subject to hard constraints on the input manipulated variables and to provide a numerically tractable algorithm for practical applications. The results are applied to a problem of online investment portfolio selection. Our approach is tested on a set of a real data from the New York Stock Exchange.

KEYWORDS

market frictions, Markov jump stochastic systems, model predictive control, portfolio selection, transaction costs

1 | INTRODUCTION

1.1 | Motivation and related studies

In recent years, considerable interest has been focused on Markov-modulated systems, which consist of a finite set of systems, also called multiple modes, and the switching among them is governed by a finite state Markov chain. The same systems have been successfully used in many real-world applications. Their applications can be found in robotics, macroeconomics, communication networks, fault-tolerant control, flight systems, immunology, automotive systems, chemical processes, solar thermal receivers, supply chain management, large flexible structures for space stations, etc.[1]

Financial engineering is also an important field of application. Models with Markovian jumps and multiplicative noises are very suitable for description of an investment portfolio dynamics when the price dynamics of the risky assets change following different states of the economy.[2-5]

There is a considerable amount of results obtained for unconstrained control strategies for Markov jump linear systems (MJLSs). In particular, Dragan and Morozan,[6] Costa and de Paulo,[7] Hou et al,[8] Costa and de Oliveira[9] investigate unconstrained discrete-time linear systems subject to Markov jump parameters and multiplicative noises. Control of constrained MJLS by using linear matrices inequalities (LMI) techniques has been considered in the works of Vargas et al[10] and Costa et al.[11] In the work of Vargas et al[10] the constraints are imposed on the second moment of both the system state and control vector. Costa et al[11] consider the quadratic optimal control problem of a discrete-time MJLS, subject to symmetric constrains on the state and control variables.

In recent years, considerable interest has been focused on model predictive control (MPC), also known as receding horizon control. MPC proved to be an appropriate and effective technique to solve the dynamic control problems subject to constraints. Extensive reviews of the literature on MPC and its applications can be found, for instance, the works of in Mayne,[12] Goodwin et al,[13] and Farina et al.[14]

In the work of Hernandez-Medjias et al,[15,16] a solution of the MPC for Markov-jump linear systems subject to state and control constraints is proposed. Stochastic MPC (MPC based on scenario generation) for constrained nonlinear Markovian switching systems consisting of a family of subsystems where subsystems are allowed to be nonlinear time-invariant deterministic was studied in the work of Patrinos et al.[17] Notably, all the aforementioned works deal with Markov jump systems where individual mode dynamics are deterministic and the mode matrices are assumed to be time invariant.

Nonetheless, there are only few results on MPC for constrained stochastic Markovian jump systems where individual mode dynamics are stochastic. Dombrovskii and Obedko[18] investigate the MPC problem of discrete-time MJLS with multiplicative noise in control in which only the input matrix is determined by a Markov process subject to hard constraints on the control variables. Similar approaches exist for random parameter systems.[19] In the work of Tonne et al,[20] the MPC approach for MJLS with additive disturbance is proposed where the control sequence for the prediction horizon is determined by minimizing the cost of the predicted expectancies of the system states.

1.2 | Main contributions

The main contribution of this paper is twofold. First, we propose a solution to the MPC problem for a class of constrained discrete-time Markov-modulated systems defined in (1), which consist of a finite set of linear subsystems with stochastic multiplicative noise in both the state and control. We consider that the state transition, input matrices, and stochastic terms are determined by a Markov process and the jump parameter defining the active mode is governed by a finite state homogeneous Markov chain. It is allowed also that hard constraints are imposed on the input manipulated variables.

We consider a generalized performance criterion defined in (3), which is composed of a weighted sum of a linear combination of the (a) expected value of quadratic forms of state and control vectors, (b) quadratic forms of the expected value of the state vector, and (c) a linear part in the expected value of the state vector. The motivation for adopting this type of criterion is that, in several situations, we provide solutions for two special cases. The first one (Problem 1 in Section 2) is MPC for the quadratic criterion defined in (4), and the second one (Problem 2 in Section 2) is MPC for the mean-variance criterion defined in (5). Note that this cost function is not traditionally used in MPC theory. This type of generalized criteria was considered in the work of Dombrovskii and Obedko[19] for the MPC of stochastic systems with serially correlated parameters and is very relevant to problems that exist, for instance, in finance. This approach to cost function formulation is based on an idea proposed in the work of Costa and Araujo[3] where a generalized multiperiod mean-variance portfolio selection problem is considered without constraints.

The goal of the present paper is to design optimal control strategies subject to hard constraints on the input manipulated variables and to provide a numerically tractable algorithm for practical applications. We derive an exact expression for the predicted performance criterion as an explicit function of predicted input variables that can be optimized online by minimizing over the vector of predicted input variables. This method offers a substantial computational advantage in that it requires only finite-time solutions of quadratic programs (QPs). To the best of our knowledge, there is no other work handling this kind of problem in the literature. Our method is an extension and generalization of the MPC framework in the work of Dombrovskii and Obedko[18,19] to a class of constrained discrete-time stochastic MJLS with multiplicative noise in both the state and control and Markov jumps in both the state transition and input matrices.

The second contribution is the application of the obtained results to a problem of investment portfolio selection. Financial market time series comprise nonstationary dynamic stochastic systems with high volatility and irregular movements. Therefore, investment portfolio is a highly nonstationary dynamic stochastic system that calls for an appropriate sophisticated model and algorithm for designing trading strategies. The portfolio management problem includes a set of major problems associated with the control of complex dynamic systems with stochastic parameters under constraints. Thus, the optimal investment problems are well suited to be tackled by stochastic control methods (see, eg, the work of Barmish and Primbs[21] and the references therein).

In this study, we consider the dynamic portfolio selection problem subject to hard constraints on trading amounts, when the dynamics of the risky asset returns are governed by a discrete-time approximation of the Markov-modulated geometric Brownian motion. Additionally, the proposed portfolio model incorporates significant features that are important for practitioners: (i) explicit transaction costs, such as broker commission costs; (ii) implicit trading costs, such as the price impact of the trade (the market friction costs). As pointed out by many researchers, these features are a major concern in active online portfolio management, so it is a practically important and challenging problem to incorporate them into the model (see, for instance, the works of Gârleanu and Pedersen[22] and Kolm et al[23]). Another realistic feature that we incorporate into our model is the different rates of borrowing and lending. We introduce two dynamic portfolio optimization problems: (i) a tracking problem of a reference portfolio with desired return; (ii) a multiperiod mean-variance portfolio optimization problem. Note that the portfolio dynamics represents a tangible example of constrained discrete-time Markovian jump nonstationary stochastic system.

There are many examples of the MPC in financial applications, such as portfolio optimization and dynamic hedging. In the works of Dombrovskii et al,[24] Dombrovskii and Obedko,[18,19,25] Herzog et al,[26] and Primbs and Yamada,[27] investment portfolio optimization using MPC is considered. Dynamic hedging of basket options under soft probabilistic constraints using MPC is presented in the work of Primbs.[28] In the work of Graf Plessen et al,[29] a stochastic MPC approach to hedging derivative contracts based on scenario simulation is proposed. However, to the authors' best knowledge, this is the first work in which the dynamic portfolio selection problem under constraints in the financial market with regime switching, explicit transaction costs, and market frictions is considered.

We implement the suggested model and present numerical results based on sets of real data from the New York Stock Exchange. We propose an online adaptive data-driven implementation of feedback trading strategies. We show that the proposed MPC policies achieve good performance, even when the estimation of the input parameters is not particularly good.

This work is organized as follows. Section 2 presents the optimization problem formulation. The main results of this article are presented in Section 3 where we design the optimal control strategy for the problem under consideration. Section 4 presents the portfolio model and the investment problem formulations. In Section 5, the numerical modeling results are presented. This paper is concluded in Section 6 with some final remarks.

2 | PROBLEM FORMULATION

Assume that the plant to be controlled can be described by the following discrete-time model:

$$\begin{aligned} x(k+1) = {} & \left[A_0\left[\alpha(k+1), k+1\right] + \sum_{j=1}^{n} A_j\left[\alpha(k+1), k+1\right] w_j(k+1)\right] x(k) \\ & + \left[B_0\left[\alpha(k+1), k+1\right] + \sum_{j=1}^{n} B_j\left[\alpha(k+1), k+1\right] w_j(k+1)\right] u(k), \end{aligned} \tag{1}$$

where $x(k) \in \mathbb{R}^{n_x}$ is the vector of state, $u(k) \in \mathbb{R}^{n_u}$ is the vector of control inputs, $A_j[\alpha(k)k] \in \mathbb{R}^{n_x \times n_x}$, $B_j[\alpha(k)k] \in \mathbb{R}^{n_x \times n_u}$, $(j = \overline{0, n})$ are system and input matrices, respectively; $w_j(k)$ are independent noises with zero mean and unit variance, $E\{w_j(k) w_i(k)\} = 0, i \neq j$; $\{\alpha(k); k = 0, 1, 2, \dots\}$ is a finite-state discrete-time Markov chain taking values in $\{1, 2, \dots, \nu\}$ with transition probability matrix

$$P = \left[P_{ij}\right], (i, j \in \{1, 2, \dots, \nu\}),$$

$$P_{ij} = P\{\alpha(k+1) = j \,|\, \alpha(k) = i\}, \sum_{j=1}^{\nu} P_{ij} = 1,$$

and initial distribution

$$p_i = P\{\alpha(0) = i\}, \left(i = \overline{1, \nu}\right); \sum_{i=1}^{\nu} p_i = 1.$$

We assume that $\alpha(k)$ and $w_j(k)$ are mutually independent and at the instant of decision making the Markov state $\alpha(k)$ observable.

The system matrices $A_j[\alpha(k), k]$ and the input matrices $B_j[\alpha(k), k]$ take values from finite sets in accordance with the state of Markov chain $\alpha(k)$

$$A_j\left[\alpha(k), k\right] \in \left\{A_j^{(i)}(k) \in \mathbb{R}^{n_x \times n_x} : i = \overline{1, \nu}\right\}, j = \overline{0, n},$$

$$B_j\left[\alpha(k), k\right] \in \left\{B_j^{(i)}(k) \in \mathbb{R}^{n_x \times n_u} : i = \overline{1, \nu}\right\}, j = \overline{0, n},$$

respectively.

