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Linear homeomorphic classification of spaces of continuous functions defined on S_A ☆

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ABSTRACT

For a subset A of the real line $\mathbb{R}$, modification of the Sorgenfrey line S_A is a topological space whose underlying points set is the reals $\mathbb{R}$ and whose topology id defined as follows: points from A are given the neighbourhoods of the right arrow while remaining points are given the neighbourhoods of the Sorgenfrey line $\mathbb{S}$ (or left arrow). A necessary and sufficient condition under which the space $C_p(S_A)$ is linearly homeomorphic to $C_p(\mathbb{S})$ is obtained.

1. Introduction

We use the following notation: $\mathbb{N}$ is the set of positive integers; $\mathbb{R}$ is the space of real numbers endowed with the usual Euclidean topology τ_E; $\mathbb{S}$ is the Sorgenfrey line (also known as the "arrow" or the "left arrow" space), that is, the set of real numbers with the topology generated by the base $\{(a, b] : a, b \in \mathbb{R}, a < b\}$. The topology of the space $\mathbb{S}$ is denoted by τ_0. The symbol $\mathbb{S}_{\rightarrow}$ denotes the set of real numbers with the topology generated by the base $\{[a, b) : a, b \in \mathbb{R}, a < b\}$. Obviously, $\mathbb{S}$ is homeomorphic to $\mathbb{S}_{\rightarrow}$. The topological space $\mathbb{S}_{\rightarrow}$ is called the "right arrow", the topology of this space is denoted by τ_S.

Let $A \subset \mathbb{R}$. The symbol S_A denotes a topological space in which the base of neighbourhoods of the point x is defined as follows:

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$$B_x = \{(x-\varepsilon, x] : \varepsilon > 0\}, \text{ if } x \in \mathbb{R} \setminus A;$$
$$B_x = \{[x, x+\varepsilon) : \varepsilon > 0\}, \text{ if } x \in A.$$

The space S_A is called the modification of the Sorgenfrey line, and the topology on this space is denoted by τ_A. It is easy to show that the modification of the Sorgenfrey line S_A, like the Sorgenfrey line $\mathbb{S}$, is perfectly normal, hereditarily Lindelöf, hereditarily separable, hereditarily Baire space [3]. Given any set $X \subset \mathbb{R}$, by $\overline{X}$ we denote its closure in the space $(\mathbb{R}, \tau_E)$. Similarly, for any set $X \subset Y$, $\operatorname{int}_Y X$ denotes the interior of X in the space Y. If $\{A_i\}_{i\in J}$ is a family of pairwise disjoint sets, then we denote the union of these sets by $\bigsqcup\limits_{i\in J} A_i$ rather than $\bigcup\limits_{i\in J} A_i$.

The following asseration was proved in [3].

Theorem 1. *For any subset A of real line, the following conditions are equivalent:*

(1) the spaces S_A and $\mathbb{S}$ are homeomorphic;
(2) there exists no set $\emptyset \neq V \subset A$ which is closed in A and satisfies the condition $\overline{V} = \overline{\overline{V} \setminus V}$;
(3) the set A is both F_σ and G_δ in $\mathbb{R}$.

All topological spaces are assumed to be completely regular. By $C_p(X)$ we denote the set of all continuous functions from X to $\mathbb{R}$ endowed with the topology of pointwise convergence.

The main result of this paper is the following theorem.

Theorem 2. *For any set $A \subset \mathbb{R}$, the following conditions are equivalent:*

(i) the spaces $\mathbb{S}$ and S_A are homeomorphic;
(ii) the spaces $C_p(\mathbb{S})$ and $C_p(S_A)$ are linearly homeomorphic.

2. Proof of the main result

The standard base of the space $C_p(X)$ consists of sets of the form $W(x_1, \ldots, x_n, U_1, \ldots, U_n)$, where $x_1, \ldots, x_n \in X$, $U_1, \ldots, U_n$ are open sets in $\mathbb{R}$, $n \in \mathbb{N}$, and

$$W(x_1, \ldots, x_n, U_1, \ldots, U_n) = \{f \in C(X) \colon f(x_i) \in U_i, i = \overline{1, n}\}.$$

For each $x \in X$ we consider the functional $\delta_x \colon C_p(X) \to \mathbb{R}$ defined by $\delta_x(f) = f(x)$. It is well known that δ_x is a linear continuous functional. In what follows, we identify the functional δ_x with x.

