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MATHEMATICAL PROCESSING OF PHYSICS EXPERIMENTAL DATA

ASYMPTOTIC STATIONARY PROBABILITY DISTRIBUTION OF TOTAL AMOUNT OF PHYSICS EXPERIMENTAL DATA

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A mathematical model has been constructed for processing of physics experimental data in the form of a two-phase resource queuing system with MMPP arrivals, general service time, and customers copying at the second phase. It is proved that the joint stationary probability distribution of the total amount of resources occupied at each block of the system converges to a three-dimensional Gaussian distribution under the asymptotic condition of the increasing intensity of the arrival process. The parameters of this asymptotic distribution are obtained.

Keywords: experimental data processing, resource queuing system.

INTRODUCTION

A Cyber-Physical System (CPS) represents a complex distributed system controlled or monitored by computer algorithms based on the technology of the Internet of Things (IoT) [1]. Its main difference is a very close interaction between computing and physical processes; therefore, it can be stated that the CPS is a complex system of computing and physical elements that constantly obtains data from the environment and uses them for subsequent optimization of control processes. Smart electrical power supply networks and control systems of smart transport and smart cities can be referred to them. The cyber-physical systems control a large amount of data obtained from gauges. Computing processing must be efficient and well-timed because physical processes proceed irrespective of the results of computations. To meet these requirements, the CPS should possess throughput or power necessary for instantaneous processing, that is, should have an amount of resource sufficient for processing of these data. It is most logical to consider the resource queuing systems (RQS) as mathematical models of a wide class of technical devices and information-communication and cyber-physical systems.

The methods of queuing theory are widely used to describe the process of information transfer. In classical RQS, devices play the role of resources necessary for servicing, but except the devices and-or waiting places, applications may require an additional amount of resources that occupies random volumes during information processing, transfer, and storage. In addition, because of heterogeneity of services (phone calls, texts, video-, and audio-messages, and Internet), the total amount of information transfer must be taken into account [2–4]. This necessitates the development of new resource models in terms of the RQS that would allow the estimation of the amount of employed resources.

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A great number of publications are devoted to the simulation of wireless communication systems based on the RQS; their most comprehensive review is presented in [5–9]. However, they mainly analyze various schemes of resource allocations in systems with deterministic or discrete resource applications [10, 11].

The present paper studies the characteristics of queuing systems with a random amount of occupied resources of wireless networks with customers splitting (copying). Unlike the existing models, the examined models allow the estimation of the amount of reserved resources for the traffic of the Internet of Things and the development of resource allocation strategy with competing for traffic.

1. PROBLEM STATEMENT

Let us consider the queuing system $\mathrm{MMPP}^{(\nu)}|\mathrm{GI}|\infty$ with customers copying at the second phase and MMPP arrivals $k(t) = 1, 2, \ldots, K$ assigned by the generator matrix $\mathbf{Q} = \|q_{\nu k}\|$, $\nu, k = 1, \ldots, K$ and the diagonal matrix of the conditional intensities $\boldsymbol{\Lambda} = \mathrm{diag}\{\lambda_1, \ldots, \lambda_K\}$. The service discipline is defined as follows. An arrival takes any free server at the first phase, requires a random resource amount $G_0(\nu)$ for servicing, and is served during the random time with distribution function $B_0(x)$. After the servicing at the first phase, the customer goes to the second phase, is served at the first block during the random time with distribution function $B_1(x)$, and is copied to the second block, where it is served during time with distribution function $B_2(x)$. The application also requires a certain amount of resource in the first block with distribution function $G_1(\nu)$ and in the second block with distribution function $G_2(\nu)$. After the service ended, the application leaves the system, freeing all occupied devices and resources. The amount of occupied resources and the service time are independent from each other.

Let $V_i(t)$ be the total amount of the system resources occupied at time t, where $i = 0, 1, 2$. The problem is to find the characteristics of the multidimensional random process $\{V_0(t), V_1(t), V_2(t)\}$. Note that the process under study is non-Markovian. We investigate it using the multidimensional dynamic screening method [12, 13].