We impose the following constraints on the decision variables (elementwise inequality):

$$u_{\min}(k) \le S(k)\,u(k) \le u_{\max}(k), \tag{2}$$

where $S(k) \in \mathbb{R}^{p\times n_u}$; $u_{\min}(k), u_{\max}(k) \in \mathbb{R}^p$.

We use the MPC methodology in order to define the optimal control strategy. The main concept of MPC is to solve an open-loop constrained optimization problem at each time instant and to implement only the initial optimizing control action of the solution.

We define the following cost function with receding horizon, which is to be minimized at every time k:

$$\begin{aligned} J(k+m|k) = {} & \sum_{i=1}^{m} E\left\{x^T(k+i)R_1(k+i)x(k+i)\,|x(k),\alpha(k)\right\} \\ & - \sum_{i=1}^{m} E\left\{x^T(k+i)\,|x(k),\alpha(k)\right\} R_2(k+i) E\left\{x(k+i)\,|x(k),\alpha(k)\right\} \\ & - \sum_{i=1}^{m} R_3(k+i)E\left\{x(k+i)\,|x(k),\alpha(k)\right\} + \sum_{i=0}^{m-1} E\left\{u^T(k+i|k)R(k+i)u(k+i|k)|x(k),\alpha(k)\right\}, \end{aligned} \tag{3}$$

on trajectories of system (1) over the sequence of predictive control inputs $u(k \mid k), \ldots, u(k+m-1|k)$ dependent on information up to time k, under constraints (2), where $R_1(k+i) \ge 0$, $R_2(k+i) \ge 0$, and $R(k+i) > 0$ are given symmetric weight matrices of corresponding dimensions, $R_3(k+i)$ is a weight vector of corresponding dimension; m is the prediction horizon and $E\{a|b\}$ is the conditional expectation of a with respect to b. We assume $R_1(k+i) \ge R_2(k+i)$. Under this assumption, the following inequality holds:

$$E\left\{x^T(k+i)R_1(k+i)x(k+i)\,|x(k),\alpha(k)\right\} \ge E\left\{x^T(k+i)\,|x(k),\alpha(k)\right\} R_2(k+i)E\left\{x(k+i)\,|x(k),\alpha(k)\right\}.$$

The proof of the above inequality is quite straightforward. Thus, we have the convex optimization problem because the fourth term in (3) is positive. Only the first control vector $u(k|k)$ is actually used for control. Thereby, we obtain control $u(k)$ as a function of $\alpha(k)$ and $x(k)$, ie, the feedback control. This optimization process is solved again at the next time instant $k+1$ to obtain control $u(k+1)$.

The motivation for adopting a generalized performance criterion defined in (3) is that, after setting the coefficients $R_1(k+i)$, $R_2(k+i)$, and $R_3(k+i)$ to some appropriate values, we provide solutions for two different problems.

Problem 1. Taking $R_2(k+i) = 0$, we have the MPC problem with quadratic criterion

$$\begin{aligned} J(k+m|k) = {} & \sum_{i=1}^{m} E\left\{x^T(k+i)R_1(k+i)x(k+i)\,|x(k),\alpha(k)\right\} \\ & - \sum_{i=1}^{m} R_3(k+i)E\left\{x(k+i)\,|x(k),\alpha(k)\right\} + \sum_{i=0}^{m-1} E\left\{u^T(k+i|k)R(k+i)u(k+i|k)|x(k),\alpha(k)\right\}. \end{aligned} \tag{4}$$

The above is a control problem to minimize a quadratic cost function, composed by a linear combination of quadratic and linear parts. The trade-off between these two conflicting contributions is balanced by the weights. The choice of particular values of the weight coefficients R_2, R, and R_3 is defined in accordance with the specific real problem that should be solved. In portfolio optimization, these parameters have a natural interpretation. The performance criterion for this problem is composed by a linear combination of a quadratic error between the portfolio value and the value of a reference portfolio, and a linear part representing an expected error between the portfolio value and the benchmark, which is desired to overcome. The trade-off between these two conflicting contributions is balanced by the weights. In Section 4, we present more detailed clarification on the choice of this type of criterion for portfolio optimization.

This type of criterion was considered in the work of Costa and de Paulo[7] for optimal control of unconstrained MJLSs with multiplicative noise.

Problem 2. Let system (1) have a scalar output $y(k) = L(k)x(k)$, where $L(k)$ is a vector of appropriate dimension. Taking

$$R_1(k+i) = R_2(k+i) = \mu(k+i)L^T(k+i)L(k+i),$$
$$R_3(k+i) = \rho(k+i)L(k+i),\ i = \overline{1,m},$$

where $\mu(k+i) \geq 0$, $\rho(k+i) \geq 0$ are scalar values, we have the MPC problem with mean-variance criterion

$$\begin{aligned} J(k+m|k) = & \sum_{i=1}^{m} \mu(k+i) E\left\{x^T(k+i)L^T(k+i)L(k+i)x(k+i)|x(k),\alpha(k)\right\} \\ & - \sum_{i=1}^{m} \mu(k+i) E\left\{x^T(k+i)|x(k),\alpha(k)\right\} L^T(k+i)L(k+i)E\{x(k+i)|x(k),\alpha(k)\} \\ & - \sum_{i=1}^{m} \rho(k+i)L(k+i)E\{x(k+i)|x(k),\alpha(k)\} \\ & + \sum_{i=0}^{m-1} E\left\{u^T(k+i|k)R(k+i)u(k+i|k)|x(k),\alpha(k)\right\}, \end{aligned}$$

which can be represented in the form

$$\begin{aligned} J(k+m|k) = & \sum_{i=1}^{m} \mu(k+i) E\left\{(y(k+i) - E\{y(k+i)|x(k),\alpha(k)\})^2|x(k),\alpha(k)\right\} \\ & - \sum_{i=1}^{m} \rho(k+i) E\{y(k+i)|x(k),\alpha(k)\} \\ & + \sum_{i=0}^{m-1} E\left\{u^T(k+i|k)R(k+i)u(k+i|k)|x(k),\alpha(k)\right\}. \end{aligned} \tag{5}$$

The input parameters $\mu(k+i)$ and $\rho(k+i)$ can be seen as risk aversion coefficients, giving a trade-off between the expected system output and the associated risk (variance) level. The mean-variance–type objective is commonly used in portfolio optimization. In Section 4, we show how to use this type of criteria for portfolio optimization problem.

Optimal mean-variance control for discrete-time linear systems with Markovian jumps and multiplicative noises without constraints on input variables was considered in the work of Costa and de Oliveira.[9]

3 | MODEL PREDICTIVE CONTROL STRATEGY DESIGN

The purpose of this section is to derive an exact expression for the predicted performance criterion as an explicit function of predicted input variables that can be optimized online by minimizing over the vector of predicted input variables. Finally, using this expression, we argue that the dynamic optimization problem is directly reduced to a sequence of static convex optimization problems, each of which can be solved with standard optimization techniques.

The discrete-time Markov chain, taking values in $\{1, 2, \ldots, \nu\}$, with transition probability matrix P admits the following representation in the state space[30]:

$$\theta(k+1) = P\theta(k) + \upsilon(k+1), \tag{6}$$

where $\theta(k) = [\delta(\alpha(k),1), \ldots, \delta(\alpha(k),\nu)]^T$, $\delta(\alpha(k), j)$ is a Kronecker function; and $\{\upsilon(k)\}$ is a sequence of martingale increments.

Taking (6) into consideration, Equation (1) can be represented as follows:

$$x(k+1) = \left[A_0\left[\theta(k+1), k+1\right] + \sum_{j=1}^{n} A_j\left[\theta(k+1), k+1\right] w_j(k+1)\right] x(k) + \left[B_0\left[\theta(k+1), k+1\right] + \sum_{j=1}^{n} B_j\left[\theta(k+1), k+1\right] w_j(k+1)\right] u(k), \quad (7)$$

$$A_j\left[\theta(k+1), k+1\right] = \sum_{i=1}^{\nu} \theta_i(k+1) A_j^{(i)}(k+1), \quad (8)$$

$$B_j\left[\theta(k+1), k+1\right] = \sum_{i=1}^{\nu} \theta_i(k+1) B_j^{(i)}(k+1), \left(j = \overline{0, n}\right), \quad (9)$$

where $\theta_i(k+1)$,$(i = \overline{1, \nu})$ are the components of the vector $\theta(k+1)$.

The objective function (3) can be written as

$$J(k+m|k) = \sum_{i=1}^{m} E\left\{x^T(k+i) R_1(k+i) x(k+i) \,|x(k), \theta(k)\right\} - \sum_{i=1}^{m} E\left\{x^T(k+i) \,|x(k), \theta(k)\right\} R_2(k+i) E\left\{x(k+i) \,|x(k) \,|\theta(k)\right\} - \sum_{i=1}^{m} R_3(k+i) E\left\{x(k+i) \,|x(k), \theta(k)\right\} + \sum_{i=0}^{m-1} E\left\{u^T(k+i|k) R(k+i) u(k+i|k) |x(k), \theta(k)\right\}. \quad (10)$$

Next, we provide an exact expression for the predicted performance criterion (10) as an explicit function of predicted input variables. We consider the following expressions:

$$J_1(k+m|k) = E\left\{\sum_{i=1}^{m}\left[x^T(k+i) R_1(k+i) x(k+i) - R_3(k+i) x(k+i) + u^T(k+i-1|k) R(k+i-1) u(k+i-1|k)\right] |x(k), \theta(k)\right\}, \quad (11)$$

and

$$J_2(k+m|k) = \sum_{i=1}^{m} E\left\{x^T(k+i) \,|x(k), \theta(k)\right\} R_2(k+i) E\left\{x(k+i) \,|x(k), \theta(k)\right\}. \quad (12)$$

It is obvious that $J(k{+}m|\,k) = J_1(k{+}m|\,k) - J_2(k{+}m|\,k)$.

