Relaxations of the Maximum Flow Minimum Cut Property for Ideal Clutters

by

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Author's Declaration

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

Given a family of sets, a covering problem consists of finding a minimum cost collection of elements that hits every set. This objective can always be bound by the maximum number of disjoint sets in the family, we refer to this as the covering dual, since when we allow covers to be fractional and relax the notion of disjoint sets, the natural Linear Programming (LP) formulations become duals and the optimal objective values of the two LPs match. A consequence of the Edmonds-Giles theorem about Totally Dual Integral systems is that if we assume the covering dual always has an optimal integer solution for every cost function, then we can always find an optimal integer cover. The converse does not hold in general, but a still standing conjecture from the mid-1970s states that the existence of an optimal integer cover for every cost function implies the existence of a $\frac{1}{4}$-integer optimal solution to the dual for every cost function. In this thesis we discuss weaker versions of the conjecture and build tools allowing us to approach them.

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Dedication

I dedicate this thesis to my family and my friends who have supported me throughout my academic career.

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Chapter 1

Introduction

In this thesis we will be interested in the general form of covering problems. Covering is a widely used and studied model for optimization problems. Despite its simplicity, it is general enough so that finding a shortest s-t path, a minimum cost s-t cut or a minimum cost perfect matching can all be formulated as finding a minimum cover for a family of sets.

Covering problems are usually studied over clutters instead of arbitrary families of sets, as this structure simplifies the discussion without losing on generality. This is explained in Section 1.1. Along with the rest of Chapter 1, this constitutes most of the required prerequisites to study clutters and the covering problem. We note that the thesis is written to be mostly self-contained and accessible to anyone familiar with basic concepts of linear programming. In the rest of this section, we give a high-level overview of the content of the thesis, and all the concepts mentioned will be rigorously defined as they are introduced in later chapters.

A close relative of covering is the set packing problem, which is also studied over clutters, and is defined in Section 2.1. Both these problems have straightforward IP formulations, whose LP relaxations are not guaranteed to be exact, so the LP optimum can be different than the IP optimum. A well known result is that integrality of the set-packing polytope implies that the corresponding set-packing system is TDI. However, the analogous statement fails for the covering problem. There are multiple classes of clutters where the covering polytope is integral but the covering system is not TDI. However, it is conjectured that a weaker statement holds, namely that whenever the covering polytope is integral, the covering system must have an optimal dual solution that is $\frac{1}{4}$-integer. This is known as the $\frac{1}{4}$ conjecture, due to Seymour, and is still standing from the mid 1970s. This conjecture appears to still be unapproachable today, as even for very special cases, for example considering the clutter of T-joins, we are not able to prove nor refute the conjecture. This pushes us to instead work with a relaxed version of the statement.

Various relaxations are discussed in Chapter 2, but the rest of thesis is focused on Conjecture 26, and investigates the following question: "Does there exist a constant $\lambda > 0$, such that if the covering polytope is integral, then there exists a dual variable of the covering system that can be set to a value greater or equal to λ in an optimal dual solution".

In Section 2.2 we touch on a fundamental difference between the behavior of dual variables in set-packing and covering. In the former, if the set-packing system is TDI, then any dual variable that can be set to a non-zero value in an optimal dual solution can be set to a non-zero *integer* value in an optimal dual solution. However, even if the covering system is TDI, there are examples where a dual variable can be only be set to an arbitrarily small value in an optimal dual solution.

Chapter 3 gives multiple characterizations of the range of values that can be assigned to a specific dual variable in an optimal dual solution. While the main result of sections 3.1 and 3.2 is already known, the proof given in Section 3.1 is original. The results in Section 3.3 all follow from applying widely known linear programming tools to the covering system.

Chapter 4 moves away from studying the possible values that a *specific* dual variable can take in an optimal dual solution to our original aim of studying the possible values that can be taken by *some* dual variable in an optimal dual solution. We discuss multiple exact IP formulations and the strength of their natural LP relaxations. While proving a non-exact relaxation is feasible does not imply that Conjecture 26 holds, it constitutes additional incentive to investigate the conjecture further. Additionally, if the relaxation is not very loose, then such a result would be evidence towards accepting the conjecture. On the other hand, if we're able to construct a family of examples where the relaxation is not feasible, this would constitute a proof that Conjecture 26 is incorrect, which would also disprove the $\frac{1}{4}$ conjecture. All the results in this chapter are original work.

Finally, Chapter 5 builds on Section 4.2 to strengthen the weak LP relaxation using lift and project methods. Section 5.5 then specializes the developed machinery to the setting of T-joins. All the results of this chapter are original findings, with the exception of Theorem 89 and Corollary 90, which are known results, but the presented proof is original work.

1.1 Covering on Clutters

A *clutter*, usually referred to $\mathbb{F}$, defined over a *ground set* $V(\mathbb{F})$, is a collection of sets of subsets $S_1, S_2, \ldots, S_k \subseteq V(\mathbb{F})$ such that no set is included in another. We call these S_i the *elements* of the clutter.[1] We usually refer to the ground set $V(\mathbb{F})$ as V when there is no ambiguity, and it is

[1]These are also referred to as edges or members of the clutter in the literature.

commonly assumed to be $\{1,\dots,n\}$.

A *covering problem* can be defined over any arbitrary family of sets $S_1, S_2, \dots, S_k \subseteq V$, and consists of the following: Given a cost vector $c \geq 0$, find a minimum cost [2] collection of elements B such that $B \cap S_i \neq \emptyset$ for all $i \in [k]$[3]. We refer to such sets as covers of $S_1, \dots, S_k$, and we say that a set B *covers* S if $B \cap S \neq \emptyset$.

Since any set that covers S covers any superset of S, covers of $S_1, \dots, S_k$ correspond exactly to the covers of the clutter whose elements are the minimal sets among $S_1, \dots, S_k$. This is why clutters are the fundamental structure to look at when studying the covering problem. We will denote the minimum cost of a cover by $\tau(\mathbb{F}, c)$, and refer to it as the *covering number*. When the clutter and cost vector are unambiguous, we simply refer to it by τ.

A *max-capacity problem* can be defined over any arbitrary family of sets $S_1, \dots, S_k \subseteq V$, and consists of the following: Given a capacity vector $c \geq 0$, find a maximum size multiset[4] $M = \{S_{i_1}, \dots, S_{i_t}\}$ such that each $j \in V$ does not belong to more than c_j sets in M. We call such a multiset a *c-packing*. Note that when $c = \mathbf{1}$, M must be a family of pairwise disjoint sets.

Given any c-packing M that contains S, and any Q that is a subset of S, $M' := (M - \{S\}) + \{Q\}$[5] is also a c-packing of same size as M, and hence we also study this problem over clutters instead of arbitrary families. We will denote the maximum size of a c-packing by $\nu(\mathbb{F}, c)$, and refer to it as the *packing number*. When the clutter and cost vector are unambiguous, we simply refer to it by ν.

Let us examine some examples. Consider a graph $G = (V, E)$ and vertices $s, t \in V$. Let $\mathbb{F}$ be the clutter whose ground set is E and whose elements are the s, t paths in G. The covering problem on $\mathbb{F}$ with an integer cost vector c is then equivalent to finding a minimum cost set $X \subseteq E$ such that X contains an edge from every s, t path, and hence corresponds to finding a minimum c-cost s, t cut in G. The max-capacity problem translates to finding a collection of s, t paths such that no edge e is used in more than c_e of these paths, which corresponds to finding a maximum value s, t flow with edge capacities given by c. This leads to the fact that the packing number and the covering number are equal in this case due to the max-flow min-cut theorem [25].

Note that this min-max relation does not hold for arbitrary clutters. For example, if we consider the clutter $\Delta_2 = \{\{1,2\}, \{2,3\}, \{1,3\}\}$ with $c = \mathbf{1}$, then $\tau = 2$, since we need at least 2 elements to cover all three sets, but $\nu = 1$ as no 2 elements of the clutter are disjoint. However,

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[2] $c(B) := \sum_{t \in B} c_t$.

[3] $[k] := \{1, 2, \dots, k\}$

[4] The size of a multiset is the sum of multiplicities of its members.

[5] M' is the multiset obtained from M by removing a copy of S and adding a copy of Q.

we can still always obtain a weaker result, namely that the covering number is at least as big as the packing number.

Remark 1. *Let $\mathbb{F}$ be a clutter and $c \geq 0$, then $\tau(\mathbb{F}, c) \geq \nu(\mathbb{F}, c)$.*

This result can be proven with a short combinatorial argument. If we consider any cover B and any c-packing M, then every set $S \in M$ must intersect B at least once, and hence :

$$|M| = \sum_{S \in M} 1 \leq \sum_{S \in M} |S \cap B| = \sum_{e \in B} \sum_{S \in M} |S \cap \{e\}| \leq \sum_{e \in B} c_e = c(B).$$

This is also a consequence of weak duality. If we denote by $M(\mathbb{F})$ the matrix whose rows are the characteristic vectors of the elements of $\mathbb{F}$, that is $\mathrm{row}_i(M(\mathbb{F})) = \chi_{S_i}$[6], then we can define the covering problem as follows (C_I) :

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & M(\mathbb{F})x \geq \mathbf{1}, \quad (C_I) \\ & x \geq 0, \\ & x \text{ integer.} \end{array}$$

As well as the max capacity problem (D_I):

$$\begin{array}{ll} \text{maximize} & \mathbf{1}^T y \\ \text{subject to} & M(\mathbb{F})^T y \leq c, \quad (D_I) \\ & y \geq 0, \\ & y \text{ integer.} \end{array}$$

For every cover B of $\mathbb{F}$ we have that χ_B is a solution to (C_I) of matching cost. Similarly any c-packing M yields a solution $\bar{y}$ to (D_I) where $\bar{y}_S$ is the number of occurrences of S in M. We can then define the LP relaxations (C) and (D) of these problems as follows:

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & M(\mathbb{F})x \geq \mathbf{1}, \quad (C) \\ & x \geq 0. \end{array}$$

$$\begin{array}{ll} \text{maximize} & \mathbf{1}^T y \\ \text{subject to} & M(\mathbb{F})^T y \leq c, \quad (D) \\ & y \geq 0. \end{array}$$

[6]For a set $S \subseteq [n]$, we denote by χ_S the n dimensional vector where $\chi_{S\,i} = 1$ if $i \in S$ and 0 otherwise. The choice of n is usually clear from context.

We define fractional covers of $\mathbb{F}$ to be the set of feasible points for (C), and fractional c-packings of $\mathbb{F}$ to be the set of feasible points to (D).

As (C) is obtained from (C_I) by omitting the integrality constraint, every feasible point of (C_I) is also feasible for (C). Since both problems have the same objective function, we must have that $\tau(\mathbb{F}, c)$, which is the optimal value for (C_I) is at least as large as the optimal objective value for (C). Following the same logic we can deduce that $\nu(\mathbb{F}, c)$, which is the optimal value for (D_I), cannot surpass the optimum value for (D). Notice that (C) and (D) are exactly the duals of one another so we can apply weak duality to obtain the following result:

$$\tau(\mathbb{F}, c) \geq_{(1)} \tau^*(\mathbb{F}, c) \geq_{(\star)} \nu^*(\mathbb{F}, c) \geq_{(2)} \nu(\mathbb{F}, c)$$

where $\tau^*(\mathbb{F}, c), \nu^*(\mathbb{F}, c)$ denote the optimums of (C) and (D) respectively. Note that while $(\star)$ always holds with equality due to strong duality, (1) (respectively (2)) holds with equality if and only if (C) (respectively (D)) has an optimal solution that is integer.

Given this observation, we ask the following natural questions:

Question 2.

(a) How big can the gap between τ and ν get ?

(b) Can we characterize the clutters for which (1) holds with equality for all non-negative integer vectors c ?

(c) Can we characterize the clutters for which (2) holds with equality for all non-negative integer vectors of c ?

We know that for (a) both the additive and multiplicative gaps can be arbitrarily big. Take $\mathbb{F}$ to be all the 2-element subsets of $[2n]$ and $c = \mathbf{1}$. We then have $\tau = 2n - 1$ since any cover of $\mathbb{F}$ cannot exclude any pair of ground elements as otherwise it would not intersect their union which is an element of $\mathbb{F}$. We also have $\nu = n$ since we cannot find more than n disjoint elements, as every element of $\mathbb{F}$ has size 2 while $|V(\mathbb{F})| = 2n$. This gives an additive gap of order $\Theta(n)$ between τ and ν. We can take it further by considering $\mathbb{F}$ to be all the n-element subsets of $[n]^2$ and $c = \mathbf{1}$ yields $\tau = n^2 - n + 1$ but $\nu = n$ for similar arguments as before: any cover cannot miss any collection of n ground elements since they form an element of the clutter, and we cannot have more than n disjoint elements since every element of $\mathbb{F}$ has size n. Hence a multiplicative gap of order $\Theta(n)$. The natural follow-up question is if we can characterize the best possible asymptotic bound on ν as a function of both τ and $|V(\mathbb{F})|$, but to our knowledge, the answer is still unknown.

(b) and (c) are a lot more difficult to answer. Although we are able to identify some classes that always satisfy (1) and/or (2) with equality, it is in general NP-hard to tell if an arbitrary clutter satisfies either of the properties [11]. An interesting fact is that the condition in (c) implies its counterpart in (b). This is an application of a theorem due to Edmonds and Giles but to state it we first need some quick definitions.

Definition 3. *A pointed* [7]*polyhedron* $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ *is said to be* integral *if every extreme point is integer, or equivalently* $P = \mathrm{conv}(P \cap \mathbb{Z}^n)$[8].

Definition 4. *A linear system of inequalities* $Ax \leq b$ *is said to be* totally dual integral *(TDI) if for any integer vector* c *such that* $\min\{c^T x : Ax \leq b\}$ *has an optimal solution, the dual has an optimal solution that is integer.*

Theorem 5 (Edmonds & Giles theorem [13])**.** *If the system* $\{Ax \leq b\}$ *is TDI and* b *is integer, then the polyhedron* $\{x \in \mathbb{R}^n : Ax \leq b\}$ *is integral.*

Clearly, the condition in (c) is equivalent to $\{M(\mathbb{F})x \geq 1, x \geq 0\}$ being TDI, since the existence of an optimal solution to (D) that is integer is equivalent to $\nu = \nu^*$. We also have that integrality of (C) is equivalent to the condition in (b) due to the following classical result [27]:

Theorem 6. $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ *is integral if and only if for all* c *such that* $\min\{c^T x : x \in P\}$ *is finite there exists an integer* $\bar{x} \in P$ *for which* $c^T \bar{x} = \min\{c^T x : x \in P\}$*. This means that whenever* $P \neq \emptyset$ *is the feasible region of an LP that is not unbounded, the optimal value is achieved by an integer point in* P*.*

1.2 Idealness and the MFMC property

Definition 7. *A clutter* $\mathbb{F}$ *is called* ideal *if the corresponding covering polyhedron (C):* $\{x \in \mathbb{R}^n : M(\mathbb{F})x \geq \mathbf{1},\ x \geq 0\}$ *is integral, or equivalently* $\tau(\mathbb{F}, c) = \tau^*(\mathbb{F}, c)$ *for all* $c \geq 0$.

Definition 8. *A clutter* $\mathbb{F}$ *is said to have the* Max-flow Min-cut *property (MFMC) if* $\nu(\mathbb{F}, c) = \nu^*(\mathbb{F}, c)$ *for all integer* $c \geq 0$.

Since the covering problem is unbounded if and only if c has a negative entry, the condition of $\nu(\mathbb{F}, c) = \nu^*(\mathbb{F}, c)$ for all non-negative integer c corresponds exactly to the system in (C) being TDI, and so the Edmonds Giles theorem applied to the covering problem translates to the following result:

[7] A non-empty polyhedron is pointed if it has an extreme point, or equivalently if it does not contain a line.

[8] $\mathrm{conv}(X)$ denotes the convex hull of X which is defined to be the intersection of all convex sets containing X, and is known to be equal to the set of all points that can be written as a convex combination of some elements in X.

Theorem 9. *If a clutter $\mathbb{F}$ has the MFMC property, then it is ideal.*

We note that the reverse implication is false. We give the example of the clutter $\mathbb{Q}_6 = \{\{1,2,3\},\{1,5,6\},\{2,4,6\},\{3,4,5\}\}$, which can be shown to be ideal[9] but it does not have the MFMC property, as $\nu(\mathbb{Q}_6, \mathbf{1}) = 1$ but $\tau(\mathbb{Q}_6, \mathbf{1}) = 2$.

We now ask some additional questions regarding these properties.

Question 10.

(a) Can we find other conditions on $\mathbb{F}$ that are equivalent to idealness or the MFMC property?

(b) Within the class of ideal clutters, is there a bound on the gap between τ and ν ?

(c) Does idealness imply a weaker version of the MFMC property ? Namely can we always find an optimal dual solution that is not 'far from integer' ?

Examples of MFMC clutters include clutters of s-t paths [25], s-t dipaths [17] and s-t cuts [12]. The class of ideal clutters includes all of the previously mentioned clutters, as well as clutters of s-t dicuts [24], acyclic T-joins [16], and odd circuits of planar graphs [19].

While the MFMC property is quite hard to certify and we do not have any particularly useful characterization, there are multiple results characterizing when idealness holds.

1.2.1 Blockers

First let us look at the extreme points of the covering polyhedron. $C = \{x \in \mathbb{R}^n : M(\mathbb{F})x \geq \mathbf{1}, x \geq 0\}$ is integral if and only if all its extreme points are integer. Hence $\mathbb{F}$ is ideal when the extreme points of the polyhedron correspond exactly to the minimal covers of $\mathbb{F}$. These covers form a clutter structure, and it turns out that studying the resulting clutter gives a lot of insight on properties of $\mathbb{F}$. This motivates the following definition:

Definition 11. *Given a clutter $\mathbb{F}$, we define the* blocker of $\mathbb{F}$ *to be the clutter on the same ground set, whose elements are the minimal covers of $\mathbb{F}$, and denote by it by $b(\mathbb{F})$.*

One might wonder if this construction is injective and/or surjective. The answer is even stronger than a simple yes to both questions. This construction is not only bijective but satisfies the involution property [14, 20] :

[9] The easiest way to verify this is to find all the extreme points of the covering polytope and check they all integer.

Theorem 12. *Let $\mathbb{F}$ be a clutter. Then $b(b(\mathbb{F})) = \mathbb{F}$.*

We also have a surprising result linking idealness of $\mathbb{F}$ and $b(\mathbb{F})$ due to Lehman [22]:

Theorem 13. *$\mathbb{F}$ is ideal if and only if $b(\mathbb{F})$ is ideal.*

We note that such equivalence does not hold for the MFMC property. We again use the example of $\mathbb{Q}_6$, which is an ideal clutter that does not have the MFMC property but its blocker $b(\mathbb{Q}_6) = \{\{1,2,3\},\{1,5,6\},\{2,4,6\},\{3,4,5\},\{1,4\},\{2,5\},\{3,6\}\}$ does. The clutter and its blocker as a pair give another characterization of idealness, namely the width-length inequality also due to Lehman [22].

Theorem 14 (Width-Length Inequality)**.** *Let $\mathbb{F}$ be a clutter. Then $\mathbb{F}$ and $b(\mathbb{F})$ are ideal if and only if the following inequality holds for all vectors $w, l \in \mathbb{R}^V_{\geq 0}$:*

$$\tau(\mathbb{F}, w) \times \tau(b(\mathbb{F}), l) \leq w^T l.$$

To illustrate this we provide two examples. First consider $C_3^1 = \{\{1\},\{2\},\{3\}\}$ and its blocker $b(C_3^1) = C_3^3 = \{1,2,3\}$. Both clutters are ideal and verify $\tau(C_3^1, c) = c_1 + c_2 + c_3$ and $\tau(C_3^3, c) \geq \min_{i\in[3]} c_i$. The width-length inequality then translates to :

$$(w_1 + w_2 + w_3) \times (\min_{i\in[3]} l_i) \leq w_1 l_1 + w_2 l_2 + w_3 l_3$$

which clearly holds for all $w, l \geq 0$.

To showcase the flip side, consider $C_3^2 = \{\{1,2\},\{1,3\},\{2,3\}\}$. We have that $b(C_3^2) = C_3^2$. This clutter is not ideal as for $c = \mathbf{1}$ we have $\tau(\mathbb{F}, \mathbf{1}) = 2$ while the point $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ is a fractional cover achieving an objective value of $\frac{3}{2} < 2$. Now consider $w = l = \mathbf{1}$, then we have:

$$\tau(C_3^2, \mathbf{1}) \times \tau(C_3^2, \mathbf{1}) = 2 \times 2 > 3 = \mathbf{1}^T \mathbf{1}.$$

So the width-length inequality fails.

In fact, an interesting result states that every self-blocking clutter, i.e. satisfying $b(\mathbb{F}) = \mathbb{F}$, is either non ideal, or must be the trivial clutter on a single element, that is $\mathbb{F} = \{\{1\}\}$, up to renaming the ground element [2].

1.2.2 Minors

We also have a result stating that when checking for idealness we do not need to verify that $\tau = \tau^*$ for all $c \geq 0$ as we can restrict ourselves to a smaller subset of cost vectors.

Theorem 15. *[21] $\mathbb{F}$ is ideal if and only if $\tau(\mathbb{F}, c) = \tau^*(\mathbb{F}, c)$ for all $c \in \{0, 1, \infty\}^V$.*

The forward direction is trivial as it follows from the equivalent definition of idealness, while the proof for the reverse direction can be found in [21]. We note that this condition can be rephrased in terms of minors of the clutter. To do that, we first introduce the notion of minors.

For each element of the ground set $j \in V(\mathbb{F})$, we define the *deletion minor* $(\mathbb{F} \setminus j)$ and *contraction minor* $(\mathbb{F}/j)$ to be the clutters on groundset $V \setminus \{j\}$, where the elements of $(\mathbb{F} \setminus j)$ are $\{S : j \notin S, S \in \mathbb{F}\}$ and the elements of $(\mathbb{F}/j)$ are the minimal elements in $\{S \setminus \{j\}, S \in \mathbb{F}\}$. These operations commute with themselves and with each other, so we have $(\mathbb{F}\setminus j)\setminus k = (\mathbb{F}\setminus k)\setminus j$, $(\mathbb{F}/j)/k = (\mathbb{F}/k)/j$, and $(\mathbb{F} \setminus j)/k = (\mathbb{F}/k) \setminus j$ for all ground elements $j \neq k \in V(\mathbb{F})$. A minor of $\mathbb{F}$ is any clutter that be obtained from $\mathbb{F}$ by a sequence of deletions and contractions. Since the order of the operations does not matter, given I, J disjoint subsets of $V(\mathbb{F})$, we denote the minor obtained by deleting elements in I and contracting element in J by $\mathbb{F} \setminus I/J$.

