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On convergence of Newton's method for the characteristic equation of 4th order pairwise comparison matrix in AHP

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On convergence of Newton's method for the characteristic equation of 4th order pairwise comparison matrix in AHP

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In this article, we apply Newton's method to the characteristic polynomial of 4th order pairwise comparison matrices. We will see that convergent limit of the sequence generated by the iteration of Newton's method depends on the choice of an initial value. We seek the viability of possible choice of the initial value corresponding to the consistency of the pairwise comparison.

Key words:AHP, pairwise comparison matrix, Newton's method, convergence, first-order condition.

1 Introduction

In AHP(Analytic Hierarchy Process), Saaty employed eigenvalue method to estimate the priority of objectives [6]. Especially, the maximum eigenvalue of a pairwise comparison matrix plays a central role. By simple transformation of the maximum eigenvalue into so-called consistency index, denoted by CI, AHP obtained wide popularity in the area of multiple criteria decision making [2, 4, 12]. This is because that Saaty's consistency criterion of CI ≤ 0.1 is simple and easy to use.

In our prior study [8], we considered 3rd order pairwise comparison matrix in AHP. We focused on the 3rd order matrix for a reason why the contradictory revelation of decision maker where $A > B, B > C$ but $C > A$ is a source of inconsistency of pairwise comparison. To calculate the maximum eigenvalue of 3rd order matrix, we applied Newton's method [5] to the characteristic equations. Newton's method has an applicability in the context of AHP. Indeed in [1], Newton's method was used to find the minimum value of the Perron eigenvalue of incomplete pairwise comparison matrices. With the aid of the favorable properties of the 3rd order pairwise comparison matrix, we verified that the sequence generated by Newton's method always converges to the maximum eigenvalue if we set the initial value of the iteration equal to 3 which is the order dimension of the matrix [8].

One of the most favorable properties of 3rd order pairwise comparison matrix is the fact that the solution of the characteristic equation is unique [7, 8]. When we intend to generalize to 4th order matrix, we immediately face to the difficulty. The number of real-numbered solutions becomes two because the order of the characteristic equation is four and other two solutions become complex-numbers [9]. So the limit of the generated sequence by Newton's method becomes to depend on the initial value of the iteration. We first considered that the initial value is appropriate to be set 4, because the maximum eigenvalue of n-th order pairwise comparison matrix is proven theoretically to be greater than n [2, 4, 6, 12]. The example we indicate below(Section 3.2) show that this intuition is false.

In this paper, we seek the viability of possible choice of the initial value corresponding to the consistency of the pairwise comparison. If A is near consistent, the number of the dimension 4 is appropriate as an initial value(Section 3.1). If A is hard inconsistent, the situation changes. As we noted above, the generated sequence may converge to the minimum eigenvalue of A if one choose the initial value equals to 4.

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Here we change our point of view. As we will see in Section 2, the characteristic polynomial takes its global minimum at unique point. This point can be calculated easily by Newton's method. However the global minimum point cannot be taken as a candidate of the initial value of Newton's method, because it satisfies the first-order condition. So if we add some perturbation $\alpha > 0$ to this point for taking as an initial value, then we observe that the generated sequence converges to the maximum eigenvalue (Section 3.2).

This paper is organized as follows. In Section 2, we consider the minimum solution of the characteristic polynomial. If we apply Newton's method, we easily obtain the minimum point. In Section 3, we consider applicability of Newton's method to obtain the maximum eigenvalue. We observe two cases of examples. One is the pairwise comparison matrix is near consistent i.e. $\mathrm{CI} \leq 0.1$. The other is the pairwise comparison matrix is hard inconsistent where CI is far from Saaty's criterion 0.1. As a conclusion, we mention the validity of use of 4 as an initial value based on the fact that most experts decides so that the pairwise comparison is near consistent.