Before presenting the main result, we first propose the following two lemmas which will be used in the proof of our main result.

Lemma 1. *The expression (11) for $J_1(k{+}m|\,k)$ can be represented in the form*

$$J_1(k+m|k) = C^{(1)}\left[x(k)\right] + \left[2x^T(k) G^{(1)}(k) - F(k)\right] U(k) + U^T(k) H^{(1)}(k) U(k), \quad (13)$$

where

$$C^{(1)}\left[x(k)\right] = x^T(k) \sum_{i_1=1}^{\nu} \sum_{j=0}^{n} \left(A_j^{(i_1)}(k+1)\right)^T Q^{(i_1)}(k) A_j^{(i_1)}(k+1) x(k) - \sum_{i_1=1}^{\nu} Q_2^{(i_1)}(k) A_0^{(i_1)}(k+1) x(k),$$

$U(k) = [u^T(k|k), \ldots, u^T(k+m\text{-}1|k)]^T$ is the set of predictive controls.

Blocks of the matrices $H^{(1)}(k), G^{(1)}(k), F(k)$ satisfy the following equations:

$$H_{t,t}^{(1)}(k) = \sum_{i_t=1}^{\nu}\sum_{j=0}^{n}\left(B_j^{(i_t)}(k+t)\right)^T Q^{(i_t)}(k) B_j^{(i_t)}(k+t) + R(k+t-1),$$

$$H_{t,s}^{(1)}(k) = \sum_{i_t=1}^{\nu}\left(B_0^{(i_t)}(k+t)\right)^T \sum_{i_{t+1}=1}^{\nu}\cdots\sum_{i_{s-1}=1}^{\nu}\left(A_0^{(i_{t+1})}(k+t+1)\right)^T\cdots\left(A_0^{(i_{s-1})}(k+s-1)\right)^T$$

$$\times\sum_{j=0}^{n}\sum_{i_s=1}^{\nu}\left(A_j^{(i_s)}(k+s)\right)^T Q^{(i_t,\ldots,i_s)}(k) B_j^{(i_s)}(k+s), s>t,$$

$$H_{t,s}^{(1)}(k) = \left(H_{s,t}^{(1)}(k)\right)^T, s<t,$$

$$G_t^{(1)}(k) = \sum_{i_1=1}^{\nu}\cdots\sum_{i_{t-1}=1}^{\nu}\left(A_0^{(i_1)}(k+1)\right)^T\cdots\left(A_0^{(i_{t-1})}(k+t-1)\right)^T$$

$$\times\sum_{j=0}^{n}\sum_{i_t=1}^{\nu}\left(A_j^{(i_t)}(k+t)\right)^T Q^{(i_1,i_2,\ldots,i_t)}(k) B_j^{(i_t)}(k+t),$$

$$F_t(k) = \sum_{i_t=1}^{\nu} Q_2^{(i_t)}(k) B_0^{(i_t)}(k+t).$$

Matrices $Q^{(i_t,\ldots,i_s)}(k), Q_2^{(i_t,\ldots,i_s)}(k), (s,t=\overline{1,m})$ are defined by the following recursive equations:

$$Q^{(i_t,\ldots,i_s)}(k) = \Theta^{(i_t,\ldots,i_s)}(k) R_1(k+s) + \sum_{j=0}^{n}\sum_{i_{s+1}=1}^{\nu}\left(A_j^{(i_{s+1})}(k+s+1)\right)^T Q^{(i_t,\ldots,i_{s+1})}(k) A_j^{(i_{s+1})}(k+s+1), t=\overline{1,m-2}, t<s<m, \tag{14}$$

$$Q^{(i_t)}(k) = E_{i_t}P^t\theta(k) R_1(k+t) + \sum_{j=0}^{n}\sum_{i_{t+1}=1}^{\nu}\left(A_j^{(i_{t+1})}(k+t+1)\right)^T Q^{(i_t,i_{t+1})}(k) A_j^{(i_{t+1})}(k+t+1), t=\overline{1,m-1}, \tag{15}$$

$$Q_2^{(i_t,\ldots,i_s)}(k) = R_3(k+s)\Theta^{(i_t,\ldots,i_s)}(k) + \sum_{i_{s+1}=1}^{\nu} Q_2^{(i_t,\ldots,i_{s+1})}(k) A_0^{(i_{s+1})}(k+s+1), \quad t=\overline{1,m-2}, t<s<m, \tag{16}$$

$$Q_2^{(i_t)}(k) = R_3(k+t) E_{i_t}P^t\theta(k) + \sum_{i_{t+1}=1}^{\nu} Q_2^{(i_t,i_{t+1})}(k) A_0^{(i_{t+1})}(k+t+1), t=\overline{1,m-1}, \tag{17}$$

starting with

$$Q^{(i_m)}(k) = E_{i_m}P^m\theta(k) R_1(k+m),$$

$$Q^{(i_t,\ldots,i_m)}(k) = \Theta^{(i_t,\ldots,i_m)}(k) R_1(k+m), t=\overline{1,m-1},$$

$$Q_2^{(i_m)}(k) = E_{i_m}P^m\theta(k) R_3(k+m),$$

$$Q_2^{(i_t,\ldots,i_m)}(k) = \Theta^{(i_t,\ldots,i_m)}(k) R_3(k+m), t=\overline{1,m-1},$$

where

$$\Theta^{(i_t,\ldots,i_s)}(k) = P_{i_s,i_{s-1}}P_{i_{s-1},i_{s+1}}\cdots P_{i_{t+1},i_t}\theta_{i_t}(k+t|k), t=\overline{1,m-1}, s>t, \tag{18}$$

$\theta_{i_t}(k+t|k)$ is the component of the forecast vector

$$\theta(k+t|k) = E\{\theta(k+t)|\theta(k)\} = P^t\theta(k),$$
$$E_{i_t} = [0, \ldots, 0, 1, 0, \ldots, 0]_{1\times\nu}, i_t = \overline{1,\nu}, t = \overline{1,m}.$$

A proof of this Lemma is reported in the Appendix.

Lemma 2. *The expression (12) for $J_2(k+m|k)$ can be represented in the form*

$$J_2(k+m|k) = C^{(2)}\left[x(k)\right] + 2x^T(k)G^{(2)}(k)U(k) + U^T(k)H^{(2)}(k)U(k), \tag{19}$$

where

$$C^{(2)}\left[x(k)\right] = x^T(k)\Psi^T(k)\Delta(k+1)\Psi(k)x(k),$$
$$G^{(2)}(k) = \Psi^T(k)\Delta(k+1)\Phi(k),$$
$$H^{(2)}(k) = \Phi^T(k)\Delta(k+1)\Phi(k),$$
$$\Delta(k+1) = \text{diag}\{R_2(k+1), \ldots, R_2(k+m)\},$$

and blocks of the matrices $\Phi(k)$,$\Psi(k)$ are of the form

$$\Phi_{t,t}(k) = \sum_{i_t=1}^{\nu} B_0^{(i_t)}(k+t)\Theta^{(i_t)}(k), t = \overline{1,m}, \tag{20}$$

$$\Phi_{s,t}(k) = \sum_{i_t=1}^{\nu}\sum_{i_{t+1}=1}^{\nu}\cdots\sum_{i_s=1}^{\nu} A_0^{(i_s)}(k+s)\cdots A_0^{(i_{t+1})}(k+t+1)B_0^{(i_t)}(k+t)\Theta^{(i_t,\ldots,i_s)}(k), s > t, \tag{21}$$

$$\Phi_{s,t}(k) = 0, s < t, \tag{22}$$

$$\Psi_t(k) = \sum_{i_1=1}^{\nu}\sum_{i_2=1}^{\nu}\cdots\sum_{i_t=1}^{\nu} A_0^{(i_t)}(k+t)\cdots A_0^{(i_1)}(k+1)\Theta^{(i_1,\ldots,i_t)}(k), t = \overline{1,m}, \tag{23}$$

where $\Theta^{(i_t,\ldots,i_s)}(k)$ is defined by (18), and $\Theta^{(i_t)}(k) = E_{i_t}P^t\theta(k)$.

A proof of this lemma is reported in the Appendix.

Based on Lemmas 1 and 2, we propose the following theorem to solve the MPC optimization problem under criterion (10) as a convex QP.

Theorem 1. *Let the system dynamics be given by (7)-(9) under constraints (2). Then, the MPC policy with receding horizon m, based on the criterion (10), for each instant k is defined by*

$$u(k) = \left[I_{n_u} \; 0_{n_u} \; \cdots \; 0_{n_u}\right]U(k),$$

where I_{n_u} is a n_u-dimensional identity matrix, 0_{n_u} is a n_u-dimensional zero matrix, and U(k) is obtained by solving the quadratic programming problem with criterion

$$Y(k+m|k) = \left[2x^T(k)G(k) - F(k)\right]U(k) + U^T(k)H(k)U(k), \tag{24}$$

under constraints

$$U_{\min}(k) \le \overline{S}(k)U(k) \le U_{\max}(k), \tag{25}$$

where

$$G(k) = G^{(1)}(k) - G^{(2)}(k),$$
$$H(k) = H^{(1)}(k) - H^{(2)}(k),$$
$$\overline{S}(k) = \operatorname{diag}\{S(k), \ldots, S(k+m-1)\},$$
$$U_{\min}(k) = \left[u_{\min}^T(k), \ldots, u_{\min}^T(k+m-1)\right]^T, U_{\max}(k) = \left[u_{\max}^T(k), \ldots, u_{\max}^T(k+m-1)\right]^T.$$

Proof. Taking into account the formulas (13) and (19) in Lemmas 1 and 2, criterion (10) can be expressed as:

$$J(k+m|k) = C^{(1)}\left[x(k)\right] - C^{(2)}\left[x(k)\right] + 2x^T(k)\left[G^{(1)}(k) - G^{(2)}(k)\right]U(k) - F(k)U(k) + U^T(k)\left[H^{(1)}(k) - H^{(2)}(k)\right]U(k). \tag{26}$$

It is obvious that the problem of minimizing the criterion (26) is equivalent to the problem of minimizing the criterion (24) (note that, in (24), we eliminated terms that are independent on the sequence of predicted control inputs). Thus, we have that the problem of minimizing the criterion (10) over the set of predictive controls $U(k)$ is equivalent to the quadratic programming problem with criterion (24). This completes the proof. □

Remark 1 (Feasibility conditions). The conditions $R_1(k+i) \geq R_2(k+i)$ and $R(k+i) > 0$ guarantee that the criterion (24) is convex. It follows from the fact that the criterion (24) was derived by the convexity preserving transformation of the convex criterion (10). Thus, the solution of the quadratic programming task with the criterion (24) exists and is unique if the constraints (25) are consistent. In that case, the designed policy is feasible, ie, optimal predictive control sequence exists such that the constraints are satisfied. Thus, the dynamic optimization problem is directly reduced to a sequence of static convex optimization problems, each of which can be solved with standard static optimization techniques: QPs.