Remark 16. *Let $\mathbb{F}$ be a clutter, then we have:*

- *If $\mathbb{F}$ is ideal then all of its minors are ideal.*
- *If $\mathbb{F}$ has the MFMC property then all of its minors have the MFMC property.*

We also note that these operations behave nicely with respect to taking the blocker, as we have $b(\mathbb{F} \setminus j) = b(\mathbb{F})/j$ and $b(\mathbb{F}/j) = b(\mathbb{F}) \setminus j$. This yields the following result:

Remark 17. *Let $\mathbb{F}$ be a clutter, $j \in V(\mathbb{F})$, and $c \geq 0$. Let $\bar{c}$ be the restriction of c on $V(\mathbb{F}) \setminus \{j\}$, then the following hold:*

- *If $c_j = 0$ then $\tau(\mathbb{F}, c) = \tau(\mathbb{F} \setminus j, \bar{c})$.*
- *If j does not belong to every c-optimal $\mathbb{F}$ cover, then $\tau(\mathbb{F}, c) = \tau(\mathbb{F}/j, \bar{c})$*

Proof. For the first result, notice that when B is a cover of $\mathbb{F}$, $B \setminus j$[10] is a cover of $\mathbb{F} \setminus j$ so $\tau(\mathbb{F}, c) - c_j \geq \tau(\mathbb{F} \setminus j, \bar{c})$. If B is a cover of $\mathbb{F} \setminus j$, then $B \cup \{j\}$ is a (not necessarily minimal)

[10] If $j \notin B$ then $B \setminus j = B$.

cover of $\mathbb{F}$, hence $\tau(\mathbb{F} \setminus j) + c_j \geq \tau(\mathbb{F}, c)$. When $c_j = 0$, these two inequalities imply that $\tau(\mathbb{F}, c) = \tau(\mathbb{F}', \bar{c})$. For the second point, notice that every cover B of $\mathbb{F}/j$ is a cover of $\mathbb{F}$ that does not contain j, so we have $c(B) = \bar{c}(B)$ hence $\tau(\mathbb{F}, c) \leq \tau(\mathbb{F}', \bar{c})$. Now if we consider an optimal cover B of $\mathbb{F}$ such that $j \notin B$, we obtain that $\tau(\mathbb{F}, c) = c(B) = \bar{c}(B) \geq \tau(\mathbb{F}/j, \bar{c})$, combining these two inequalities yields the desired result. □

This shows that we can think of deleting j as setting its cost to 0, while contracting it corresponds to setting its cost to an arbitrarily high value. Therefore, we can reformulate Theorem 15 in terms of minors namely in the following way:

Theorem. *Let $\mathbb{F}$ be a clutter such that $\tau(\mathbb{F}', \mathbf{1}) = \tau^*(\mathbb{F}', \mathbf{1})$ for all $\mathbb{F}'$ minors of $\mathbb{F}$, then $\mathbb{F}$ is ideal.*

We note that it is conjectured that an analogous version of this theorem holds for the MFMC property, namely the following:

Conjecture 18 (Replication conjecture)**.** *[8] Let $\mathbb{F}$ be a clutter such that $\tau(\mathbb{F}, c) = \nu(\mathbb{F}, c)$ for all vectors $c \in \{0, 1, \infty\}^V$. Then $\mathbb{F}$ has the MFMC property.*

1.2.3 Further discussion

Going back to question 10-b concerning for the gap between τ and ν within the class of ideal clutters, we can still find instances with arbitrarily large additive gap. For example, define n clutters $\mathbb{Q}_{6i}$, $i \in [n]$, that are 'independent' copies of $\mathbb{Q}_6$, with respective ground sets $\{1_i, 2_i, 3_i, 4_i, 5_i, 6_i\}$. Then consider $\mathbb{F} = \mathbb{Q}_6^n = \cup_{i=1}^n \mathbb{Q}_{6i}$, so the elements of $\mathbb{Q}_6^n$ correspond to the rows of the block diagonal matrix with n identical blocks equal to $M(\mathbb{Q}_6)$. We have $\tau(\mathbb{Q}_6^n, \mathbf{1}) = 2n$ as any minimal cover corresponds to the union of n sets $C_i = \{a_i, b_i\} \subseteq \{1_i, 2_i, 3_i, 4_i, 5_i, 6_i\}$, where each C_i is a cover of a $\mathbb{Q}_{6i}$, while $\nu(\mathbb{Q}_6^n, \mathbf{1}) = n$ as we cannot have any 2 elements belonging to the same $\mathbb{Q}_{6i}$ in a packing. We are not currently aware of any example with asymptotically better than constant ratio gap, nor of any non-trivial bounds on the worst case scenario ratio bound.

The answer to 10-c will be the focus of this thesis. In the next chapter, we discuss potential relaxations of the MFMC property and formalize multiple conjectures as potential answers to this question.

Chapter 2

Weakenings of the MFMC property

Since idealness does not imply the existence of optimal packing, we can ask if it implies the existence of some ”nice” fractional packings. No general positive result is currently known, but there are two main conjectures of interest in this direction. First is the natural soft relaxation of the integrality condition.

Conjecture 19 ($\frac{1}{k}$ conjecture)**.** *There exists an integer $k \geq 1$ such that for all ideal clutters $\mathbb{F}$ and all integer $c \geq 0$, $\mathbb{F}$ admits an optimal c-packing y such that ky is integral.*

Simply working with $\mathbb{Q}_6$ we realize that if such a k exists it must be a multiple of 2, and there are reasonably small examples of ideal clutters without a $\frac{1}{2}$ optimal packing, so the smallest still standing possible value of k is 4, and the hypothesis that the $\frac{1}{k}$ conjecture holds with $k = 4$ is commonly referred to as the $\frac{1}{4}$ conjecture [28].

The second conjecture, due to Seymour in 1975[1], is more intriguing, as it claims the existence of an optimal solution of a special form :

Conjecture 20 (Dyadic conjecture)**.** *Let $\mathbb{F}$ be an ideal clutter and $c \geq 0$. Then $\mathbb{F}$ admits an optimal fractional c-packing y such that $2^k y$ is integral for some integer k.*

This conjecture is perhaps inspired by a common duplication technique used to prove that many classes of ideal clutters always admit a half integer packing. We will see an example of this technique at the end of this chapter, and again in Section 5.5. The hope is that such an

[1]To our knowledge, this conjecture never appeared in a publication by Seymour, but is it mentioned by Schrijver in [26], section 79.6e.

argument can be extended in the general case resulting in a dyadic packing.[2] We note that it is proven that for all ideal clutters with $\tau(\mathbb{F}, c) \geq 2$, there exists a dyadic c-packing y with $y^T\mathbf{1} \geq 2$ hence the conjecture holds for pairs $(\mathbb{F}, c)$ with $\tau(\mathbb{F}, c) = 2$. [1]

But how can we start to approach these conjectures, and what would be reasonable intermediate result to support them? A good place to start would be to look at a close relative of covering, namely the packing problem, for which the equivalent notion of idealness implies the TDI condition. Perhaps this would give us an idea on what type of properties are useful to have when trying to certify a weaker version of the MFMC property.

2.1 The packing problem.

A packing problem defined over any arbitrary family of subsets $S_1, \ldots, S_k \subseteq V$ consists of the following : Given a cost vector $c \geq 0$, find a maximum cost collection of elements B such that no two elements belong to a common set S_i. Such a collection is called a packing.

It also makes sense to study the packing problem over clutters, as any pair of elements that belong to the same S, also both belong to any superset of S. A notable difference between the covering and packing problems defined on an arbitrary family of sets is that for the covering problem it is sufficient to restrict ourselves to the clutter formed by the minimal elements of the family, while for the packing problem we can restrict to the clutter of maximal elements in the family. Note that when $c = \mathbf{1}$, the packing problem on $\mathbb{F} = \{S_1, \ldots, S_m\}$ where $V(\mathbb{F}) = [n]$ is equivalent to the maximum capacity problem on $\mathbb{F}'$ on ground set $[m]$ whose elements are the maximal sets in $\{S'_1, \ldots, S'_n\}$ where $S'_i = \{j \in [m] : i \in S_j\}$. In other words, the elements of $\mathbb{F}'$ correspond to maximal characteristic vectors of the columns of $M(\mathbb{F})$.

The IP formulation is straightforward. We write the packing problem as (P_I):

$$\begin{array}{ll} \text{maximize} & c^T x \\ \text{subject to} & M(\mathbb{F})x \leq \mathbf{1}, \quad (P_I) \\ & x \geq 0, \\ & x \text{ integer.} \end{array}$$

We will denote by (P) its LP relaxation, and its dual is given by (D):

$$\begin{array}{ll} \text{minimize} & \mathbf{1}^T y \\ \text{subject to} & M(\mathbb{F})^T y \geq c, \quad (D) \\ & y \geq 0. \end{array}$$

[2] A vector x is said to be dyadic if every entry is of the form $\frac{1}{2^k}$ for some integer k.

Since all these problems are bounded when $c \geq 0$, weak duality implies that we have:

$$\mathrm{opt}(P_I) \leq_{(1)} \mathrm{opt}(P) \leq_{(*)} \mathrm{opt}(D) \leq_{(2)} \mathrm{opt}(D_I).^3$$

where (D_I) is obtained from (D) by imposing an integrality constraint on y. Note that $(*)$ always holds with equality due to strong duality, and we can ask the same questions that we did for the covering problem, namely the following:

Question 21.

(a) How big can the gap between $\mathrm{opt}(P_I)$ *and* $\mathrm{opt}(D_I)$ *get ?*

(b) Can we characterize the clutters for which (1) *holds with equality for all non-negative integer vectors* c *?*

(c) Can we characterize the clutters for which (2) *holds with equality for all non-negative integer vectors of* c *?*

While the answer to (a) is the same as the covering case, that is, we can find arbitrarily large gaps between the two values, the answers to the (b) and (c) are very different. In fact, (b) and (c) turn out to be equivalent to each other, and we can identify if they hold in polynomial time. The recognition algorithm is due to a result stating that the packing polytope is integer if and only if $M(\mathbb{F})$ is the clique vertex matrix of a perfect graph [10], and there exists polynomial time algorithms to recognize clique-vertex matrices [10] and to recognize perfect graphs [6]. Similarly as for the covering case, the condition in (c) implies its counterpart in (b) due to Theorem 5. The reverse direction is the interesting part about this result, so we formalize it:

Theorem 22. *Let* $\mathbb{F}$ *be a clutter. If the system* $\{M(\mathbb{F})x \leq \mathbf{1}, x \geq 0\}$ *is integral, then it is TDI.*

The proof for this result relies on two main ideas. First we make sure that we can set a dual variable y_S to 1 (or higher) value in an optimal dual solution, then we use induction on $opt(P)$, by considering the covering problem on the same clutter with cost vector $c - \chi_S$.

Lemma 23. *Let* y *be an optimal solution to* (D')*, then for every* S *such that* $y_S > 0$*, there exists an optimal solution* y' *such that* $y'_S \geq 1$.

Proof. Let c be integer and consider $\gamma = \max\{c^T x : M(\mathbb{F}) \leq \mathbf{1}, x \geq 0\}$ and $\gamma' = \max\{c'^T x : M(\mathbb{F})x \leq \mathbf{1}, x \geq 0\}$ where $c' = c - \chi_S$. We claim that $\gamma' = \gamma - 1$ and that this is sufficient to

[3] Given an optimization problem (P), we denote by $\mathrm{opt}(P)$ its optimal value outcome.

imply the result. Indeed, given y' that satisfies $M(\mathbb{F})^T y \geq c - \chi_S$ and $\mathbf{1}^T y' = \gamma - 1$, we have $M(\mathbb{F})^T(y' + e_S) \geq c$, $\mathbf{1}^T(y' + e_S) = \gamma = \mathrm{opt}(P)$ and $(y' + e_S)_S \geq 1$. Now to actually prove the equality, note that the two problems have the same feasible region, and for any feasible x, $c^T x \geq c'^T x = c^T x - \chi_S^T x \geq c^T x - 1$, and hence $\gamma \geq \gamma' \geq \gamma - 1$. Since the polyhedron in integral and c' is integer, γ' must be integer and hence $\gamma' \in \{\gamma, \gamma - 1\}$. We claim that $\gamma' \neq \gamma$. Were this to be the case, every optimal solution $\bar{x}$ for the c' objective must satisfy $c'^T \bar{x} = c^T \bar{x}$, and hence $\bar{x}$ must also be optimal for the c objective and satisfy $\chi_S^T \bar{x} = 0$, which violates the complementary slackness conditions(C.S.) with respect to y_S. We give a quick reminder of the C.S. conditions for the sake of completeness. □

Theorem 24 (Complementary Slackness for a maximization primal)**.** *[27] Let (P) be $\max\{c^T x : Ax \leq b, x \geq 0\}$ and its dual (D) be $\min\{b^T y : A^T y \geq c, y \geq 0\}$ where $A \in \mathbb{R}^{m \times n}$. For x, y feasible for (P) and (D) respectively, x and y are both optimal if and only if both of the following conditions hold:*

- *For all $i \in [n]$, $x_i = 0$ or $(A^T y - c)_i = 0$,*
- *For all $j \in [m]$, $y_j = 0$ or $(Ax - b)_j = 0$.*

The full proof of Theorem 22 then follows by a simple inductive argument on $\mathrm{opt}(P)$. First assume $\mathrm{opt}(P) = 0$, then $y = 0$ is an integer optimal solution to (D). Now if $\mathrm{opt}(P) \geq 1$, consider an optimal dual solution y and $S \in \mathbb{F}$ such that $y_S > 0$. By Lemma 23 there must exist an optimal solution y' with $y'_S \geq 1$. Now consider (P'):

$$\begin{array}{ll} \text{maximize} & (c - \chi_S)^T x \\ \text{subject to} & M(\mathbb{F})x \leq \mathbf{1}, \quad (P') \\ & x \geq 0. \end{array}$$

Then $\mathrm{opt}(P') = \mathrm{opt}(P) - 1$, so we can obtain an integer optimal solution to the dual of (P'), that is an integer $y' \geq 0$ such that $\mathbf{1}^T y' = \mathrm{opt}(P) - 1$ and $M(\mathbb{F})^T y' \geq c - \chi_S$. Therefore, $y' + e_S$ will be an optimal integer solution to (D) finishing the proof.

In the next section we go back to the covering problem while keeping in mind Lemma 23 that was the key to obtaining TDI-ness.

2.2 Further relaxations

The takeaway from the proof is that the key to obtaining the TDI property for the packing problem was the ability to 'push' the dual variables. That is, as long as there exists a dual optimal solution

y with $y_S > 0$, then there must exists an optimal dual solution y' with $y'_S \geq 1$. This property does not hold for ideal clutters, and does not even hold for MFMC clutters. As an example, consider the s, t paths clutter on the graph $G = (V = \{s, t, a, b\}, E = \{(s, a), (s, b), (a, b), (a, t), (b, t)\})$. The members are $S_1 = \{(s, a), (a, t)\}, S_2 = \{(s, b), (b, t)\}, S_3 = \{(s, a), (a, b), (b, t)\}$, $S_4 =$

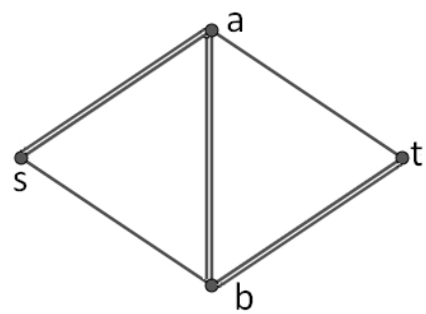

Figure 2.1: The graph G. S_3 is represented by the double edges.

$\{(s, b), (a, b), (a, t)\}$. An illustration of this graph is given in Figure 2.1. Note that S_3 and S_4 do appear with non-zero coefficients in the optimal dual solution $y = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ but can never be achieve a higher coordinate value in any other optimal solution, since the set of optimal solutions to the dual is $\mathrm{conv}(\{(1, 1, 0, 0), (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})\})$.

Therefore, to construct a weaker version of this property for ideal clutters, we relax it in a natural way as follows:

Conjecture 25 (Packing analog conjecture)**.** *There exists an integer $k > 0$ such that for all ideal clutters $\mathbb{F}$ and non-negative integer vectors c, if $y_S > 0$ for some optimal dual solution y for the covering problem, then there exists an optimal dual solution y' such that $y'_S \geq \frac{1}{k}$.*

Even this weaker version turns out to be false, we will prove this in Chapter 3 via Proposition 49 after we build up the necessary machinery. This leads us to abandon the hope of a universal bound on how far we can push these values in an optimal solution. Thus, we relax the condition to no longer impose it for every $S \in \mathbb{F}$.

Conjecture 26 (Weak $\frac{1}{k}$ conjecture)**.** *There exists an integer $k \geq 0$ such that for all ideal clutters $\mathbb{F}$ and non-negative integer vectors c, there exists a set $S \in \mathbb{F}$ such that $y_S \geq \frac{1}{k}$ for some optimal dual solution y for the covering problem.*

This property would be immediately implied by the $\frac{1}{k}$ conjecture for the same k, so we can ask whether we can obtain a positive result in the other direction.

Conjecture 27 (Weak to strong $\frac{1}{k}$ conjecture)**.** *For every $\bar{k} \geq 1$, there exists an integer $f(\bar{k})$ such that every ideal clutter $\mathbb{F}$ satisfying the weak $\frac{1}{k}$ conjecture condition with $k = \bar{k}$ admits a $\frac{1}{f(\bar{k})}$-integer packing all positive integer vectors c.*

Note that Conjecture 27 is true for $\bar{k} = 1$, with $f(1) = 1$, so the MFMC property is equivalent to the weak $\frac{1}{k}$ conjecture condition holding with $k = 1$.

The reasoning behind this is similar to the packing problem where we use induction on $\tau(\mathbb{F}, c)$. Consider an optimal dual solution $\bar{y}$ with a $\bar{y}_S \geq 1$ for some S. We must have that $\tau(\mathbb{F}, c) \geq \tau(\mathbb{F}, c - \chi_S) + 1$ since every x that is feasible for the primal satisfies $\chi_S^T x \geq 1$. We also have that $\bar{y} - e_S$ is feasible for $\{M(\mathbb{F})^T y \geq c - \chi_S, y \geq 0\}$, and hence $\nu^*(\mathbb{F}, c - \gamma) \geq \nu^*(\mathbb{F}, c) - 1$. Since $\tau = \tau^* = \nu^*$, we must then have that the objective value for the Primal/Dual pair decreased by exactly 1. So any optimal integer $c - \chi_S$ packing augmented via adding a copy of S yields an optimal integer c packing.

For all other values of k however, this reasoning does not work since $c - \frac{1}{k}$ is not an integer vector, so to apply the weaker $\frac{1}{k}$ conjecture again we would have to scale it back up, but then the 'size' of c increases instead of decreasing. Not all hope is yet lost, since the conjecture remains open even if we impose that $f(k) = k$, which means that there is still a possibility that the $\frac{1}{k}$ conjecture is equivalent to its weak version.

In the next chapter, we formalize this notion of 'maximal dual value' for both $\mathbb{F}$ and each of its elements and attempt to understand its properties. We note that in the rest of this document, $\mathbb{F}$ is always taken to be an ideal clutter, unless otherwise specified.

Chapter 3

The dual potential

Let us first recall the LP relaxation for the covering problem and its dual :

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & M(\mathbb{F})x \geq \mathbf{1}, \quad (C) \\ & x \geq 0. \end{array}$$

$$\begin{array}{ll} \text{maximize} & \mathbf{1}^T y \\ \text{subject to} & M(\mathbb{F})^T y \leq c, \quad (D) \\ & y \geq 0. \end{array}$$

Definition 28. *Given a covering problem* $(\mathbb{F}, c)$ *and a set* $S \in \mathbb{F}$*, we define*

$$\lambda_{\mathbb{F},c}(S) = \max\{y_S : y \textit{ is an optimal solution of } (D)\}$$

and we call this quantity the dual potential *of* S*.*

For convenience, we will also write this quantity as $\lambda_c(S)$ when the clutter $\mathbb{F}$ is unambiguous, or even $\lambda(S)$ when c is taken to be the all 1 vector. We also define :

$$\lambda_c(\mathbb{F}) = \max_{S \in \mathbb{F}}(\lambda_{\mathbb{F},c}(S))$$

and again omit c when it is equal to the all 1 vector.

These quantities allow us to rephrase the conjectures from the previous section in a more convenient form, namely in the following way:

Conjecture (Weak $\frac{1}{k}$ conjecture)**.** *There exists an integer k such that for all ideal clutters $\mathbb{F}$, $\lambda_c(\mathbb{F}) \geq \frac{1}{k}$ for all non-negative integer vectors c.*

Conjecture (Weak to strong $\frac{1}{k}$ conjecture)**.** *For every $k \geq 1$, there exists an integer $f(k)$ such that every ideal clutter $\mathbb{F}$ satisfying $\lambda_c(\mathbb{F}) \geq \frac{1}{k}$ for all positive integer vectors c admits a $\frac{1}{f(k)}$-integer packing all non-negative integer vectors c.*

First, we try to characterize when $\lambda_c(S) > 0$. The answer to this is immediately given by the strict complementary slackness conditions.