In the linear space $C_pC_p(X)$ we consider the subspace of linear functionals

$$L_p(X) = \left\{ \sum_{i=1}^{n} \alpha_i x_i \in C_pC_p(X) \colon x_1, \ldots, x_n \in X, \alpha_1, \ldots, \alpha_n \in \mathbb{R}, n \in \mathbb{N} \right\},$$

algebraically generated by the set X. For $f \in L_p(X)$, let $\mathfrak{l}(f) = n$ if $f = \sum\limits_{i=1}^{n} \alpha_i x_i$, where $\alpha_i \neq 0$ for each $i = \overline{1, n}$, and $\mathfrak{l}(f) = 0$ if $f \equiv 0$. It is known [1], that X is a closed subspace of the space $L_p(X)$, $L_p(X) = (C_p(X))^*$, $C_p(X)$ and $C_p(Y)$ are linearly homeomorphic if and only if $L_p(X)$ and $L_p(Y)$ are linearly homeomorphic.

Let X and Y be topological spaces and let $T \colon C_p(Y) \to C_p(X)$ be a linear continuous map. The conjugate map $T^* \colon L_p(X) \to L_p(Y)$ is defined by $T^*(F) = F \circ T$ for $F \in L_p(X)$. For each point $x \in X$, $T^*\delta_x = T^*x \in$

$L_p(Y)$ holds and, therefore, $T^*x = \sum_{i=1}^{n} \alpha_i y_i$, where $\alpha_i \neq 0$ for all $i = \overline{1,n}$. The subset $\{y_1, \ldots, y_n\}$ is called the support of x and it is denoted by $supp(x)$. If $T^*y \equiv 0$, then $supp(y) = \emptyset$. The length of the support is the value of $|supp(x)|$. For any function $f \in C_p(Y)$ we have the following equality: $T(f)(x) = (\delta_x \circ T)(f) = (T^*\delta_x)(f) = \sum_{i=1}^{n} \alpha_i f(y_i)$.

Let $T\colon C_p(Y) \to C_p(X)$ be a continuous map and take $x \in X$ such that $supp(x) \neq \emptyset$. Then the multi-valued map $\varphi\colon X \to 2^Y$ defined by $\varphi(x) = supp(x)$ is lower semicontinuous [4].

We set $L_p^n(X) = \{z \in L_p(X); \mathfrak{l}(z) \leq n\}$ and $M_p^n(X) = \{z \in L_p(X); \mathfrak{l}(z) = n\} = L_p^n(X) \setminus L_p^{n-1}(X)$. It is known [1], that $L_p^n(X)$ is closed in $L_p(X)$ for each $n \in \mathbb{N}$ and the following theorem about the base of the point $y \in M_p^n(X)$ in $L_p^n(X)$ holds.

Theorem 3. *Let X be a topological space and $y = \sum_{i=1}^{n} \alpha_i x_i \in M_p^n(X)$. Then the family of the sets $O(V_1, \ldots, V_n, \varepsilon) = \{y' = \sum_{i=1}^{n} \alpha_i' x_i' : |\alpha_i' - \alpha_i| < \varepsilon, (\alpha_i - \varepsilon, \alpha_i + \varepsilon) \in \mathbb{R} \setminus \{0\}, x_i' \in V_i, i = \overline{1,n}\}$ is the base of the point y in the space $L_p^n(X)$, where V_i are open, pairwise disjoint sets in X and $\varepsilon > 0$.*

Recall that a topological space X is called perfect if any open subset of X has type F_σ in X. It is easy to see that a difference of two closed subsets of a perfect space is an F_σ-set.

Lemma 1. *Let X and Y be topological spaces, X be a perfect space, $T\colon C_p(Y) \to C_p(X)$ be a linear continuous map, $X_n = \{x \in X : |supp(x)| = n\}$, where $n \in \mathbb{N}$, and $X_0 = \{x \in X : T^*x = 0\}$. Then $T^*(X_n) \subset M_p^n(Y)$ and the set X_n is an F_σ in X.*