We get the solution of Problem 1 taking $R_2(k+i) = 0$. Taking $R_1(k+i) = R_2(k+i) = \mu(k+i)L^T(k+i)L(k+i), R_3(k+i) = \rho(k+i)L(k+i), i = \overline{1,m}$, we get the solution of the Problem 2. In our formulation, we can attain efficiency in MPC by striking a balance between the two conflicting objectives of maximizing the expected system output and minimizing the associated risk level. These two conflicting objectives are balanced by the weights.

4 | THE INVESTMENT PORTFOLIO MODEL

This article builds upon and generalizes the portfolio model[24] and makes the model closer to reality. Compared with that in the work of Dombrovskii et al,[24] the portfolio model discussed in the present work describes investment portfolio dynamics in the financial market with switching modes when the price dynamics of the risky assets (both growth rate and volatility) change following different states of the economy.

Consider an investment portfolio consisting of n risky assets and one risk-free asset (eg, a bank account). We assume that the decision time horizon is composed of a large number of periods (eg, trading days). During each such period, the decision-maker (investor) obtains new information about the assets prices and reacts to a new market situation by selling some assets and acquiring some other assets.

Let $\eta_i(k+1)$ $(i = \overline{1,n})$ denote the (simple) return of the ith risky asset per period $[k,k+1]$. It is stochastic unobservable at time k with the value defined as

$$\eta_i(k+1) = \frac{K_i(k+1) - K_i(k)}{K_i(k)}, \left(i = \overline{1,n}\right),$$

where $K_i(k)$ denotes the market value of the ith risky asset at time k.

Let $x_i(k)$ $(i = \overline{1,n})$ denote the amount of the wealth invested in the ith risky asset at time k; $x_{n+1}(k) \geq 0$ is the amount invested in a risk-free asset; $u_i^+(k) \geq 0$ is the amount of money by which an investor buys the ith risky asset in period k; $u_i^-(k) \geq 0$ is the amount of money by which an investor sells the ith risky asset in period k. If $x_i(k) < 0$ $(i = \overline{1,n})$, then the short position is used with the amount of shorting$|x_i(k)|$. The investor also can borrow capital in case of need. The volume of borrowing of a risk-free asset is equal to $x_{n+2}(k) \geq 0$; $v(k) \geq 0$ is the amount of borrowing that transferred from borrowing account to bank account in period k, $v(k) < 0$ means a credit repayment in an amount $|v(k)|$ in period k. Let

$r_1(k+1)$ be the riskless lending rate over time period $(k,k+1]$, $r_2(k+1)$ is the riskless borrowing rate over time period $(k,k+1]$, and $r_1(k+1) < r_2(k+1)$.

The ith stock holding $x_i(k)$ $(i=\overline{1,n})$ satisfies the following stochastic difference equation:

$$x_i(k+1) = \left[1+\eta_i(k+1)\right]\left[x_i(k) + u_i^+(k) - u_i^-(k)\right]. \tag{27}$$

Suppose the investor pays fractions λ^+ and λ^- of the amount transacted on purchase and sell of the ith stock, respectively. We assume that the transaction costs are deducted from the bank account (risk-free asset). Then, the dynamics of the bank account is given by

$$x_{n+1}(k+1) = \left[1+r_1(k+1)\right]\left[x_{n+1}(k) + v(k) - \left(1+\lambda^+\right)\sum_{i=1}^{n} u_i^+(k) + \left(1-\lambda^-\right)\sum_{i=1}^{n} u_i^-(k)\right]. \tag{28}$$

The evolution of the borrowing account is the following:

$$x_{n+2}(k+1) = \left[1+r_2(k+1)\right]\left[x_{n+2}(k) + v(k)\right]. \tag{29}$$

As $x_{n+1}(k+1) \geq 0$, $x_{n+2}(k+1) \geq 0$, the following inequalities should be satisfied:

$$x_{n+1}(k) + v(k) - \left(1+\lambda^+\right)\sum_{i=1}^{n} u_i^+(k) + \left(1-\lambda^-\right)\sum_{i=1}^{n} u_i^-(k) \geq 0, \tag{30}$$

$$x_{n+2}(k) + v(k) \geq 0. \tag{31}$$

We suppose that the amounts of the short sale are restricted by $d_i(k) \geq 0$; therefore, the following inequality holds:

$$x_i(k) + u_i^+(k) - u_i^-(k) \geq -d_i(k), \left(i=\overline{1,n}\right), \tag{32}$$

if the short sale is prohibited, then $d_i(k) = 0$. The borrowing amount is also restricted by $d_0(k) \geq 0$, so that

$$x_{n+2}(k) + v(k) \leq d_0(k). \tag{33}$$

It should be noted that $d_i(k)$ $(i=\overline{0,n})$ may be both constant and varying dependent on the portfolio capital $V(k)$, eg, $d_i(k) = c_i V(k)$, where $c_i > 0$ is a constant coefficient.

The wealth process $V(k)$ satisfies

$$V(k) = \sum_{i=1}^{n+1} x_i(k) - x_{n+2}(k). \tag{34}$$

We assume that the returns of assets $\eta_i(k), (i=\overline{1,n})$ in which we are able to invest are described by a discrete-time approximation of the geometric Brownian motion with parameters dependent on the state of the Markov chain $\theta(k)$ (here, we use the state-space representation of Markov chain defined by Equation (6))

$$\eta_i\left[\theta(k),k\right] = \mu_i\left[\theta(k),k\right] + \sum_{j=1}^{n} \sigma_{ij}\left[\theta(k),k\right] w_j(k), \tag{35}$$

where $\mu_i[\theta(k),k]$ is the expected return of the ith risky asset; $\sigma[\theta(k),k] = (\sigma_{ij}[\theta(k),k])$, $i,j=1,..,n$ is the volatility matrix; the sequences $\{w_j(k); k=0,1,\ \dots;j=1,\ \dots\ ,n\}$ are independent noises with zero mean and unit variance.

The expected return, variance, and covariance of the assets at time k are affected by local or global factors, which are represented by the market operation mode $\theta(k)$. When the market operation mode is j, then $\mu_i[\theta(k),k] = \mu_i^{(j)}$, $(j=\overline{1,\nu}, i=\overline{1,n})$ represents the expected return of the ith risky asset, while $\sigma[\theta(k),k] = \sigma^{(j)}$ is the volatility matrix of the risky returns.

The Markovian chain defines the state of a market, eg, a market in a state of high or low volatility and/or a market in a state of ascending or descending trend. We assume that, at the instant of decision making, the current state of the market is known, ie, the Markov state $\theta(k)$ is observable. When practical problems are solved, indicators of a market state can be market indices.

Taking (35) into account, Equation (27) is expressed as

$$x_i(k+1) = [1+\mu_i\left[\theta(k+1), k+1\right] + \sum_{j=1}^{n}\sigma_{ij}\left[\theta(k+1), k+1\right] w_j(k+1)]\left[x_i(k) + u_i^+(k) - u_i^-(k)\right]. \tag{36}$$

Let us introduce the following definitions: $x(k) = [x_1(k), x_2(k), \dots, x_{n+2}(k)]^T$ is the state of the portfolio at time k, and $u(k) = \begin{bmatrix} v(k) & u_1^+(k) & \cdots & u_n^+(k) & u_1^-(k) & \cdots & u_n^-(k)\end{bmatrix}^T$ is the vector of input (manipulated) variables. Then, taking in to account Equations (28), (29), (36), the evolution of the portfolio over time can be described by the following discrete-time state-space representation:

$$\begin{aligned} x(k+1) = &\left[A_0\left[\theta(k+1), k+1\right] + \sum_{j=1}^{n} A_j\left[\theta(k+1), k+1\right] w_j(k+1)\right] x(k) \\ &+ \left[B_0\left[\theta(k+1), k+1\right] + \sum_{j=1}^{n} B_j\left[\theta(k+1), k+1\right] w_j(k+1)\right] u(k), \end{aligned}$$

where

$$A_0\left[\theta(k), k\right] = \operatorname{diag}\{b_0[\theta(k), k], 1+r_1(k), 1+r_2(k)\},$$

$$A_j\left[\theta(k), k\right] = \operatorname{diag}\left\{\sigma_{1j}\left[\theta(k), k\right], \dots, \sigma_{nj}\left[\theta(k), k\right], 0, 0\right\},$$

$$B_0\left[\theta(k), k\right] = \begin{bmatrix} \overline{0}_n^T & b_0\left[\theta(k), k\right] & -b_0\left[\theta(k), k\right] \\ 1+r_1(k) & -\left(1+\lambda^+\right) b_1(k) & \left(1-\lambda^-\right) b_1(k) \\ 1+r_2(k) & \overline{0}_n & \overline{0}_n \end{bmatrix},$$

$$B_j\left[\theta(k), k\right] = \begin{bmatrix} \overline{0}_n^T & \overline{b}_j\left[\theta(k), k\right] & -\overline{b}_j\left[\theta(k), k\right] \\ 0 & \overline{0}_n & \overline{0}_n \\ 0 & \overline{0}_n & \overline{0}_n \end{bmatrix}, \left(j = \overline{1, n}\right),$$

$$b_0\left[\theta(k), k\right] = \operatorname{diag}\{1+\mu_1[\theta(k), k], \dots, 1+\mu_n[\theta(k), k]\},$$

$$b_1(k) = \left[1+r_1(k)\right] e_n,$$

$$\overline{b}_j\left[\theta(k), k\right] = \operatorname{diag}\left\{\sigma_{1j}[\theta(k), k], \dots, \sigma_{nj}[\theta(k), k]\right\}, \left(j = \overline{1, n}\right),$$

$$\overline{0}_n = [0, \dots, 0]_n, e_n = [1, \dots, 1]_n,$$

diag{.} forms a block diagonal matrix from its arguments.