Let us depict four parallel time axes. The MMPP axis will display the arrival process, axis 0 will correspond to the first screened process, axis 1 – to the second screened process, and axis 2 – to the third screened process. We now fix the time T. Let there be a set of functions $S_0(t), S_1(t), S_2(t),$ and $S_{12}(t)$ whose values lie in the range [0, 1] and possess the property $S_0(t) + S_1(t) + S_2(t) + S_{12}(t) \le 1$ for any t. An arriving process can be screened only on one of the three axes 0, 1, or 2 or on no one. The probability that the process arriving at the system at time $t > t_0$:

– will generate an arrival point on axis 0, that is, by time T will not end service at the first phase, is

$$S_0(t) = 1 - B_0(T - t);$$

– will generate an arrival point on axis 1, that is, by time T will end service at the first phase, in the second block of the second phase, and will not end service at the first block of the second phase, is

$$S_1(t) = (B_2 * B_0)(T - t) - \int_0^{T-t} B_1(T - t - x) B_2(T - t - x) dB_0(x);$$

– will generate an arrival point on axis 2, that is, by time T will end service at the first phase, at the second phase in the first block, and will not end service at the second phase in the second block, is

$$S_2(t) = (B_1 * B_0)(T - t) - \int_0^{T-t} B_1(T - t - x) B_2(T - t - x) dB_0(x);$$

– will generate an arrival point on axes 1 and 2, that is, by time T will end service at the first phase and will be present at the second phase in two blocks simultaneously, is

$$S_{12}(t) = B_0(T-t) - (B_1 * B_0)(T-t) - (B_2 * B_0)(T-t) + \int_0^{T-t} B_1(T-t-x)B_2(T-t-x)dB_0(x).$$

Let us designate by $W_i(t)$ the total amount of the occupied resource screened by customers on the ith axis ($i = 0, 1, 2$). It can easily be shown that

$$P\{V_0(T) < z_0, V_1(T) < z_1, V_2(T) < z_2\} = P\{W_0(T) < z_0, W_1(T) < z_1, W_2(T) < z_2\} \tag{1}$$

for any $z_0 > 0$, $z_1 > 0$, and $z_2 > 0$. We use equality (1) to investigate the process $\boldsymbol{V}(t)$ in terms of the process $\boldsymbol{W}(t)$.

2. SYSTEM OF THE KOLMOGOROV INTEGRO-DIFFERENTIAL EQUATIONS

We now add the state of the Markov chain at time t $k(t)$. Then the resulting four-dimensional process $\{k(t), W_0(t), W_1(t), W_2(t)\}$ will be Markovian. We introduce the following notation for its probability distribution:

$$P(k, z_0, z_1, z_2, t) = P\{k(t) = k, W_0(t) < z_0, W_1(t) < z_1, W_2(t) < z_2\}.$$

For this distribution, using the Δt method, we can write for $k = 1, \ldots, K$, $z_0 > 0$, $z_1 > 0$, and $z_2 > 0$ the following system of the Kolmogorov integro-differential equations:

$$\frac{\partial P(k, z_0, z_1, z_2, t)}{\partial t} = \lambda_k S_0(t)\left[\int_0^{z_0} P(k, z_0 - y, z_1, z_2, t)dG_0(y) - P(k, z_0, z_1, z_2, t)\right]$$

$$+\lambda_k S_{12}(t)\left[\int_0^{z_1}\int_0^{z_2} P(k, z_0, z_1 - y_1, z_2 - y_2, t)dG_1(y_1)dG_2(y_2) - P(k, z_0, z_1, z_2, t)\right]$$

$$+\lambda_k S_1(t)\left[\int_0^{z_1} P(k, z_0, z_1 - y, z_2, t)dG_1(y) - P(k, z_0, z_1, z_2, t)\right]$$

$$+\lambda_k S_2(t)\left[\int_0^{z_2} P(k, z_0, z_1, z_2 - y, t)dG_2(y) - P(k, z_0, z_1, z_2, t)\right] + \sum_{\nu \neq k} q_{\nu k} P(\nu, z_0, z_1, z_2, t). \tag{2}$$

The initial condition for its solution $P(k, z_0, z_1, z_2, t)$ at time t_0 we take in the form

$$P(k, z_0, z_1, z_2, t_0) = \begin{cases} r(k), & z_0 = z_1 = z_2 = 0, \\ 0, & \text{otherwise}, \end{cases}$$

where $r(k)$ is the stationary probability distribution of states of the Markov chain $k(t)$.