Theorem 29 (Strict Complementary Slackness for a minimization primal)**.** *[27]*

Let $(P) = \min(c^T x, Ax \geq b, x \geq 0)$ and its dual $(D) = \max(b^T y, A^T y \leq c, y \geq 0)$ where $A \in \mathbb{R}^{m \times n}$ such that both are feasible, then following are equivalent for all $i \in [n]$:

- *There exists an optimal solution x to (P) with $x_i > 0$;*
- *For all optimal dual solutions y to (D), we have $(A^T y - c)_i = 0$.*

Similarly, for all $j \in [m]$ the following are equivalent:

- *There exists an optimal solution y to (D) with $y_j > 0$;*
- *For all optimal primal solution x to (P), we have $(Ax - b)_j = 0$.*

Applying this to our setting, we obtain the following result:

Lemma 30. *Let $\mathbb{F}$ be an ideal clutter and $c \geq 0$. Then $\lambda_c(S) > 0$ if and only if $|S \cap B| = 1$ for all covers $B \in b(\mathbb{F})$ satisfying $c(B) = \tau(\mathbb{F}, c)$.*

Proof. Since $\mathbb{F}$ is ideal, the set of optimal solutions to (C) is $\Gamma = \operatorname{conv}\{\chi_B : B \in b(\mathbb{F}), c(B) = \tau(\mathbb{F}, c)\}$. Strict complementary slackness then states that $\lambda_c(S) > 0$ if and only if $(M(\mathbb{F})x - \mathbf{1})_S = 0$ for all $x \in \Gamma$. We observe that $(M(\mathbb{F})x - \mathbf{1})_S \geq 0$ holds for all $B \in b(\mathbb{F})$, hence for every point in Γ as well. A convexity argument shows that equality is attained by every point in Γ if and only if it is attained at every extreme point of Γ, and we give a quick proof of this in proposition 31 for the sake of completeness. Therefore $\lambda_c(S) > 0$ if and only if $(M(\mathbb{F})\, \chi_B - \mathbf{1})_S = 0$ for all $B \in b(\mathbb{F})$ with $c(B) = \tau(\mathbb{F}, c)$, directly yielding the desired result. $\square$

Proposition 31. *Let X be a bounded closed convex set such that $a^T x \geq b$ holds for all $x \in X$. Then $a^T x = b$ for all $x \in X$ if and only if $a^T x = b$ for every extreme point of X.*

Proof. The forward direction is trivial and so we focus on the reverse direction. Let $x \in X$, if x is an extreme point, then the equality is already satisfied via the assumption. Otherwise, since X is a closed, x can be written as a convex combination of a finite number of extreme points of X, namely $x = \sum \lambda_i x^i$, where $\lambda \geq 0$, $\sum \lambda_i = 1$ and each x^i is an extreme point of X, hence we have:

$$a^T x = a^T(\sum \lambda_i x^i) = \sum \lambda_i (a^T x^i) = \sum \lambda_i b = b \sum \lambda_i = b.$$

$\square$

These elements play the most significant role in any discussion regarding optimal dual solutions, and hence the following definition.

Definition 32. *Let $\mathbb{F}$ be a clutter and $c \geq 0$, we define*

$$\overline{\mathbb{F}}_c = \{S \in \mathbb{F} : |S \cap B| = 1 \text{ for all } B \in b(\mathbb{F}) \text{ satisfying } c(B) = \tau(\mathbb{F}, c)\}$$

and refer to it as the c-core of $\mathbb{F}$.

Now we try to find an expression for $\lambda_c(\mathbb{F})$. The next two sections arrive at the same result but utilize different approaches.

3.1 An LP formulation

Consider a set S with $\lambda_c(\mathbb{F}) > 0$. The most straightforward approach to determine $\lambda_c(S)$ would be to write the natural LP for the problem. Let $\tau = \tau(\mathbb{F}, c)$ and then consider (M):

$$\begin{array}{ll} \text{maximize} & y_S \\ \text{subject to} & M(\mathbb{F})^T y \leq c, \quad (M) \\ & -\mathbf{1}^T y = -\tau, \\ & y \geq 0. \end{array}$$

$M(\mathbb{F})^T y \leq c, y \geq 0$ imply that y is feasible for the dual, while $1^T y = \tau$ implies that it's optimal, so $\lambda_c(S) = opt(M)$. To find this optimal value, we instead solve dual (N):

$$\begin{array}{ll} \text{minimize} & c^T x - \tau\gamma \\ \text{subject to} & M(\mathbb{F})x \geq \gamma\mathbf{1} + e_S, \quad (N) \\ & x \geq 0, \gamma \text{ free.} \end{array}$$

The first observation to make is that if (x, γ) is an optimal solution to (N) and $\gamma' > \gamma$, then there exists x' such that (x', γ') is also optimal for (N). To show this simply consider $x' = x + (\gamma' - \gamma)\,\chi_B$ where $B \in b(\mathbb{F})$ with $c(B) = \tau(\mathbb{F}, c)$. Then we have $x' \geq 0'$ and:

$$M(\mathbb{F})x' = M(\mathbb{F})x + M(\mathbb{F})(\gamma'-\gamma)\,\chi_B \geq \gamma\mathbf{1} + e_S + (\gamma'-\gamma)M(\mathbb{F})\,\chi_B \geq \gamma\mathbf{1} + e_S + (\gamma'-\gamma)\mathbf{1} = \gamma'\mathbf{1} + e_S.$$

And hence (x', γ') is feasible for (N) and we have $c^T[x + (\gamma' - \gamma)\,\chi_B] - \tau\gamma' = c^T x - \tau\gamma$.

Now to solve this LP, we can fix γ and solve $\min\{c^T x : M(\mathbb{F})x \geq \gamma\mathbf{1} + e_S, x \geq 0\}$ and then deduce the optimal value of (N). Treating γ as a parameter is particularly useful since for all $\gamma > 0$, we can scale x by $\frac{1}{\gamma}$ and look a the following LP (N_γ):

$$\begin{aligned} \text{minimize} \quad & \gamma c^T x - \tau\gamma \\ \text{subject to} \quad & M(\mathbb{F})x \geq \mathbf{1} + \frac{1}{\gamma} e_S, \quad (N_\gamma) \\ & x \geq 0. \end{aligned}$$

Note that by our initial observation we can disregard the cases of $\gamma \leq 0$, as we're interested in $\mathrm{opt}(N) = \lim_{\gamma \to \infty} \mathrm{opt}(N_\gamma)$, so for the rest of the discussion, we will assume $\gamma > 1$. A nice property of (N_γ) is that its feasible region corresponds to the feasible region of the covering problem (C) (given at the start of Chapter 3) with the inequality $\chi_S^T\, x \geq 1$ strengthened to $\chi_S^T\, x \geq 1 + \frac{1}{\gamma}$.

This gives an alternative proof of Lemma 30, as if we assume the existence of an optimal cover B with $|B \cap S| > 1$, then we have χ_B is feasible for (N_γ) and yields an objective value of $\gamma(c^T\,\chi_B) - \tau\gamma = \gamma(c(B) - \tau) = 0$, implying $\lambda_c(S) = \mathrm{opt}(M) = \mathrm{opt}(N) = 0$.

Now we assume $|S \cap B| = 1$ for all $b \in b(\mathbb{F})$ satisfying $c(B) = \tau$. we claim that $\chi_S^T\, x_\gamma = 1 + \frac{1}{\gamma}$ for all x_γ that is optimal for (N_γ). Indeed assume that x is feasible for (N_γ) with $\chi_S^T\, x > 1 + \frac{1}{\gamma}$, and consider $x' = (1 - \epsilon)x + \epsilon\,\chi_B$, where $B \in b(\mathbb{F})$ satisfies $c(B) = \tau$ and $\epsilon > 0$ is chosen small enough. x' is also a feasible solution (N_γ) with strictly smaller objective value.

We also know that the optimal value is always achieved at an extreme point of (N_γ), and we can easily characterize the extreme points of (N_γ) given the extreme points of (C) via the following result:

Lemma 33. *Let $P \subseteq \mathbb{R}^n$ be a pointed polyhedron and $Q = \{x : x \in P, c^T x \leq \alpha\}$ so Q is obtained from P by adding one more inequality. Then every extreme point of Q is either an extreme point of P or can be written as a convex combination of two extreme points of P, or as $x + r$ where x is an extreme point of P and r is an extreme ray of P.*

Proof. Let $\bar{x}$ be an extreme point of Q, $d = \dim(Q)$ and $d + t = \dim(P)$. Note that $t \geq 0$ as $Q \subseteq P$. First let us assume $t = 0$. $\bar{x}$ must satisfy d different facet constraints of Q with equality. But every facet of Q either corresponds to $c^T x \leq \alpha$ or a facet of P, and hence $\bar{x}$ must satisfy at least $d - 1$ facet constraints of P with equality, and hence the minimal face F of P containing it must have $\dim(F) \leq \dim(P) - (d - 1) = 1$. If F has dimension 0 then $\bar{x}$ is an extreme point of P. Otherwise, since P is pointed, its dimension 1 faces are either segments or rays. If F is a segment then each point in F can be written as a convex combination of its endpoints which are extreme points of P, and if F is a ray then $F = \{x + \lambda r, \lambda \geq 0\}$ where x is an extreme point of P and r is an extreme ray of P. To obtain the result of the theorem it is sufficient to notice that for all $\lambda > 0$, λr is also an extreme ray of P. For $t \geq 1$, the proof remains largely the same. Every implicit equality of Q that is not an implicit equality of P also corresponds to either a facet of P or to $c^T x \leq \alpha$. Hence $\bar{x}$ again must satisfy a total of at least $d + t - 1$ facet constraints of P with equality, which proves that the minimal face of F containing it satisfies $\dim(F) \leq \dim(P) - (d + t - 1) = 1$. □

Now consider an extreme point of $\{x \in \mathbb{R}^n : M(\mathbb{F})x \geq \mathbf{1} + \frac{1}{\gamma} e_S,\ x \geq 0\}$. Since the covering polyhedron is pointed, Lemma 33 implies that it is an extreme point of (C) or can either be written as the convex combination of 2 extreme points of the covering polyhedron or as $x + r$ where x is an extreme point and r is an extreme ray of the covering polyhedron. The extreme points of (C) are exactly $\chi_B, B \in b(\mathbb{F})$ while the extreme rays are $\alpha e_i, i \in V(\mathbb{F}), \alpha \geq 0$.

Since every optimal point must satisfy $\chi_S^T x = 1 + \frac{1}{\gamma}$, we can dismiss the case that $x = \chi_B$ given $\gamma > 1$. Consider the case where an optimal extreme point is of the form $x = \chi_B + \alpha e_i$. We then must have $i \in S$ and $\chi_S^T \chi_B = 1$ as otherwise $x' = \chi_B$ will also be a feasible solution to (N_γ) of smaller objective value. Clearly α must be equal to $\frac{1}{\gamma}$ and any choice of B and $i \in S$ yields a feasible solution to (N_γ). To minimize the objective value we must choose B such that $c(B) = \tau$ and i such that $c_i = \min\{c_j : j \in S\}$. This yields an objective value for (N_γ) equal to $\gamma(c^T x - \tau) = \gamma(\tau + \frac{1}{\gamma} c_i - \tau) = \min\{c_j : j \in S\}$.

Now consider $x = \lambda \chi_B + (1 - \lambda) \chi'_B$, $0 < \lambda < 1$. Since we know the optimal x satisfies $\chi_S^T x = 1 + \frac{1}{\gamma}$ and $\gamma > 1$, we must have, up to permuting B and B', $|B \cap S| > 1$, $|B' \cap S| = 1$ and $\lambda = \frac{1}{\gamma(|B \cap S| - 1)}$. Note that every choice of B' yields a feasible solution, so in order to minimize the objective function it is clear that we should pick B' to satisfy $c(B) = \tau$. This yields an objective value of:

$$\gamma(c^T x - \tau) = \gamma(\frac{1}{\gamma(|B \cap S| - 1)} c(B) + (1 - \frac{1}{\gamma(|B \cap S| - 1)})\tau - \tau) = \frac{c(B) - \tau}{|B \cap S| - 1}.$$

The optimal objective value of (N_γ), which is equal to the objective value of (N) for all $\gamma > 1$, is therefore given by the minimum of these quantities, proving the following result:

Theorem 34. *Let $\mathbb{F}$ be an ideal clutter, $c \geq 0$ and $S \in \mathbb{F}$ such that $|B \cap S| = 1$ for every c-optimal cover B. Then $\lambda_c(S) = \min(m_1, m_2)$ where :*

$$m_1 = \min_{e \in S}(c_e) \ \textit{ and } \ m_2 = \min_{B \in b(\mathbb{F}), |B \cap S| > 1} \frac{c(B) - \tau(\mathbb{F}, c)}{|B \cap S| - 1}.$$

This result will be the backbone of the following chapters, as it gives us a clear way to certify that $\lambda_c(S) \geq \lambda$ for any given value $\lambda < 1$: we simply check that $\frac{c(B)-\tau(\mathbb{F},c)}{|B \cap S|-1} \geq \lambda$ for all $B \in b(\mathbb{F}), |B \cap S| > 1$, and if we want to certify that $\lambda_c(S) < \lambda$ then we just need to exhibit $B \in b(\mathbb{F})$ such that $|B \cap S| > 1$ and $\frac{c(B)-\tau(\mathbb{F},c)}{|B \cap S|-1} < \lambda$. This is exactly what we use to disprove Conjecture 25 in Section 3.4, but first we explore alternative ways to arrive at this result.

3.2 Mate conditions

In this section we will be giving an alternative proof of Theorem 34 that's obtained by tinkering with the objective function. We start with the following result.

Theorem 35. *Let $\mathbb{F}$ be an ideal clutter, $c \geq 0$ and $S \in \mathbb{F}$. Then for all $\lambda \leq \min_{e \in S}(c_e)$, the following are equivalent:*

- $\lambda_c(S) \geq \lambda$;
- $\nu^*(\mathbb{F}, c - \lambda\, \chi_S) = \nu^*(\mathbb{F}, c) - \lambda$;
- $\tau(\mathbb{F}, c - \lambda\, \chi_S) = \tau(\mathbb{F}, c) - \lambda$.

Note that the condition $\lambda \leq \min_{e \in S}(c_e)$ ensures that $c - \lambda\, \chi_S \geq 0$ and that τ is finite.

Proof. Let $S \in \mathbb{F}$ satisfy $\lambda_c(S) > 0$. For every dual solution y for the c covering problem such that $y_S \geq \lambda$ we have that $y' = y - \lambda e_S$ is a feasible dual solution for the $c - \lambda\, \chi_S$ covering problem on $\mathbb{F}$. We claim that if y is optimal for the c dual, then y' must be optimal for the $c - \chi_S$ dual. This is easy to see since for any feasible $\bar{y}$ for the $c - \lambda\, \chi_S$ dual, $\bar{y}' = \bar{y} + \lambda y_S$ is a feasible solution for the c dual with objective value $\mathbf{1}^T \bar{y} + \lambda$, therefore $\nu^*(\mathbb{F}, c) \leq \nu^*(\mathbb{F}, c - \lambda\, \chi_S) + \lambda$. This proves our claim since if we assume that y is optimal we will have $\mathbf{1}^T y' = \mathbf{1}^T y - \lambda = \nu^*(\mathbb{F}, c) - \lambda \leq \nu^*(\mathbb{F}, c - \lambda\, \chi_S)$. Therefore it becomes clear that the existence of a dual optimal solution with $y_S \geq \lambda$ is equivalent to $\nu(\mathbb{F}, c) - \lambda = \nu^*(\mathbb{F}, c - \lambda\, \chi_S)$. The third characterization follows immediately from the 2nd via strong duality and the fact that the covering polyhedron is integral. $\square$

We then characterize exactly when $\tau(\mathbb{F}, c - \lambda\,\chi_S) = \tau(\mathbb{F}, c) - \lambda$ holds via the notion of λ-mates.

Definition 36. *For a set $S \in \mathbb{F}$, we say that $B \in b(\mathbb{F})$ is a λ-mate of S if $c(B) - \lambda|S \cap B| < \tau(\mathbb{F}, c) - \lambda$.*

Theorem 37. *Let $S \in \mathbb{F}$, and $\lambda \leq c_e$ for all $e \in S$, then the following are equivalent :*

- *$\lambda_c(S) \geq \lambda$;*
- *S has no λ mate.*

Proof. We have shown that $\lambda_c(S) \geq \lambda$ is equivalent to $\tau(\mathbb{F}, c - \lambda\,\chi_S) = \tau(\mathbb{F}, c) - \lambda$ $(*)$. From the proof of Theorem 35 we already know $\tau(\mathbb{F}, c - \lambda\,\chi_S) \leq \tau(\mathbb{F}) - \lambda$, so we only need to ensure that $\tau(\mathbb{F}, c - \lambda\,\chi_S) \geq \tau(\mathbb{F}) - \lambda$. We know $\mathbb{F}$ is ideal, so $\tau(\mathbb{F}, c - \lambda\,\chi_S) = \min_{B \in b(\mathbb{F})}((c - \lambda\,\chi_S)^T\,\chi_B) = \min_{B \in b(\mathbb{F})}(c^T\,\chi_B - \lambda|B \cap S|)$. Clearly every set B with $|B \cap S| = 1$ achieves value $c(B) - \lambda \geq \tau(\mathbb{F}, c) - \lambda$. For each element B with $|B \cap S| > 1$, we must have $c^T\,\chi_B - \lambda|B \cap S| \geq \tau(\mathbb{F}, c) - \lambda$ or equivalently B is not a λ-mate of S. $\square$

The result of Theorem 34 follows immediately as corollary of this result, simply noting that $\lambda_c(S) = \max(\lambda : S \text{ has no } \lambda\text{-mate})$.

We can also deduce that these conditions are equivalent to ensuring that every optimal c cover is also an optimal $c - \lambda\,\chi_S$ cover, giving another characterization of $\lambda_c(S)$ that we explore in the next section.

3.3 The cone of tight constraints

Let $b(\mathbb{F})_c$ be the set of elements in $b(\mathbb{F})$ that are c-optimal, i.e. $b(\mathbb{F})_c = \{B \in b(\mathbb{F}), c(B) = \tau(\mathbb{F}, c)\}$. Then since $\mathbb{F}$ is ideal, we must have that $b(\mathbb{F})_c$ are the extreme points of a face F_c of the covering polyhedron. As we have seen in the previous section, $\lambda_c(S) \geq \lambda \iff b(\mathbb{F})_c$ remain optimal for $c' = c - \lambda\,\chi_S$, which is equivalent that F_c is contained within the optimal face for c'. Therefore, we can use the cone of tight constraints at F_c to characterize the dual potential. First we recall its definition and some of its key properties.

Definition. *Let F be a face of a polyhedron $\{x : Ax \geq b\}$, then we define the set of tight constraint at F and denote it by $T_F = \{j \in [m] : A_j x = b_j$ for all $x \in F\}$. We also define $\text{cone}(F) = \{\sum_{j \in T_F} \lambda_j (A_j)^T : \lambda \geq 0\}$[1] and refer to it by the cone of tight constraints at F[2].*

[1]Note that this is different from the usual definition of $\text{cone}(S) = \{\sum_{B \in S} \lambda_B\,\chi_B,\ \lambda \geq 0\}$,

[2]This is also sometimes referred to as the normal cone in the literature.

Proposition 38. *1. F is optimal for c if and only if $c \in \text{cone}(F)$.*

2. Let $\bar{x}$ be a point in the relative interior of F, then $T_F = \{j \in [m] : A_j\bar{x} = b_j\}$.

3. If F is bounded then $T_F = \{j \in [m] : A_j x = b_j$ for all x that are extreme points of $F\}$.

Proof. To prove the first statement, consider $c \in \text{cone}(F)$. Then we have $c^T = \sum_{j \in T_F} \lambda_j A_j$, $\lambda \geq 0$. Consider the problem $(LP) = \min\{c^T x : Ax \geq b\}$ and it's dual $(LD) : \max\{b^T y : A^T y = c, y \geq 0\}$. We claim $y \in \mathbb{R}^m$ defined by $y_{T_F} = \lambda$ and $y_j = 0$ for all $j \notin T_F$ is feasible for (LD). Indeed we have for all $i \in [n]$: $(A^T y)_i = \sum_j A_{i,j} y_j = \sum_{j \in T_F} A_{i,j}\lambda_j = c_i$. We also have for all $x \in F$ that $c^T x = \sum_{j \in T_F} \lambda_j A_j x = \sum_{j \in T_F} \lambda_j b_j = b^T y$ and hence x and y are both optimal and F is contained within the optimal face of (LP). The reverse direction follows from the complementary slackness conditions. Given c such that every $x \in F$ is optimal for $(LP) = \min\{c^T x : Ax \geq b\}$, consider y optimal for the dual (LD). Then for each $j \notin T_F$, we know there exists $x \in F$ such that $A_j x < b$, since x is also optimal this means that $y_j = 0$ by the C.S. conditions. This means that $c = A^T y = \sum_{j \in T_F} y_j A_j$ and $y \geq 0$.

The proof of the 2nd statement follows from simple arguments from polyhedral theory. Let $\bar{x}$ be in the relative interior of F. Clearly $T_F \subseteq \{j \in [m], A_j\bar{x} = b_j\}$. For the reverse inclusion, consider $j \in [m]$ such that $A_j\bar{x} = b_j$. The set of optimal solutions of $\min(A_j^T x, x \in P)$ is a face of P, this is because it can be written as $\{x, x \in P, A_j x = \mathit{opt}\}$. Since $A_j x \geq b_j$ is valid for P, then we must have $\min(A_j x, x \in P) \geq b_j$, and this is satisfied with equality for $\bar{x}$, and hence the set of optimal solutions must be a face containing $\bar{x}$. But the minimal face containing $\bar{x}$ is F as $\bar{x}$ was taken to be in the relative interior, and hence all $x \in F$ must be optimal, therefore $A_j x = b_j$ for all $x \in F$.

The third result follows from a convexity argument. If F is bounded then every point in F can be written as a convex combination of its extreme points, so ensuring $A_j x = b_j$ for all extreme points of F implies the result for all $x \in F$. □

Now we apply this result for the covering polyhedron.

Definition 39. *Let $\mathbb{F}$ be a clutter, $c \in \mathbb{R}^V_{\geq 0}$ and $V_c = \cup b(\mathbb{F})_c$ is the set of ground elements that appear in an optimal cover.*[3] *Then we define:*

$$\text{cone}(b(\mathbb{F})_c) = \{\sum_{S \in \bar{\mathbb{F}}_c} \gamma_S \, \chi_S + \sum_{i \notin V_c} \alpha_i e_i, \gamma, \alpha \geq 0\}$$

and refer to it as the cone of tight constraints *at c.*

[3] We denote by $\cup X = \cup_{Y \in X} Y$ and $\cap X = \cap_{Y \in X} Y$

Proposition 40. *Let $S \in \mathbb{F}$, and $0 \leq \lambda \leq c_e$ for all $e \in S$. Then the following are equivalent :*

(1) $\lambda_c(S) \geq \lambda$*;*

(2) S *has no* λ *mate;*

(3) $c - \lambda\, \chi_S \in \mathrm{cone}(b(\mathbb{F})_c)$.

In order to get more out of (3) we try to understand more about the cone of tight constraints at c. Since we have defined the cone via its generators, we ask if we could also find a polyhedral formulation. The answer is yes, and the next result gives such a formulation.