2 The first-order condition for the characteristic polynomial

In this paper, we treat 4th order pairwise comparison matrix:

$$A = \begin{pmatrix} 1 & a_{12} & a_{13} & a_{14} \\ \frac{1}{a_{12}} & 1 & a_{23} & a_{24} \\ \frac{1}{a_{13}} & \frac{1}{a_{23}} & 1 & \frac{1}{a_{34}} \\ \frac{1}{a_{14}} & \frac{1}{a_{24}} & \frac{1}{a_{34}} & 1 \end{pmatrix}.$$

Its characteristic polynomial has the following form:

$$P_A(\lambda) = \lambda^4 - 4\lambda^3 + c_3\lambda + \det A,$$

where

$$c_3 = \sum_{i<j<k} \Big(2 - \big(\frac{a_{ij}a_{jk}}{a_{ik}} + \frac{a_{ik}}{a_{ij}a_{jk}}\big)\Big).$$

See [2, 4, 11]. From the well-known inequality between the arithmetic mean and the geometric mean, we have

$$c_3 \leq 0.$$

The pairwise comparison matrix A is said to be **consistent** provided that $a_{ik}a_{kj} = a_{ij}$ for all i, k, j. Concept of consistency relates Saaty's consistency index defined by

$$\mathrm{CI} = \frac{\lambda_{\max} - n}{n - 1},$$

where $\lambda_{\max}$ stands for the maximum eigenvalue of n-th order pairwise comparison matrix. It is well known that $\mathrm{CI} = 0$ is equivalent to the consistency of the matrix [2, 4, 12]. It has been shown that consistency of A is also equivalent to $c_3 = 0$ [11]. Below, we treat inconsistent case where $c_3 < 0$ holds. By taking the first-order and second-order derivatives of $P_A(\lambda)$, we have

$$\begin{aligned} P_A'(\lambda) &= 4\lambda^3 - 12\lambda^2 + c_3, \\ P_A''(\lambda) &= 12\lambda^2 - 24\lambda = 12\lambda(\lambda - 2). \end{aligned}$$

Thus $P_A'(\lambda)$ takes its local maximum at $\lambda = 0$ and local minimum at $\lambda = 2$. Since

$$\begin{aligned} P_A'(0) &= c_3 < 0, \\ P_A'(2) &= -16 + c_3 < 0, \\ \lim_{\lambda \to -\infty} P_A' &= -\infty, \\ \lim_{\lambda \to \infty} P_A'(\lambda) &= \infty, \end{aligned}$$

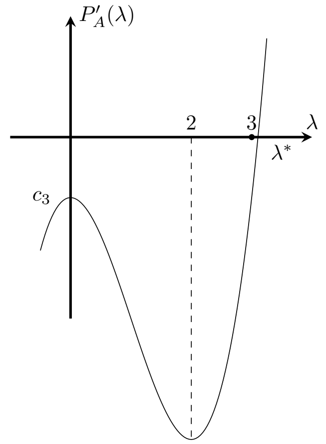

Figure 1: Graph of $P_A'(\lambda)$ in inconsistent case

the equation $P_A'(\lambda) = 0$ has a unique solution, say $\lambda^* > 0$. We indicate the shapes of the graph of $P_A'(\lambda)$ in Figure 1.

If the matrix is inconsistent, the first-order condition for the characteristic polynomial holds on the only unique point. This determines the shapes of the graph of $P_A(\lambda)$. See Figures 2 and 3. It takes global minimum at $\lambda^* > 0$. In the area of the left-hand side of λ^*, the graph is monotone decreasing. In the area of the right-hand side of λ^*, the graph is monotone increasing. We note that if the matrix is consistent, the point $\lambda = 0$ is a stationary point.

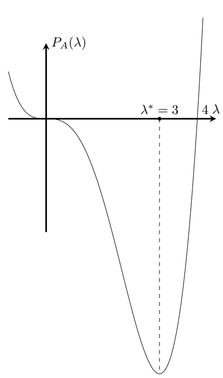

Figure 2: Graph of $P_A(\lambda)$ in consistent case

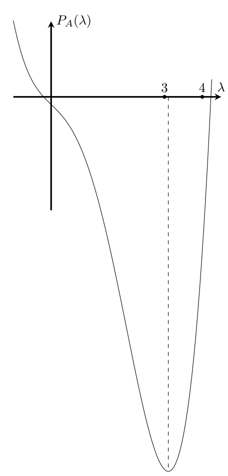

Figure 3: Graph of $P_A(\lambda)$ in near consistent case

The inequality $\lambda^* > 3$ also follows from

$$4{\lambda^*}^3 - 12{\lambda^*}^2 = 4{\lambda^*}^2(\lambda^* - 3) = -c_3 > 0.$$

We generate a sequence by Newton's method as follows:

- $\lambda_0 = 3$
- $\lambda_{n+1} = \lambda_n - \dfrac{P_A'(\lambda_n)}{P_A''(\lambda_n)}$

As we see immediately below, this sequence converges to λ^* where the first-order condition $P'_A(\lambda^*) = 0$ holds.

