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On a maximum eigenvalue of 3rd order pairwise comparison matrices in AHP and convergence of Newton's method

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On a maximum eigenvalue of 3rd order pairwise comparison matrices in AHP and convergence of Newton's method

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In this short note, we apply the Newton's method to the characteristic polynomial of the 3rd order pairwise comparison matrices. We show the convergence of the sequence generated from the initial value 3, to the maximum eigenvalue.

Key words:AHP, pairwise comparison matrix, Newton's method, convergence

1 Introduction

In the context of AHP(Analytic Hierarchy Process), evaluating the principal eigenvalue of pairwise comparison matrices is a key ingredient. The associated eigenvector is used as a priority vector [1, 3, 7, 12]. To find the maximum eigenvalue and the associated eigenvector, one usually uses 'Power Iteration'. For the beginners of AHP, the calculation is not an easy task. So we constructed WWW systems for wide users[4, 5]. On the other hand, to calculate the maximum eigenvalue by Excel, we proposed Newton's method[11]. In this note, we focus on 3rd order pairwise comparison matrices. With the aid of the favorable properties of them, we will show that the sequence generated by Newton's method converges to the maximum eigenvalue.

2 Existence of the solution

We consider the 3rd order pairwise comparison matrix:

$$A = \begin{pmatrix} 1 & a_{12} & a_{13} \\ \frac{1}{a_{12}} & 1 & a_{23} \\ \frac{1}{a_{13}} & \frac{1}{a_{23}} & 1 \end{pmatrix}. \tag{1}$$

Its characteristic polynomial has the following form [9, 10]:

$$P_A(\lambda) \quad = \quad \lambda^3 - 3\lambda^2 - \det A.$$

We can calculate determinant of the pairwise comparison matrix as follows.

$$\begin{aligned} \det A &= \begin{vmatrix} 1 & a_{12} & a_{13} \\ \frac{1}{a_{12}} & 1 & a_{23} \\ \frac{1}{a_{13}} & \frac{1}{a_{23}} & 1 \end{vmatrix} \\ &= \frac{a_{12}a_{23}}{a_{13}} + \frac{a_{13}}{a_{12}a_{23}} - 2 \geq 0. \end{aligned}$$

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The last inequality follows from the relationship between the arithmetic mean and the geometric mean. We say the matrix A is **consistent** if $a_{ik}a_{kj} = a_{ij}$. For the 3rd order pairwise comparison matrix, it is obvious that the consistency of A and $\det A = 0$ is equivalent. If A is consistent, the characteristic polynomial is $P_A(\lambda) = \lambda^3 - 3\lambda^2 = \lambda^2(\lambda - 3)$. In this case, solution of the characteristic equation is $\lambda = 3, 0$ (0 is a multiple root).

Below, we treat the case when A is inconsistent(i.e. $\det A > 0$). Taking the derivatives of $P_A(\lambda)$, we have $P_A'(\lambda) = 3\lambda(\lambda - 2)$. So the point $\lambda = 0$ takes a local maximum and the point $\lambda = 2$ takes a local minimum. The vertical axis section is $-\det A < 0$. We indicate the shapes of the graph of $P_A(\lambda)$ in Figure1.

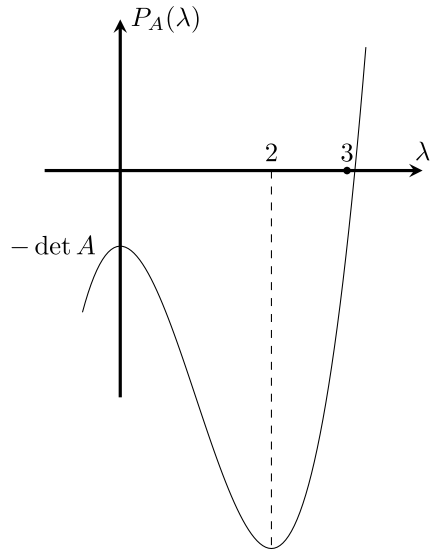

Figure 1: Inconsistent case

Easy calculation shows the followings:

$$\begin{aligned} P_A(3) &= 3^3 - 3 \cdot 3^2 - \det A = -\det A < 0, \\ P_A(3 + \det A) &= (3 + \det A)^3 - 3(3 + \det A)^2 - \det A \\ &= 27 + 27 \det A + 9(\det A)^2 + (\det A)^3 - (27 + 18 \det A + 3(\det A)^2) - \det A \\ &= 8 \det A + 6(\det A)^2 + (\det A)^3 > 0. \end{aligned}$$

We immediately have the following existence theorem of the solution for the characteristic equation.