Let us introduce the partial characteristic functions in the form

$$h(k,v_0,v_1,v_2,t)=\int_0^\infty e^{jv_0z_0}\int_0^\infty e^{jv_1z_1}\int_0^\infty e^{jv_2z_2}P(k,dz_0,dz_1,dz_2,t),$$

where $j=\sqrt{-1}$ is the imaginary unit. Then Eq. (2) can be written in the form of the following differential equations:

$$\frac{\partial h(k,v_0,v_1,v_2,t)}{\partial t}=h(k,v_0,v_1,v_2,t)\Big[\lambda_k S_0(t)\big(G_0^*(v_0)-1\big)+\lambda_k S_1(t)\big(G_1^*(v_1)-1\big)$$

$$+\lambda_k S_2(t)\big(G_2^*(v_2)-1\big)+\lambda_k S_{12}(t)\big(G_1^*(v_1)G_2^*(v_2)-1\big)\Big],$$

$k = 1, \ldots, K$, where $G_i^*(v_i)=\int_0^\infty e^{jv_iy}dG_i(y)$. Let us re-write this system in the matrix form:

$$\frac{\partial \boldsymbol{h}(v_0,v_1,v_2,t)}{\partial t}=\boldsymbol{h}(v_0,v_1,v_2,t)\Big\{\boldsymbol{\Lambda}\Big[S_0(t)\big(G_0^*(v_0)-1\big)+S_1(t)\big(G_1^*(v_1)-1\big)$$

$$+S_2(t)\big(G_2^*(v_2)-1\big)+S_{12}(t)\big(G_1^*(v_1)G_2^*(v_2)-1\big)\Big]+\boldsymbol{Q}\Big\} \tag{3}$$

with the initial condition

$$\boldsymbol{h}(v_0,v_1,v_2,t_0)=\boldsymbol{r}, \tag{4}$$

where $\boldsymbol{h}(v_0,v_1,v_2,t)=\big[h(1,v_0,v_1,v_2,t),\ldots,h(K,v_0,v_1,v_2,t)\big]$, $\boldsymbol{r} = [r(1), \ldots, r(K)]$ is the vector of the stationary probability distribution of states of the Markov control chain $k(t)$ satisfying to the system

$$\begin{cases}\boldsymbol{rQ}=\boldsymbol{0},\\ \boldsymbol{re}=1,\end{cases}$$

and $\boldsymbol{e}$ is the unit vector-column.

3. ASYMPTOTIC ANALYSIS METHOD

Because analytical solution of Eq. (3) is impossible, to solve problem (3)–(4), we use the method of asymptotic analysis [9] for infinitely increasing intensity of arrivals and extremely frequent changes of the states of the Markov chain. Let us substitute $\boldsymbol{\Lambda}=N\overline{\boldsymbol{\Lambda}}$ and $\boldsymbol{Q}=N\overline{\boldsymbol{Q}}$ in Eq. (3), where $N\to\infty$ is the parameter used for asymptotic analysis. Then with initial condition (4), we can write

$$\frac{1}{N}\frac{\partial \boldsymbol{h}(v_0,v_1,v_2,t)}{\partial t}=\boldsymbol{h}(v_0,v_1,v_2,t)\Big\{\overline{\boldsymbol{\Lambda}}\Big[S_0(t)\big(G_0^*(v_0)-1\big)+S_1(t)\big(G_1^*(v_1)-1\big)$$

$$+S_2(t)\big(G_2^*(v_2)-1\big)+S_{12}(t)\big(G_1^*(v_1)G_2^*(v_2)-1\big)\Big]+\overline{\boldsymbol{Q}}\Big\}. \tag{5}$$

Theorem 1. The first-order asymptotic characteristic function of the four-dimensional stochastic process $\{k(t), W_0(t), W_1(t), W_2(t)\}$ has the following form:

$$\boldsymbol{h}(v_0, v_1, v_2, t) = \boldsymbol{r} \exp\left\{ N\lambda \left[jv_0 a_1^{(0)} \int_{t_0}^{t} S_0(\tau) d\tau + jv_1 a_1^{(1)} \int_{t_0}^{t} S_1(\tau) d\tau \right.\right.$$

$$\left.\left. + \; jv_2 a_1^{(2)} \int_{t_0}^{t} S_2(\tau) d\tau + \left(jv_1 a_1^{(1)} + jv_2 a_1^{(2)} \right) \int_{t_0}^{t} S_{12}(\tau) d\tau \right] \right\},$$

where $\lambda = \boldsymbol{r}\overline{\boldsymbol{\Lambda}}\boldsymbol{e}$ is the intensity of the arrival process, and $a_1^{(i)}$ is the mean of the total resources in the ith system block.