Proof. Let us show that for each $n \in \mathbb{N}$, the sets $Z_n = X \setminus \left(\bigsqcup_{i=0}^{n} X_i \right)$ are open in X. It is clear that X_0 is closed in X and, therefore, Z_0 is open in X. Let $x_0 \in Z_n$. Then $T^*(x_0) = \sum_{i=1}^{m} \alpha_i(x_0) y_i(x_0)$, where $m > n$. Hence, for each $i = \overline{1,m}$ there exist pairwise disjoint neighbourhoods U_i of points $y_i(x_0)$. The multi-valued map $\varphi = supp$ maps x_0 to the set $\{y_i(x_0)\}_{i=1}^m$. Since this map φ is lower semicontinuous, it follows that there exists a neighbourhood O_{x_0} of point x_0 such that for each point $x \in O_{x_0}$ we have $\varphi(x) \cap U_i \neq \emptyset$ for all $i = \overline{1,m}$. Thus, $O_{x_0} \subset Z_n$ and, therefore, Z_n is open in X for each $n \in \mathbb{N} \cup \{0\}$ and $\bigsqcup_{i=0}^{n} X_i$ are closed for each $n \in \mathbb{N} \cup \{0\}$.

Because X is a perfect space, $X_n = \left(\bigsqcup_{i=0}^{n} X_i \right) \cap Z_{n-1}$ is an F_σ-set in X. □

Corollary 1. *Let X and Y be topological spaces, X be a perfect space and $T\colon C_p(Y) \to C_p(X)$ be a linear continuous map. If the set V is closed in X, then $V = \bigsqcup_{n=1}^{\infty} \bigcup_{i=1}^{\infty} F_i^n$, where F_i^n is closed in X and $T^*(F_i^n) \subset M_p^n(Y)$.*

Proof. According to the previous lemma, $X_n = \bigcup_{i=1}^{\infty} \Phi_i^n$, where the sets Φ_i^n are closed in X. Hence, the sets $F_i^n = V \cap \Phi_i^n$ are closed in X for $n, i \in \mathbb{N}$ and $V = \bigsqcup_{n=1}^{\infty} \bigcup_{i=1}^{\infty} F_i^n$. □

Lemma 2. *Let X and Y be topological spaces, $T\colon C_p(Y) \to C_p(X)$ be a linear continuous map, and $X_n = \{x \in X : |supp(x)| = n\}$, where $n \in \mathbb{N}$, i.e., $T^*(x) = \sum_{i=1}^{n} \alpha_i(x) y_i$ for each $x \in X_n$. Then the sets $X_n^k = \{x \in X_n; |\alpha_i(x)| \le k, i = \overline{1,n}\}$ are closed in X_n for each $k \in \mathbb{N}$.*

Proof. Let $x_0 \in X_n$ be the limit point for X_n^k. Since $x_0 \in X_n$, it follows that $T^* x_0 = \sum_{i=1}^{n} \alpha_i(x_0) y_i^0$. Suppose that $|\alpha_j(x_0)| > k$ for some $j = \overline{1,n}$. Consider a neighbourhood $O(U_1, \dots, U_n, \varepsilon)$ of the point $T^* x_0$ in $L_p^n(Y)$ for $\varepsilon < |\alpha_j(x_0)| - k$. Because the map T^* is continuous, there exists a neighbourhood U_{x_0} such that $T^* U_{x_0} \subset O(U_1, \dots, U_n, \varepsilon)$. Hence $|\alpha_i(x_0) - \alpha_i(x)| < \varepsilon$ for $x \in X_n^k \cap U_{x_0}$ and all $i = \overline{1,n}$. Due $\varepsilon > |\alpha_j(x_0) - \alpha_j(x)| \ge |\alpha_j(x_0)| - |\alpha_j(x)| \ge |\alpha_j(x_0)| - k$ we obtain a contradiction. □

The idea of the proof of the following lemma is borrowed from [2] for the space $\mathbb{S}$.

Lemma 3. *Let X be a subset of S_A such that eighter $X \subset A$ or $X \subset S_A \setminus A$ and let the map $f\colon X \to \mathbb{S}$ be continuous. Then X can be represented as a countable union of closed subsets of X, on each of which the map f is non-increasing or non-decreasing, respectively.*

Proof. Let $X \subset A$. For each $n \in \mathbb{N}$ we set

$$F_n = \left\{ x \in X; x \le y < x + \frac{1}{n} \Rightarrow f(y) \le f(x) \right\}.$$

