A critical review of ‘optimal’ annuitization strategies

by

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A thesis
presented to the University of Waterloo
in fulfillment of the
thesis requirement for the degree of
Masters of Mathematics
in
Actuarial Science

Waterloo, Ontario, Canada, 2020

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Author's Declaration

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

This paper is an analysis of different self-annuitization strategies advised to a retiree. At the time of retirement, an individual has the choice between annuitizing immediately with their wealth or delaying until a future date while making the most from returns earned through financial markets. This is a similar structure to how a Defined Contribution (DC) pension plan works. Various researchers have assisted retirees on this dilemma of whether or not to annuitize. They designed strategies that work under general market settings and any individual preferences that maximize the income generation. However, these studies assume that individuals are making decisions as if they are maximizing or behaving optimally. In reality, what might be "optimal" in a general sense may not match the optimality defined in a normative sense of giving advice. The effect of the gap between the prescribed and descriptive nature of this advice is not well discussed in the current literature.

Our motivation comes from examining the prescriptive nature of the strategies with respect to downside risk. In this paper, we choose three different self-annuitization strategies, from three different papers: Milevsky (2001), Milevsky and Young (2007) and Giacinto and Vigna (2012). The strategies proposed in these papers provide normative advice to retirees on *if and when* to annuitize in retirement years. While an attempt is made to minimize the risk of outliving their own assets and maximize the risk-reward trade-off. The main concern should be the elicited risk in adopting these strategies and doing poorly. In the retirement years, it could be more catastrophic to an individual if he/she runs out of money by following the normative solutions proposed through these strategies. For an average retiree who has limited financial literacy or deteriorating cognitive abilities, it is highly challenging to have their future secured without the fear of inadequate income. Thus, we use risk measures to review the given literature on various plausible explanations of their working strategy and assess the risk of compromising income security. We implement modern world parameters into these strategies to assess if they are truly delivering sound advice.

Acknowledgements

I would like to thank my supervisor, Prof. Mary Hardy for her guidance and support in making this thesis possible. Her support in this work is commendable and most importantly her belief in me has helped me get through it. It was a great time working with her.

I thank my parents and my grandfather for believing in me and encouraging me to pursue my interest. I would like to dedicate this thesis to them.

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Chapter 1

Introduction

At the age of retirement, most individuals have to deal with the choice between voluntary annuitization and discretionary management of their assets. The challenge lies with the retiree to manage these options to provide themselves a desirable income stream. For instance, buying a life annuity assures lifelong guaranteed income, but they are perceived to be costly and monetarily inflexible. The alternative option, to create an own consumption stream, offers flexibility, but also carries the risk that a constant standard of living will not be attainable.

Recent studies including Zhang et al. (2018) and Ramsay and Oguledo (2018) show that retirees profoundly undervalue life annuities. This suggests that voluntary annuitization is not very common in practice, nor it is well understood. In order to assist retirees in this aspect, researchers have formulated various optimal annuitization strategies that attempt to minimize the financial risks and maximize the risk-reward trade-off. Financial risk here can be split into two parts: Annuity risk and Investment risk. Annuity risk is observed mainly in the retirement phase, where the possibility of lower interest rates can lead to lower than expected pension income. It is combined with dissipation risk i.e. the risk of running out of adequate income still being alive. Investment risk is initially observed in the accumulation phase, where lower than expected investment returns from the market portfolio affects the accumulated wealth available at retirement. This risk also extends itself to the de-cumulation phase when the retiree defers annuitization at retirement and continues to invest funds in a market portfolio holding stocks and bonds, drawing down periodic withdrawals, until annuitization occurs, if ever.

Our main motivation is to walk-through different strategies proposed by researchers answering the investment-consumption-annuitization problem. The main findings in this paper interest a current retiree on the elicited risk in adopting these strategies. One of the common approaches in solving the "annuity puzzle" has been to express an objective function in terms of expected discounted utility for a utility maximizing retiree. This approach can be found in Yaari (1965), Richard (1975), Milevsky et al. (2006).