Theorem 41. *Let $\mathbb{F}$ be an ideal clutter and $c \geq 0$, then :*

$$\mathrm{cone}(b(\mathbb{F})_c) = \{d : d \geq 0, (\chi_{B'} - \chi_B)^T d \geq 0 \textit{ for all } B \in b(\mathbb{F})_c \textit{ and } B' \in b(\mathbb{F})\}.$$

Proof. $d \in \mathrm{cone}(b(\mathbb{F})_c)$ if and only if the face defined by $b(\mathbb{F})_c$ is optimal for $(LP) : \min\{d^T x : M(\mathbb{F}) \geq 1, x \geq 0\}$. The extreme points of this face are exactly the elements in $b(\mathbb{F})_c$ hence it is sufficient to check that every element in $b(\mathbb{F})_c$ is optimal for (LP). Since the optimal value of (LP) is always achieved at an extreme point, proving that an element $B \in b(\mathbb{F})_c$ is optimal for (LP) is the same as ensuring that it yields an objective value at least as low as any other extreme point of the covering polyhedron. $\mathbb{F}$ is ideal, and so the extreme points of the covering polyhedron are exactly χ_S, $S \in b(\mathbb{F})$. Therefore $d^T \chi_{B'} \geq d^T \chi_B$ for all $B' \in b(\mathbb{F})$ implies that B is optimal for (LP). □

This formulation has a lot of redundant inequalities, and we can cut down on many of them by simply considering a point in the interior of $b(\mathbb{F})_c$, giving a more compact formulation.

Theorem 42. *Let $\bar{x} \in \mathrm{relint}(\mathrm{conv}(\{\chi(B), B \in b(\mathbb{F})_c\})$[4], then we have :*

$$\mathrm{cone}(b(\mathbb{F})_c) = \{d : d \geq 0, (\chi_B - \bar{x})^T d \geq 0 \textit{ for all } B \in b(\mathbb{F})\}.$$

Proof. One of the directions is easy to prove. Indeed, if $d \in \mathrm{cone}(b(\mathbb{F})_c$ then clearly $d \geq 0$ and $\bar{x}^T d = d^T \bar{x} = \tau(\mathbb{F}, d) \leq d(B) = \chi_B^T d$ for all $B \in b(\mathbb{F})$. For the reverse direction, let $d \geq 0$ satisfy $(\chi_B - \bar{x})^T d \geq 0$ for all $B \in b(\mathbb{F})$. Since $d \geq 0$, we know the covering problem is not unbounded and therefore admits an optimum. Since $\mathbb{F}$ is ideal the extreme points are exactly χ_B, $B \in b(\mathbb{F})$, so to prove $d \in b(\mathbb{F})_c$ it is sufficient to prove $d(B') \geq d(B)$ for all $B' \in b(\mathbb{F})$ and $B \in b(\mathbb{F})_c$. Let $\bar{x} = \sum_{i=1}^k \lambda_i\, \chi_{B_i}, \lambda \geq 0, \sum \lambda_i = 1$. Then we have $(\chi_{B_i} - \bar{x})^T d \geq$

[4] relint denotes the relative interior of a set, with the convention that $\mathrm{relint}(\{t\}) = \{t\}$

0 for all $1 \leq i \leq k$, implying $\lambda_i(\chi_{B_i} - \bar{x})^T d \geq 0$. Summing the LHS over all i we obtain $(\sum_i \lambda_i \chi_{B_i} - \bar{x})^T d = (\bar{x} - \bar{x})^T d = 0^T d = 0$, so all the inequalities must hold at equality, therefore we have $d(B_i) = \bar{x}^T d$ for all i, and thus $d(B) = \bar{x}^T d$ for all $B \in b(\mathbb{F})_c$, since the minimal face containing all of $B_1, \ldots, B_k$ is $b(\mathbb{F})_c$. Therefore, we immediately obtain $d(B') \geq d(B)$ for all $B' \in b(\mathbb{F})$ and $B \in b(\mathbb{F})_c$ as desired. □

A big step towards understanding the properties of the cone would be to understand its facets: exactly when do the non-negativity constraints correspond to facets, and which elements in $B \in b(\mathbb{F})$ yield facet defining inequalities. A small inconvenience is that this formulation depends on a choice of $\bar{x}$, and perhaps this choice might affect which elements of the blocker yield facets and which correspond to equality constraints. In the next two propositions, we first dispel these concerns, then we characterize the equality constraints.

Proposition 43. *Let $x^1, x^2 \in \operatorname{relint}(\operatorname{conv}(\{\chi(B), B \in b(\mathbb{F})_c\})$, and $C_i = \{d : d \geq 0, (\chi_B - x^i)^T d \geq 0$ for all $B \in b(\mathbb{F})\}$. We then have :*

1. *$(\chi_B - x^1)^T d \geq 0$ is a facet of C_1 if and only if $(\chi_B - x^2)^T d \geq 0$ is a facet of C_2.*
2. *$(\chi_B - x^1)^T d = 0$ for all $d \in C_1$ if and only if $(\chi_B - x^2)^T d = 0$ for all $d \in C_2$.*
3. *$d_i \geq 0$ is a facet of C_1 if and only if $d_i \geq 0$ is a facet of C_2.*
4. *$d_i = 0$ for all $d \in C_1$ if and only if $d_i = 0$ for all $d \in C_2$*

Proof. Items 3 and 4 are immediate since $C_1 = C_2$ and item 2 follows from $(x^1)^T d = (x^2)^T d$ for all $d \in C_i$. To prove item 1, we observe that if $(\chi_B - x^1)^T d \geq 0$ is a facet of C_1, then it must hold at equality for $r-1$ affinely independent points of C_1 (where $r = \dim(C_i)$) and hence $(\chi_B - x^2)^T d \geq 0$ must also hold at equality for $r-1$ affinely independent points in C_2 by simply considering the same set of points, and thus it must correspond to a facet of C_2. □

Proposition 44. *Let $C = \operatorname{cone}(b(\mathbb{F})_c)$, then we have :*

1. *$(\chi_B - \bar{x})^T d = 0$ for all $d \in C$ if and only if $B \in b(\mathbb{F})_c$.*
2. *$d_i = 0$ for all $d \in C$ if and only if $i \in \cup b(\mathbb{F})_c$ and $i \notin \cup \overline{\mathbb{F}}_c$.*

Proof. The proof for item 1 follows from the same argument as for Theorem 42, while the 2nd item follows from the initial definition of the cone. □

The most straightforward condition that implies $B \in b(\mathbb{F})$ yields a facet defining inequality for $\mathrm{cone}(b(\mathbb{F})_c)$ is the non-redundancy of the constraint, that is if we remove $(\chi_B - x)^T d \geq 0$ from the formulation in Theorem 42 we would obtain a different cone. This leads us to the following result:

Theorem 45. *Let $S \in \mathbb{F}$ satisfy $\lambda(S) < \min\{c_e : e \in S\}$ and let $M(S) = \{B \in b(\mathbb{F}) : \lambda_c(S) = \frac{c(B) - \tau(\mathbb{F}, c)}{|B \cap S| - 1}\}$, then the following holds:*

- *If $M(S) = \{B\}$, then $(\chi_B - x)^T d \geq 0$ is a facet of* $\mathrm{cone}(b(\mathbb{F})_c)$.
- *If $|M(S)| \geq 2$, then at least 2 different elements $B, B' \in M(S)$ generate facet defining inequalities of the cone.*

Proof. We have that $c \in \mathrm{cone}(b(\mathbb{F})_c), c - \lambda_c(S)\, \chi_S \in \mathrm{cone}(b(\mathbb{F})_c)$ but $c - (\lambda_c(S) + \epsilon) \notin \mathrm{cone}(b(\mathbb{F})_c)$ for any strictly positive ϵ, hence we must have that $c - \lambda_c(S)\, \chi_S$ satisfies at least one facet inequality with equality that c did not. Since $\lambda_c(S) < \min\{c_e : e \in S\}$ no non-negativity constraint became tight. If $M(S) = \{B\}$ then $(\chi_B - x)^T d \geq 0$ is the only inequality satisfied at equality by $c - \lambda_c(S)\, \chi_S$ but not c, hence it must correspond to a facet of the cone. In the case that $|M(S)| \geq 2$, then we consider two cases. First, if $c - \lambda_c(S)\, \chi_S$ lies in the relative interior of a facet, then all the elements in $M(S)$ must correspond to this same facet. Otherwise, if $c - \lambda_c(S)\, \chi_S$ lies on a face of dimension less than or equal to $\dim(b(\mathbb{F})_c) - 2$, then we must it must lie on the intersection of at least 2 facets, and hence at least two different elements of $M(S)$ yield different facet defining inequalities. $\square$

We end this section with a small discussion of the consequences of every facet corresponding to a non-negativity constraint.

Simple Conic Clutters

Suppose that all the facets of $\mathrm{cone}(b(\mathbb{F})_c)$ correspond to a non negativity constraint.[5] This implies that $\mathrm{cone}(b(\mathbb{F})_c)$ is the intersection of a linear space and the non-negative orthant, and hence the idea behind the definition of simple conic clutters.

Definition 46. *A clutter $\mathbb{F}$ is* simple conic *if* $\mathrm{cone}(b(\mathbb{F})_c)$ *is the intersection of a linear space, and the non-negative orthant for all $c \geq 0$.*

[5]Note that this does not imply that no blocker element inequality defines a facet, but in the case they do, it coincides with a non negativity facet.

It turns out that this property is very strong, and implies that an integer optimal dual solution can always be found and can be constructed in an almost arbitrary fashion, as the clutter satisfies the packing-analog conjecture with $k = 1$.

Theorem 47. *Let $\mathbb{F}$ be an ideal simple conic clutter, then $\mathbb{F}$ has the MFMC property.*

Proof. The reasoning behind this result is quite simple. Consider an arbitrary set $S \in \mathbb{F}$ with $\lambda_c(\mathbb{F}) > 0$, and consider $c' = c - \chi_S$. If $c' \notin \mathrm{cone}(b(\mathbb{F})_c)$, then it must violate at least one facet constraint. However, all facets correspond to non negativity constraints, so as long as $c_S \geq \mathbf{1}_S$, then c' remains in the cone and hence $y_S \geq 1$. Therefore $\mathbb{F}$ satisfies the packing analog conjecture with $k = 1$, and thus has the MFMC property. □

However, being simple conic is a very hard condition to satisfy, and it is plausible that no ideal clutter is simple conic. This leads us to relax this condition as follows :

Definition 48. *Given an ideal clutter $\mathbb{F}$ and non-negative vector c, we say clutter $\mathbb{F}$ is* c-simple conic *if* $\mathrm{cone}(b(\mathbb{F})_{c'})$ *is the intersection of a linear space, and the non-negative orthant for all $0 \leq c' \leq c$, c' integer.*

This becomes a more reasonable property to satisfy, and while we can no longer certify the MFMC property, we can still certify that $\tau(\mathbb{F}, c) = \nu(\mathbb{F}, c)$ using a similar proof to the previous theorem, since $c' = c - \chi_S \leq c$ and hence we can repeat the argument on c' to construct an integer optimal packing.

3.4 Next steps

Now we use our characterization of $\lambda_{\mathbb{F},c}(S)$ to disprove Conjecture 25.

Proposition 49. *For every $k \geq 1$, there exists a clutter $\mathbb{F}$ that has the MFMC property, $c \geq 0$ integer and $S \in \mathbb{F}$ such that $0 < \lambda_c(S) < \frac{1}{k}$.*

Proof. We prove this by construction for each $k \geq 3$ (The result will then also be implied for $k \in \{1, 2\}$). Consider $\mathbb{F}$ the clutter of s-t paths in the graph $G_k = (s \cup A \cup B \cup t, E)$, where A is a clique of k vertices $a_i, i \in [k]$, B is a clique of k vertices $b_i, i \in [k]$. These clique edges will have some arbitrarily high cost M. We also have edges (s, a_i) and (b_i, t) with costs 2 for all $2 \leq i \leq k - 1$, and edges $(s, a_1), (s, a_k), (b_1, t), (b_k, t)$ with cost 1. We also have edges (a_i, b_i) for all $i \in [k]$, and (a_i, b_{i+1}) for all $i \in [k - 1]$, both with cost 1. An illustration

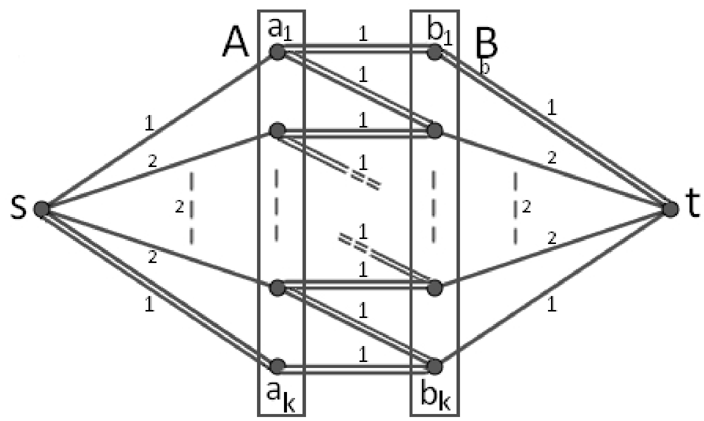

Figure 3.1: The graph G_k, edges in $E(A), E(B)$ of cost M are not drawn. Edges in P are represented by double lines.

of this graph is given in Figure 3.1.There are only 3 cuts of size less than M: $B_1 = \delta(s)$ with $c(B_1) = 2k-2$, $B_2 = \delta\{s \cup A\}$ with $c(B_2) = 2k-1$, and $B_3 = \delta(s \cup A \cup B)$ with $c(B_3) = 2k-2$.We have $\tau(\mathbb{F}, c) = 2k-2$, with only B_1, B_3 being optimal covers. Now consider the path $P \in \mathbb{F}$ that starts from s and in order visits a_k,b_k,a_{k-1},b_{k-2},...,a_2, b_2, a_1, b_1, t. Clearly $|P \cap B_1| = |P \cap B_3| = 1$ hence $\lambda(\mathbb{F}) > 0$. Meanwhile, B_2 gives a strong bound of $\lambda(P)$. We have $|P \cap B_2| = 2k-1$, and thus we have $\lambda(P) \leq \frac{c(B)-\tau}{|B \cap S|-1} = \frac{2k-1-(2k-2)}{2k-1-1} = \frac{1}{2k-2}$. The path P then satisfies $0 < \lambda(P) \leq \frac{1}{2k-2} < \frac{1}{k}$ for all $k \geq 3$. Since $\mathbb{F}$ is in the family of clutters of s-t paths it has the MFMC property, concluding our proof. $\square$

This result is a crucial point in our strategy when tackling our original question of how to approach relaxations of the MFMC property. We now know a lot about $\lambda_{\mathbb{F},c}(S)$, including the fact being non zero is not enough to certify any universal lower bound. This shows that there is a fundamental difference between the behavior of dual variables in the packing problem on perfect clutters and the covering problem on ideal and even MFMC clutters. If we want to continue with the idea of mimicking the packing problem, it is then hopeless to consider arbitrary sets $S \in \mathbb{F}$ with $\lambda(S) \geq 0$: While we may still be able to prove a lower bound on $\lambda_{\mathbb{F},c}(S)$ as function of $\tau, ||c||_1$, or other parameters, it is unlikely that they would be very helpful. This leads us to consider special elements of the clutter and try to find a universal lower bound on their λ-values. There are multiple options to consider. We could focus on sets $S \in \mathbb{F}$ of minimal cardinality as these are intuitively more likely to appear in an optimal packing. There are other intuitively justifiable choices, such as sets whose members have a high average c-cost, but we forego these ideas in favor of the most straightforward approach: Pick the set S^* such that $\lambda_{\mathbb{F},c}(S^*) = \max_{S\in\mathbb{F}}\{\lambda_{\mathbb{F},c}(S)\}$, and check if we can find a universal lower bound on $\lambda_{\mathbb{F},c}(S^*)$. This corresponds exactly to Conjecture 26, as $\lambda_{\mathbb{F},c}(S^*)$ is the definition of $\lambda_c(\mathbb{F})$. This will be the focus of the rest of the thesis.

Chapter 4

LP Formulations

Our objective in this section is to investigate Conjecture 26. While we do have an expression for $\lambda_c(\mathbb{F})$, namely

$$\lambda_c(\mathbb{F}) = \max_{S\in\mathbb{F}}\{\lambda_{\mathbb{F},c}(S)\} = \max_{S\in\mathbb{F}}\left\{\min_{B\in b(\mathbb{F}),|B\cap S|>1}\left\{\frac{c(B)-\tau(\mathbb{F},c)}{|B\cap S|-1}\right\}\right\}.$$

it is quite difficult to untangle the max-min function and the formula is generally difficult to work with. Contrary to the formula for $\lambda_c(S)$ given in Theorem 34, where every element of the blocker provides an upper bound, a single (S,B) pair does not yield an upper nor lower bound on $\lambda_c(\mathbb{F})$. Therefore, we attempt to find a reasonable proxy for $\lambda_c(\mathbb{F})$, say $f(\mathbb{F},c)$, and examine the behavior of this proxy instead. The reasoning behind this is simple. If we can prove that $f(\mathbb{F},c) > \frac{1}{k}$ for some constant k, then this would be a nice result supporting Conjecture 26. On the other hand, if we can construct examples where $f(\mathbb{F},c)$ is arbitrarily small, then this would be a good argument against the conjecture, and in the case that $f(\mathbb{F},c)$ is an upper bound to $\lambda_c(\mathbb{F})$, it would completely refute it.

Our approach will be mainly LP based. If we can write an integer program (IP_λ) whose solution sets correspond exactly to sets $S \in \mathbb{F}$ satisfying $\lambda_{\mathbb{F},c}(S) \geq \lambda$, then we would have $\lambda_c(\mathbb{F}) = \max\{\lambda : (IP_\lambda)$ is feasible$\}$. But since integer programs are difficult to work with, we can consider the LP relaxations (LP_λ), and consider $f(\mathbb{F},c) = \max\{\lambda : (LP_\lambda)$ is feasible$\}$ as a proxy for $\lambda_c(\mathbb{F})$. Note that with this approach, f will always be an upper bound for $\lambda_c(\mathbb{F})$.

In this chapter, we consider two possible formulations. The first is a high dimensional formulation with indicator variables for each element of the clutter, while the second is more compact with variables corresponding to ground elements.

4.1 Set variable LP

We introduce a variable y_S for every set of the clutter (note that we can restrict this to sets in $\overline{\mathbb{F}}_c$, i,e, sets satisfying $\lambda_c(S) > 0$), we consider the polyhedron $L_c(\lambda)$:

$$\left.\begin{aligned}
&\sum_{S\in\mathbb{F}} y_S = 1, && (1)\\
&\sum_{S\in\mathbb{F}} |B\cap S| y_S \leq \frac{1}{\lambda}(c(B)-\tau)+1, \text{ for all } B\in b(\mathbb{F}), && (2)\\
&y \geq 0. && (3)
\end{aligned}\right\} \quad L_c(\lambda)$$

Due to (1) any integer point in $L_c(\lambda)$ corresponds to e_S for some $S \in \mathbb{F}$, and (2) implies that $\lambda_c(S) \geq \lambda$ due to Theorem 34. We try to gauge how strong this relaxation is. First we note that this polyhedron is always non empty as long as we impose a very lenient upper bound on λ.

Proposition 50. *Let $\mathbb{F}$ be a clutter, $c \geq 0$ and $\lambda \leq \tau(\mathbb{F}, c)$, then $L_c(\lambda)$ is non-empty.*

Proof. We prove this by exhibiting a feasible point. Let $\bar{y}$ be an optimal dual solution for the covering problem, that is $\mathbf{1}^T\bar{y} = \tau(\mathbb{F}, c)$ and $\sum_{S:e\in S} \bar{y}_S \leq c_e$. Let $y' = \frac{\bar{y}}{\tau}$, we claim y' is feasible for $(L_c(\lambda))$. We clearly have $\mathbf{1}^T y' = 1$, and we also have :

$$\sum_{S\in\mathbb{F}} |B\cap S| y'_S = \sum_{e\in B}\sum_{S:e\in S} y'_S \leq \sum_{e\in B}\frac{c_e}{\tau} \leq \frac{c(B)}{\tau} = \frac{1}{\tau}(c(B)-\tau)+1$$

and therefore y' is feasible for $L_c(\tau(\mathbb{F}, c))$. Since (2) only gets more relaxed the more we decrease λ, y' is feasible for all smaller values of λ as well. $\square$

Despite this being a positive result, it being this general reduces any hope of non-emptiness implying strong properties $\lambda(\mathbb{F})$ and in particular, we can certify the following:

Proposition 51. *There does not exists a constant $\alpha > 0$ such that for all ideal clutters $\mathbb{F}$, $L_c(\lambda)$ is non-empty $\Rightarrow \lambda_c(\mathbb{F}) \geq \min(\frac{\lambda}{\alpha}, 1)$.*

Proof. The proof follows from the simple fact that we can set τ to be arbitrarily large while still having a bound on $\lambda(\mathbb{F})$. If we consider $\mathbb{F} = \mathbb{Q}_6^n$[1], then we have $\tau = 2n$ but $\lambda(\mathbb{F}) = \frac{1}{2}$. $\square$

[1]Recall $\mathbb{Q}_6^n$ is the clutter obtained by combining n independent copies of $\mathbb{Q}_6$, which we have already encountered in section 1.2.3

This directs us towards strengthening the relaxation if we want to obtain a positive result. We can consider the Chvátal-Gomory closure of $L_c(\lambda)$ and check whether we can leverage more out of its feasibility. This is a well-studied operation in polyhedral theory [7], and we give a quick reminder of its construction and some of the main results.

Definition 52. *Given a polyhedron $P = \{x : Ax \leq b\}$, and $\alpha^T x \leq \beta$ a valid inequality for P where α is integer, the inequality $\alpha^T x \leq \lfloor \beta \rfloor$ is valid for the integer hull of P, that is $P_I = \mathrm{conv}(\{x \in P : x \text{ integer.}\})$ and is called Chvátal-Gomory cut of P, also referred to as a CG-cut.*

Definition 53. *For a polyhedron P, we define:*

$$CG(P) = \{x \in P : \alpha^T x \leq \lfloor \beta \rfloor \text{ for all CG cuts of } P\}$$

and refer to it as the Chvátal-Gomory closure of P.