Let us prove that the set F_n is closed in X. Indeed, if $x_0 \in X$ is a limit point of F_n, then there exists a sequence $\{x_i\}_{i=1}^{\infty} \subset F_n$ such that $x_{i+1} < x_i$ for every $i \in \mathbb{N}$ and $\lim_{i \to \infty} x_i = x_0$. For each point $y \in \left(x_0, x_0 + \frac{1}{n} \right)$, there exists a number $i_0 \in \mathbb{N}$ such that $x_i < y$ for all $i > i_0$. Since $x_i \in F_n$ and $x_i \le y < x_0 + \frac{1}{n} < x_i + \frac{1}{n}$, it follows that $f(y) \le f(x_i)$ for each $i > i_0$ and, therefore, $f(y) \le f(x_0)$. This proves the closedness of F_n in X. It is easy to see that $X = \bigcup_{n=1}^{\infty} F_n$. Now, let $\{I_n^k\}_{k=1}^{\infty}$ be a countable family of closed intervals shorter than $\frac{1}{n}$ which forms a cover of the set F_n. Then $F_n^k = F_n \cap I_n^k$ are closed subsets of X, and $X = \bigcup_{n,k \in \mathbb{N}} F_n^k$. Let us show that the map f is non-increases on each of the sets F_n^k. Take $x, y \in F_n^k$, $x < y$. Since $x, y \in I_n^k$, it follows that $x < y < x + \frac{1}{n}$, and since $x \in F_n$, it follows that $f(y) \le f(x)$.

In the case $X \subset S_A \setminus A$, we consider the subsets

$$F_n = \left\{ x \in X; x - \frac{1}{n} < y \le x \Rightarrow f(y) \le f(x) \right\}$$

of X and, as in the previous case, prove that the map f is non-decreasing on each F_n^k in the same way as in the case considered above. □

Let $T\colon C_p(\mathbb{S}) \to C_p(S_A)$ be a linear homeomorphism. Because $\mathbb{R}$ is a linearly ordered space, for each point $t \in S_A$ and its support $supp(t) = \{q_1, \dots, q_n\}$ we assume $q_1 < \ldots < q_n$, i.e. the numbering of the points in the support corresponds to the natural order in the set of real numbers.

For each $n \in \mathbb{N}$ we define the map $\pi_j\colon M_p^n(\mathbb{S}) \to \mathbb{S}$ by $\pi_j \left(\sum_{i=1}^{n} \alpha_i q_i \right) = q_j$. By Theorem 3, it is easy to see that the map π_j is continuous on $M_p^n(\mathbb{S})$ for each $j = \overline{1,n}$.

Lemma 4. *Let $T\colon C_p(\mathbb{S}) \to C_p(S_A)$ be a linear homeomorphism, a set $B \subset S_A$ be such that $|B| \geq \aleph_0$ and $T^*(B) \subset M_p^n(\mathbb{S})$. Then there exists a number $1 \leq j \leq n$ such that $|(\pi_j \circ T^*)(B)| \geq \aleph_0$.*

Proof. Suppose the contrary, i.e. for each $j = \overline{1,n}$ the set $(\pi_j \circ T^*)(B)$ is finite. Since the set $(\pi_1 \circ T^*)(B)$ is finite, there exists a set $B_1 \subset B$ such that $|B_1| \geq \aleph_0$ and $|(\pi_1 \circ T^*)(B_1)| = 1$, i.e., there exists a point $q_1^0 \in \mathbb{S}$ such that for each $b \in B_1$ we have $T^*(b) = \alpha_1(b)q_1^0 + \sum_{i=2}^{n} \alpha_i(b)q_i(b)$. Repeating a similar reasoning for each $j \leq n$ we get a sets $B_1, \dots, B_n$ such that $B_{j+1} \subset B_j$, $|B_j| \geq \aleph_0$, $|(\pi_j \circ T^*)(B_j)| = 1$ for all $j = \overline{1,n}$. This means that there are points $q_j^0 \in \mathbb{S}$, $q_1^0 < \ldots < q_n^0$, such that for any point $b \in B_n$ we have $T^*(b) = \sum_{i=1}^{n} \alpha_i(b)q_i^0$. Because B_n is infinite, there are pairwise different points $\{b_1, \dots, b_{n+1}\} \subset B_n$. Since T^* is a linear bijection and the set $\{b_1, \dots, b_{n+1}\} \subset L_p(S_A)$ is linearly independent, it follows that the set $\{T^*(b_1), \dots, T^*(b_{n+1})\} \subset L_p(\mathbb{S})$ is also linearly independent. Then the vectors $\alpha(b_k) = (\alpha_1(b_k), \dots, \alpha_n(b_k)) \in \mathbb{R}^n$ for $k = \overline{1, n+1}$ are linearly independent, which is impossible. □

Now, let us prove the Theorem 2.

Proof. The implication *(i)* ⇒ *(ii)* is obvious.