Theorem 2. The second-order asymptotic characteristic function of the multidimensional stochastic process $\{k(t), W_0(t), W_1(t), W_2(t)\}$ has the following form:

$$\boldsymbol{h}(v_0, v_1, v_2, t) \approx \boldsymbol{r} \exp\left\{ N\lambda \left[jv_0 a_1^{(0)} \int_{t_0}^{t} S_0(\tau) d\tau + jv_1 a_1^{(1)} \int_{t_0}^{t} \left(S_1(\tau) + S_{12}(\tau) \right) d\tau \right.\right.$$

$$\left. + jv_2 a_1^{(2)} \int_{t_0}^{t} \left(S_2(\tau) + S_{12}(\tau) \right) d\tau \right] + \frac{(jv_0)^2}{2} \left[N\lambda a_2^{(0)} \int_{t_0}^{t} S_0(\tau) d\tau + N\kappa \left(a_1^{(0)} \right)^2 \int_{t_0}^{t} S_0^2(\tau) d\tau \right]$$

$$+ \frac{(jv_1)^2}{2} \left[N\lambda a_2^{(1)} \int_{t_0}^{t} \left(S_1(\tau) + S_{12}(\tau) \right) d\tau + N\kappa \left(a_1^{(1)} \right)^2 \int_{t_0}^{t} \left(S_1(\tau) + S_{12}(\tau) \right)^2 d\tau \right]$$

$$+ \frac{(jv_2)^2}{2} \left[N\lambda a_2^{(2)} \int_{t_0}^{t} \left(S_2(\tau) + S_{12}(\tau) \right) d\tau + N\kappa \left(a_1^{(2)} \right)^2 \int_{t_0}^{t} \left(S_2(\tau) + S_{12}(\tau) \right)^2 d\tau \right]$$

$$+ j^2 v_0 v_1 a_1^{(0)} a_1^{(1)} N\kappa \left[\int_{t_0}^{t} S_0(\tau) \left(S_1(\tau) + S_{12}(\tau) \right) d\tau \right] + j^2 v_0 v_2 a_1^{(0)} a_1^{(2)} N\kappa \left[\int_{t_0}^{t} S_0(\tau) \left(S_2(\tau) + S_{12}(\tau) \right) d\tau \right]$$

$$\left. + j^2 v_1 v_2 a_1^{(1)} a_1^{(2)} \left[N\lambda \int_{t_0}^{t} S_{12}(\tau) d\tau + N\kappa \int_{t_0}^{t} \left(S_1(\tau) + S_{12}(\tau) \right) \left(S_2(\tau) + S_{12}(\tau) \right) d\tau \right] \right\}.$$

Here $\kappa = 2\boldsymbol{g}\left(\overline{\boldsymbol{\Lambda}} - \lambda \boldsymbol{I} \right)\boldsymbol{e}$; $a_1^{(i)}$ and $a_2^{(i)}$ are the first- and second-order initial moments of the random variables with the probability distribution function $G_i(y)$.

Proof. If we write the function $\boldsymbol{h}(v_0, v_1, v_2, t)$ in the form

$$\boldsymbol{h}(v_0, v_1, v_2, t) = \boldsymbol{h}_2(v_0, v_1, v_2, t) \exp\left\{ N\lambda \left[jv_0 a_1^{(0)} \int_{t_0}^{t} S_0(\tau) d\tau + jv_1 a_1^{(1)} \int_{t_0}^{t} S_1(\tau) d\tau \right.\right.$$

$$+ jv_2 a_1^{(2)} \int_{t_0}^{t} S_2(\tau) d\tau + \left(jv_1 a_1^{(1)} + jv_2 a_1^{(2)} \right) \int_{t_0}^{t} S_{12}(\tau) d\tau \Bigg] \Bigg\} , \tag{6}$$

we obtain the following equation for the function $\boldsymbol{h}_2(v_0, v_1, v_2, t)$:

$$\frac{1}{N} \frac{\partial \boldsymbol{h}_2(v_0, v_1, v_2, t)}{\partial t} + \lambda \boldsymbol{h}_2(v_0, v_1, v_2, t) \Big[S_0(t) jv_0 a_1^{(0)} + \big(S_1(t) + S_{12}(t)\big) jv_1 a_1^{(1)} + \big(S_2(t) + S_{12}(t)\big) jv_2 a_1^{(2)} \Big]$$

$$= \boldsymbol{h}_2(v_0, v_1, v_2, t) \Big\{ \bar{\boldsymbol{\Lambda}} \Big[S_0(t) \big(G_0^*(v_0) - 1 \big) + S_1(t) \big(G_1^*(v_1) - 1 \big)$$

$$+ S_2(t) \big(G_2^*(v_2) - 1 \big) + S_{12}(t) \big(G_1^*(v_1) G_2^*(v_2) - 1 \big) \Big] + \bar{\boldsymbol{Q}} \Big\} . \tag{7}$$

By making substitutions

$$\frac{1}{N} = \varepsilon^2, \; v_0 = \varepsilon x_0, \; v_1 = \varepsilon x_1, \; v_2 = \varepsilon x_2, \; \boldsymbol{h}_2(v_0, v_1, v_2, t) = \boldsymbol{f}_2(x_0, x_1, x_2, t, \varepsilon) , \tag{8}$$

and using designations (8), we can write Eq. (7) in the form

$$\varepsilon^2 \frac{\partial \boldsymbol{f}_2(x_0, x_1, x_2, t, \varepsilon)}{\partial t} + \boldsymbol{f}_2(x_0, x_1, x_2, t, \varepsilon) \lambda \Big[j\varepsilon x_0 S_0(t) + j\varepsilon x_1 S_1(t) + j\varepsilon x_2 S_2(t)$$

$$+ (j\varepsilon x_1 + j\varepsilon x_2) S_{12}(t) \Big] = \boldsymbol{f}_2(x_0, x_1, x_2, t, \varepsilon) \Big\{ \bar{\boldsymbol{\Lambda}} \Big[\big(G_0^*(\varepsilon x_0) - 1 \big) S_0(t)$$

$$+ \big(G_1^*(\varepsilon x_1) - 1 \big) S_1(t) + \big(G_2^*(\varepsilon x_2) - 1 \big) S_2(t) + \big(G_1^*(\varepsilon x_1) G_2^*(\varepsilon x_2) - 1 \big) S_{12}(t) \Big] + \bar{\boldsymbol{Q}} \Big\} . \tag{9}$$

Let us find an asymptotic solution of this problem for $\varepsilon \to 0$: $\boldsymbol{f}_2(x_0, x_1, x_2, t) = \lim_{\varepsilon \to 0} \boldsymbol{f}_2(x_0, x_1, x_2, t, \varepsilon)$.

Step 1. Proceeding to the limit as $\varepsilon \to 0$ in Eq. (9), we obtain $\boldsymbol{f}_2(x_0, x_1, x_2, t, \varepsilon) \bar{\boldsymbol{Q}} = \mathbf{0}$. Write $\boldsymbol{f}_2(x_0, x_1, x_2, t)$ in the form

$$\boldsymbol{f}_2(x_0, x_1, x_2, t) = \boldsymbol{r} \Phi_2(x_0, x_1, x_2, t) , \tag{10}$$

where $\Phi_2(x_0, x_1, x_2, t)$ is a scalar differentiable function satisfying to the condition $\Phi_2(x_0, x_1, x_2, t_0) = 1$.

Step 2. We express a solution of Eq. (9) in the form of the decomposition

$$\boldsymbol{f}_2(x_0, x_1, x_2, t) = \Phi_2(x_0, x_1, x_2, t) \Big\{ \boldsymbol{r} + \boldsymbol{g} \Big[j\varepsilon x_0 S_0(t) + j\varepsilon x_1 S_1(t) + j\varepsilon x_2 S_2(t)$$

$$+ (j\varepsilon x_1 + j\varepsilon x_2) S_{12}(t) \Big] + O(\varepsilon^2) \Big\} , \tag{11}$$

where $\boldsymbol{g}$ is a row vector. Having substituted decomposition (11) in Eq. (9), we obtain the matrix equation for the vector $\boldsymbol{g}$: $\boldsymbol{g}\bar{\boldsymbol{Q}} = \boldsymbol{r}\left(\lambda \boldsymbol{I} - \bar{\boldsymbol{\Lambda}}\right)$.