- In his seminal paper, Yaari (1965) showed that, assuming complete markets with actuarially fair annuity prices, uncertain lifetimes, and no bequest motives, utility-maximizing individuals would need to annuitize all of their wealth at retirement.
- Similarly, Richard (1975) generalizes the model of Merton (1971) in a continuous time framework to obtain Yaari (1965)'s results, with a known distribution of lifetime and assumption of constant relative risk aversion for consumption and bequest utility.
- Milevsky et al. (2006) develop a metric to capture the loss from annuitizing prematurely under various institutional restrictions. They maximize the expected utility of lifetime consumption with age-dependent force of mortality and power utility function.

However, according to Vigna (2009) optimal portfolios derived through expected utility maximization of CRRA[1] utility function have proved to be not efficient in the mean-variance setting . As the name suggests "expected" utility subdues the potential of downside risk with a greater upside gain. In this context, downside risk is the potential for income being lower than expected, or in the case of no risk, lower than desired could be achieved through immediate annuitization. Additionally, studies from R.J.Thomson (2003) and Suhonen (2007) use expected utility theory for an individual subscribed to certain set of axioms in order to make normative recommendations between investment channels. These axioms are based on assumptions that individuals are rational and have well-defined preferences. A part of this paper highlights the measures of risk for opting expected utility theorem in the context of retirement planning.

A "probability-based" methodology is used in Milevsky et al. (2006), Milevsky (2001) and Milevsky and Robinson (2000), who minimize the *probability of consumption shortfall or lifetime ruin*. The main idea is driven by the spread between interest rate credited to life annuity, and the rate available in open market. This allows us to compute the probability of beating the rate of return from a life annuity, providing a probabilistic approach to *if and when* to annuitize.

Different from the papers mentioned earlier, Gerrard et al. (2012) and Giacinto and Vigna (2012) provide closed-form solutions to the optimization problem in the presence of

[1]Refer to Chapter 3

target-dependent quadratic loss functions, to produce optimal portfolios that are efficient in a mean-variance setting (Højgaard and Vigna (2007)).

In this paper, we closely investigate three models to analyze their credibility in a real world setting.

The models used in this paper are:

- Probability-based methodology from Milevsky (2001).
- Utility-based methodology from Milevsky and Young (2007).
- Target-dependent quadratic loss function optimization from Gerrard et al. (2012).

For an average retiree who has limited financial literacy or deteriorating cognitive abilities with older age, what might be "optimal" in general sense may not match the optimality defined in the mathematical sense. Zhang et al. (2018) outlines the differences in the normative and descriptive behavior models in retirement decision making context. They investigate the implied optimal behavior prescribed to a retiree, and prove that normatively plausible solutions do not exist. The motivation of the work is to assess the retiree's risk of inadequate income or of compromising income security. On similar lines, our work closely examines the prescriptive nature of these models with respect to downside risk.

The word "normative" is widely used in this paper. The term expresses how people *should* behave when they are confronting risky decisions. While the "descriptive" point of view implies how people *actually* make decisions in real life. The substantial differences in the implications of the two lines of work can be noticed in the premise of this work. Researchers believe that people will make decisions that maximize their financial choices. However, the models used in our study centers around the normative optimality of annuitization under certain assumptions. Milevsky (2001) advises retirees that deferring annuitization would prove beneficial, through his probabilistic line of approach. Milevsky and Young (2007) believe that a utility maximizing retiree should maximize their expected discounted utility in order to benefit from financial market, while being able to defer annuitization. Their normative advice points out to the upside gain in the benefits of annuitizing at a later date. Giacinto and Vigna (2012) also discusses the same, under better optimal conditions, and leaves the choice to its' reader on whether or not to adopt their model as a decision making tool. Something very common to these models is the fact that, people are expected to make decisions as if they are maximizing their expected discount utility of consumption, but it is not always true. This is because, in real life, downside risk is of great concern to a retiree (Zhang et al. (2018)), and utility model, fails to capture this, hence they can not adequately replicate decisions in this context.