Lemma 1 *For all $n \geq 1$, we have $\lambda_n > \lambda^*$.*

Proof. We prove by induction. Set $n = 1$. $\lambda_1 = 3 - \frac{c_3}{36} > 3$ and $P'_A(\lambda_1) = c_3^2\Big(\frac{1}{18} - \frac{4c_3}{36^3}\Big) > 0$. From the monotonicity of $P'_A(\lambda)$, we have $\lambda_1 > \lambda^*$.

Assume $\lambda_n > \lambda^*$ for the induction. From this assumption, we can take $\lambda^* \leq \lambda < \lambda_n$. We can deduce $P'''_A(\lambda) = 12(\lambda - 1) > 0$, hence $P''_A(\lambda)$ is monotone increasing for $\lambda > 1$. Thus we have

$$P''_A(\lambda) < P''_A(\lambda_n).$$

This implies

$$\int_{\lambda^*}^{\lambda_n} P''_A(\lambda) d\lambda < \int_{\lambda^*}^{\lambda_n} P''_A(\lambda_n) d\lambda,$$

and

$$P'_A(\lambda_n) - P'_A(\lambda^*) \quad < \quad P''_A(\lambda_n)(\lambda_n - \lambda^*). \tag{1}$$

Taking account into the fact that $P'_A(\lambda^*) = 0$, the inequality (1) says that

$$\lambda^* \leq \lambda_n - \frac{P'_A(\lambda_n)}{P''_A(\lambda_n)} = \lambda_{n+1}.$$

//

Lemma 2 *The generated sequence λ_n is monotone decreasing for $n \geq 1$.*

Proof. From lemma 1, we have $3 < \lambda^* < \lambda_n$. Since $P'_A(\lambda)$ is monotone increasing for $3 < \lambda$, we have $P'_A(\lambda_n) > P'_A(\lambda^*) = 0$. It is obvious that $P''_A(\lambda_n) > 0$. Thus the sequence λ_n is monotone decreasing. //

From Lemmas 1 and 2, the sequence is monotone decreasing and bounded below. So it converges [3].

Theorem 1 *The sequence λ_n converge to λ^*.*

Proof. From lemmas 1 and 2, the sequence is monotone decreasing and bounded below. So it converges to, say $\bar{\lambda} \geq \lambda^*$. Taking the limit of the iteration in

$$\lambda_{n+1} = \lambda_n - \frac{P'_A(\lambda_n)}{{P_A}''(\lambda_n)},$$

we have

$$\bar{\lambda} = \bar{\lambda} - \frac{P'_A(\bar{\lambda})}{{P_A}''(\bar{\lambda})}.$$

Since $\bar{\lambda} \geq \lambda^* > 3$, the inequality ${P_A}''(\bar{\lambda}) > 0$ holds. So we have $P'_A(\bar{\lambda}) = 0$. Because the equation $P'_A(\lambda) = 0$ has a unique solution for $\lambda > 3$, $\bar{\lambda}$ is identical to λ^*. //

3 Examples and convergence of Newton's method

3.1 Near consistent case

In the context of AHP the maximum eigenvalue of pairwise comparison matrix plays an important role as consistency index [2, 4, 6, 12]. We denote the maximum eigenvalue by $\lambda_{\max}$. Here we employ the Newton method again. The iteration is as follows.

- Set λ_0,
- $\lambda_{n+1} = \lambda_n - \dfrac{P_A(\lambda_n)}{P'_A(\lambda_n)}$.