Theorem 1 *For the inconsistent pairwise comparison matrix A, the characteristic equation has a unique real-valued solution in the interval* $(3, 3 + \det A)$. *This solution is the maximum eigenvalue.*

Proof. By the intermediate value theorem[2], the existence of real solution is obvious. From the shape of the graph, this solution is unique real-valued solution. In general, other two conjugate complex-valued solutions exist, say $a \pm bi$, $b \neq 0$.

From the general theory of eigenvalues and the trace of the matrix[7], we have $\lambda + (a + bi) + (a - bi) = 3$. So we have

$$a = \frac{3 - \lambda}{2}.$$

Since $a + bi$ is the solution of the characteristic equation, we have

$$(a + bi)^3 - 3(a + bi)^2 - \det A = 0.$$

By expanding the left-hand-side of the formula, we have

$$\begin{aligned} 0 &= a^3 - 3ab^2 - 3a^2 + 3b^2 - \det A, \\ 0 &= 3a^2 b - b^3 - 6ab. \end{aligned}$$

Since $b \neq 0$, from the last formula, we have

$$3a^2 - b^2 - 6a = 0.$$

So we finally have

$$b^2 = 3a^2 - 6a.$$

Thus we have

$$\begin{aligned} a^2 + b^2 &= \left(\frac{3-\lambda}{2}\right)^2 + 3\left(\frac{3-\lambda}{2}\right)^2 - 6\left(\frac{3-\lambda}{2}\right) \\ &= \lambda^2 - 3\lambda < \lambda^2. \end{aligned}$$

This means λ is maximum. //

3 Convergence of Newton's method

From the general theory of Newton's method, convergence is obvious[6]. Here we give the elementary proof. We set initial point $\lambda_0 = 3$ and generate a sequence by the following iteration.

- $\lambda_0 = 3$,
- $\lambda_{n+1} = \lambda_n - \dfrac{P_A(\lambda_n)}{P'_A(\lambda_n)}$.

λ_{max} denotes the maximum eigenvalue guranteed by Theorem 1. So $P_A(\lambda_{max}) = 0$.

Lemma 1 *For all $n \geq 1$, we have $\lambda_n > \lambda_{max}$.*

Proof. We prove by induction. Set $n = 1$. $\lambda_1 = 3 + \frac{\det A}{9} > 3$ and $P_A(\lambda_1) = \frac{2}{27}(\det A)^2 + (\frac{\det A}{9})^3 > 0$. So we have $\lambda_1 > \lambda_{max}$.

From the assumption of the induction, we can take $\lambda_{max} \leq \lambda < \lambda_n$. Since $P'_A(\lambda)$ is monotone increasing for $\lambda > 2$, we have

$$P'_A(\lambda) < P'_A(\lambda_n).$$

Thus we have

$$\int_{\lambda}^{\lambda_n} P'_A(\lambda) d\lambda < \int_{\lambda}^{\lambda_n} P'_A(\lambda_n) d\lambda.$$

So we conclude

$$P_A(\lambda_n) - P_A(\lambda) \quad < \quad P'_A(\lambda_n)(\lambda_n - \lambda). \tag{2}$$

Set $\lambda = \lambda_{max}$. Taking account into $P_A(\lambda_{max}) = 0$, we have

$$\lambda_{max} < \lambda_n - \frac{P_A(\lambda_n)}{P'_A(\lambda_n)} = \lambda_{n+1}.$$

//

Lemma 2 *The generated sequence λ_n is monotone decreasing for $n \geq 1$.*

Proof. From Lemma1, we have $3 < \lambda_{max} < \lambda_n$. Since $P_A(\lambda)$ is monotone increasing for $\lambda > 3$, we have $P_A(\lambda_n) > P_A(\lambda_{max}) = 0$. Obviously $P'_A(\lambda_n) > 0$ and $P_A(\lambda_n) > 0$, so we have $\lambda_n > \lambda_{n+1}$. //

From Lemmas 1 and 2, the sequence is monotone decreasing and bounded below. So it converges[2].

Theorem 2 *The sequence λ_n converge to λ_{max}.*

Proof. Obvious. //

4 Conclusion

In this note, we fully use the favorable properties from which A is 3rd order. For the matrices for 4 and more order dimension, the followings are open problems.

- If one happens to find the solution of the characteristic equation and can verify $\lambda \geq n$, then is it the maximum eigenvalue?
- If we set initial value $\lambda_0 = n$, does Newton's method generate a convergence sequence.

We leave them to the further reseach.

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