In this thesis we explore, numerically, the strategies from the three papers. We first use parameters from the papers chosen, and then update to current market parameters. Throughout the thesis,

- We use a fixed life annuity with a continuous payout for the duration of retiree's life for the cost of annuitization.
- The force of mortality is assumed to follow two-parameter Gompertz distribution. The two parameters are, m, the modal lifetime and b, the scaling factor.
- We assume that the retiree holds a lump sum at retirement, which may be invested in a risk-free asset paying interest at a fixed rate and a risky asset that follows Geometric Brownian motion. The proportion of investment in the risky asset is based on retiree's risk preferences.

We analyse the models through Monte Carlo simulations, based on these factors, to allow for fund movement and consumption changes. We then assess the outcomes through graphical representation, and interpret the results. For easier understanding we adopt consistent notation throughout the paper.

List of Notation:

- x: The retirement age of an individual
- W_t: Wealth of the investing retiree at time t
- c_t: Consumption by the retiree at time t. If consumption is fixed then it is denoted by c.
- r: Risk free rate
- k: Rate of return for deterministic case (Chapter 2)
- $\overline{a}_x$: Price of the annuity paying 1$ continuously to an individual aged x.
- l: Insurance loading
- S_t: Price of risky asset at time t.
- X_t: Price of risk-free asset at time t.
- μ_x: Force of mortality of at age x.

- ${}_tp_x$: The conditional survival probability of an individual aged x living up to age $x + t$.
- T: Final time or optimal time to annuitize.
- t^*: Time of ruin
- $V(W_t, t)$: Value function associated with the expected utility of discounted lifetime consumption at time t.
- ρ: Subjective discount rate
- μ and σ: Market parameters for drift and volatility of the risky asset.
- π_t: Proportion of wealth invested in risky asset at time t.
- $u(\cdot)$: Power utility function.
- $J(t, W_t)$: Expected loss criterion quadratic function at time t of an individual holding wealth W_t.
- δ: Constant force of mortality.

The remainder of the paper is organised as follows. Chapter 2 explores the work of Milevsky (2001). We introduce their strategy and present their idea through simulations. We provide the results of their strategy in Section 2.3, through our simulations and introduce our parameters to implement with their strategy. We review and conclude the outcomes of the risk measures and present our simulation graphs.

In Chapter 3, we highlight the strategy derived in Milevsky and Young (2007). We list out the differences and similarities to notice between Chapter 2 and Chapter 3. We present their proposed strategy, and introduce the necessary elements to implement it. In Section 3.2.1 we present the results of implementing the strategy using their parameters as well as updated ones. We review the strategy and make concluding statements from the results obtained.

In Chapter 4 we highlight the work of Giacinto and Vigna (2012) and identify the differences with Chapter 2 and 3. The strategy is explained, and the results presented and reviewed.

Chapter 5 concludes the thesis.

Chapter 2

Optimal Annuitization Policies by Milevsky (2001)

2.1 Highlights

In this paper, the author illustrates the importance of life annuities in a retiree's choice, using a simple micro-economic consumer choice model. He provides evidence that retirees with negligible bequest motives tend to benefit from self-annuitizing around the age of 75-80. In the process, Milevsky (2001) leaves its readers with normative advice, focusing on the *probability of consumption shortfall*, in other terms, the probability of beating a life annuity rate through a portfolio designed to replicate annuity income. The author terms this strategy as *consume term and invest the difference*.

For a person reluctant to annuitize at retirement, he suggests considering an "own consumption" stream which provides similar income to a life annuity. He uses the above strategy under discrete and continuous-time setting, with deterministic and stochastic investment returns. Through this approach, the author advises his readers on *if and when* to annuitize. From his simulations, Milvesky claims that a reasonably high chance of replicating the same annuity at a later date in retirement exists.

One of the interesting observations in the paper is the interest spread between the rate credited to life annuity and rate available in the market. The greater the difference, the higher is the probability of beating the life annuity. Milevsky (2001) mentions that the sensitivity to current levels of interest rates and insurance loadings plays a major role. In this chapter, we first describe the strategy and subsequently test the strategy using Monte Carlo simulations and up-to-date parameters.