Constraints $u_i^+(k) \geq 0, u_i^-(k) \geq 0$ and (30)-(33) can be written in the matrix form (elementwise inequality) as follows:

$$D(k) \leq S(k) u(k), \tag{37}$$

where

$$S(k) = \begin{bmatrix} \overline{0}_n^T & I_n & 0_n \\ \overline{0}_n^T & 0_n & I_n \\ \overline{0}_n^T & I_n & -I_n \\ 1 & -\left(1+\lambda^+\right) e_n & \left(1-\lambda^-\right) e_n \\ 1 & \overline{0}_n & \overline{0}_n \\ -1 & \overline{0}_n & \overline{0}_n \end{bmatrix}, D(k) = \begin{bmatrix} \overline{0}_n^T \\ \overline{0}_n^T \\ \overline{X}(k) \\ -x_{n+1}(k) \\ -x_{n+2}(k) \\ x_{n+2}(k) - d_0(k) \end{bmatrix}, \overline{X} = \begin{bmatrix} -x_1(k) - d_1(k) \\ \cdots \\ -x_n(k) - d_n(k) \end{bmatrix}.$$

We define two portfolio control problems.

Problem 3. Our objective is to control the investment portfolio, via dynamic asset allocation among the n stocks and a risk-free asset, by tracking a desired deterministic reference trajectory

$$V^0(k+1) = [1+\mu_0]\, V^0(k), \tag{38}$$

where μ_0 is a given parameter representing the growth factor and the initial state is $V^0(0) = V(0)$.

Therefore, we have a dynamic tracking problem of a reference portfolio (38) with desired return μ_0 subject to constraints (37) with criterion

$$\begin{aligned} J(k+m|k) = E\Bigg\{ \sum_{i=1}^{m} \left[V(k+i) - V^0(k+i)\right]^2 - \rho(k+i)\left[V(k+i) - V^0(k+i)\right] \\ + u^T(k+i-1|k)\, R(k+i-1)\, u(k+i-1|k) | V(k), \theta(k) \Bigg\}, \end{aligned} \tag{39}$$

where m is the prediction horizon, $u(k+i|k) = [v(k+i|k), u_1{}^{+}(k+i|k), \ldots, u_n{}^{+}(k+i|k), u_1{}^{-}(k+i|k), \ldots, u_n{}^{-}(k+i|k)]^T$ is the predictive control vector, $R(k+i) > 0$ is a positive symmetric matrix of control cost coefficients, and $\rho(k+i) \geq 0$ is the weight coefficient. The performance criterion (39) is composed by a linear combination of a quadratic part, representing the conditional mean-square error between the investment portfolio value and the value of a reference (benchmark) portfolio, and a linear part, penalizing wealth values that are less than the desired value. The trade-off between these two terms is balanced by the weight $\rho(k+i)$. The third term may be interpreted as follows. It penalizes for transaction costs associated with the trading amount $u(k+i|k)$. We allow for trading to incur quadratic transaction costs, which we take into account directly in our objective function. Quadratic transaction costs are appropriate to model market impact costs, which arise when the investor makes large trades that distort market prices. A common assumption in the literature is that market price impact is linear to the amount traded,[22,23] and thus, market impact costs are quadratic.

An important advantage of the approach of tracking a reference portfolio is its capability to predict the trajectory of portfolio wealth growth, which closely follows the deterministic (given by the investor) benchmark or beats it, which makes it possible to obtain a smooth curve for the growth of the portfolio wealth over the entire investment horizon. This is one of the basic requirements for the trading strategies of investors in financial markets. The growth factor μ_0 is selected by the investor based on an analysis of the local financial market.

Criterion (39) can be transformed into the equivalence form

$$\begin{aligned} J(k+m|k) = \sum_{i=1}^{m} \Big[E\left\{V^2(k+i) | V(k), \theta(k)\right\} - \left[2V^0(k+i) + \rho(k+i)\right] E\{V(k+i) | V(k), \theta(k)\} \\ + E\left\{u^T(k+i-1|k)\, R(k+i-1)\, u(k+i-1|k) | V(k), \theta(k)\right\}\Big], \end{aligned} \tag{40}$$

where we eliminated the term that is independent on control variables.

We can establish a relationship between (40) and (4) by taking $V(k) = Lx(k), R_1(k) = L^T L, L = [1, \ldots, 1, -1]_{n+2}$, and $R_3(k) = [2V^0(k) + \rho(k)]L$.

Problem 4. We define a multiperiod mean-variance portfolio optimization problem with criterion

$$\begin{aligned} J(k+m|k) = \sum_{i=1}^{m} \Big[E\left\{(V(k+i) - E\{V(k+i) | V(k), \theta(k)\})^2 | V(k), \theta(k)\right\} \\ - \rho(k+i) E\{V(k+i) | V(k), \theta(k)\} + E\left\{u^T(k+i-1|k)\, R(k+i-1)\, u(k+i-1|k) | V(k), \theta(k)\right\}\Big], \end{aligned} \tag{41}$$

where the input parameter $\rho(k+i)$ denotes the level of risk aversion, giving a trade-off between the expected portfolio value and the associated risk (variance) level at time k.

We can establish a relationship between (41) and (5) by taking $V(k) = y(k), \mu(k) = 1$. It is obvious that the results presented in Theorem 1 can be applied to solve Problems 3 and 4.

5 | REAL DATA EXAMPLES

In contrast to related results in multiperiod portfolio optimization with Markov switching parameters,[3,7,9] we take into account a variety of constraints and transaction costs. The incorporation of transaction costs and constraints substantially complicates the portfolio model, and standard control design methods cannot be applied directly to solve the portfolio selection problem. Compared with other control methods, the MPC has the advantage of being able to explicitly deal with constraints and we give the computationally tractable technique to solve this new complicated portfolio optimization problem. The purpose of this chapter is to demonstrate the efficiency and the powerful practical potential of the proposed algorithms using the example of stochastic system as complex as an investment portfolio.

In this section, our approach is tested on a set of real stocks. We want to assess the performance of our model under real market conditions by computing the portfolio wealth over a long period. The data used for these examples are taken from the New York Stock Exchange (www.finance.yahoo.com). They include the daily closing stock prices of the largest companies, as well as the values of the Dow Jones Index.

We consider the situation of an investor who has to allocate one unit of wealth over the investment horizon among risky assets and one risk-free asset (bank account). The risk-free asset return is assumed to be of $r_1 = 0.00001$ (0.001% per day). The investor is allowed to borrow capital in case of need. The borrowing interest rate is assumed to be of $r_2 = 0.0001$ (0.01% per day).

We assume that, at zero time, all the portfolio capital is invested in the risk-free asset; therefore, initial values are as follows:

$$x_i(0) = x_{n+2}(0) = 0, \left(i = \overline{1, n}\right),$$
$$x_{n+1}(0) = V(0) = V^0(0) = 1.$$

We introduce two dynamic portfolio optimization problems: (i) a tracking problem of a reference portfolio with desired return; (ii) a multiperiod mean-variance portfolio optimization problem. The updating of the portfolio based on the MPC is executed once every trading day. The weight matrix is set as $R(k+i) = \text{diag}\{10^{-3}, \dots, 10^{-3}\}$ for all k,i. For the online finite horizon MPC problems, we used a horizon of $m = 20$, and numerically solved it in MATLAB by using the quadprog.m function.

During the numerical modeling, we proposed an approach to online adaptive data-driven implementation of the obtained trading strategies. Our approach allows us to capture the trend and volatility changes due to changes of Markov chain state.

In the set of experiments, we used risky assets traded on the New York Stock Exchange: AT&T Inc, Apple Inc, Bank of America, Caterpillar Inc, Cisco Systems Inc, Citigroup Inc, Coca-Cola Company, ExxonMobil, Goldman Sachs Group Inc, Google Inc, IBM, Microsoft Corp, and JPMorgan Chase & Co. All investment portfolios were composed of six risky assets.

We assume that the market parameters depend on the market mode that switches according to a Markov chain among two states ($\nu = 2$). The motivation for such assumption is that a large number of empirical analyses show that the market is characterized by two regimes: one with high volatility and the other with low volatility (see, for instance, the work of Levy and Kaplanski[4] and references there in).

Let State 1 represent low market volatility and let State 2 represent high market volatility. We use Dow Jones Index to observe the current Markov state and to estimate the transition probability matrix. Whenever the daily volatility of the index was below 0.01, we defined that day as low volatility and set $\sigma_{ii}{}^{(1)} = 0.005$, $(i = \overline{1, 6})$, whereas the daily volatility of the index was above 0.01, we defined the day as a high-volatility and set $\sigma_{ii}{}^{(2)} = 0.015$, $(i = \overline{1, 6})$. The "daily volatility of the index" defines degree of variation of an index during a trading day and it is estimated as a sample variation of daily index returns. These values of volatility were obtained by analyzing a real American market behavior. We do not take into account cross-sectional correlation between different assets, ie, $\sigma_{ij}^{(1)} = \sigma_{ij}^{(2)} = 0, i \neq j$. We have experimented with more sophisticated scheme, under assumption that cross-correlation between assets is presented. However, we found its impact on the tracking performance quite negative. This is expected, since we need to estimate a large number of parameters that introduces «estimation uncertainty» into the portfolio optimization strategy.