(ii) ⇒ *(i)*. Let $T\colon C_p(\mathbb{S}) \to C_p(S_A)$ be a linear homeomorphism. Then the map $T^*\colon L_p(S_A) \to L_p(\mathbb{S})$ is also a linear homeomorphism. Since S_A is closed in $L_p(S_A)$, $T^*(S_A)$ is closed in $L_p(\mathbb{S})$. Now, suppose that S_A is not homeomorphic to $\mathbb{S}$. Then, by Theorem 1 there exists a set $V \subset \mathbb{R}$ closed in A such that $\overline{V} = \overline{\overline{V} \setminus V}$. Let $\overline{V}' = \overline{V} \setminus E$, where E is the set of τ_A-isolated points of $\overline{V}$. Because $\overline{V}'$ is closed in S_A, by Corollary 1, $\overline{V}' = \bigsqcup_{n=1}^{\infty} \bigcup_{m=1}^{\infty} F_m^n$, where F_m^n are closed in S_A and $T^*(F_m^n) \subset M_p^n(\mathbb{S})$.

Two mutually exclusive cases are possible:

1. Either there exists an interval (a,b) such that $(a,b) \cap \overline{V}' \subset V$, or there exists an interval (a,b) such that $(a,b) \cap \overline{V}' \subset \overline{V} \setminus V$.
2. For each interval (a,b), $B = (a,b) \cap \overline{V}' \cap V \neq \emptyset$ if and only if $C = (a,b) \cap \overline{V}' \cap (\overline{V} \setminus V) \neq \emptyset$.

Case 1. Since the set (a,b) is open in S_A and $\overline{V}'$ is Baire (because $\overline{V}'$ is closed in the hereditary Baire space S_A), then the set $(a,b) \cap \overline{V}'$ is Baire. Then $(a,b) \cap F_m^n$ are closed in $(a,b) \cap \overline{V}'$ for $n, m \in \mathbb{N}$. Therefore, there are $n, m \in \mathbb{N}$ and an interval $I \subset (a,b)$ such that $I \cap \overline{V}' \subset F_m^n$ and, therefore, $T^*(I \cap \overline{V}') \subset M_p^n(\mathbb{S})$.

Let the interval (a,b) be such that $(a,b) \cap \overline{V}' \subset V$.

Because $I \cap \overline{V}'$ is Baire, then by Lemma 3, there exists an interval $I_1 \subset I$ such that $\pi_1 \circ T^*|_{I_1 \cap \overline{V}'}$ is non-increasing. For the same reasons, there exists an interval $I_2 \subset I_1$ such that $\pi_2 \circ T^*|_{I_2 \cap \overline{V}'}$ is non-increasing. Continuing this process for each $j = \overline{3,n}$ we get an interval $I_n \subset I_{n-1}$ such that the map $\pi_j \circ T^*|_{I_n \cap \overline{V}'}$ are non-increasing for each $j = \overline{1,n}$. Let

$$R_k = \left\{ t \in I_1 \cap \overline{V}'; T^*(t) = \sum_{i=1}^{n} \alpha_i(t)q_i(t), |\alpha_i(t)| \leq k, i = \overline{1,n} \right\}.$$

The sets R_k are closed in X_n by Lemma 2 and $I_1 \cap \overline{V}' = \bigcup_{k=1}^{\infty} R_k$. Since $I_1 \cap \overline{V}'$ is Baire, there are $k \in \mathbb{N}$ and the interval I_2 such that $I_2 \cap \overline{V}' \subset R_k$.