2.2 Strategy: Consume term and invest the difference

We introduce the model from Milevsky (2001) for self-annuitizing under **continuous time with deterministic investment returns**. Here are some key variables:

x = The retirement age used in this paper is 65.

W_t = Wealth of the investing retiree at time t.

c = Desired fixed consumption amount per month.

r = Risk-free interest rate.

l = Loading percentage on the actuarial premium charged by the insurance company. This is a separate loading factor other than the one usually imposed on mortality credits.

$\overline{a}_x$ = Market price of a one-dollar per year life annuity for an individual age x, shown in equation (2.1).

$$\overline{a}_x = (1+l)\int_0^\infty e^{-rt}\,{}_tp_x\,dt \tag{2.1}$$

k = Rate of return for deterministic case.

${}_tp_x$ = Conditional probability that a life aged x survives up to age $x+t$. Table 2.1 provides Milevsky's figures of survival probabilities adjusted using Individual Annuity Mortality (IAM) 2000 table, published by Society of Actuaries. To update these figures, we dynamically adjusted probabilities using scale G and IAM 2012 table from Subcommittee (2011), for comparison and consistency. Values are provided in the same table under "Updated" columns.

μ_x = The force of mortality is assumed to follow two-parameter Gompertz distribution as in equation (2.2). The future lifetime distribution of life aged 65 for both male and female under this assumption can be seen in Table 2.1.

$$\mu_x = \frac{1}{b}\,exp\left(\frac{x-m}{b}\right) \tag{2.2}$$

Where m stands for modal lifetime and b stands as scaling factor. The author uses parameters adjusted and smoothed by the above-mentioned Gompertz specification. Parameters used for male individual are, $m = 88.18$ and $b = 10.5$, and female individuals are, $m = 92.63$ and $b = 8.78$.

Table 2.1: Survival probabilities: Using Gompertz fit to IAM 2000 table for a life currently aged 65 from Milevsky (2001) and updated probabilities fitted to the IAM 2012 table.

Survival to Age	Male	Female	**Updated (male)**	**Updated (female)**
70	0.935	0.967	0.952	0.962
75	0.839	0.913	0.883	0.906
80	0.705	0.823	0.772	0.819
85	0.533	0.686	0.603	0.672
90	0.339	0.497	0.374	0.455

Sample annuity prices were computed using different risk-free rates and the survival probabilities obtained from Table 2.1. The loading factor assumed in the calculation is $l = 10\%$.

Milevsky (2001) points out that a retiree holding wealth $W_0 = w$ has two options:

- Full annuitization: The retiree annuitizes all of her liquid wealth in a life annuity quoted at $\overline{a}_x$, and will be able to purchase $c = \frac{w}{\overline{a}_x}$ dollars of consumption per year of lifetime.

- No annuitization: Alternatively, the retiree can choose to invest her wealth in a portfolio earning a continuously compounded rate of k and replicating the same amount obtained by purchasing a life annuity. The wealth process of the retiree would thus obey the following differential equation:

$$dW_t = (kW_t - c)dt \tag{2.3}$$

The equation (2.3) observes that the wealth is invested and grows at a deterministic rate (k) minus the consumption made by the individual in the time interval t. Solving the ordinary differential equation in (2.3) we get:

$$W_t = \begin{cases} \frac{c}{k} + \left(w - \frac{c}{k}\right) e^{kt} & t < t^*, W_0 = w \\ 0 & t \geq t^* \end{cases} \tag{2.4}$$

where t^* is the time at which wealth hits zero. This eventually happens when $wk < c$ i.e. the interest earned increases at a slower rate than the consumption itself. Thus we