The transition probability matrix was estimated by the maximum likelihood method using the past 500 daily closing values of the Dow Jones Index prior to the tracking period.

The estimation of transition probability matrix was

$$P = \begin{bmatrix} 0.8966 & 0.1034 \\ 0.0150 & 0.9850 \end{bmatrix}.$$

The Markov process is assumed to be a stationary multiperiod process over the investment horizon.

We computed the expected returns using l-day simple averaging of past historical return data as follows:

$$\hat{\mu}(k) = \frac{1}{l\left[\theta(k)\right]} \sum_{i=1}^{l[\theta(k)]} \eta(k-i+1), l\left[\theta(k)\right] \in \left\{l^{(1)}, l^{(2)}\right\},$$

where the parameter l depends on the state of Markov chain $\theta(k)$. In case of low volatility in the market, we assume that there exists long-term market trend and set $l[\theta(k)] = l^{(1)} = 40$, whereas in case of high volatility, short trends are more relevant and we set $l[\theta(k)] = l^{(2)} = 15$. We also assume that the expected returns remain constant over the predictive horizon m. We use the adjusted procedure, updating the estimates at each decision time k, to adapt the portfolio to price changes on the market incorporating of newly arrived information.

We impose hard constraints on the portfolio problem. The borrowing amounts are restricted by $d_0(k) = 3\ V(k)$. The short selling is assumed to be prohibited, ie, $d_i(k) = 0$, $(i = \overline{1,6})$. Transactions costs fractions are equal to $\lambda^+ = \lambda^- = 0.001$.

For the case of tracking a reference portfolio (Strategy 1), we set the tracking target to return 0.1% per day ($\mu_0 = 0.001$) and the weight $\rho(k+i) = 0.02$. For the case of mean-variance approach (Strategy 2), we set the weight as $\rho(k+i) = 0.1$.

The dynamics of actual portfolio was calculated from Equations (27)-(29) and (34) using realized returns at the next trading day.

We present the typical results of the experiments on Figures 1 to 7. In the pictures, the portfolio was composed of risky assets: AT&T Inc, Apple Inc, IBM, Caterpillar Inc, Microsoft Corp, and JPMorgan Chase & Co. The investment period was from November 2009 to July 2016. Figures 1 and 2 plot the tracking portfolio and a reference portfolio values for Strategy 1 and a mean-variance performance for Strategy 2. In Figures 3 to 6, we have investments and control actions for the risky asset AT&T Inc for two control strategies. Figure 7 illustrates the Dow Jones Index daily returns and the estimated states of the Markov chain.

Figures 1 and 2 show that the tracking a reference portfolio strategy allows us to obtain a smoother curve of growth compared with the mean-variance strategy. The advantage of the control according to the quadratic criterion is that it is possible to predict the trajectory of the growth of portfolio wealth, which should follow as close as possible to the deterministic benchmark given by the investor.

Several observations can be gathered from the examples illustrated above. From Figure 7, it can be observed that there are periods with high and low volatility in the markets. As seen from Figures 1 and 2, both strategies provide the same portfolio capital approximately at the final point; however, obviously the tracking portfolio strategy allows to obtain a

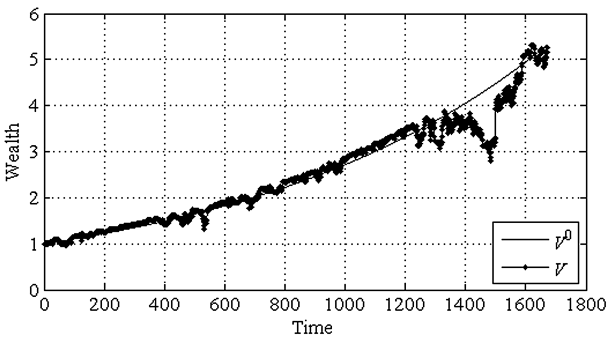

FIGURE 1 Tracking portfolio and reference portfolio values (Strategy 1)

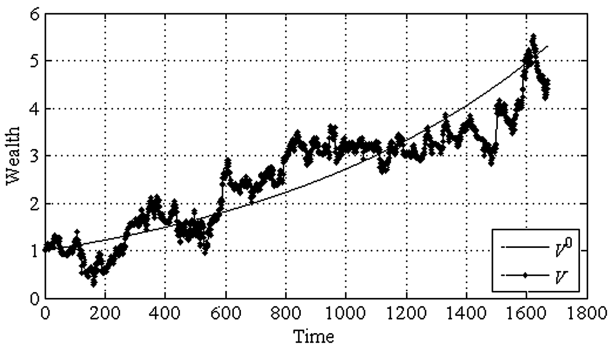

FIGURE 2 Mean-variance portfolio values (Strategy 2). Note that the reference portfolio value is provided in this figure for illustrative purpose to compare the portfolio capitals for the two proposed strategies

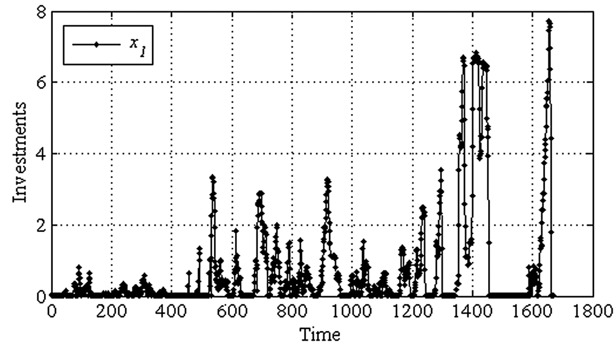

FIGURE 3 Wealth invested in AT&T Inc for tracking a reference portfolio approach

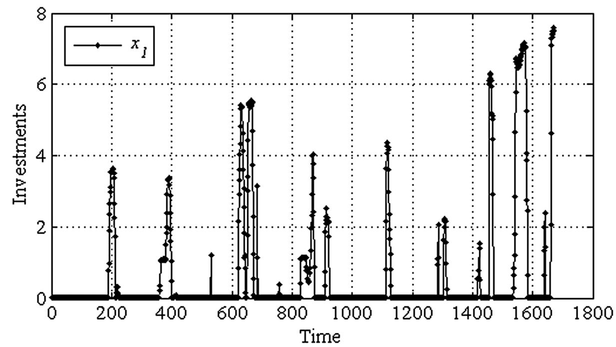

FIGURE 4 Wealth invested in AT&T Inc for mean-variance portfolio approach

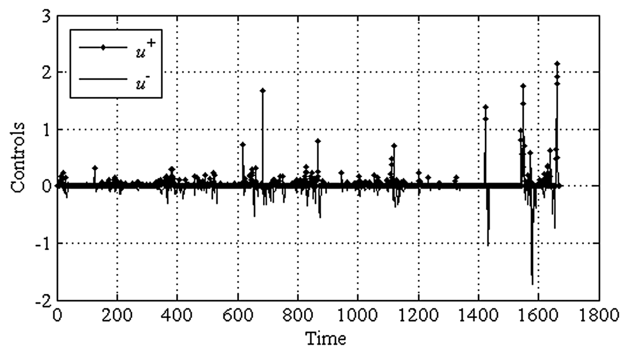

FIGURE 5 Controls on asset AT&T Inc for tracking a reference portfolio approach

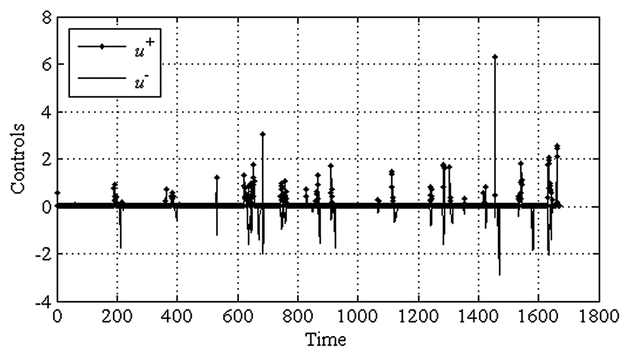

FIGURE 6 Controls on asset AT&T Inc for mean-variance portfolio approach

smoother growth path compared to the mean-variance one. Significant fluctuations are observed for the mean-variance strategy and it is almost impossible to obtain a measurable stable growth by tuning risk-aversion parameters. Tracking strategy is preferable to the investors, and due to the less risk, there is of capital losses as the tracking trajectory is predictable compared to the mean-variance performance.

Our approach does not require a heavy reliance on parametric estimation based on past data; instead, it focuses on trying to capture the dynamic changes of the market and reacts accordingly.

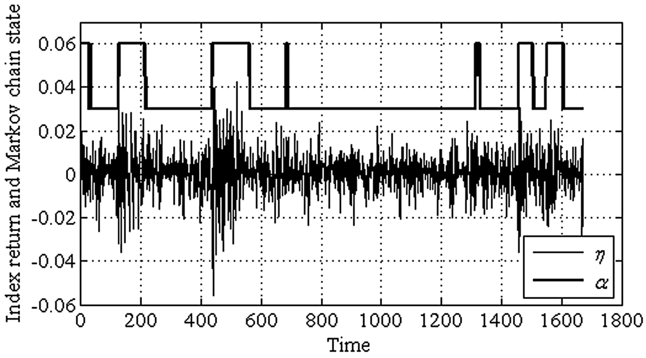

FIGURE 7 Daily Dow Jones Index return and the state of Markov chain

6 | CONCLUSION

In this paper, we have offered a predictive control strategy for a class of discrete-time Markovian jump systems with multiplicative noises subject to hard constraints on the input manipulated variables under a generalized criterion. Setting the weight coefficients to some appropriate values we provide solutions for two special cases: (a) MPC for quadratic criterion and (b) MPC for mean-variance criterion. The synthesis of predictive control strategies leads to the sequence of computationally tractable quadratic programming problems.