Proposition 54 (Properties of the CG closure)**.** *[7]*

1. *$CG(P)$ is a polyhedron.*
2. *$P_I \subseteq CG(P) \subseteq P$.*
3. *If $P \subseteq Q$ then $CG(P) \subseteq CG(Q)$.*
4. *If A is rational, then there exists a finite integer $k \geq 0$ such that $P_I = CG^k(P)$[2]. The smallest such k is referred to as the **CG rank** of P.*

Much to our surprise, it turns out that the CG closure of $L_c(\lambda)$ corresponds exactly to its integer hull. This follows from rewriting constraint (2) as :

$$0 \leq \sum_{S \in \mathbb{F}} y_S \left(\frac{1}{\lambda}(c(B) - \tau) + 1 - |B \cap S|\right), \quad \forall B \in b(\mathbb{F}). \quad (2)'$$

Then we utilize the following lemma :

Lemma. *Let $P \subseteq \{x : \sum_{i \in [n]} a_i x_i \geq 0 (i),\ \sum_{i \in [n]} x_i = 1 (ii),\ x \geq 0\}$. Then for all $a_i < 0$, we have that $x_i = 0$ is valid for $CG(P)$.*

[2] $CG^k(P)$ denotes the k-th iteration of the CG closure, that is $CG^k(P) = CG(CG^{k-1}(P))$ with the convention that $CG^0(P) = P$.

Proof. Let $T = \{i \in [n] : a_i > 0\}$, and let $a_j < 0$. Then : $\sum_{i \in T} a_i x_i + a_j x_j \geq 0$ is valid for P as we are simply adding non negativity constraints to (i). Normalizing by $|a_j|$ and adding (ii) we get that $\sum_{i \in T}(\frac{a_i}{|a_j|} + 1)x_i + \sum_{i \in [n] \setminus T \setminus \{j\}} x_i \geq 1$ is valid for P. By simply adding non negativity constraints we can increase all the coefficients to match the maximal coefficient, so we get $M \sum_{i \in [n] \setminus \{j\}} x_i \geq 1$. This implies that $\sum_{i \in [n] \setminus \{j\}} x_i \geq 1$ is valid for the $CG(P)$, which along with (ii) implies $x_j = 0$ is valid for $CG(P)$. □

Applying this to $L_c(\lambda)$, we have that $(\frac{1}{\lambda}(c(B) - \tau) + 1 - |B \cap S|) < 0$ if B is a λ-mate of S, and hence $y_S = 0$ is valid for $CG(L_c(\lambda))$ as long as S has a λ-mate, while any set without a λ-mate yields a feasible integer point, and therefore $CG(L_c(\lambda)) = \mathrm{conv}(e_S, S$ has no λ-mate).

This leads us to abandon the CG closure, and instead attempt to manually find valid inequalities for the integer hull that we can add to $L_c(\lambda)$ while still avoiding an exact formulation.

We start by noticing that we can impose a stronger version of (2) for any subset of $\mathbb{F}$, namely: $\sum_{S \in K} |B \cap S| y_S \leq (\frac{1}{\lambda}(c(B) - \tau) + 1) \sum_{S \in K} y_S$ for any $B \in b(\mathbb{F})$ and $K \subseteq \mathbb{F}$. Finding out which pairs B, K are useful to impose is a very interesting question, but it is very hard to assess an individual pair's contribution. Thus we opt to first omit any sets not in $\overline{\mathbb{F}}_c$, then add many related inequalities at once, and compare the different LPs we obtain. For this we define $L_c^q(\lambda)$:

$$\sum_{S \in \overline{\mathbb{F}}_c} y_S = 1, \tag{1}$$
$$\sum_{S \in K} |B \cap S| y_S \leq (\frac{1}{\lambda}(c(B) - \tau) + 1) \sum_{S \in K} y_S, \ \forall B \in b(\mathbb{F}), \ \forall K \subseteq \overline{\mathbb{F}}_c, \ |K| = q, \tag{2^q}$$
$$y \geq 0. \tag{3}$$

Writing $g_\lambda(B, S) := -|B \cap S| + (\frac{1}{\lambda}(c(B) - \tau) + 1)$, we simplify the LP and write in the following form :

$$\begin{aligned} &\sum_{S \in \overline{\mathbb{F}}_c} y_S = 1, && (1) \\ &\sum_{S \in K} g_\lambda(B, S) y_s \geq 0, \ \forall B \in b(\mathbb{F}), \ \forall K \subseteq \overline{\mathbb{F}}_c, \ |K| = q, && (2^q) \quad (L_c^q(\lambda)) \\ &y \geq 0. && (3) \end{aligned}$$

We then obtain a series of nested polyhedra $L_c^1(\lambda) \subseteq L_c^2(\lambda) \subseteq ... \subseteq L_c^{|\overline{\mathbb{F}}_c|}(\lambda)$ where $L_c^{|\overline{\mathbb{F}}_c|}(\lambda) = L_c(\lambda)$ and $L_c^1(\lambda)$ corresponds to the integral hull due to $y_S = 0$ being valid for any set S with $g_\lambda(B, S) < 0$ for some $B \in b(\mathbb{F})$. As expected, pinpointing the values of λ and q for which these polyhedra stop being non-empty is quite difficult, but in compensation we are able to find a positive result as a consequence of their feasibility. Before we proceed, we introduce some notation to simplify the discussion.

Definition 55. *For a clutter $\mathbb{F}$ and $c \geq 0$, we define $\lambda_c(q) = \max\{\lambda : L_c^q(\lambda)$ is non-empty$\}$ for $q \in [|\overline{\mathbb{F}}_c|]$,*
we also define $q_c(\lambda) = \min\{q \in [|\mathbb{F}|] : L_c^q(\lambda)$ is non-empty$\}$ for $0 \leq \lambda \leq \tau$.[3] We will omit c from the subscript when clear from context.

We then have $\lambda_c(\mathbb{F}) = \lambda_c(1) \leq \lambda_c(2) \leq \cdots \leq \lambda_c(|\overline{\mathbb{F}}_c|) = \tau$, while $q_c(\lambda)$ is an increasing function of λ, with $q(0) = q(\lambda_c(\mathbb{F})) = 1$ and $q_c(\tau) = |\overline{\mathbb{F}}|_c$. It is quite hard to pinpoint the values of these functions, however checking small examples suggests that the decrease does not jump from τ to $\lambda_c(\mathbb{F})$ between 1 point and the next, and is more or less uniform. For example, if we consider $\mathbb{F} = \mathbb{Q}_6$ and $c = \mathbf{1}$, then we have $\tau = 2$ and $|\overline{\mathbb{F}}_c| = |\mathbb{F}| = 4$, with $\lambda(1) = \frac{1}{2}$, $\lambda(2) = 1$, $\lambda(3) = \frac{3}{2}$, $\lambda(4) = 2$. A weak version of this uniform decrease follows from the next result :

Theorem 56. *Assume $L_c^q(\lambda)$ is non-empty, then $\lambda(\mathbb{F}) \geq \min(\frac{\lambda}{q}, \min_{e \in V} c_e)$*

Proof. Let y be a feasible point for $(L_c^q(\lambda))$, and S^* be a set maximizing y_S. For all $B \in b(\mathbb{F})$, we have the following: for each $K \subseteq \overline{\mathbb{F}}_c \setminus \{S^*\}$ with size $q-1$, we have:

$$g_\lambda(B, S^*)y_{S^*} + \sum_{S \in K} g_\lambda(B, S)y_s \geq 0.$$

Let $F = |\overline{\mathbb{F}}_c|$, we can sum this over all possible choices of K to obtain :

$$\binom{F-1}{q-1} g_\lambda(B, S^*)y_{S^*} + \sum_{S \neq S^*} \binom{F-2}{q-2} g_\lambda(B, S)y_s \geq 0$$

$$\Rightarrow (F-1)g_\lambda(B, S^*)y_{S^*} \geq -\sum_{S \neq S^*} (q-1)g_\lambda(B, S)y_S$$

$$= (q-1)\sum_{S \neq S^*} (|B \cap S| - 1 - \frac{1}{\lambda}(c(B) - \tau))y_S$$

$$\geq \sum_{S \neq S^*} (-\frac{1}{\lambda}(c(B) - \tau))y_S$$

$$\geq (F-1)(q-1)(-\frac{1}{\lambda}(c(B) - \tau))y_{S^*}$$

$$\Rightarrow g_\lambda(B, S^*) \geq -(q-1)(\frac{1}{\lambda}(c(B) - \tau))$$

[3] The reason we only define this up to τ is simply because $\lambda_c(|\overline{\mathbb{F}}_c|) = \tau$.

$$\Rightarrow \frac{1}{\lambda}(c(B)-\tau)+1-|B\cap S^*| \geq -(q-1)(\frac{1}{\lambda}(c(B)-\tau))$$
$$\Rightarrow \frac{q}{\lambda}(c(B)-\tau)+1-|B\cap S^*| \geq 0.$$

Since this holds for all $B \in b(\mathbb{F})$ we obtain:

$$\lambda(\mathbb{F}) \geq \lambda_c(S^*) \geq \min(\frac{\lambda}{q}, \min_{e\in S^*} c_e) \geq \min(\frac{\lambda}{q}, \min_{e\in V} c_e).$$

□

This immediatly shows that $\frac{\lambda}{q(\lambda)} \leq \lambda_c(\mathbb{F})$, hence we have $q(\lambda) \geq \frac{\lambda}{\lambda_c(\mathbb{F})}$ and similarly $\lambda_c(q) \leq q\lambda_c(\mathbb{F})$. While these hold at equality for $\mathbb{Q}_6$, there is no guarantee that this is the case in general.

4.2 Ground element LP

Next, we try our second LP formulation, where now we have a variable for every element of the clutter, and integer solutions correspond exactly to χ_S, where $\lambda_c(S) \geq \lambda$. Thus we define $P_{(\mathbb{F},c)}(\lambda)$:

$$\begin{array}{lll} x(B) \geq 1, \text{for all } B \in b(\mathbb{F}), & (1) & \\ x(B) \leq \frac{1}{\lambda}(c(B)-\tau)+1, \text{for all } B \in b(\mathbb{F}), & (2) & P_{(\mathbb{F},c)}(\lambda) \\ 0 \leq x \leq \mathbf{1}. & (3) & \end{array}$$

As usual, we will omit $\mathbb{F}$ and/or c when they are clear from context.

Proposition 57. *If $x = \chi_S$ is an integer point in $P_c(\lambda)$ and $\lambda \leq c_e$ for all $e \in S$, then there exists $S' \subseteq S$ such that $S' \in \mathbb{F}$ and $\lambda_c(S') \geq \lambda$. Note that $x' = \chi_{S'}$ will also be an integer point in $P_c(\lambda)$.*

Proof. Given x that is integer in $(P_c(\lambda))$, (3) implies that $x = \chi_S$ for some $S \subseteq V(\mathbb{F})$. (1) then implies that S is a (not necessarily minimal) cover of $b(\mathbb{F})$, and hence there exists $S' \subseteq S$ such that $S' \in \mathbb{F}$, as the minimal covers of $b(\mathbb{F})$ are exactly the elements in $b(b(\mathbb{F})) = \mathbb{F}$. Finally note that for all $B \in b(\mathbb{F})$ we have $|B \cap S'| \leq |B \cap S| = x(B) \leq \frac{1}{\lambda}(c(B)-\tau)+1$ which implies $\lambda \leq \frac{c(B)-\tau}{|B\cap S'|-1}$, and since $\lambda \leq c_e$ for all $e \in S'$, we have $\lambda_c(S') \geq \lambda$. □

This formulation has some interesting features, most surprising of which is that for nice values of λ, the polytope is the intersection of 2 integral polyhedra.

Proposition 58. *Let $k \in \mathbb{N}$, $\mathbb{F}$ an ideal clutter and $c \geq 0$. Then $P_{(\mathbb{F},c)}(\frac{1}{k})$ is the intersection of two integral polyhedra.*

Proof. Let $P^1 1 = \{x \geq 0 : x(B) \geq 1 \text{ for all } B \in b(\mathbb{F})\}$ and $P^2 = \{y \geq 0 : y(B) \leq k(c(B) - \tau) + 1, \text{ for all } B \in b(\mathbb{F})\}$. We have $P^1 = \operatorname{conv}(\chi_S, S \in \mathbb{F}) + \operatorname{cone}(e_1, e_2, \ldots, e_{|V(\mathbb{F})|})$ as it is the covering polytope of $b(\mathbb{F})$ which is ideal, and hence the extreme points are elements of $b(b(\mathbb{F})) = \mathbb{F}$. To show P^2 is integral, we relate it back to P^1. We have :

$$y(B) \leq k(c(B) - \tau) + 1, \text{ for all } B \in b(\mathbb{F})$$

$$\iff (k\mathbf{1} - y)(B) \geq k\tau - 1 \text{ for all } B \in b(\mathbb{F})$$

$$\iff \frac{1}{k\tau - 1}(k\mathbf{1} - y) \geq 1 \text{ for all } B \in b(\mathbb{F}).$$

We claim that if y is an extreme point of P^2 then $y' = \frac{1}{k\tau-1}(k\mathbf{1} - y)$ is an extreme point for P^1. Clearly y' is feasible for P^1, so assume it is not an extreme point and write it as $y' = \gamma x^1 + (1 - \gamma)x^2, x^1, x^2 \in P^1$. Then we will have $y^i = k\mathbf{1} - (k\tau - 1)x^i$ is feasible for P^2. The non negativity constraints can be satisfied as long as we choose x^1, x^2 close enough to y', and all the B induced constraints can be obtained by reversing the equivalence chain from above. We then obtain that y is a convex combination of y^1, y^2, contradicting that it's an extreme point. Hence we have that P^2 only has integer extreme points since given x integer $k\mathbf{1} - (k\tau - 1)x$ is also integer. To finish the proof, we simply write $P_{(\mathbb{F},c)}(\frac{1}{k}) = P_1' \cap P_2$, where $P_1' = \{x \in P_1, x \leq 1\}$ is also integral with extreme points corresponding exactly to $\chi_S + \chi_M$, where $S \in \mathbb{F}$ and $M \subseteq V(\mathbb{F}) \setminus S$. □

Similarly to the previous formulation in section 4.1, it turns out that $P_{(\mathbb{F},c)}(\tau(c, \mathbb{F}))$ is always non-empty for any clutter (even non ideal). To prove this we simply consider $x = \frac{c}{\tau(c,\mathbb{F})}$. For (1) we have $x(B) = \frac{c(B)}{\tau} \geq 1$ for all $B \in b(\mathbb{F})$ by definition of τ, while (2) is always satisfied with equality. This leads to a similar result, namely that feasibility can't imply any bound on $\lambda_c(\mathbb{F})$.

Proposition 59. *There does not exists a constant $\alpha > 0$ such that for all ideal clutters $\mathbb{F}$, $P_c(\lambda)$ is non-empty $\Rightarrow \lambda_c(\mathbb{F}) \geq \min(\frac{\lambda}{\alpha}, 1)$.*

The proof is similar to that of proposition 51.

This suggests that the current formulation is too weak, and hence we move to stronger relaxations via common lift and project methods that we explore in the next chapter.

Chapter 5

Convex lifts

In this section we discuss different possibilities to strengthen $P_c(\lambda)$. We use a common lifting technique and consider the system $S(P_c(\lambda))$:

$$\begin{aligned}
&Y = \begin{bmatrix} 1 & x^T \\ x & X \end{bmatrix}{}^{1} && (1) \\
&\mathrm{diag}(Y) = Ye_0 && (2) \\
&Ye_i \in \mathrm{cone}(\begin{bmatrix} 1 \\ P_c(\lambda) \end{bmatrix}) \text{ for all } i \in \{0, \dots, n\}, && (3) \quad (S(P_c(\lambda))) \\
&Ye_0 - Ye_i \in \mathrm{cone}(\begin{bmatrix} 1 \\ P_c(\lambda) \end{bmatrix}) \text{ for all } i \in \{0, \dots, n\}, && (4) \\
&\mathrm{rank}(Y) = 1. && (5)
\end{aligned}$$

While this is a commonly used technique to lift subsets of $\{0, 1\}^n$ to the space of $(n+1)\times(n+1)$ matrices, there is quite a bit to unpack in this definition. The main idea is that given Y feasible for $S(P_c(\lambda))$, the corresponding x is an integer point of $P_c(\lambda)$.

Consider $Y \in (S(P_c(\lambda)))$. (5) forces Y to be of the form $\begin{bmatrix} 1 \\ x \end{bmatrix} \begin{bmatrix} 1 \\ x \end{bmatrix}^T$, this along with (2) forces x to be integer as $x_i = (Ye_0)_i = \mathrm{diag}(Y)_i = x_i^2$. Finally (3) implies that $\begin{bmatrix} 1 \\ x \end{bmatrix} = Ye_0 \in \mathrm{cone}(\begin{bmatrix} 1 \\ P_c(\lambda) \end{bmatrix})$, which implies that $x \in P_c(\lambda)$. (3) and (4) are trivially satisfied by matrices of

[1] We index the rows and columns of $Y \in S(P_c(\lambda))$ by $i \in \{0, 1, \dots, n\}$ where $n = |V(\mathbb{F})|$, so $Y_{0,i} = x_i$ for all $i \in [n]$.

the form $\begin{bmatrix} 1 \\ \chi_S \end{bmatrix}^T \begin{bmatrix} 1 \\ \chi_S \end{bmatrix}$, as Ye_i is either a copy of Ye_0 or the all 0 vector. However, they become more interesting once we omit (5), as the set of feasible points of the system becomes convex.

Let's consider a matrix Y that satisfies (1),(2),(3) and (4). If $x_i = 1$ then $Ye_0 - Ye_i \in \text{cone}(\begin{bmatrix} 1 \\ P_c(\lambda) \end{bmatrix})$ implies that $Ye_i = Ye_0$, and if $x_i = 0$ then $Ye_0 \in \text{cone}(\begin{bmatrix} 1 \\ P_c(\lambda) \end{bmatrix})$ implies Ye_i must be the all 0 vector. Finally if $0 < x_i < 1$ and $Ye_i = \begin{bmatrix} x_i \\ Xe_i \end{bmatrix}$), (3) implies that $x^1 := \frac{1}{x_i} Xe_i \in P_c(\lambda)$ while (4) implies $x^2 := \frac{1}{1-x_i}(x - Xe_i) \in P_c(\lambda)$. x is therefore a convex combination of x^1 and x^2, where $x^1, x^2 \in P_c(\lambda)$ and $(x^1)_i = \frac{1}{x_i} x_i = 1$, while $(x^2)_i = \frac{1}{1-x_i}(x_i - x_i) = 0$.

This immediately brings to mind the Balas, Ceria, Cornejols (BCC) operator, and it turns out that (1),(2),(3) and (4) correspond exactly to a lift of $BCC(P_c(\lambda))$. We will give a quick reminder of the BCC operator and formalise this result in the next section.

5.1 LS_0, LS_s, and LS_+.

As we have discussed in the introduction to the chapter, $\text{rank}(Y) = 1$ is the main culprit of enforcing all points in $S(P_c(\lambda))$ to be integer. Since our objective is to find a non exact formulation of finding a set with a target $\lambda_c(S)$ value, relaxing this constraint is the natural way to go, and we discuss multiple ways to do this. The simplest way to relax a constraint is to simply omit it, and by omitting $\text{rank}(Y) = 1$ we obtain the following system S_0 :

$$
\begin{aligned}
&Y = \begin{bmatrix} 1 & x^T \\ x & X \end{bmatrix} && (1) \\
&\text{diag}(Y) = Ye_0 && (2) \\
&Ye_i \in \text{cone}(\begin{bmatrix} 1 \\ P_c(\lambda) \end{bmatrix}) \text{ for all } i \in \{0, \ldots, n\}, && (3) \quad (S_0(P_c(\lambda))) \\
&Ye_0 - Ye_i \in \text{cone}(\begin{bmatrix} 1 \\ P_c(\lambda) \end{bmatrix}) \text{ for all } i \in \{0, \ldots, n\}. && (4)
\end{aligned}
$$

To formalize the main result about this relaxation, we first recall the BCC operator.

Definition 60. *[5] For a polyhedron $P \subseteq [0,1]^n$, we define:*

$$BCC_i(P) = \text{conv}([P \cap \{x_i = 0\}] \cup [P \cap \{x_i = 1\}]).$$

$$BCC(P) = \cap_i BCC_i(P).$$

Proposition 61. *For a polyhedron $P \subseteq [0,1]^n$, we have* $\operatorname{proj}_x(S_0(P)) = BCC(P)$.[2]

Proof. We have essentially proven the forward containment in the discussion at the start of chapter 5, the reverse containment follows from a simple constructive argument. If $x \in BCC(P)$ then for all $i \in [n]$ there exists $z^{i,0}, z^{i,1}$, such that x is a convex combination of $z^{i,0}, z^{i,1}$ and they are both in P with $z_i^{i,0} = 0$ and $z_i^{i,1} = 1$. Clearly we must have that $x = x_i z^{i,1} + (1-x_i)z^{i,0}$ for the i-th coordinate to match. Then we simply construct Y by taking $Ye_0 = \begin{bmatrix} 1 \\ x \end{bmatrix}$, $Ye_i = \begin{bmatrix} x_i \\ x_i z^{i,1} \end{bmatrix}$ and we will verify $Ye_0 - Ye_i = \begin{bmatrix} 1 - x_i \\ (1-x_i)z^{i,0} \end{bmatrix}$, so we can easily verify Y is feasible for $S_0(P)$. $\square$

We also discuss alternative ways to relax condition (5). Since xx^T is always symmetric we can opt to relax $\operatorname{rank}(Y) = 1$ constraint with a simple symmetry condition, and we obtain the following S_s:

$$
\begin{array}{lll}
Y = \begin{bmatrix} 1 & x^T \\ x & X \end{bmatrix} & (1) & \\
\operatorname{diag}(Y) = Ye_0 & (2) & \\
Ye_i \in \operatorname{cone}(\begin{bmatrix} 1 \\ P_c(\lambda) \end{bmatrix}) \text{ for all } i \in \{0, \ldots, n\}, & (3) & (S_s(P_c(\lambda))) \\
Ye_0 - Ye_i \in \operatorname{cone}(\begin{bmatrix} 1 \\ P_c(\lambda) \end{bmatrix}) \text{ for all } i \in \{0, \ldots, n\}, & (4) & \\
Y \in \mathbb{S}^{n+1}.^{3} & (5) &
\end{array}
$$

We can also observe that feasible points of $S(P_c(\lambda))$ are all of the form $\begin{bmatrix} 1 \\ x \end{bmatrix} \begin{bmatrix} 1 \\ x \end{bmatrix}^T$, which always yields a positive semidefinite matrix, so we can also consider the system S_+:

$$
\begin{array}{lll}
Y = \begin{bmatrix} 1 & x^T \\ x & X \end{bmatrix} & (1) & \\
\operatorname{diag}(Y) = Ye_0 & (2) & \\
Ye_i \in \operatorname{cone}(\begin{bmatrix} 1 \\ P_c(\lambda) \end{bmatrix}) \text{ for all } i \in \{0..n\}, & (3) & (S_+(P_c(\lambda))) \\
Ye_0 - Ye_i \in \operatorname{cone}(\begin{bmatrix} 1 \\ P_c(\lambda) \end{bmatrix}) \text{ for all } i \in \{0, \ldots, n\}, & (4) & \\
Y \in \mathbb{S}_+^{n+1}.^{4} & (5) &
\end{array}
$$

[2]By $\operatorname{proj}_x(S)$ we mean the projection of the set of feasible points for S.
[3]$\mathbb{S}^{n+1}$ is the set of all $(n+1) \times (n+1)$ matrices that are symmetric.