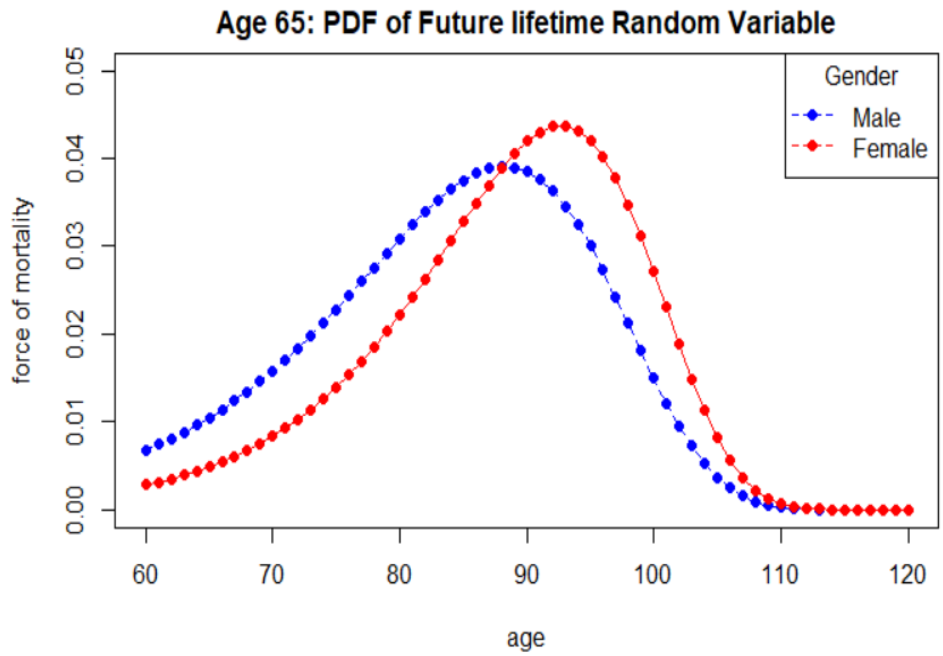

Figure 2.1: Probability density function of future lifetime of individual aged 65 under Gompertz specification

compute for the time at which the individual is ruined, by setting equation (2.4) to 0. We obtain:

$$t^* = \begin{cases} \frac{-\ln(1-k\,\overline{a}_x)}{k} & k < (\overline{a}_x)^{-1} \\ \infty & k \geq (\overline{a}_x)^{-1} \end{cases}$$

The wealth function is conditionally bound to ruin or shortfall if $k < (\overline{a}_x)^{-1}$, given the individual is still alive. We obtain the lifetime probability of consumption shortfall by computing the probability of surviving until time t^* i.e. ${}_{t^*}p_x$.

For illustrative purposes, let $w =$ \$100,000 with $l = 10\%$ and $r = 3\%$, then $\overline{a}_{65}$ (female) $= 18.08$, and the consumption rate is $c^* = w/\overline{a}_{65} =$ \$5530.97 per year. So, if the individual decides to consume the same amount while investing at a rate of $k = 4\%$, then her time of ruin would be 32.11 years from now. There is still a 0.20 chance of being alive, which implies a 0.20 probability of ruin by using this strategy. Figure 2.2 shows wealth movement for different rates of returns.