The proposed approach was applied to the control of stochastic system as complex as an investment portfolio under explicit transactions costs and trading constraints. The problem is stated as (a) a dynamic tracking problem of a reference portfolio with desired return and (b) a mean-variance problem. The effectiveness of the proposed approach is demonstrated on a real data from the New York Stock Exchange.

Some possible further research topics regard the cases as follows: (1) when dynamics of system is modulated by unobservable Markov chain (systems with hidden Markov parameters); (2) when constraints are probabilistic, also named as chance constraints. In addition, offered approach can be extended to the case when constraints are defined by convex functions. In this case, synthesis of predictive control strategies leads to the sequence of nonlinear programming problems.

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APPENDIX

Proof of Lemma 1. Let us consider the expression (11) for $J_1(k+m|\,k)$. It is obvious that the following equation holds:

$$J_1(k+s|k) = J_1(k+s-1|k) + E\{[u^T(k+s)R_1(k+s)x(k+s) - R_3(k+s)x(k+s) + u^T(k+s-1|k)R(k+s-1)u(k+s-1|k)]|x(k),\theta(k)\Big\},\ s=\overline{1,m}, J_1(k|k)=0.$$

Let us consider

$$J_1(k+1|k) = E\{[x^T(k+1)R_1(k+1)x(k+1) - R_3(k+1)x(k+1) + u^T(k|k)R(k)u(k|k)]\mid x(k),\theta(k)\}. \tag{A1}$$

Taking (8), (9) into consideration, Equation (7) can be written as

$$x(k+1)=\sum_{i_1=1}^{\nu}\theta_{i_1}(k+1)\left[A_0^{(i_1)}(k+1)x(k)+\sum_{j=1}^{n}A_j^{(i_1)}(k+1)w_j(k+1)x(k)+\right.$$
$$\left.+B_0^{(i_1)}(k+1)u(k)+\sum_{j=1}^{n}B_j^{(i_1)}(k+1)w_j(k+1)u(k)\right]. \tag{A2}$$

Substituting $x(k+1)$ from (A2) into (A1) by taking $u(k)=u(k|k)$, we obtain

$$J_1(k+1|k)=E\left\{\left[x^T(k)\sum_{i_1=1}^{\nu}\sum_{j_1=1}^{\nu}\left(A_0^{(i_1)}(k+1)\right)^T R_1(k+1)A_0^{(j_1)}(k+1)\theta_{i_1}(k+1)\theta_{j_1}(k+1)x(k)\right.\right.$$
$$+x^T(k)\sum_{i_1=1}^{\nu}\sum_{j_1=1}^{\nu}\sum_{j=1}^{n}\sum_{l=1}^{n}\left(A_j^{(i_1)}(k+1)\right)^T R_1(k+1)A_l^{(j_1)}(k+1)w_j(k+1)w_l(k+1)\theta_{i_1}(k+1)\theta_{j_1}(k+1)x(k)$$
$$+u^T(k|k)\sum_{i_1=1}^{\nu}\sum_{j_1=1}^{\nu}\left(B_0^{(i_1)}(k+1)\right)^T R_1(k+1)B_0^{(j_1)}(k+1)\theta_{i_1}(k+1)\theta_{j_1}(k+1)u(k|k)$$
$$+u^T(k|k)\sum_{i_1=1}^{\nu}\sum_{j_1=1}^{\nu}\sum_{j=1}^{n}\sum_{l=1}^{n}\left(B_j^{(i_1)}(k+1)\right)^T R_1(k+1)$$
$$\times B_l^{(j_1)}(k+1)w_j(k+1)w_l(k+1)\theta_{i_1}(k+1)\theta_{j_1}(k+1)u(k|k)+2x^T(k)\sum_{i_1=1}^{\nu}\sum_{j_1=1}^{\nu}\left(A_0^{(i_1)}(k+1)\right)^T R_1(k+1)$$
$$\times B_0^{(j_1)}(k+1)\theta_{i_1}(k+1)\theta_{j_1}(k+1)u(k|k)+2x^T(k)\sum_{i_1=1}^{\nu}\sum_{j_1=1}^{\nu}\sum_{j=1}^{n}\sum_{l=1}^{n}\left(A_j^{(i_1)}(k+1)\right)^T R_1(k+1)$$
$$\times B_l^{(j_1)}(k+1)w_j(k+1)w_l(k+1)\theta_{i_1}(k+1)\theta_{j_1}(k+1)u(k|k)$$
$$-R_3(k+1)\sum_{i_1=1}^{\nu}\theta_{i_1}(k+1)\left[A_0^{(i_1)}(k+1)x(k)+B_0^{(i_1)}(k+1)u(k|k)\right]$$
$$\left.\left.+u^T(k|k)R(k)u(k|k)\right]|x(k),\theta(k)\right\}. \tag{A3}$$

Note that $\theta_{i_1}(k+1)\theta_{j_1}(k+1)=0$ when $i_1\neq j_1$, and $\theta_{i_1}^2(k+1)=\theta_{i_1}(k+1)$. Taking mathematical expectation of (A3) given the representation of the Markov chain (6) and combining similar terms, $J_1(k+1|k)$ can be rewritten as follows:

$$J_1(k+1|k)=x^T(k)\sum_{i_i=1}^{\nu}\sum_{j=0}^{n}\left(A_j^{(i_1)}(k+1)\right)^T R_1(k+1)A_j^{(i_1)}(k+1)E_{i_1}P\theta(k)x(k)$$
$$+u^T(k|k)\sum_{i_1=1}^{\nu}\sum_{j=0}^{n}\left(B_j^{(i_1)}(k+1)\right)^T R_1(k+1)B_j^{(i_1)}(k+1)E_{i_1}P\theta(k)u(k|k)$$
$$+2x^T(k)\sum_{i_1=1}^{\nu}\sum_{j=0}^{n}\left(A_j^{(i_1)}(k+1)\right)^T R_1(k+1)B_j^{(i_1)}(k+1)E_{i_1}P\theta(k)u(k|k)$$
$$-R_3(k+1)\sum_{i_1=1}^{\nu}E_{i_1}P\theta(k)\left[A_0^{(i_1)}(k+1)x(k)+B_0^{(i_1)}(k+1)u(k|k)\right]+u^T(k|k)R(k)u(k|k).$$

Proceeding to the next step, we obtain

$$J_1(k+2|k) = J_1(k+1|k) + E\left\{\left[x^T(k+2)R_1(k+2)x(k+2) - R_3(k+2)x(k+2) + u^T(k+1|k)R(k+1)u(k+1|k)\right]|x(k),\theta(k)\right\}, \tag{A4}$$

$$\begin{aligned} x(k+2) = &\sum_{i_2=1}^{\nu}\theta_{i_2}(k+2)\left[A_0^{(i_2)}(k+2)\left[\sum_{i_1=1}^{\nu}\theta_{i_1}(k+1)\left(A_0^{(i_1)}(k+1)x(k)\right.\right.\right. \\ &\left.\left.+\sum_{j=1}^{n}A_j^{(i_1)}(k+1)w_j(k+1)x(k) + B_0^{(i_1)}(k+1)u(k) + \sum_{j=1}^{n}B_j^{(i_1)}(k+1)w_j(k+1)u(k)\right)\right] \\ &+\sum_{j=1}^{n}A_j^{(i_2)}(k+2)w_j(k+2)\left[\sum_{i_1=1}^{\nu}\theta_{i_1}(k+1\left(A_0^{(i_1)}(k+1)x(k) + \sum_{j=1}^{n}A_j^{(i_1)}(k+1)w_j(k+1)x(k)\right.\right. \\ &\left.\left.+B_0^{(i_1)}(k+1)u(k) + \sum_{j=1}^{n}B_j^{(i_1)}(k+1)w_j(k+1)u(k)\right)\right] \\ &\left.+B_0^{(i_2)}(k+2)u(k+1) + \sum_{j=1}^{n}B_j^{(i_2)}(k+2)w_j(k+2)u(k+1)\right]. \end{aligned} \tag{A5}$$

Substituting $x(k+2)$ from (A5) into (A4) by taking $u(k) = u(k|k)$, $u(k+1) = u(k+1|k)$, we obtain the following expression:

$$\begin{aligned} J_1(k+2|k) = &x^T(k)\sum_{i_1=1}^{\nu}\sum_{j=0}^{n}\left(A_j^{(i_1)}(k+1)\right)^T Q^{(i_1)}(k)A_j^{(i_1)}(k+1)x(k) \\ &+2x^T(k)\sum_{i_1=1}^{\nu}\sum_{j=0}^{n}\left(A_j^{(i_1)}(k+1)\right)^T Q^{(i_1)}(k)B_j^{(i_1)}(k+1)u(k|k) \\ &+2x^T(k)\sum_{i_1=1}^{\nu}\left(A_0^{(i_1)}(k+1)\right)^T\sum_{i_2=1}^{\nu}\sum_{j=0}^{n}\left(A_j^{(i_2)}(k+2)\right)^T Q^{(i_1,i_2)}(k)B_j^{(i_2)}(k+2)u(k+1|k) \\ &+u^T(k|k)\sum_{i_1=1}^{\nu}\sum_{j=0}^{n}\left(B_j^{(i_1)}(k+1)\right)^T Q^{(i_1)}(k)B_j^{(i_1)}(k+1)u(k|k) \\ &+u^T(k+1|k)\sum_{i_2=1}^{\nu}\sum_{j=0}^{n}\left(B_j^{(i_2)}(k+2)\right)^T Q^{(i_2)}(k)B_j^{(i_2)}(k+2)u(k+1|k) \\ &+2u^T(k|k)\sum_{i_1=1}^{\nu}\left(B_0^{(i_1)}(k+1)\right)^T\sum_{i_2=1}^{\nu}\sum_{j=0}^{n}\left(A_j^{(i_2)}(k+2)\right)^T Q^{(i_1,i_2)}(k)B_j^{(i_2)}(k+2)u(k+1|k) \\ &-\sum_{i_1=1}^{\nu}Q_2^{(i_1)}(k)A_0^{(i_1)}(k+1)x(k) - \sum_{i_1=1}^{\nu}Q_2^{(i_1)}(k)B_0^{(i_1)}(k+1)u(k|k) - \sum_{i_2=1}^{\nu}Q_2^{(i_2)}(k)B_0^{(i_2)}(k+2)u(k+1|k) \\ &+u^T(k|k)R(k)u(k|k) + u^T(k+1|k)R(k+1)u(k+1|k), \end{aligned}$$

where the matrices $Q^{(i_t)}(k)$, $Q^{(i_t,\dots,i_s)}(k)$, $Q_2^{(i_t)}(k)$ are defined by Equations (14)-(18).