Step 3. Multiplying Eq. (9) by the vector $\boldsymbol{e}$, using Eq. (11) and the decomposition $e^{j\varepsilon x} = 1 + j\varepsilon x + \frac{(j\varepsilon x)^2}{2} + O\left(\varepsilon^3\right)$, after simple transformations we obtain

$$\frac{\partial \Phi_2(x_0,x_1,x_2,t)}{\partial t} = \Phi_2(x_0,x_1,x_2,t)\Bigg[\frac{(jx_0)^2}{2}\left(\lambda a_2^{(0)} S_0(t) + \kappa\left(a_2^{(0)}\right)^2 S_0^2(t)\right)$$

$$+\frac{(jx_1)^2}{2}\left[\lambda a_2^{(1)}\left(S_1(t)+S_{12}(t)\right) + \kappa\left(a_1^{(1)}\right)^2\left(S_1(t)+S_{12}(t)\right)^2\right]$$

$$+\frac{(jx_2)^2}{2}\left[\lambda a_2^{(2)}\left(S_2(t)+S_{12}(t)\right) + \kappa\left(a_1^{(2)}\right)^2\left(S_2(t)+S_{12}(t)\right)^2\right]$$

$$+j^2 x_0 x_1 a_1^{(0)} a_1^{(1)} \kappa\left[S_0(t)\left(S_1(t)+S_{12}(t)\right)\right] + j^2 x_0 x_2 a_1^{(0)} a_1^{(2)} \kappa\left[S_0(t)\left(S_2(t)+S_{12}(t)\right)\right]$$

$$+j^2 x_1 x_2 a_1^{(1)} a_1^{(2)}\left[\lambda S_{12}(t) + \kappa\left(S_{12}^2(t) + S_1(t)S_2(t)S_1(t)S_{12}(t) + S_2(t)S_{12}(t)\right)\right]\Bigg],$$

where $\kappa = 2\boldsymbol{g}\left(\bar{\boldsymbol{\Lambda}} - \lambda \boldsymbol{I}\right)\boldsymbol{e}$. With allowance for the initial condition, the solution of this equation assumes the form

$$\Phi_2(x_0,x_1,x_2,t) = \exp\Bigg\{\frac{(jx_0)^2}{2}\left(\lambda a_2^{(0)}\int_{t_0}^{t} S_0(\tau)d\tau + \kappa\left(a_2^{(0)}\right)^2\int_{t_0}^{t} S_0^2(\tau)d\tau\right)$$

$$+\frac{(jx_1)^2}{2}\left[\lambda a_2^{(1)}\int_{t_0}^{t}\left(S_1(\tau)+S_{12}(\tau)\right)d\tau + \kappa\left(a_1^{(1)}\right)^2\int_{t_0}^{t}\left(S_1(\tau)+S_{12}(\tau)\right)^2 d\tau\right]$$

$$+\frac{(jx_2)^2}{2}\left[\lambda a_2^{(2)}\int_{t_0}^{t}\left(S_2(\tau)+S_{12}(\tau)\right)d\tau + \kappa\left(a_1^{(2)}\right)^2\int_{t_0}^{t}\left(S_2(\tau)+S_{12}(\tau)\right)^2 d\tau\right] \qquad (12)$$

$$+j^2 x_0 x_1 a_1^{(0)} a_1^{(1)}\kappa\int_{t_0}^{t} S_0(\tau)\left(S_1(\tau)+S_{12}(\tau)\right)d\tau + j^2 x_0 x_2 a_1^{(0)} a_1^{(2)}\kappa\int_{t_0}^{t} S_0(\tau)\left(S_2(\tau)+S_{12}(\tau)\right)d\tau$$

$$+j^2 x_1 x_2 a_1^{(1)} a_1^{(2)}\left[\lambda\int_{t_0}^{t} S_{12}(\tau)d\tau + \kappa\int_{t_0}^{t}\left(S_1(\tau)+S_{12}(\tau)\right)\left(S_2(\tau)+S_{12}(\tau)\right)d\tau\right].$$

Substituting Eq. (12) in Eq. (10) and using substitutions inverse to Eqs. (6) and (8), we obtain the approximate equality for the characteristic function $\boldsymbol{h}\left(v_0,v_1,v_2,t\right)$ that coincides with the equality of theorem 2; thus, Theorem 2 has been proved.