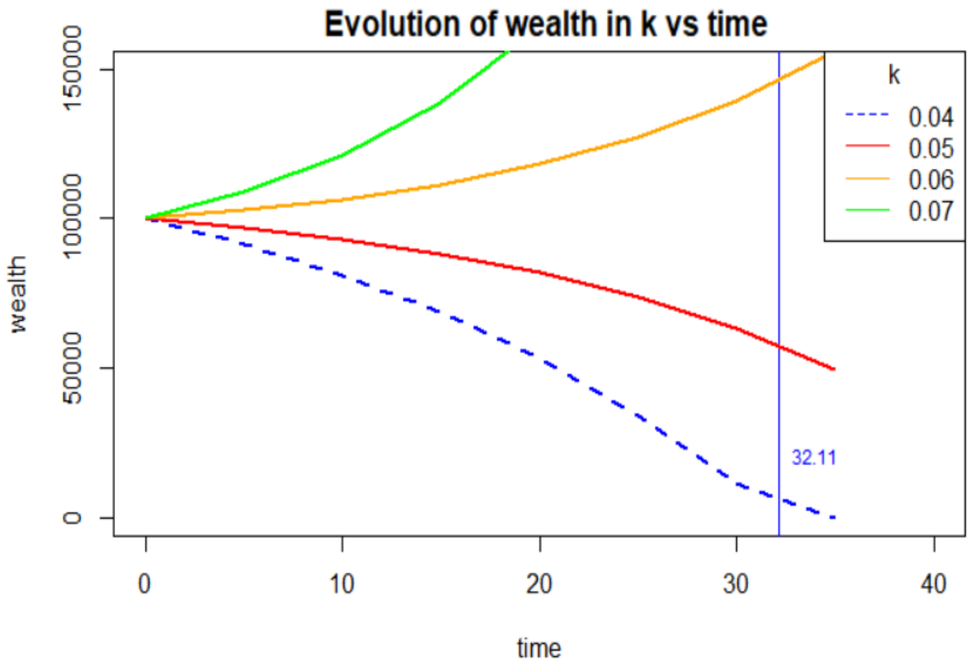

Figure 2.2: Deterministic returns case: Evolution of wealth vs k in time (for female aged 65)

The strategy so far answers "if" an individual should annuitize at a later date in retirement. The need to annuitize at some point leads to the question of "when" to annuitize. The author formulates the optimization problem in a simple function, as in equation (2.5).

$$\max_{0 \leq T \leq \infty} \{T\} \text{ s.t. } \quad \frac{W_T}{\overline{a}_{x+T}} \geq c \tag{2.5}$$

The conditional future time T, is the maximum time up to when the retiree can choose to defer annuitizing before the original consumption becomes no longer affordable in the annuity market. The retiree has to stop investing and switch to life annuity with the remaining wealth once they hit the time T. Still using deterministic assumptions and equation (2.5), the following equation (2.6) for time T is obtained:

$$T = \begin{cases} \frac{1}{k} \ln \left[\frac{1/k - \overline{a}_{x+T}}{1/k - \overline{a}_x} \right] & k < (\overline{a}_x)^{-1} \\ \infty & k \geq (\overline{a}_x)^{-1} \end{cases} \tag{2.6}$$

Continuing with the example above of a female retiree aged 65 and using equation (2.6), we can calculate the maximum time T at which she has to annuitize. Given the lifetime ruin occurs after 32.11 years of retirement, the maximum time by which she should annuitize would be 25.08 years or at the age of 90.08, assuming that annuity prices are known. Figure 2.3 supports equation (2.5) to show the consumption movement along time with respect to the wealth and annuity rate. The green line is the benchmark point at which the consumer drops to original consumption rate c. This is the maximum time point allowed for a retiree to annuitize to meet the original consumption rate.