We then define $LS_\star(P_c(\lambda)) = \mathrm{proj}_x(S_\star(P_c(\lambda)))$ where $\star \in \{0, s, +\}$. These are known as the Lovàsz-Schrivjer operators [4, 23], and satisfy $P_I \subseteq LS_+(P) \subseteq LS_s(P) \subseteq LS_0(P) \subseteq P$. We note that while LS_0 and LS_s are polyhedral operators, LS_+ is not. In the following sections we will be exploring properties of these operators, their relative strength, and how we can utilize them to learn more about properties of ideal clutters.

5.2 Feasibility

Contrary to $P_c(\lambda)$ which is always non-empty for $\lambda \leq \tau(\mathbb{F}, c)$ even when $\mathbb{F}$ is not ideal, $LS_0(P_c(1))$ is always empty for some c if $\mathbb{F}$ is not ideal.

Theorem 62. *Let $\mathbb{F}$ be a non-ideal clutter, $\mathbb{F}'$ be a minimally non-ideal minor of $\mathbb{F}$ [5], and let $J, I \subsetneq V$ such that $\mathbb{F}' = \mathbb{F} \setminus J/I$. Then for c given by $c_i = 0$ for $i \in J$, $c_i = \infty$ for $i \in I$ and $c_i = 1$ otherwise, we have that $LS_0(P_{(\mathbb{F},c)}(1))$ is empty.*

Proof. We have $\tau(\mathbb{F}, c) = \tau(\mathbb{F}', \mathbf{1})$, and so we have $proj_{V\setminus(I\cup J)}(P_{(\mathbb{F},c)}(\lambda)) \subseteq P_{\mathbb{F}',\mathbf{1}}(\lambda)$. For this we require a strong result about minimally non-ideal clutters that can be found in [10], which is that $P_{\mathbf{1}}(\lambda)$ contains no element with an integer coordinate. To obtain the result of the theorem, it suffices to consider an element $e \in V \setminus (J \cup I)$, then $proj_{V\setminus I\cup J}(BCC_i(P_{(\mathbb{F},c)}(\lambda))) \subseteq BCC_i(P_{\mathbb{F}',\mathbf{1}}(\lambda)) = \emptyset$ and hence $LS_\star \subseteq LS_0 \subseteq BCC_i(P_{(\mathbb{F},c)}(\lambda)) = \emptyset$. $\square$

This behavior is different from what happens in the ideal case, where we can always certify these objects are non-empty.

Theorem 63. *Let $\mathbb{F}$ be an ideal clutter and $c \geq 0$ integer such that $\tau(\mathbb{F}, c) \geq 1$. Then $LS_+(P_{(\mathbb{F},c)}(1))$ is non-empty.*[6]

In order to prove this theorem, we will use a minimal counter example argument, where we prove the result for a particular class of pairs $(\mathbb{F}, c)$, then try to reduce an arbitrary case to one of these pairs.

Definition 64. *For a clutter $\mathbb{F}$ and $c \geq 0$ such that $\tau(\mathbb{F}, c) \geq 1$, we say that $\mathbb{F}$ is c-*critical *if every element $e \in V(\mathbb{F})$ appears in a c-optimal cover. We can write this condition as $\cup b(\mathbb{F})_c = V(\mathbb{F})$.*

[4] $\mathbb{S}_+^{n+1}$ is the set of all $(n+1) \times (n+1)$ matrices that are symmetric and positive semidefinite.

[5] $\mathbb{F}$ is said to be minimally non-ideal if it is non-ideal but all of its proper minors are ideal.

[6] Note this also proves that LS_s, LS_0 are non-empty.

Lemma 65. *Let $\mathbb{F}$ be an ideal clutter and $c \geq 0$ integer such that $\mathbb{F}$ is c-critical, then $LS_+(P_{(\mathbb{F},c)}(1))$ is non-empty.*

Proof. Let y be an optimal dual solution for the c-covering problem. Then since every element in $V(\mathbb{F})$ appears in some c-optimal cover, we must have that $\sum_{S,e\in S} y_S = c_e$ for all $e \in V$ due the C.S. conditions. Now consider $Y = \frac{1}{\tau(\mathbb{F},c)} \sum_{S\in\mathbb{F}} y_S \begin{bmatrix} 1 \\ \chi_S \end{bmatrix} \begin{bmatrix} 1 & \chi_S^T \end{bmatrix}$. We claim this is feasible for LS_+(and hence for all the other relaxations). Y is the sum of rank 1 symmetric matrices so is clearly symmetric and positive semidefinite. it also satisfies $\operatorname{diag}(Y) = Ye_0$ as this condition is satisfied for each of the rank 1 integer matrices. We also have $Ye_0 = \begin{bmatrix} 1 \\ \frac{c}{\tau} \end{bmatrix}$) and we have already proven that $\frac{c}{\tau} \in P_c(1)$. For the final constraints, we have

$$Ye_i = \frac{1}{\tau} \sum_{S:i\in S} y_S \begin{bmatrix} 1 \\ \chi_S \end{bmatrix} = \frac{c_e}{\tau} \begin{bmatrix} 1 \\ \frac{1}{c_e} \sum_{S:e\in S} y_s \, \chi_S \end{bmatrix}$$

So to verify if $Ye_i \in \operatorname{cone}(\begin{bmatrix} 1 \\ P_c(1) \end{bmatrix})$, we have to verify that $\frac{1}{c_e} \sum_{S:e\in S} y_s \, \chi_S \in P_c(1)$. For this we simply check every inequality. Let $B \in b(\mathbb{F})$, then we have :

$$\frac{1}{c_e} \sum_{S:e\in S} y_s |B \cap S| \geq \frac{1}{c_e} \sum_{S:e\in S} y_s = \frac{1}{c_e} c_e = 1$$

$$\frac{1}{c_e} \sum_{S:e\in S} y_s |B \cap S| = \frac{1}{c_e} (\sum_{S:e\in\mathbb{F}} y_s |B \cap S| - \sum_{S:e\notin S} y_s |B \cap S|)$$
$$\leq \frac{1}{c_e} (c(B) - \sum_{S:e\notin S} y_s) = \frac{1}{c_e} (c(B) - \tau + c_e) \leq c(B) - \tau + 1$$

A similar argument also shows that $Ye_0 - Ye_i$ is in the cone, and completes the proof. $\square$

Hence the assumption that $\mathbb{F}$ is c-critical will be our base case condition. Now let us consider a pair $(\mathbb{F}, c)$ that is not critical. The natural operation to perform in order to move closer toward every element appearing in a minimum cover is to simply decrease the cost of the missing elements. When $c_e = 1$, decreasing c_e to 0 corresponds to deleting element e, which leads us to explore how these operators behave with respect to minor operations.

5.3 Feasibility from smaller instances

In this section, we will assume that $c = \mathbf{1}$, and study the relation between $LS_\star(P_{\mathbb{F}}(1))$ and $LS_\star(P_{\mathbb{F}'}(1))$[7] where $\mathbb{F}' = \mathbb{F} \setminus \{k\}$ or $\mathbb{F}/\{k\}$, and we start with the deletion minor. Since all we know about $LS_\star$ is that it is the projection of $S_\star$, most of the proofs will be in the language of $S_\star$ and square matrices.

Let $k \in V(\mathbb{F})$, $\mathbb{F}' = \mathbb{F} \setminus \{k\}$, $\tau' = \tau(\mathbb{F}', \mathbf{1})$ and $\tau = \tau(\mathbb{F}, \mathbf{1})$. Assume $Y' = \begin{bmatrix} 1 & x^T \\ x & X \end{bmatrix}$ is feasible for $S_\star(P_{\mathbb{F}'}(1))$, and we wish to extend it to $Y = \begin{bmatrix} 1 & x^T & x_k \\ x & X & z \\ x_k & z'^T & x_k \end{bmatrix}$ feasible for $S_\star(P_{\mathbb{F}}(1))$.

The first condition we will look at is $\begin{bmatrix} x_k \\ z \\ x_k \end{bmatrix} \in \text{cone}(\begin{bmatrix} 1 \\ P_{\mathbb{F}}(1) \end{bmatrix})$. If we pick $x_e \neq 0$, then this means having to find a feasible solution $\bar{x}$ to $P_{\mathbb{F}}(1)$ with $\bar{x}_e = 1$, which we cannot guarantee the existence of. Hence we choose $x_k = 0$ and attempt to prove that $Y = \begin{bmatrix} 1 & x^T & 0 \\ x & X & 0 \\ 0 & 0 & 0 \end{bmatrix}$ is feasible. This also aligns with our intuition behind the relaxation: If x is aiming to represent an element in $\mathbb{F}'$, then $\begin{bmatrix} x \\ 0 \end{bmatrix}$ should represent the same element in $\mathbb{F}$, and as we are looking for elements of the clutter that have small intersections with blocker elements, adding e, even with a small weight, only hurts feasibility.

Next we look at $Ye_0 \in \text{cone}(\begin{bmatrix} 1 \\ P_{\mathbb{F}}(1) \end{bmatrix})$. For every $B \in b(\mathbb{F})$, we have to verify the following:

$$1 \leq_{(1)} \chi_B^T x \leq_{(2)} |B| - \tau + 1.$$

First thing to note is that we do not have to worry about (1), since $B \setminus \{k\}$ is always a cover of $\mathbb{F} \setminus \{k\}$, we already know that $1 \leq \chi_B^T x$ holds by feasibility of Y' for $P_{\mathbb{F}'}$. For (2), we have $\chi_B^T x \leq |B \setminus \{k\}| - \tau' + 1$, which means we only risk violating the constraint if $k \notin B$ and $\tau' < \tau$. Thus we try to characterize when $\tau = \tau'$. Since the covers of $\mathbb{F}'$ are exactly $B \setminus \{k\}$ where B is a cover of $\mathbb{F}$, $\tau = \tau'$ is equivalent to k not belonging to any minimum $\mathbb{F}$ cover. This is quite convenient for us since we were mainly interested in deleting elements that do no appear in any minimum cover.

[7] We will drop all the c subscripts to reduce notation, as the cost vector is usually clear from context.

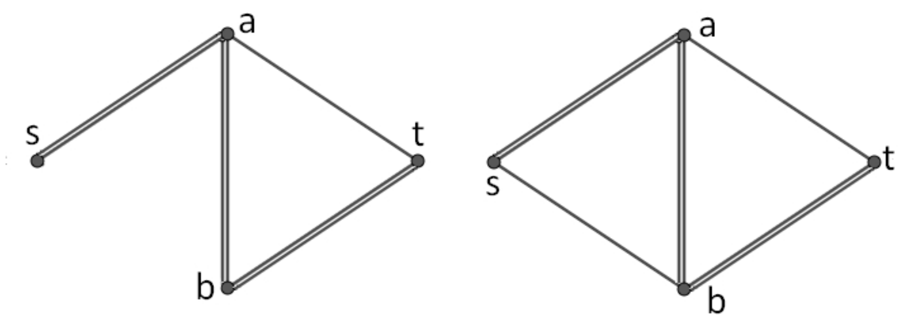

Figure 5.1: On the left : $G \setminus \{(s, b)\}$ with $\tau = 1$, on the right G with $\tau = 2$. S consisting of the double edges has $\lambda_{\mathbb{F}\setminus\{k\}}(S) = 1$ but $\lambda_{\mathbb{F}}(S) = \frac{1}{2}$.

It turns out that this is actually a sufficient condition to verify all the other constraints as well, and hence we obtain the following theorem:

Theorem 66. *Let $\mathbb{F}$ be an ideal clutter, $k \in \mathbb{F}$ such that k does not appear in any $\mathbf{1}$-optimal cover, and $\mathbb{F}' = \mathbb{F} \setminus \{e\}$. If x is feasible for $LS_\star(P_{\mathbb{F}'}(1))$ then $\begin{bmatrix} x \\ 0 \end{bmatrix}$ is feasible for $LS_\star(P_{\mathbb{F}}(1))$, for all $\star \in \{0, s, +\}$.*

Proof. It is sufficient to prove our earlier claim that whenever Y' is feasible for $S_\star(P_{\mathbb{F}'}(1))$, the 0 row and column extension Y is feasible for $S_\star(P_{\mathbb{F}}(1))$. We have already shown that $Ye_0 \in \text{cone}(\begin{bmatrix} 1 \\ P_{\mathbb{F}}(1) \end{bmatrix})$. The proofs for Ye_i and $Ye_0 - Ye_i$ belonging to the cone are also very similar, and follow directly from feasibility of Y'. It is also clear that Y is symmetric if and only if Y' is symmetric, and Y is p.s.d. if and only if Y' is p.s.d. $\square$

It remains to discuss what happens if $\tau \neq \tau'$ (which is equivalent to $\tau' = \tau - 1$), can we somehow augment the solution? The answer is no, even if x is integer. This also aligns with our objective of modelling the behavior $\lambda(S)$, since it is generally false that an arbitrary $S \in \mathbb{F} \setminus \{k\}$ with $\lambda_{\mathbb{F}\setminus\{k\}}(S) \geq \lambda$ would necessarily satisfy $\lambda_{\mathbb{F}}(S) \geq \lambda$. Figure 5.1 shows a small example where $\mathbb{F}$ is a clutter of s-t paths and $\lambda = 1$.

Going back to where the proof fails for $\tau' = \tau - 1$, the main problem is that we are unable to verify that $\chi_B^T x \leq |B| - \tau + 1$. However, this is only an issue when $k \notin B$, as otherwise we already have $\chi_B^T x \leq |B \setminus \{k\}| - \tau' + 1 = |B| - \tau + 1$. When thinking about how to strengthen these inequalities, keeping in mind that we only need to consider elements of the blocker that do not contain k, the contraction clutter comes to mind, and it turns out that the key does indeed lie there, as illustrated by the following theorem :

Theorem 67. *Let $\mathbb{F}$ be an ideal clutter, $\mathbb{F}' = \mathbb{F} \setminus \{k\}$ and $\mathbb{F}'' = \mathbb{F}/\{k\}$. If Y' is feasible for both $S_\star(P_{\mathbb{F}'}(1))$ and $S_\star(P_{\mathbb{F}''}(1))$, then $Y = \begin{bmatrix} Y' & 0 \\ 0^T & 0 \end{bmatrix}$ is feasible for $S_\star(P_{\mathbb{F}}(1))$.*

Proof. Let $\tau = \tau(\mathbb{F}, \mathbf{1})$, and similarly $\tau' = \tau(\mathbb{F}', \mathbf{1}) \geq \tau - 1$, and $\tau'' = \tau(\mathbb{F}'', \mathbf{1}) \geq \tau$ since the covers of $\mathbb{F}''$ are covers of $\mathbb{F}$ that do not contain k. Assume x is feasible for for both $P_{\mathbb{F}'}(1)$ and $P_{\mathbb{F}''}(1)$. For all $B \in b(\mathbb{F})$ that do contain k, we use feasibility for $\mathbb{F}'$ to obtain $\chi_B^T x \leq |B \setminus \{k\}| - \tau' + 1 \leq |B| - \tau + 1$, and for B that do not contain k, B is a cover of $\mathbb{F}''$ and hence $\chi_B^T x \leq |B| - \tau'' + 1 \leq |B| - \tau + 1$. This shows that $Ye_0 \in \operatorname{cone}(\begin{bmatrix} 1 \\ P_{\mathbb{F}}(1) \end{bmatrix})$, and the other constraints follow using very similar arguments. □

This result can be easily generalized to arbitrary target value of λ instead of 1, yielding the following result :

Proposition 68. *Let $\mathbb{F}$ be an ideal clutter. Then for all $\star \in \{0, s, +\}$ and $\lambda \geq 0$ we have:*

$$S_\star(P_{\mathbb{F}\setminus k}(\lambda)) \cap S_\star(P_{\mathbb{F}/k}(\lambda)) \subseteq S_\star(P_{\mathbb{F}}(\lambda)).^8$$

While we would like to say that this implies $LS_\star(P_{\mathbb{F}\setminus k}(\lambda)) \cap LS_\star(P_{\mathbb{F}/k}(\lambda)) \subseteq LS_\star(P_{\mathbb{F}}(\lambda))$, we are not guaranteed that the certificates of feasibility [9] of a vector $x \in LS_\star(P_{\mathbb{F}\setminus k}(\lambda)) \cap LS_\star(P_{\mathbb{F}/k}(\lambda))$ are equal for both lifts, so the result is slightly out of reach. However, we can still obtain the result when x is integer, since there is a unique candidate for its certificate matrix in both lifts, implying the following result:

Proposition 69. *Let $\mathbb{F}$ be an ideal clutter, $k \in V(\mathbb{F})$ and $S \in \mathbb{F} \cap (\mathbb{F}/k)$, then we have:*

$$\lambda_{\mathbb{F},1}(S) \geq \min(\lambda_{\mathbb{F}\setminus k,\mathbf{1}}(S), \lambda_{\mathbb{F}/k,\mathbf{1}}(S)).$$

This bound is quite interesting, and it gives rise to many questions:

Question 70. *Let $\mathbb{F}$ be an ideal clutter, $k \in V(\mathbb{F})$.*

(a) Can we characterise when $S \in \mathbb{F} \cap (\mathbb{F}/k)$ satisfies Proposition 69 with equality ?

[8] Here we are identifying $S_\star(P)$ with its feasible region.
[9] We say that Y is a certificate of feasibility for $x \in LS_\star$ if $proj(Y) = x$ and Y is feasible for $S_\star$.

(b) Can we obtain a similar result for $k \notin S \in \mathbb{F} \setminus (\mathbb{F}/k)$?

(c) Can we obtain a similar bound for $\lambda_{\mathbb{F},c}(S)$? I.e. an upper bound that that depends on $\lambda_{\mathbb{F},c'}(S)$ for some values of c' related to c ?

The answer to any of these questions would be very helpful if we are trying to utilize feasibility of smaller instances in a more general setting. However, we will not be delving deeper as we already have all we need to prove Theorem 63.

As we discussed near the end of the previous section, given a non c-critical clutter $\mathbb{F}$, the natural operation to perform to move towards being critical is considering $c' = c - e_k$ where k does not belong to any c-optimal cover. However, this only works if $c_k \geq 1$, as otherwise c' will have a negative entry. Thus, we handle both cases separately.

Lemma 71. *Let $\mathbb{F}$ be an ideal clutter, $c \geq 0$ such that $\mathbb{F}$ is not c-critical, and $k \notin \cup b(\mathbb{F})_c$ satisfying $c_k = 0$. Let $\mathbb{F}' = \mathbb{F} \setminus k$, and c' be the restriction of c on $V(\mathbb{F}')$. If $x \in LS_\star(P_{(\mathbb{F}',c')}(\lambda))$, then $\begin{bmatrix} x \\ 0 \end{bmatrix} \in LS_\star(P_{(\mathbb{F},c)}(\lambda))$ for all $\star \in \{0, s, +\}$.*

Lemma 72. *Let $\mathbb{F}$ be an ideal clutter, $c \geq 0$ integer such that $\mathbb{F}$ is not c-critical, and $k \notin \cup b(\mathbb{F})_c$ satisfying $c_k \geq 1$. Let $c' = c - e_k$. If $x \in LS_\star(P_{(\mathbb{F},c')}(\lambda))$, then $x \in LS_\star(P_{(\mathbb{F},c)}(\lambda))$ for all $\star \in \{0, s, +\}$.*

Proof. To prove both lemmas we simulate the proof of Theorem 66. For the first lemma, we notice that $\tau' = \tau(\mathbb{F}', c') = \tau(\mathbb{F}, c)$ and then use feasibility of x to obtain that for every $B \in b(\mathbb{F})$ we have $\chi_B^T \begin{bmatrix} x \\ 0 \end{bmatrix} = \chi_{B \setminus k}^T x \leq \frac{1}{\lambda} c(B \setminus k) - \tau' + 1 = c(B) - \tau + 1$. For the second lemma the argument is even easier, as $\tau(\mathbb{F}, c') = \tau(\mathbb{F}, c)$ and $c'(B) \leq c(B)$ for all $B \in \mathbb{F}$, so every constraint either remains the same or gets more relaxed when moving from c' to c. ☐

The proof of Theorem 63 now follows from a classic inductive argument. We will call $(\mathbb{F}, c)$ a minimal counterexample if :

(i) $|V(\mathbb{F})|$ is minimal among all the counterexamples,

(ii) $||c||_1$ is minimal among all counterexamples satisfying (i)[10].

[10] $||c||_1$ is the usual 1-norm, equal to $\sum_i |c_i|$.

We show that there are no counterexamples by showing that there are no minimal counterexamples. Assume $(\mathbb{F}, c)$ is a minimal counterexample, then clearly $\mathbb{F}$ cannot be c-critical as that would violate Theorem 65. So we know there exists $k \in V(\mathbb{F}) \setminus \cup b(\mathbb{F})_c$. If $c_k = 0$ then by Lemma 71 with $\lambda = 1$, we obtain another counterexample $(\mathbb{F}', c')$ with $|V(\mathbb{F}')| = |V(\mathbb{F})| - 1$, and if $c_k \geq 1$ then we use Lemma 72 instead to obtain a counterexample $(\mathbb{F}, c')$ where $||c'||_1 = ||c|| - 1$. In both cases, we have violated the minimality condition, so it must be that Theorem 63 holds for all ideals clutters $\mathbb{F}$ and integer $c \geq 0$ satisfying $\tau(\mathbb{F}, c) \geq 1$.

A point of intrigue is that decreasing c by e_k seems to behave very similarly to the deletion operation when $c_k = 1$. This is not a coincidence, since we can convert it to a deletion operation. If we were to identify $(\mathbb{F}, c)$ with the clutter $\mathbb{F}'$ where every element k is duplicated c_k times with each copy given cost 1, then deleting one of these copies corresponds to decreasing the cost by 1. We formalize this in the following result.