Corollary. Setting $t = T$, $t_0 \to -\infty$, and taking into account Eq. (1), we obtain expressions for the asymptotic characteristic function of the stationary three-dimensional probability distribution function of the total amount of resources at each phase of the system which has the form of the Gaussian distribution with the mean vector

$$\boldsymbol{a} = N\lambda\begin{bmatrix} a_1^{(0)}b_0 & a_1^{(1)}b_1 & a_1^{(2)}b_2 \end{bmatrix}$$

and the covariance matrix

$$\boldsymbol{K} = N\begin{bmatrix} \lambda a_2^{(0)}b_0 + \kappa\left(a_1^{(0)}\right)^2\beta_0 & \kappa a_1^{(0)}a_1^{(1)}\beta_{01} & \kappa a_1^{(0)}a_1^{(2)}\beta_{02} \\ \kappa a_1^{(0)}a_1^{(1)}\beta_{01} & \lambda a_2^{(1)}b_1 + \kappa\left(a_1^{(1)}\right)^2\beta_1 & \lambda a_1^{(1)}a_1^{(2)}b_{12} + \kappa a_1^{(1)}a_1^{(2)}\beta_{12} \\ \kappa a_1^{(0)}a_1^{(2)}\beta_{02} & \lambda a_1^{(1)}a_1^{(2)}b_{12} + \kappa a_1^{(1)}a_1^{(2)}\beta_{12} & \lambda a_2^{(2)}b_2 + \kappa\left(a_1^{(2)}\right)^2\beta_2 \end{bmatrix},$$

where

$$b_0 = \int_0^\infty \left(1 - B_0(\tau)\right)d\tau, \beta_0 = \int_0^\infty \left(1 - B_0(\tau)\right)^2 d\tau,$$

$$b_1 = \int_0^\infty \left(B_0(\tau) - \left(B_1 * B_0\right)(\tau)\right)d\tau, \beta_1 = \int_0^\infty \left(B_0(\tau) - \left(B_1 * B_0\right)(\tau)\right)^2 d\tau,$$

$$b_2 = \int_0^\infty \left(B_0(\tau) - \left(B_2 * B_0\right)(\tau)\right)d\tau, \beta_2 = \int_0^\infty \left(B_0(\tau) - \left(B_2 * B_0\right)(\tau)\right)^2 d\tau,$$

$$\beta_{01} = \int_0^\infty \left(1 - B_0(\tau)\right)\left(B_0(\tau) - \left(B_1 * B_0\right)(\tau)\right)d\tau, \ \beta_{02} = \int_0^\infty \left(1 - B_0(\tau)\right)\left(B_0(\tau) - \left(B_2 * B_0\right)(\tau)\right)d\tau,$$

$$b_{12} = \int_0^\infty \left(B_0(\tau) - (B_1 * B_0)(\tau) - (B_2 * B_0)(\tau) + \int_0^\tau B_1(\tau - x)B_2(\tau - x)dB_0(x) \right)d\tau,$$

$$\beta_{12} = \int_0^\infty \left(B_0(\tau) - \left(B_1 * B_0\right)(\tau)\right)\left(B_0(\tau) - \left(B_2 * B_0\right)(\tau)\right)d\tau.$$

4. NUMERICAL ANALYSIS OF THE ACCURACY OF THE ASYMPTOTIC RESULTS

Let the MMPP arrivals be assigned by the following matrices:

TABLE 1. Kolmogorov's Distances

N	1	3	5	7	10	20	50	100
Δ_0	0.372	0.118	0.065	0.044	0.031	0.019	0.012	0.008
Δ_1	0.374	0.102	0.056	0.040	0.030	0.020	0.013	0.009
Δ_2	0.367	0.115	0.062	0.042	0.028	0.018	0.011	0.008
Δ_{12}	0.374	0.118	0.068	0.046	0.031	0.019	0.012	0.008