Since $I_2 \cap \overline{V} \neq \emptyset$, it follows that there exists $x_0, x_1 \in I_2 \cap (\overline{V} \setminus V) \subset \overline{V} \setminus A$, $x_0 < x_1$. The last inclusion is true due to the closedness of the set V in A. Because $I_2 \cap \overline{V}' \subset V$, then $x_0, x_1 \in E$, i.e., they are isolated in τ_A. This means that $(x_0 - \varepsilon, x_0] \cap \overline{V} = \{x_0\}$ and $(x_1 - \varepsilon, x_1] \cap \overline{V} = \{x_1\}$ holds for a sufficiently small $\varepsilon > 0$. Since all points in (V, τ_E) are condensation points, there exists a point $t_0 \in I_2 \cap \overline{V}'$ such that $x_0 < t_0 < x_1$ and a decreasing sequence $\{t_l\}_{l=1}^{\infty} \subset I_2 \cap \overline{V}'$, $t_l > x_1$, converging in the Euclidean topology to the point x_1. This means that the sequence $\{t_l\}_{l=1}^{\infty}$ in S_A is divergent. Because the points $t_l \in R_k$, then $T^*(t_l) = \sum_{i=1}^{n} \alpha_i(t_l) q_i(t_l)$ and $|\alpha_i(t_l)| \le k$ for all $i, l \in \mathbb{N}$. Since the map $(\pi_j \circ T^*)|_{I_1 \cap \overline{V}'}$ is non-increasing and the sequence $\{t_l\}_{l=1}^{\infty}$ is decreasing, it follows that the sequences $\{q_j(t_l)\}_{l=1}^{\infty}$ are increasing and $q_j(t_l) \le q_j(t_0)$ for all $j = \overline{1,n}$ and $l \in \mathbb{N}$. Therefore, the sequences $\{q_j(t_l)\}_{l=1}^{\infty}$ converge to points q_j in $\mathbb{S}$ for each $j = \overline{1,n}$. Due to the fact that $|\alpha_1(t_l)| \le k$, there is a subsequence $\left\{t_l^{(1)}\right\}_{l=1}^{\infty}$ such that there exists $\alpha_1 = \lim_{l\to\infty} \alpha_1\left(t_l^{(1)}\right)$. Next, choose a subsequence $\{t_l^{(2)}\}_{l=1}^{\infty} \subset \{t_l^{(1)}\}_{l=1}^{\infty}$ such that $\lim_{l\to\infty} \alpha_2(t_l^{(2)}) = \alpha_2$. At the n-th step, we get the subsequence $\{t_l^{(n)}\}_{l=1}^{\infty}$, for which there are $\lim_{l\to\infty} \alpha_i(t_l^{(n)}) = \alpha_i$ for each $i = \overline{1,n}$. It is easy to see that the sequence $\left\{T^*\left(t_l^{(n)}\right)\right\}_{l=1}^{\infty}$ converges to the point $\sum_{i=1}^{n} \alpha_i q_i$. Thus, we obtain a contradiction to the fact that the map T^* is a homeomorphism.

The case when there exists an interval (a, b) such that $(a, b) \cap \overline{V}' \subset \overline{V} \setminus V$ is proved similarly, considering the increasing sequence $\{t_l\}_{l=1}^{\infty} \subset I_2 \cap \overline{V}'$, $t_l < x_0$, converging on the Euclidean topology to the point $x_0 \in I_2 \cap V$.

Case 2. Since S_A is hereditarily Baire, the set $\overline{V}' \subset S_A$ is Baire, and therefore there are $n, m \in \mathbb{N}$ and the interval $I = (p, q)$ such that $I \cap \overline{V}' \subset F_m^n$ and $T^*(I \cap \overline{V}') \subset M_p^n(\mathbb{S})$.

Because I is open in S_A and $\overline{V}'$ is Baire, $I \cap \overline{V}'$ is Baire by [5, p. 228]. In this case, it follows from the assertion of case 2 that $I \cap \overline{V}' = B \sqcup C$, where $B, C \neq \emptyset$. Therefore, at least one of the sets B or C is of the second category.

Let B be of the second category. By Lemma 3, $B = \bigcup_{k=1}^{\infty} B_k$, where B_k are closed subsets of B and the maps $(\pi_j \circ T^*)|_{B_k}$ are non-increasing for each $j = \overline{1,n}$. Since B is of the second category, it follows that there exists a number $k \in \mathbb{N}$ and an interval I_1 such that $\emptyset \neq I_1 \cap B \subset B_k$. Let $t \in I_1 \cap C$. Then there is an increasing sequence $\{t_l\}_{l=1}^{\infty}$, $t_l \in I_1 \cap B$, converging to t in S_A. By Lemma 4, there exists a number $j = \overline{1,n}$ such that the set $\{(\pi_j \circ T^*)(t_l)\}_{l=1}^{\infty}$ is infinite. Since the mapping $\pi_j \circ T^*$ is non-increasing and the sequence $\{t_l\}_{l=1}^{\infty}$ is increasing, it follows that it turns into a decreasing sequence $\{(\pi_j \circ T^*)(t_l)\}_{l=1}^{\infty}$, i.e., it diverges in $\mathbb{S}$. Thus, we obtain a contradiction, because the map $\pi_j \circ T^*$ is continuous.

The case when the set C is of the second category is proved similarly. □

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