Definition 73. *Let $\mathbb{F}$ be a clutter and $c \geq 0$ integer. We define $\mathbb{F}$, the replication clutter of $\mathbb{F}$ by c, to be the clutter on ground set $V(\mathbb{F}') = \cup_{e \in V(\mathbb{F})}\{e_1, e_2, \ldots, e_{c_e}\}$ whose elements are exactly the copies of sets in $\mathbb{F}$, i.e. sets S' of the form $S' = \{(e^1)_{\alpha_1}, \ldots, (e^k)_{\alpha_k}\}$ where $\{e^1, \ldots, e^k\} \in \mathbb{F}$, and $1 \leq \alpha_i \leq c_{e^i}$.*

In other words, $\mathbb{F}'$ is obtained by replacing every ground element e with c_e copies, and then considering the sets that do not contain any two copies of the same element and are clones of elements in $\mathbb{F}$, where a clone of $S = \{e^1, e^2, \ldots, e^k\}$ is a set of the form $S' = \{(e^1)_{\alpha_1}, \ldots, (e^k)_{\alpha_k}\}$ where $1 \leq \alpha_i \leq c_{e^i}$.

Proposition 74. *Let $\mathbb{F}$ be a clutter on ground set $V(\mathbb{F}) = [n]$, $c \geq 0$ integer, and let $\mathbb{F}'$ be the replication clutter of $\mathbb{F}$ by c.*

(a) $b(\mathbb{F}') = \{B' : B' = \cup_{e \in B}\{e_1, e_2, ..., e_{c_e}\}, B \in b(\mathbb{F})\}$.

(b) $\mathbb{F}$ is ideal if and only if $\mathbb{F}'$ is ideal.

(c) $\tau(\mathbb{F}, c) = \tau(\mathbb{F}', \mathbf{1})$.

(d) If $c = \mathbf{1} - e_k$, then $\mathbb{F}' = \mathbb{F} \setminus k$[11].

(e) For $S \in \mathbb{F}$ and $S' \in \mathbb{F}'$ is a clone of S, we have $\lambda_{\mathbb{F}',\mathbf{1}}(S') = \min\{1, \lambda_{\mathbb{F},c}(S)\}$.

(f) For all $\lambda \leq 1$, $P_{(\mathbb{F},c)}(\lambda)$ is non-empty if and only if $P_{(\mathbb{F}',\mathbf{1})}(\lambda)$ is non-empty.

[11] Up to relabelling the ground set.

Proof. (a) Holds from the simple observation that a cover of $\mathbb{F}'$ containing e_i either contains all copies $e_j, j \in [c_e]$ or can be made smaller by omitting e_i from the cover. (b) follows from (a) and the width-length inequality(Theorem 14) by adding up all the weights of the clones for $\mathbb{F}'$ and taking the minimum weight in $b(\mathbb{F}')$, and so does (c). (d) follows from the definition.

To prove (e), we consider $S' = \{(e^1)_{\alpha_1}, \dots, (e^k)_{\alpha_k}\} \in \mathbb{F}'$ a clone of $S = \{e^1, \dots, e^k\} \in \mathbb{F}$. Let $x = \chi_S \in \mathbb{R}^{V(\mathbb{F})}$ and $x' = \chi_{S'}$. We claim x' is feasible for $P_{(\mathbb{F}',\mathbf{1})}(\lambda)$ if and only if x is feasible for $P_{(\mathbb{F},c)}(\lambda)$. From (a), we know that there exists a one-to-one correspondence between elements in $b(\mathbb{F})$ and $b(\mathbb{F}')$, so $B' = \cup_{e \in B}\{e_1, e_2, ..., e_{c_e}\}$. Note that we have $x'(B') = x(B)$, and $\frac{c(B)-\tau(\mathbb{F},c)}{|B \cap S|-1} = \frac{|B'|-\tau(\mathbb{F}',1)}{|B' \cap S'|-1}$, and hence x' satisfies the two constraints given by B' if and only if x satisfies the two constraints given by B.

The forward direction of (f) is a simple generalization of (e). Given x feasible for $P_{(\mathbb{F},c)}(\lambda)$, then we can simply construct x' coordinate-wise by $x'_{e_1} = x_e$ and $x'_{e_i} = 0$ for all $j \geq 2$, and we would have x' is feasible for $P_{(\mathbb{F}',\mathbf{1})}(\lambda)$. For the reverse direction, given x' feasible for $P_{(\mathbb{F}',\mathbf{1})}(\lambda)$ then x given by $x_e = \sum_{i=1}^{c_e} x'_{e_i}$ is feasible for $P_{(\mathbb{F},c)}(\lambda)$ since $x'(B') = x(B)$ just like in the proof of (e). We note that if $c_e = 0$ then x_e is taken to be 0 in this construction. □

This means that we can centralise our focus on $c = \mathbf{1}$, and if we can prove that for a small enough value $\bar{\lambda}$, $P_{(\mathbb{F},\mathbf{1})}(\bar{\lambda})$ is always non-empty then the result would hold for all $c \geq 0$ integer as well by deleting elements of cost 0 then considering the replication clutter of the minor.

5.4 Next iteration and LS ranks

Now that we have proven that $LS_\star(P_{(\mathbb{F},c)}(1))$ is always non-empty for ideal $\mathbb{F}$ and integer $c \geq 0$ as long as $\tau \geq 1$, we can keep iterating the operator and ask when do we reach the integer hull. To formalise this, we need a quick definition.

Definition 75. *Given an ideal $\mathbb{F}$, integer $c \geq 0$ and $\tau > \lambda > 0$. For all $\star \in \{0, s, +\}$, we define:*

$$r^{\star}_{(\mathbb{F},c)}(\lambda) := \min\{r \in \mathbb{N} : LS^r_\star(P_{(\mathbb{F},c)}(\lambda)) = (P_{(\mathbb{F},c)}(\lambda))_I\}.^{12}$$

First thing to note is that this value is always finite and is bounded above by $|V(\mathbb{F})|$. This due to the fact that for any polytope $P \subseteq [0,1]^n$ and $T \subseteq [n]$ with $|T| = k$, we have $LS^k_0(P) \subseteq BCC_T(P) = \text{conv}\{x \in P : x_T \text{ is integer}\}$, implying the following result :

$$0 \leq r^{+}_{(\mathbb{F},c)}(\lambda) \leq r^{s}_{(\mathbb{F},c)}(\lambda) \leq r^{0}_{(\mathbb{F},c)}(\lambda) \leq |V(\mathbb{F})|.$$

[12] $LS^k_\star(P)$ denotes the k-th iteration of $LS_\star$, i.e. $LS^k_\star(P) := LS_\star(LS^{k-1}_\star(P))$ with the convention that $LS^0_\star(P) := P$.

Many questions immediately come to mind, so we discuss a few:

Question 76.

(a) How big is the gap between r^0, r^s and r^+ ? Can we characterize the triplets $(\mathbb{F}, c, \lambda)$ for which they coincide ?

(b) Can we find better upper bounds on any of the $r^\star$ for certain classes of ideal clutters ? Perhaps a universal constant, a function of $|V(\mathbb{F})|$, or a more intrinsic parameter of the clutter.

(c) Can we find lower bounds for certain classes of ideal clutters ?

In all the small examples we are able to work out by hand, all the values $r^\star$ for $0 \leq \lambda \leq \tau$ are equal. Equality between r^0 and r^s could be due to the high symmetry in small examples, making it is quite plausible that a gap can manifest once the examples get bigger and less symmetric. The difference between r^+ and r^s is a lot harder to grasp: It is not clear what the p.s.d. condition is achieving. In all the examples we studied feasibility was always achieved by a convex combination of vv^T matrices which is guaranteed to satisfy the p.s.d. condition, and it proved to be quite challenging to construct a non p.s.d solution to S_s, which seems to suggest that not only $LS_+ = LS_s$ is true, but that $S_+ = S_s$ also holds.

(b) and (c) appear to be quite challenging and we are very far from finding any general bounds, but we still attempt to approach certain special cases. First we choose to study the behavior of these rank functions for $\lambda = 1$ and $c = \mathbf{1}$. We then focus on the case when $\lambda(\mathbb{F}) < 1$ mainly because we have no good tools of being able to tell if a polytope is equal to the integer hull aside from when the integer hull is empty. In this setting, we know $r^\star_{\mathbb{F}} \geq 2$ since we know $LS_\star(P_{\mathbb{F}})$ is always non-empty, so we ask if we can find a nice class of clutters for which $r^\star_{\mathbb{F}} = 2$. A step towards finding such a class is the following result:

Definition 77. *For a clutter $\mathbb{F}$ we define* $\mathrm{Gap}(\mathbb{F}) := \min\{|B| - \tau(\mathbb{F}) : B \in \mathbb{F}, |B| > \tau\}$.

Lemma 78. *Let $\mathbb{F}$ be an ideal clutter where :*

1. *$\tau = 2$, and*
2. *The minimal covers partition $V(\mathbb{F})$, and*
3. *$\mathrm{Gap}(\mathbb{F}) \geq k$.*

Then, $LS^k_+(P_{(\mathbb{F},\mathbf{1})}(1))$ is non-empty.

Proof. The result already holds for $\mathrm{Gap}(\mathbb{F}) = 1$ without any assumptions on $\mathbb{F}$. We will prove this for $\mathrm{Gap}(\mathbb{F}) = 2$ by construction. For $i \in V(\mathbb{F})$ denote by $\bar{i}$ the element such that $\{i, \bar{i}\}$ is a cover of size 2. Note that due to Assumptions 1 and 2, this element exists and is unique, and we also have $\bar{\bar{i}} = i$. We construct Y^i as follows : $Y^i e_0 = Y^i e_i = \frac{1}{2}\mathbf{1} + \frac{1}{2}(e_0 + e_i - e_{\bar{i}})$, $Y^i e_{\bar{i}} = 0$ and for any $j \notin \{0, i, \bar{i}\}$, $Y^i e_j = \frac{1}{4}\mathbf{1} + \frac{1}{4}(e_0 + e_i + e_j - e_{\bar{i}} - e_{\bar{j}})$. Note that $Y^i e_j \in \mathrm{cone}\begin{bmatrix} 1 \\ P(1) \end{bmatrix}$ since every cover of size at least 4 hits at least 2 non-zero elements. Hence the lower bound inequality is satisfied. A size ≥ 4 cover also hits at most 2 elements with entry 1 while every other entry is at most $\frac{1}{2}$. Hence, the upper bound inequality is also satisfied. This shows that $Y^i \in S_0(P(1))$ for all i, and we can also show that Y^i is symmetric p.s.d. and therefore $\frac{1}{2}\mathbf{1} + \frac{1}{2}(e_i - e_{\bar{i}}) \in LS_+(P_{\mathbb{F},\mathbf{1}}(1))$ for all $i \in [V(\mathbb{F})]$, and so $\frac{1}{2}\mathbf{1} \in LS_+^2(P(1))$. This idea can be easily generalized for arbitrary $\mathrm{Gap}(\mathbb{F})$ values by showing that $LS_+(P(1))$ has a solution with $\mathrm{Gap}(\mathbb{F}) - 1$ entries set to 1. □

Corollary 79. *Let $\mathbb{F}$ be an ideal clutter where :*

1. $\tau = 2$, *and*
2. *The minimal covers partition $V(\mathbb{F})$, and*
3. $\lambda_{\mathbf{1}}(\mathbb{F}) < 1$.

Then, $r^{\star}_{(\mathbb{F},\mathbf{1})} \geq \mathrm{Gap}(\mathbb{F}) + 1$.

Proof. This result follows directly from Lemma 78 by simply noticing that $(P_{\mathbb{F},\mathbf{1}}(1))_I = \emptyset$ for every clutter satisfying $\lambda_{\mathbf{1}}(\mathbb{F}) < 1$. □

We note however that this does not constitute a proof that $r^{+}_{\mathbb{F}}$ can get arbitrarily high, since we have yet to prove the existence of clutters with arbitrarily large gap that are ideal, let alone clutters satisfying the conditions in Corollary 79. The highest gap value we are currently able to achieve with these conditions is $\mathrm{Gap}(\mathbb{F}) = 2$, with the example of $\mathbb{Q}_{10}$ [3] given below.

$$M(\mathbb{Q}_{10}) = \begin{bmatrix} 0&0&0&0&0&1&1&1&1&1\\ 1&0&0&0&0&0&1&1&1&1\\ 1&1&0&0&0&0&0&1&1&1\\ 1&1&1&0&0&0&0&0&1&1\\ 1&1&1&1&0&0&0&0&0&1\\ 0&0&0&1&0&1&1&1&0&1\\ 0&1&1&1&0&1&0&0&0&1\\ 0&0&1&1&0&1&1&0&0&1\\ 0&1&0&0&1&1&0&1&1&0\\ 1&0&1&0&1&0&1&0&1&0\\ 0&1&1&0&1&1&0&0&1&0\\ 0&0&1&0&1&1&1&0&1&0\\ 0&1&0&1&1&1&0&1&0&0\\ 1&0&0&1&1&0&1&1&0&0\\ 1&1&0&1&1&0&0&1&0&0\\ 1&0&1&1&1&0&1&0&0&0 \end{bmatrix}$$

To complement Corollary 79, we give a large class of clutters that satisfy the conditions of the corollary with gap 1, whose $r^\star$ rank is exactly 2.

Proposition 80. *Let H_r denote the $r \times 2^r - 1$ matrix whose rows correspond to non-empty subsets of $[r]$. Denote by H_r^* the matrix of the same dimension such that $H_r + H_r^* = J$[13]. Then the clutter $\mathcal{Q}_{r,t}$ [9] given by its matrix representation :* $\begin{bmatrix} H_r & H_r^* & J & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ H_r^* & H_r & \mathbf{0} & J & \mathbf{1} & \mathbf{0} \\ J & \mathbf{0} & H_t^* & H_t & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & J & H_t & H_t^* & \mathbf{0} & \mathbf{1} \end{bmatrix}$ *is ideal with $\tau = 2$, $\lambda(\mathbb{F}) < 1$, its minimum covers partition the ground set, $\mathrm{Gap}(\mathcal{Q}_{r,t}) = 1$ and $r^0_{\mathcal{Q}_{r,t},\mathbf{1}} = 2$.*[14]

Proof. We will focus on proving that $r^0 = 2$ as the rest of the properties are verified in [9]. First we label the ground set of $\mathcal{Q}_{r,t}$ as follows :

[13] J is the all 1 matrix.

[14] This also proves that $r_s = r_+ = 2$, as we already know $1 < r_+ \leq r_s$ since $LS_+(P_{(\mathbb{F},c)}(1))$ is always non-empty.

$$\begin{array}{cccccc} L & S & P & Q & e & f \\ \end{array}$$
$$\begin{bmatrix} H_r & H_r^* & J & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ H_r^* & H_r & \mathbf{0} & J & \mathbf{1} & \mathbf{0} \\ J & \mathbf{0} & H_t^* & H_t & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & J & H_t & H_t^* & \mathbf{0} & \mathbf{1} \end{bmatrix}$$

. To show $LS_0^2(P(1))$ is empty, we will show that there are no points in $LS_0(P(1))$ with $x_e \in \{0,1\}$. Consider any feasible point x and let Y be its certificate matrix. Without loss of generality we will assume $x_e = 1$ (otherwise we will have $x_{\bar{f}} = 1$ and the argument is similar). Denote the elements in L, S, P, Q by $\{l_1, \dots, l_r\}$,$\{s_1, \dots, s_r\}$, $\{p_1, \dots, p_t\}$,$\{q_1, \dots, q_t\}$ so that $\{l_i, s_i\}$ and $\{p_i, q_i\}$ are minimum covers. We must have have $x_{l_1} + x_{s_1} = 1$ so again without loss of generality we can assume $x_{l_1} > 0$. Thus, $Ye_{l_1} \in \operatorname{cone}\begin{bmatrix} 1 \\ P(1) \end{bmatrix}$ implies that we must have a solution $x' \in P(1)$ with $x'_{l_1} = x'_e = 1$. Note that for x' to be feasible for $P(1)$, we must have that $1 \le x'. \chi_B \le |B| - 1$ for every $B \subseteq \mathbb{F}$ that is a cover (not necessarily minimal) of $\mathbb{F}$ (for non minimal covers, the inequality is implied by the minimal cover contained in them).

Now note that for every $1 \le j \le r$, $\{l_1, s_j, e\}$ is a cover, so we must have $x'_S = \mathbf{0}$, and thus $x'_L = \mathbf{1}$. We also have that $\{S, p_i, f\}$ is a cover for all $1 \le i \le t$, and hence we must have that $x'_P = 1$, and thus $x'_Q = 0$. With this, we have found every single coordinate of x', so the only potential candidate is $x' = \chi_L + \chi_P + x_e$, which is an integer vector (and corresponds to the first row of $\mathbb{F}$). However, $L \cup P \cup \{e\}$ is a cover (not necessarily minimal of $\mathcal{Q}_{r,t}$, that does not contain any cover of size 2, and hence must contain a non minimal cover B whose inequality in $P(1)$ will be violated by x'. □

This hints that $\mathrm{Gap}(\mathbb{F})$ could be closely related to the LS ranks, but characterizing this relation proved to be quite difficult to approach, especially due to the lack of examples with high gap.

Another observation we make is that in all of these examples, the central point $\frac{c}{\tau}$ seems to always belong to the last non-empty iteration of the LS operators, and so we end this discussion with a small observation regarding its certificate.

Remark 81. *Let $\mathbb{F}$ be a c-critical ideal clutter such that the covering dual LP (D) (Given at the start of Chapter 3) has a unique optimal solution. The point $x = \frac{c}{\tau} \in LS_\star(P_c(1))$ admits a unique certificate matrix in $S_\star$ for all $\star \in \{0, s, +\}$.*

Proof. Let Y be a certificate for feasibility of x, and without loss of generality assume that $\{1, \dots, \tau\}$ is a c-optimal cover. Consider $\tau Y e_j = \begin{bmatrix} 1 \\ Y^j \end{bmatrix}$ where $Y^j \in P_c(1)$. Since $P_c(1) \subseteq$

$\mathrm{conv}(\chi_S : S \in \mathbb{F})$, we must have $Y_j = \sum_{S \in cF} y_S^j \chi_S$, with $\sum y_S^j = 1$. Note that since $Y_{i,j} = 0$ for all $1 \leq i \neq j \leq \tau$, the sets appearing in these decompositions with non-zero coefficients are distinct, and thus $y' = \sum_{j=1}^{\tau} y^j$ will be an optimal dual solution. To check feasibility, we simply calculate $\sum_{S:e\in S} y_S = \sum_{S:e\in S} \sum_{j=1}^{\tau} y_S^j = \sum_{j=1}^{\tau} \sum_{S:e\in S} y_S^j = \sum_{j=1}^{\tau} Y_{j,e} = \sum_{j=1}^{\tau} Y_{e,j} = 1$. Now consider Y' induced from y' via the usual construction (given in the proof of Lemma 65). Y' agrees with Y on all the columns corresponding to $1, 2, \ldots, \tau$ but we cannot yet certify that they agree on elements $Y_{i,j}$ where $i \neq j > \tau$. So we repeat the construction but starting with a minimum cover containing the ground element $\tau + 1$. We find Y'' that agrees with Y on column $\tau + 1$. But the optimal dual solution y'' found in the process must be equal to y' (as it is unique) and hence $Y' = Y''$, and thus Y' agrees with Y on all columns $1, \ldots, \tau, \tau + 1$. By simple repetition of the argument for each index, we find that $Y = Y'$. $\square$

We might get tempted to believe that the uniqueness of an optimal dual solution is actually equivalent to the uniqueness of the certificate as every optimal dual solution yields a certificate matrix for $\frac{c}{\tau}$, but this is incorrect. Different dual solutions may give rise to the same matrix, we give the example of a uniform packing of $\{1, 2, 3\}, \{1, 5, 6\}, \{2, 4, 6\}, \{3, 4, 5\}$ and a uniform packing of $\{1, 2, 6\}, \{1, 3, 5\}, \{2, 3, 4\}\{4, 5, 6\}$ yielding the same matrix.

In the next section, we go back to focusing on the initial lift, but now specifically for the clutter of T-joins and $c = \mathbf{1}$, and attempt to gauge how the relaxation behaves for this class.

5.5 The case of T-joins

A graft is a pair (G, T) where $G = (V(G), E(G))$ is a graph and T is a subset of $V(G)$ of even cardinality. A T-join M is a collection of edges of G, such that $deg_M(v)$ is odd if and only if $v \in T$. We refer to sets $X \subseteq V$ as T-even (resp. T-odd) if $|T \cap X|$ is even (resp. odd). When T is clear from context we drop the T and refer to such sets as even or odd.

Definition 82. *Let (G, T) be a graft, we define its clutter of T-joins to be the clutter on ground-set $E(G)$ whose elements are acyclic T-joins*[15] *and refer to it as $\mathbb{F}(G, T)$.*

To reduce the notation, we define $\tau(G, T) = \tau(\mathbb{F}(G, T), \mathbf{1})$ and $P(G, T, \lambda) = P_{(\mathbb{F}(G,T)),\mathbf{1}}(\lambda)$, similarly $LS_\star(G, T, \lambda) = LS_\star(P(G, T, \lambda)(\lambda))$.

The main focus of this section is to show that $LS_\star(G, T, \lambda)$ mimics a lot of the properties satisfied by integer T-joins packings.

[15] A T-join is minimal if and only if it is acyclic.

First we note that the clutter minor operations behave well with respect to the graft minor operations. That is $\mathbb{F}(G,T)\setminus\{k\} = \mathbb{F}((G,T)\setminus\{k\})$, and similarly for the contraction operation.

We also have a nice characterization of the blocker of this clutter, which turns out to be the clutter of minimal T-cuts.

Remark 83. *Let (G,T) be a graft where G is connected. Then $b(\mathbb{F}(G,T)) = \{\delta(U) : U \subseteq V(G)$ is T-odd, $G|U$ and $G|(V \setminus U)$[16] are both connected.$\}$. These elements are referred to as T-*cuts.

Now consider $S = \delta(A)$ for a T-odd set $A \subseteq V(G)$ with $|S| = \tau$. Let $L = E(G|A)$ and $R = E(G|V - A)$, so L, R, S partition $E(G)$. Let $G' = G/R$ and $G'' = G/L$ with $T' = T \cap A \cup \{R\}, T'' = T \cap (V - A) \cup \{L\}$, and define $\mathbb{F}' = \mathbb{F}(G', T'), \mathbb{F}'' = \mathbb{F}(G'', T'')$. These will satisfy $\tau' = \tau'' = \tau$. Given disjoint T'-joins $J'_1, \ldots, J'_\tau$ and disjoint T''-joins $J''_1, \ldots, J''_\tau$, we can construct τ disjoint T-joins by 'gluing' each J'_i to J''_j where $J'_i \cap S = J''_j \cap S$, and hence $J'_i \cup J''_j$ appears in an optimum packing for (G,T). We claim that this idea extends to our relaxation, in the following sense:

Lemma 84. *Consider a graft (G,T) and $S = \delta(A)$ for an odd set $A \subseteq V(G)$ with $|S| = \tau$. Let $L = E(G|A)$ and $R = E(G|V - A)$. Let $G' = G/R$ and $G'' = G/L$. If (x_L, x_S) is feasible for $LS_\star(G', T', \lambda)$ and (x_R, x_S) is feasible for $LS_\star(G'', T'', \lambda)$, then (x_S, x_L, x_R) is feasible for $LS_\star(G, T, \lambda)$, for all $\star \in \{0, s, +\}$.*

We will prove the result for LS_+ since it is the trickiest, the proof for the other two operators is very similar.