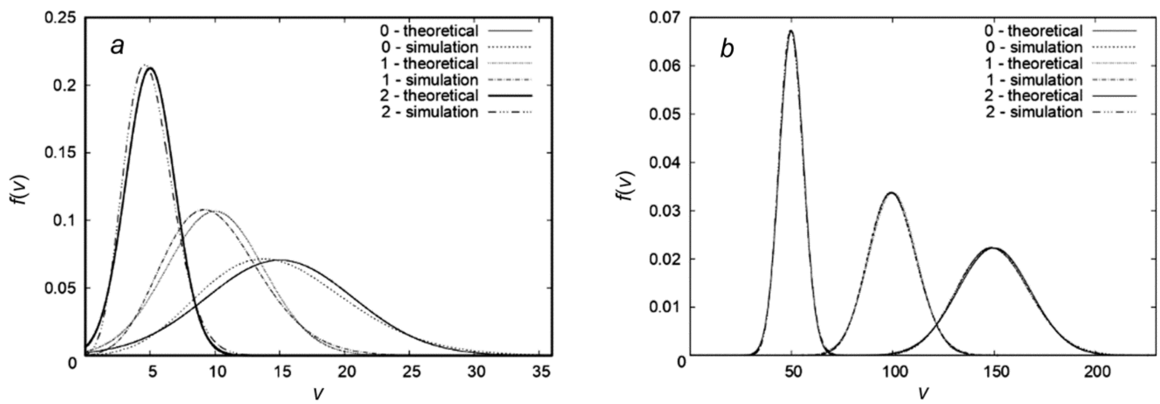

Fig. 1. Probability distribution function of the total amount of resources occupied in each system block (0, 1, 2) for $N = 10$ (a) and 100 (b).

$$\mathbf{\Lambda} = N\begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1.5 \end{bmatrix}, \ \boldsymbol{Q} = N\begin{bmatrix} -0.8 & 0.4 & 0.4 \\ 0.3 & -0.6 & 0.3 \\ 0.4 & 0.4 & -0.8 \end{bmatrix}$$

(as $N \to \infty$, we obtain the asymptotic condition of the increasing process intensity). The probability of the service time obeys a γ-distribution with the parameters $\alpha_0 = \beta_0 = 0.5$, $\alpha_1 = \beta_1 = 1.5$, and $\alpha_2 = \beta_2 = 2.5$. The resources are uniformly distributed in the following ranges: [0; 3] for the first phase, [0; 2] for the first block of the second phase; and [0; 1] for the second block of the second phase. We now carry out a series of experiments with increasing N values and compare the asymptotic distributions with empirical ones using the Kolmogorov distance $\Delta = \max_x |F(x) - A(x)|$, where $F(x)$ is the asymptotic Gaussian distribution function and $G(x)$ is the empirical distribution function.

Table 1 presents the Kolmogorov distances between the asymptotic and empirical distribution functions of total amount of resources occupied in three system blocks (Δ_0, Δ_1, Δ_2). The approximation accuracy increases with incoming process intensity N, which is also illustrated by Fig. 1. As demonstrated by values Δ_{12} from Table I and illustrated by Fig. 2, the same is true for the joint probability distribution function.

CONCLUSIONS

The resource RQS with an unlimited number of servers and customers copying at the second phase of the system and MMPP arrivals has been investigated. By means of the asymptotic analysis method, it was shown that the

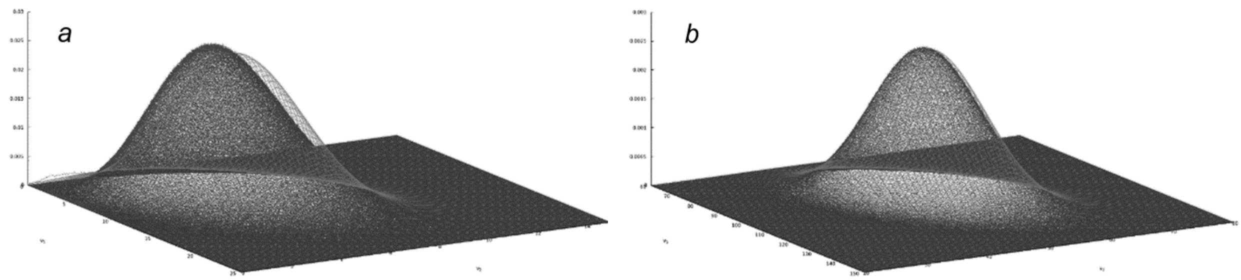

Fig. 2. Joint probability distribution function of the total amount of resources occupied at the second system phase of the first and second blocks for $N = 10$ (a) and 100 (b).

joint asymptotic distribution of the probability of total amount of resources occupied at each system phase converges to a three-dimensional Gaussian distribution under the asymptotic condition of increasing process intensity.

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