First we consider the following pairwise comparison matrix.

$$A = \begin{pmatrix} 1 & 3 & 5 & 7 \\ \frac{1}{3} & 1 & 5 & 7 \\ \frac{1}{5} & \frac{1}{5} & 1 & 3 \\ \frac{1}{7} & \frac{1}{7} & \frac{1}{3} & 1 \end{pmatrix}$$

Easy calculation shows

$$\begin{aligned} c_3 &= -3.885714285714290, \\ \det A &= -0.812698412698412, \\ P_A(4) &= -16.3555555555556, \end{aligned}$$

hence A is inconsistent but near consistent as we see later soon. Since $P_A(0) = \det A < 0$, $P_A(4) < 0$, $\lim_{\lambda \to -\infty} P_A(\lambda) = \infty$, $\lim_{\lambda \to -\infty} P_A(\lambda) = \infty$, there exist two [1] real-number eigenvalues in which larger one is greater than 4 and smaller one is negative. Table 1 displays the convergence to the maximum eigenvalue.

Table 1: Convergence to maximum eigenvalue

n	λ_n	$P_A(\lambda_n)$
0	4	-16.35555556
1	4.272074355724550	3.800335333
2	4.229363816108630	0.105222281
3	4.228112292650730	8.85852×10^{-5}
4	4.228111237233530	6.29409×10^{-11}
5	4.228111237232780	-2.74225×10^{-14}

The consistency index CI $= 0.076$ satisfies Saaty's criterion CI ≤ 0.1, so we say that the pairwise comparison is near consistent. In this case, the global minimum point is $\lambda^* = 3.10101640728595 < 4$. We think that the iteration worked well by the fact that the adopted initial value is located to the right-hand side of λ^*.

3.2 Hard inconsistent case

Next we consider the following pairwise comparison matrix [9].

$$A = \begin{pmatrix} 1 & 8 & \frac{1}{8} & 8 \\ \frac{1}{8} & 1 & 7 & 8 \\ 8 & \frac{1}{7} & 1 & 2 \\ \frac{1}{8} & \frac{1}{8} & \frac{1}{2} & 1 \end{pmatrix}$$

[1] If all the solutions of $P_A(\lambda) = 0$ is real-numbered , they must be 0 (with multiplicity 3) and 4. See [9].

Easy calculation shows

$$\begin{aligned} c_3 &= -482.479910714286, \\ \det A &= 162.38671875, \\ P_A(4) &= -1767.532924, \end{aligned}$$

hence A is inconsistent. In this case, there exist two positive solutions of $P_A(\lambda) = 0$.

If we set initial value of the iteration to be $\lambda_0 = 4$, we have the convergence result displayed in Table 2. On the other hand, if we set $\lambda_0 = 7$, we have another convergence result in Table 3. Note that the consistency index CI of A is 1.778 which is far from Saaty's consistency criterion CI ≤ 0.1. It seems to be a consequence of the existence of contradictory triad of the objectives. Objective 1 is highly preferred to objective 2 ($a_{12} = 8$) and objective 2 is highly preferred to objective 3 ($a_{23} = 7$), but objective 3 is highly preferred to objective 1 ($a_{31} = 8$).

Table 2: Convergence to minimum value

n	λ_n	$P_A(\lambda_n)$
0	4	-1767.532924
1	-0.223698387552739	270.3639772
2	0.335916364839731	0.174934909
3	0.336278037953792	-4.38848×10^{-7}
4	0.33627803704649	0

Table 3: Convergence to maximum value

n	λ_n	$P_A(\lambda_n)$
0	7	-2185.972656
1	14.249840836239000	22945.48681
2	11.598715594278600	6423.104817
3	10.048988345204200	1452.275256
4	9.434868757280800	174.7882348
5	9.338234707707280	3.899896756
6	9.335978296973030	0.002092977
7	9.335977084712800	6.04757×10^{-10}
8	9.335977084712450	-9.66338×10^{-13}

As we saw in our earlier study [9], inconsistent matrix has two real-number solution of the characteristic equation. This example shows Newton's method generates two possibilities of convergence $\lambda = 0.33627803704649$ or $\lambda_{\max} = 9.335977084712450$ according to the choice of the initial value.