Following the procedure repeated above for $J_1(k+3|k)$, $J_1(k+4|k)$, …, the expression can be obtained for $J_1(k+m|k)$

$$\begin{aligned} J_1(k+m|k) = &x^T(k)\sum_{i_1=1}^{\nu}\sum_{j=0}^{n}\left(A_j^{(i_1)}(k+1)\right)^T Q^{(i_1)}(k)A_j^{(i_1)}(k+1)x(k) \\ &+2x^T(k)\sum_{t=1}^{m}\sum_{i_1=1}^{\nu}\cdots\sum_{i_{t-1}=1}^{\nu}\left(A_0^{(i_1)}(k+1)\right)^T\cdots\left(A_0^{(i_{t-1})}(k+t-1)\right)^T \\ &\times\sum_{j=0}^{n}\sum_{i_t=1}^{\nu}\left(A_j^{(i_t)}(k+t)\right)^T Q^{(i_1,i_2,\dots,i_t)}(k)B_j^{(i_t)}(k+t)u(k+t-1|k) \end{aligned} \tag{A6}$$

$$+\sum_{t=1}^{m} u^T(k+t-1|k)\left[\sum_{i_t=1}^{v}\sum_{j=0}^{n}\left(B_j^{(i_t)}(k+t)\right)^T Q^{(i_t)}(k)B_j^{(i_t)}(k+t)+R(k+t-1)\right]u(k+t-1|k)$$
$$+\sum_{t=1}^{m-1}\sum_{s=t+1}^{m} u(k+t-1|k)\sum_{i_t=1}^{v}\left(B_0^{(i_t)}(k+t)\right)^T\sum_{i_{t+1}=1}^{v}\cdots\sum_{i_{s-1}=1}^{v}\left(A_0^{(i_t)}(k+t+1)\right)^T\cdots\left(A_0^{(i_{s-1})}(k+s-1)\right)^T$$
$$\times\sum_{j=0}^{n}\sum_{i_s=1}^{v}\left(A_j^{(i_s)}(k+s)\right)^T Q^{(i_t,\ldots,i_s)}(k)B_j^{(i_s)}(k+s)u(k+s-1|k)$$
$$-\sum_{i_1=1}^{v} Q_2^{(i_1)}(k)A_0^{(i_1)}(k+1)x(k)-\sum_{t=1}^{m}\sum_{i_t=1}^{v} Q_2^{(i_t)}(k)B_0^{(i_t)}(k+t)u(k+t-1|k),$$

where the matrices $Q^{(i_t)}(k)$, $Q^{(i_t,\ldots,i_s)}(k)$, $Q_2^{(i_t)}(k)$ are defined by Equations (14)-(18).

Note that the following equality was also used to obtain the formula (18):

$$\begin{aligned}\Theta^{(i_t,\ldots,i_s)}(k)&=E\left\{\theta_{i_t}(k+t)\cdots\theta_{i_s}(k+s)\,|\theta(k)\right\}\\&=E_{i_s}Pdiag\left\{Pdiag\left\{\cdots Pdiag\left\{P^t\theta(k)\right\}E_{i_t}{}^T\right\}E_{i_{t+1}}^T\right\}\cdots\right\}E_{i_{s-2}}^T\right\}E_{i_{s-1}}^T\\&=P_{i_s,i_{s-1}}P_{i_{s-1},i_{s+1}}\cdots P_{i_{t+1},i_t}\theta_{i_t}(k+t|k),\ t=\overline{1,m-1},s>t.\end{aligned}\tag{A7}$$

The expression (A6) can be written in the matrix form (13). This completes the proof.

Proof of Lemma 2. Let us consider the expression (12) for $J_2(k+m|k)$. Using the system dynamics (7)-(9) and representation of the Markov chain (6), we obtain by iterating system dynamics forward in time s steps

$$\begin{aligned}E\{x(k+s)\,|x(k),\theta(k)\}=&\sum_{i_1=1}^{v}\cdots\sum_{i_s=1}^{v}A_0^{(i_s)}(k+s)\cdots A_0^{(i_1)}(k+1)\Theta^{(i_1,\ldots,i_s)}(k)x(k)\\&+\sum_{t=1}^{s}\sum_{i_t=1}^{v}\sum_{i_{t+1}=1}^{v}\cdots\sum_{i_s=1}^{v}A_0^{(i_s)}(k+s)\cdots A_0^{(i_{t+1})}(k+t+1)B_0^{(i_t)}(k+t)\Theta^{(i_t,\ldots,i_s)}(k)\\&\times u(k+t-1|k),\ \left(s=\overline{1,m}\right),\end{aligned}$$

where $\Theta^{(i_1,\ldots,i_s)}(k)$ is defined by (A7) and

$$\Theta^{(i_t)}(k)=E\left\{\theta_{i_t}(k+t)\,|\theta(k)\right\}=E_{i_t}P^t\theta(k).$$

Let us introduce the extended vector

$$X(k+1)=\left[E\left\{x^T(k+1)\,|x(k),\theta(k)\right\},\ \ldots\ ,E\left\{x^T(k+m)\,|x(k),\theta(k)\right\}\right]^T.$$

The evolution of the vector $E\{x(k+s)|\, x(k),\theta(k)\}$ over a prediction horizon m, starting at k, can be described in a compact form as

$$X(k+1)=\Psi(k)x(k)+\Phi(k)U(k),\tag{A8}$$

where the matrices $\Phi(k)$, $\Psi(k)$, $U(k)$ are of the form

$$\Phi(k)=\begin{bmatrix}\sum_{i_1=1}^{v}B_0^{(i_1)}(k+1)\Theta^{(i_1)}(k)\\\sum_{i_1=1}^{v}\sum_{i_2=1}^{v}A_0^{(i_2)}(k+2)B_0^{(i_1)}(k+1)\Theta^{(i_1,i_2)}(k)\\\cdots\\\sum_{i_1=1}^{v}\sum_{i_2=1}^{v}\cdots\sum_{i_m=1}^{v}A_0^{(i_m)}(k+m)\cdots A_0^{(i_2)}(k+2)B_0^{(i_1)}(k+1)\Theta^{(i_1,\ldots,i_m)}(k)\end{bmatrix}$$

$$\left.\begin{matrix} 0 & \cdots & 0 \\ \sum_{i_2=1}^{\nu} B_0^{(i_2)}(k+2)\,\Theta^{(i_2)}(k) & \cdots & 0 \\ \cdots & \cdots & \cdots \\ \sum_{i_2=1}^{\nu}\sum_{i_3=1}^{\nu}\cdots\sum_{i_m=1}^{\nu} A_0^{(i_m)}(k+m)\cdots A_0^{(i_3)}(k+3)\,B_0^{(i_2)}(k+2)\,\Theta^{(i_2,\dots,i_m)}(k) & \cdots & \sum_{i_m=1}^{\nu} B_0^{(i_m)}(k+m)\,\Theta^{(i_m)}(k) \end{matrix}\right],$$

$$\Psi(k) = \begin{bmatrix} \sum_{i_1=1}^{\nu} A_0^{(i_1)}(k+1)\,\Theta^{(i_1)}(k) \\ \sum_{i_1=1}^{\nu}\sum_{i_2=1}^{\nu} A_0^{(i_2)}(k+2)\,A_0^{(i_1)}(k+1)\,\Theta^{(i_1,i_2)}(k) \\ \cdots \\ \sum_{i_1=1}^{\nu}\sum_{i_2=1}^{\nu}\cdots\sum_{i_m=1}^{\nu} A_0^{(i_m)}(k+m)\cdots A_0^{(i_1)}(k+1)\,\Theta^{(i_1,\dots,i_m)}(k) \end{bmatrix}, U(k) = \begin{bmatrix} u(k|k) \\ u(k+1|k) \\ \cdots \\ u(k+m-1|k) \end{bmatrix}.$$

It is obvious that the blocks of the matrices $\Phi(k)$,$\Psi(k)$ are of the form (20)-(23).

The expression (12) for $J_2(k+m|\,k)$ can also be written compactly as

$$J_2(k+m|k) = X^T(k+1)\,\Delta(k+1)\,X(k+1), \tag{A9}$$

here, the matrix $\Delta(k+1)$ is given by $\Delta(k+1) = \mathrm{diag}\{R_2(k+1), \dots, R_2(k+m)\}$..

Substituting (A8) into (A9), we obtain

$$\begin{aligned} J_2(k+m|k) = {} & x^T(k)\,\Psi^T(k)\,\Delta(k+1)\,\Psi(k)\,x(k) + 2x^T(k)\,\Psi^T(k)\,\Delta(k+1)\,\Phi(k)\,U(k) \\ & + U^T(k)\,\Phi^T(k)\,\Delta(k+1)\,\Phi(k)\,U(k). \end{aligned}$$

Let us denote

$$\begin{aligned} C^{(2)}\left[x(k)\right] &= x^T(k)\,\Psi^T(k)\,\Delta(k+1)\,\Psi(k)\,x(k), \\ G^{(2)}(k) &= \Psi^T(k)\,\Delta(k+1)\,\Phi(k), \\ H^{(2)}(k) &= \Phi^T(k)\,\Delta(k+1)\,\Phi(k). \end{aligned}$$

Given the above notations, we have (19). This completes the proof.