Proof. Let Y', Y'' be the certificates of feasibility of (x_L, x_S) and (x_R, x_S) for their respective S_+ lifts. So, we have $Y' = \begin{bmatrix} 1 & x_L^T & x_S^T \\ x_L & X_L & X_{L,S} \\ x_S & X_{S,L} & X_S \end{bmatrix}$ is feasible for $S_+(G,T,\lambda)$ and $Y'' = \begin{bmatrix} 1 & x_R^T & x_S^T \\ x_R & X_R & X_{R,S} \\ x_S & X_{S,R} & X_S \end{bmatrix}$ is feasible for $S_+(G'', T'', \lambda)$. We claim that there exists:

$$Y = \begin{bmatrix} 1 & x_L^T & x_S^T & x_R^T \\ x_L & X_L & X_{L,S} & Z \\ x_S & X_{S,L} & X_S & X_{S,R} \\ x_R & Z^T & X_{R,S} & X_R \end{bmatrix} \text{ feasible for } S_+(G,T,\lambda).$$

[16]For a graph G and a set $A \subseteq V(G)$, we denote by $G|A$ the graph G restricted to A, so $G|A = G \setminus (V - A)$

When considering possible candidates for Z, we might be tempted to pick $Z = 0$ or $Z = y_L y_R^T$ as they are the most straight forward extensions. While the former always yields a p.s.d. matrix, it can violate every other condition. The latter seems more promising: if Y', Y'' were coming from convex combinations of rank 1 solutions, y_e represents the total weight of solutions that include e, and $y_e y_f$ would be a reasonable guess for the total weight of rank 1 solutions combining e and f (this is akin to assuming an independence between the T-joins on each side). However this still can fail even for very small clutters, and is only guaranteed to work when $x_S = e_k$ for some $k \in S$. We could then attempt to prove that every solution to S_+ can be written as a convex combination of feasible matrices where $x_S = e_k$, combine these individually and find the appropriate Z by taking a convex combination of these extensions. This is unfortunately unattainable, since the S_+ for $\mathcal{Q}_6$ has a unique solution Y that satisfies $Ye_0 = \begin{bmatrix} 1 \\ \frac{1}{2}\mathbf{1} \end{bmatrix}$[17]. Fortunately, we can still extend our solutions with a choice of Z that is inspired by the integral case. For $i \in L, j \in R$, $Z_{i,j}$ should represent the total weight of rank 1 solutions sharing i and j, and since we are combining rank 1 solutions based on their S intersection, a more calculated choice would be to take $Z_{i,j} = \sum_{k \in S, x_k \neq 0} \frac{1}{x_k} X_{i,k} X_{j,k}$ (since $X_{i,k}$ would be representing the fraction from x_i of rank 1 solutions combining i and k, and and among those $\frac{1}{x_k} X_{j,k}$ use j), and so we pick $Z = X_{L,S} X_S^{-1} X_{S,R}$.[18]

First we shall prove that if $(x_L, x_S) \in P(G', T', \lambda)$ and $(x_R, x_S) \in P(G'', T'', \lambda)$ then $(x_L, x_S, x_R) \in P(G, T, \lambda)$. Let C be a T-cut of (G, T). If C is also a T' cut of (G', T') (or T'' cut of (G'', T'')), then feasibility of (x_L, x_S) (respectively (x_R, x_S)) guarantees that $1 \leq \sum_{e \in C} x_e \leq |C| - \tau + 1$. Hence we focus on cuts C where $C = \delta(A' \cup B')$ where $\emptyset \subsetneq A' \subsetneq A$ and $\emptyset \subsetneq B' \subsetneq V \setminus A$. Note that exactly 1 of A' or B' is T-odd, (we can simply assume it is A) so let $P = \delta(A')$ and $Q = \delta(B - B')$, both T-odd. Note that

$$\sum_{e \in C} x_e = \sum_{e \in P} x_e + \sum_{e \in Q} x_e - \sum_{e \in S_1} x_e + \sum_{e \in S_2} x_e$$

where $S_1 = \delta(A', B - B') \subseteq S, S_2 = \delta(A - A', B') \subseteq S$. Since S is a minimal size T'-cut, we have $\sum_{e \in S} x_e = 1$ and therefore :

$$\sum_{e \in C} x_e \geq \sum_{e \in P} x_e + \sum_{e \in Q} x_e - \sum_{e \in S} x_e \geq 1 + 1 - 1 = 1,$$

$$\sum_{e \in C} x_e \leq \sum_{e \in P} x_e + \sum_{e \in Q} x_e + \sum_{e \in S} x_e \leq \frac{1}{\lambda}(|P| - \tau) + 1 + \frac{1}{\lambda}(|Q| - \tau) + 1 + 1.$$

[17]Note that $\mathcal{Q}_6$ is the T-join clutter of $G = K_{2,3}$ where the terminals are all the vertices but a single degree 3 vertex.

[18]Whenever X is singular, we will refer by X^{-1} to its unique Moore-Penrose pseudoinverse.

$$\leq \frac{1}{\lambda}[(|P| + |Q| - \tau + 2) - \tau] + 1 \leq |C| - \tau + 1$$

As $|C| + |S| - 2 \geq |P| + |Q|$, since S must contain at least 1 edge in $\delta(A', B - B')$ and one other edge in $\delta(B', A - A')$ for C to be a minimal cut, and these edges belong to exactly one of P or Q.

To show $Ye_i \in \text{cone}(\begin{bmatrix} 1 \\ P(G, T, \lambda) \end{bmatrix})$, a similar argument applies for each $i \in S$, and now we prove it when $e \in L$ (and similarly for $e \in R$). Every cut $C \subseteq L$ must satisfy

$$1 \leq \sum_{j \in C} Y'_{i,j} \leq \frac{1}{\lambda}(|C| - \tau) + 1$$

simply from the feasibility of Y', as $Y_{i,j} = Y'_{i,j}$. But what if $C \subseteq R$? Then we have :

$$Ye_{i|\{0\} \cup S \cup R} = \begin{bmatrix} x_i \\ X_{S,i} \\ \sum_{k \in S} X_{k,i} \frac{1}{x_k} X_{k,R} \end{bmatrix} = \sum_{k \in S} \frac{1}{x_k} X_{k,i} \begin{bmatrix} x_k \\ x_{S,k} \\ X_{k,S} \end{bmatrix} = \sum_{k \in S} \frac{1}{x_k} X_{k,i} (Y'' e_k)$$

and hence the inequality follows from the fact that $Y'' e_k \in \begin{bmatrix} 1 \\ P(G'', T'', \lambda) \end{bmatrix}$. Finally when C is contained in neither R nor L, we simply combine the two tricks above by breaking it into a cut of G' and another of G'' and obtaining bounds for each using the appropriate feasibility conditions from Y', Y''. The $Ye_0 - Ye_i \in \begin{bmatrix} 1 \\ P(G, T, \lambda) \end{bmatrix}$ conditions follow by similar arguments. To verify that Y is psd, we use Schur complement on the S block. This is a common characterization for positive semi-definiteness of a block matrix [18], and we state the result for the sake of completeness.

Theorem 85. *(Schur complement conditions for positive semi-definiteness.)*

Let $M = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix}$ *be a symmetric matrix. Then, the following hold:*

- $M \succeq 0 \iff A \succeq 0$, $C - B^T A^{-1} B \succeq 0$, *and* $(I - AA^{-1})B = 0$.
- *If* $A \succ 0$, *then* $M \succeq 0 \iff C - B^T A^{-1} B \succeq 0$.

Back to our proof, we already know $X_S \succeq 0$ from $Y' \succeq 0$ as it is feasible for $S_+(G', T', \lambda)$, so we need to verify that :

$$M = \begin{bmatrix} 1 & x_L^T & x_R^T \\ x_L & X_L & Z \\ x_R & Z^T & X_R \end{bmatrix} - \begin{bmatrix} x_S^T \\ X_{L,S} \\ X_{R,S} \end{bmatrix} X_S^{-1} \begin{bmatrix} x_S & X_{S,L} & X_{S,R} \end{bmatrix} \succeq 0.$$

Since $1 = \sum_{k \in S} x_S$, and for all $e \in R \cup L$, we have $x_e - \sum_{k \in S} X_{e,k} = 0$, we have :

$$M = \begin{bmatrix} 0 & 0 & 0 \\ 0 & X_L - X_{L,S} X_S^{-1} X_{S,L} & 0 \\ 0 & 0 & X_R - X_{R,S} X_S^{-1} X_{S,R} \end{bmatrix},$$

which is p.s.d. if and only if each diagonal block is p.s.d., and that is easily verified by applying Schur complement condition on the S blocks of Y' and Y''. We do not have to worry about the last condition of Theorem 85, as X_S is a diagonal matrix, and for all $i \in S$ we have that if $X_{S i,i} = 0$, then $Ye_i = Y^T e_i = 0$. □

Next we look at another reduction. Consider 2 edges $(u, v), (u, w)$ of (G, T) and let G' be the graph obtained by deleting both and then introducing a new (v, w) edge. Note that any packing of (G', T) can be trivially extended to a packing of (G, T). We claim that a similar extension works for $LS_\star$.

Lemma 86. *Let (G, T) be a graft, and $\mathbb{F}(G, T)$ be the clutter of T-joins. Let $n, n+1$ be edges of G that are incident to the same vertex w, such that G' the graph obtained by deleting the edges (w, n) and $(w, n+1)$, then adding the edge $(n, n+1)$, satisfies $\tau(G', T) = \tau(G, T)$. If $(x_1, x_2, \dots, x_n)$ is feasible for $LS_\star(G', T, \lambda)$, then $(x_1, x_2, ..., x_n, x_n)$ is feasible for $LS_\star(G, T, \lambda)$.*

Proof. Let $Y' = \begin{bmatrix} 1 & x^T & x_n \\ x_n & X & \alpha \\ x_n & \alpha^T & x_n \end{bmatrix}$ be the certificate matrix for (x, x_n) in $S_\star(G', T, \lambda)$. we claim that $Y = \begin{bmatrix} 1 & x^T & x_n & x_n \\ x & X & \alpha & \alpha \\ x_n & \alpha^T & x_n & x_n \\ x_n & \alpha^T & x_n & x_n \end{bmatrix}$ is feasible for $S_\star(G, T, \lambda)$. Let $n = (u, v), n+1 = (u, w)$, and $n+2 = (v, w)$ (so $E(G) = [n+1], E(G') = [n-1] \cup \{n+2\}$). First we prove that $Ye_0 \in \text{cone}(\begin{bmatrix} 1 \\ P(G, T, \lambda) \end{bmatrix})$. Consider any T-cut $C = \delta(U)$ of (G, T). We proceed according to $S = U \cap \{u, v, w\}$.

If $|S| \in \{0,3\}$, then C is also a T-cut of (G',T), and hence by feasibility of Y' we have:

$$1 \leq \sum_{i \in C} x_i \leq \frac{1}{\lambda}(|C| - \tau) + 1.$$

If $S = \{u\}$, then we have $n, n+1 \in C$, and $C' = C \setminus \{n, n+1\}$ is a T-cut of G'. Thus:

$$1 \leq \sum_{i \in C'} x_i \leq \frac{1}{\lambda}(|C'|-\tau)+1 \Rightarrow 1 \leq 2x_n + \sum_{i \in C \setminus \{n,n+1\}} x_i \leq 2+\frac{1}{\lambda}(|C'|-\tau)+1 \leq \frac{1}{\lambda}(|C|-\tau)+1.$$

If $S = \{v\}$, then $C \cap \{n, n+1\} = \{n\}$, and $C' = C \cup \{n+2\} \setminus \{n\}$ is a T-cut of of G'. So:

$$1 \leq \alpha_0 + \sum_{i \in C' \setminus \{n+2\}} x_i \leq \frac{1}{\lambda}(|C'| - \tau) + 1 \Rightarrow 1 \leq x_n + \sum_{i \in C \setminus \{n+2\}} x_i \leq \frac{1}{\lambda}(|C| - \tau) + 1.$$

Other possibilities of S simply follow by considering $\bar{U}$ instead of U.

$Ye_i, Ye_0 - Ye_i \in \text{cone}(\begin{bmatrix} 1 \\ P(G,T,\lambda) \end{bmatrix})$ follows using the same argument from $Y'e_i, Y'e_0 - Ye_i \in \text{cone}(\begin{bmatrix} 1 \\ P(G',T,\lambda) \end{bmatrix})$ for $i \leq n-1$, and for $i \in \{n, n+1\}$, the result follows from $Y'e_{n+2}, Y'e_0 - Ye_{n+2} \in \text{cone}(\begin{bmatrix} 1 \\ P(G',T,\lambda) \end{bmatrix})$. $Y \succeq 0$ can be easily verified by Schur complement condition, since :

$$\begin{bmatrix} x_n & x_n \\ x_n & x_n \end{bmatrix} - \begin{bmatrix} x_n & \alpha^T \\ x_n & \alpha^T \end{bmatrix} \begin{bmatrix} 1 & x^T \\ x & X \end{bmatrix}^{-1} \begin{bmatrix} x_n & x_n \\ \alpha^T & \alpha^T \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} (x_n - \begin{bmatrix} x_n & \alpha^T \end{bmatrix} \begin{bmatrix} 1 & x^T \\ x & X \end{bmatrix}^{-1} \begin{bmatrix} x_n \\ \alpha^T \end{bmatrix}) \succeq 0$$

as $(x_n - \begin{bmatrix} x_n & \alpha^T \end{bmatrix} \begin{bmatrix} 1 & x^T \\ x & X \end{bmatrix}^{-1} \begin{bmatrix} x_n \\ \alpha \end{bmatrix}) \succeq 0$ follows from $Y' \succeq 0$.

□

Together, Lemmas 84 and 86 provide a very powerful series of reductions, inferring feasibility of (SDP) from smaller instances. Lemma 84 ensures that every minimal cut is of the form $\delta(t)$ for some $t \in T$, while Lemma 86 will ensure that $T = V(G)$ and $\delta(t) = \tau$ for every $t \in T$ by short-cutting through every vertex of degree $> \tau$ as long as τ does not decrease. Unfortunately, this cannot be done in general, as the degree of a vertex can decrease by 2 when reducing 2 of its incident edges, but it works if we ensure all odd cuts have the same parity, since that would imply that every non minimal odd cut has to have size at least $\tau + 2$, decreasing the size of any non minimal cut by 2 still ensures it has size at least τ. Note that the above operation preserves this parity condition, since every cut remains of its same size or decreases by exactly 2. This results in a nice structure for every minimal (G, T) for which $LS_\star$ is empty:

Definition 87. *We call a graft (G, T)* pseudo-eulerian *if every T-odd cut has the same parity.*

Lemma 88. *For every $\lambda \geq 0$, a minimal*[19] *pseudo-eulerian graft (G, T) for which $LS_\star(G, T, \lambda)$ is empty must satisfy :*

- *$V(G) = T$, and*
- *$\deg(u) = \tau$ for all $u \in T$, and*
- *if $|\delta(U)| = \tau$ for a T-odd set U, then $U \in \{\{u\}, V - \{u\}\}$ for some $u \in T$.*

One can be tempted to ignore this reduction since we have already proven in the previous section that $LS_\star(1)$ is always non-empty for every ideal clutter which includes every clutter of T-joins. However, the construction we used is robust enough to preserve some nice properties of the feasible solutions. For example, if we show that $LS_\star(G, T, \lambda)$ has an integer point for every graft satisfying the assumptions in Lemma 88, then we will obtain that $LS_\star(G, T, \lambda)$ has a feasible integer point for every pseudo-eulerian graft. This is because our constructions in Lemmas 86 and 84 preserve integrality. This leads us to the following result :

Theorem 89. *Let (G, T) be a pseudo-eulerian graft with $|T| \leq 8$, then the clutter of T-joins packs.*

Proof. Note that both of our reductions preserve the graph being pseudo-eulerian, and our constructions ensure that integral solutions of the reduced instances yield an integral solution for the original instance, and crucially do not augment any coordinate that is originally 0. Hence, ”disjoint” solutions for the reduced instances augment to ”disjoint” final solutions. Therefore, it is sufficient to prove that every graft satisfying the conditions of lemma 88 with $|T| \leq k$ to obtain the result for all pseudo-eulerian grafts with $|T| \leq k$. For these instances, we do simple induction on τ. The result is trivial for $\tau = 1$, and for $\tau \geq 2$, it suffices to obtain a perfect matching M, then consider $G' = G \setminus M$. We claim that $\tau(G', T) = \tau(G, T) - 1$, and that (G', T) is pseudo-eulerian. The latter is due to the fact that every perfect matching intersects every T-odd cut an odd number of times, and so it is sufficient to find a perfect matching M with $\tau(G \setminus M, T) = \tau(G, T) - 1$. First we prove the existence of a perfect matching. Assume

[19] So $LS_\star(\lambda)$ is non-empty for every graft minor of (G, T).

otherwise, then we must have a Tutte-Berge set certificate S and disjoint odd sets $K_1, \ldots, K_l$ such that $l > |S|$ and $\delta(K_i) \subseteq \delta(S)$. Since each K_i is odd, $\delta(K_i)$ is an odd T-cut, and hence must have size at least τ. But then we have :

$$|\delta(S)| \geq \sum_{i=1}^{l} |\delta(K_i)| \geq l\tau \geq (|S|+1)\tau$$

which is not possible since each vertex in S has degree τ. Then we claim that χ_M is feasible for $P_{\mathbb{F}(G,T),\mathbf{1}}(1)$ and hence $\lambda(\chi_M) = 1$. Indeed, consider any odd T-cut C of (G,T): if $C = \delta(v)$ for some v then clearly $|C \cap M| = 1$, otherwise $|C| \geq \tau + 2$, but since C must have at least one side containing 3 vertices, we have $|C \cap M| \leq 3 \leq |C| - \tau + 1$. $\square$

This leads to the following corollary for any T joins clutter with $|T| \leq 8$:

Corollary 90. *Let (G,T) be a graft where $|T| \leq 8$. Then the clutter of T-joins admits a half packing.*

Proof. The result follows trivially from doubling every edge then applying Theorem 89. $\square$

For $|T| \geq 10$, we do not generally expect integer solutions, but we have feasibility of $LS_\star(G,T,1)$ from the previous sections.

These reductions also show that the $\frac{1}{k}$ conjecture for T-joins is equivalent to a weaker version of the still open generalized Fulkerson Conjecture, that we state below:

Conjecture 91. *Weak Generalized Fulkerson conjecture : There exists a universal constant k such that every m-regular graph on $2n$ vertices with no odd cut of size $< m$ admits a collection of km perfect matchings that cover each edge exactly k times.*

Chapter 6

Takeaways, and what lies ahead

Let us take a step back and revisit our findings in context of the initial plan. We were looking for a proxy for $\lambda_c(\mathbb{F})$. We started by finding a formulation P_λ such that P_λ is non-empty if and only $\lambda_c(\mathbb{F}) \geq \lambda$, and then we would have $\lambda_c(\mathbb{F}) = \max(\lambda : P_\lambda \text{ is non-empty})$. We were hoping to consider a proxy of the form $f(\mathbb{F}, c) = \max(\lambda : Q_\lambda \text{ is non-empty})$, where Q_λ is a relaxation of P_λ, and to somehow find a bound of $\lambda_c(\mathbb{F})$ as a function of $f(\mathbb{F}, c)$, ideally of the form $\lambda_c(\mathbb{F}) \geq \frac{1}{\alpha} f(\mathbb{F}, c)$ for some constant α.

We tried two different IP formulations in Sections 4.1 and 4.2 and attempted to take the natural LP relaxations, but this turned out to be too weak as those were always non-empty for $\lambda = 1$. We then tried different ways to strengthen each relaxation.

For the LP relaxation in 4.1 we first tried considering the CG closure, but it turned out that this gives back the integer hull and goes back to being exact, so we attempted an alternative method by adding a certain set of valid inequalities for the integer hull. We succeeded in obtaining a nice lower bound but as a trade-off we lost track of the feasibility of these relaxations.

For the LP relaxation in 4.2, which was always non-empty even for non-ideal clutters, we considered multiple convex lifts. The first iteration always remained non-empty even for $\lambda = 1$ when $\mathbb{F}$ is ideal, but for the following iterations, it was quite difficult to understand when the relaxation becomes exact and how we could obtain a useful lower bound from its feasibility. It is also intriguing that the rank of the lift and project operators we considered seemed to depend on $\mathrm{Gap}(\mathbb{F})$, so perhaps we should investigate more about this quantity and perhaps it is tightly related to $\lambda(\mathbb{F})$. Section 5.5 shows that even when we restrict ourselves to clutters of T-joins, which are a relatively nice class, the problem remains quite hard and it at least as hard as a still, weak version of the Fulkerson conjecture.

The common point between our main two formulations is that they use a very rough averaging idea, where a feasible point could correspond to convex combination of sets $S_1, \dots, S_k \in \mathbb{F}$ such these individual sets no longer need to satisfy the blocker inequalities $|B \cap S| - 1 \leq \frac{1}{\lambda}(c(B) - \tau)$, but they do on average for every $B \in b(\mathbb{F})$. Theorem 56 is perhaps a good indicator of what's happening behind the scenes, as when we obtain a convex combination of at most k elements of $\mathbb{F}$ who on average satisfy these inequalities, we obtain a lower bound of $\frac{\lambda}{k}$ on $\lambda(\mathbb{F})$, and the example of $\mathbb{Q}_6$ shows this is best bound we can get using this method. This hints that any good bound $\lambda_c(\mathbb{F})$ requires a large number of local constraints, as averaging over the whole clutter seems to be too weak. However, it appears to be quite difficult to balance adding strong local constraints with maintaining information on the feasibility of these relaxations.

In conclusion, the $\frac{1}{k}$-conjecture is still far from our reach. While we do have very useful tools to understand $\lambda_{\mathbb{F},c}(S)$, we do not yet have any grasp on the behavior of $\lambda_c(\mathbb{F})$.

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