In the sequel, we explore what value is appropriate as an initial value. We use the point λ^* in which the first-order condition $P'_A(\lambda^*) = 0$ holds. For a positive number $\alpha > 0$, we set $\lambda_0 = \lambda^* + \alpha$. In this example, we have $\lambda^* = 6.169243126$. We execute Newton's method for $\alpha = 0.1$ and $\alpha = 3$. Tables 4 and 5 display the result. For $\alpha = 0.1$, the speed of convergence is rather slow, but it surely converges to $\lambda_{\max}$. After we knew the value of $\lambda_{\max}$, we should have taken the initial value to be probably greater than 9. If we take pretty big value $\lambda_0 = 100$, the generated sequence converges to $\lambda_{\max}$ within 14 iterations.

If we set $\alpha = 0.01$ which seems to be a pretty small perturbation, the generated sequence converges to $\lambda_{\max}$ within 23 iterations. So we may set an open problem whether Newton's method generates a convergence sequence or not for any small $\alpha > 0$.

If we set α to be minus, then the limit converges to the minimum eigenvalue. See Tables 6 and 7. Hence another research question arises. Does the limit of the sequence converge to the real-numbered maximum eigenvalue or the minimum eigenvalue whether we take the initial point greater than λ^* or smaller than it.

4 Conclusion

When we apply Newton's method, we expect initially that the starting point is appropriate to be 4 which is the order of the matrix [10]. This is based on the fact that the maximum eigenvalue satisfies $\lambda_{\max} \geq n$. In this paper, we showed this expectation is disappointing. However, most experts are expected to make their

Table 4: $\alpha = 0.1$

n	λ_n	$P_A(\lambda_n)$
0	6.269243126	-2303.249673
1	79.412213554346700	37728074.15
2	59.833868245604100	11931508
3	45.165172530193900	3771005.331
4	34.189139919570200	1190133.313
5	26.000813565363400	374340.1162
6	19.935247616875200	116791.8014
7	15.517798800980200	35714.1292
8	12.432283037020700	10367.14983
9	10.494134824844600	2604.359332
10	9.570191109609920	427.3456909
11	9.348139987068080	21.06018771
12	9.336012161595220	0.060560989
13	9.335977085005290	5.056×10^{-7}
14	9.335977084712450	-9.66338×10^{-13}

Table 5: $\alpha = 3$

n	λ_n	$P_A(\lambda_n)$
0	9.169243126	-276.5971607
1	9.34295919605378	12.07470833
2	9.33598866245479	0.019989108
3	9.33597708474435	5.50817×10^{-8}
4	9.33597708471245	-9.66338×10^{-13}

Table 6: $\alpha = -0.1$

n	λ_n	$P_A(\lambda_n)$
0	6.06924312562351	-2303.291046
1	-70.07541392690180	25524046.72
2	-52.29913099739670	8078903.945
3	-38.95657591609530	2558597.161
4	-28.92986702107190	811434.2812
5	-21.37289999950420	258194.0381
6	-15.63738654603580	82796.09146
7	-11.21260875037460	27017.12915
8	-7.67163754706579	9133.62062
9	-4.62176525122383	3243.472532
10	-1.76081670971280	1043.395832
11	0.16596323022640	82.29526788
12	0.33642016013832	-0.068742847
13	0.33627803718671	-6.78198×10^{-8}
14	0.33627803704649	0

Table 7: $\alpha = -3$

n	λ_n	$P_A(\lambda_n)$
0	3.169243126	-1393.154241
1	0.240481614892951	46.30688558
2	0.336331616023033	-0.025915346
3	0.336278037066414	-9.63732×10^{-9}
4	0.33627803704649	0

decision to be near consistent, the use of $\lambda_0 = 4$ may be valid practically. Indeed, if one can verify $\lambda^* < 4$, there is no reason to hesitate of using $\lambda_0 = 4$ as an initial value.

On the other hand, new expectation arised. If we set the initial value to be the minimum of the characteristic polynomial plus α, then we can expect the generated sequence may converge to the maximum eigenvalue. Practically, we find first the minimum λ^* of the characteristic polynomial by Newton's method. Next we add slight plus perturbation $\lambda^* + \alpha$ and apply Newton's method. Theoretical proof of convergence and the findings of appropriate size of α are left for the future research.

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