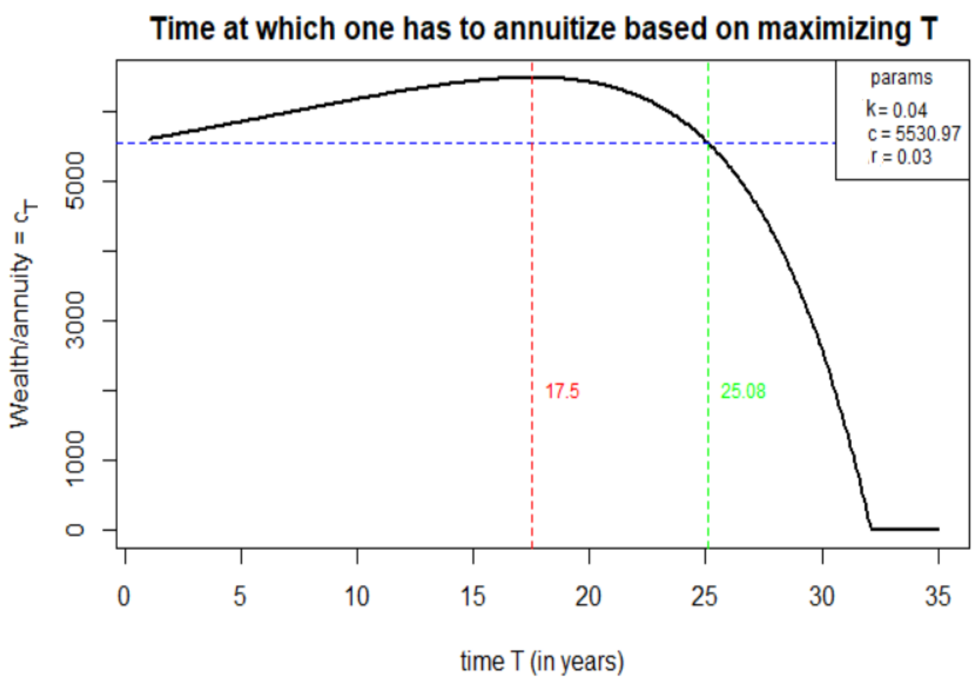

Figure 2.3: Deterministic returns case: Movement in $\frac{W_T}{\bar{a}_{x+T}}$ along time and the maximum time (T) at which one has to annuitize before they hit t^*, based on Milevsky's equation (2.5)

Another interesting aspect of Figure 2.3 is that, though the author claims that consumer should defer annuitizing until the original consumption stream is no longer affordable this waiting period does no good to the consumer when compared to the one who annuitizes immediately. Explicitly speaking, if one had an option to annuitize at age 65 and consume at the rate of \$5531 per annum while the other option is to wait for the next 25 years to earn a similar consumption rate by beating the mortality adjusted returns as long as she survives, the retiree is better off choosing to annuitize at retirement just to assure herself no risk is involved, i.e. $c_{max\,\{T\}} \neq max\,\{c_T\}$. What would be even better is the option where waiting for another couple of years gives the retiree a chance to maximize her consumption rate by annuitizing at that point. So we are looking for c_τ down the timeline at which the retiree can annuitize to obtain a maximum possible rate and continue to consume the same as long as she survives. This value in Figure 2.3 is shown using a red line. We can annuitize at time $\tau \leq T$ such that:

$$c_\tau = \max_{0 \leq T \leq \infty} \frac{W_T}{\overline{a}_{x+T}} \tag{2.7}$$

Using equation (2.4) and the equation (2.7) we obtain the conditional consumption rate in equation (2.8) that maximizes with respect to T. Assuming that future annuity prices are known, one has to annuitize before they reach their deterministic time of ruin (t^*).

$$c_\tau = \frac{(kw - c)e^{kT} - c}{k\overline{a}_{x+T}} \tag{2.8}$$

where $T \leq \frac{1}{k}\ln(\frac{c}{kw-c})$, which is also the time of ruin (t^*) the retiree needs to be aware of. From equation (2.8) we obtain that the optimal time to annuitize while maximizing benefits would be after 17.5 years of retirement. At this time the value of annuity purchasable is \$6476.80. Therefore, a retiree is able to maximize her returns by 17% more than what she would have earned initially.

A similar approach is used in the model for **continuous time with stochastic investment returns** in the paper. In a stochastic case, the need for beating life annuity rate and mortality credits might not live up to the expectations. This is due to the uncertain nature of returns earned.

The author mentions three possible sources of uncertainty that can alter the decision made in the deterministic setting. They are:

- Stochastic investment returns
- Stochastic interest rates
- Stochastic mortality rates

Milevsky (2001) introduces stochastic investment returns to account for uncertainty. In order to quantify the risk of the strategy, he uses Monte Carlo simulations to simulate the wealth process. These simulations were to meant to provide the probability of being able to successfully defer annuitization as in equation (2.9):

$$P\left[\frac{W_T}{a_{x+T}} \geq c\right] \tag{2.9}$$

The investor's portfolio follows a Geometric Brownian motion as in equation (2.10), with parameters μ and σ, where μ denotes the growth rate of the wealth invested (akin to k in deterministic case) and σ denotes the volatility of the wealth. The wealth is modelled as a geometric Brownian motion, with parameters in Milevsky (2001): $\mu = 0.13$ and $\sigma = 0.17$.

$$\begin{cases} dS_t = \mu S_t dt + \sigma S_t dB_t \\ S_t = S > 0 \end{cases} \tag{2.10}$$

Here, S_t is the price of the risky asset. We can write the stochastic differential equation for the wealth process as the following equation (2.11).

$$dW_t = [\mu W_t - c]dt + \sigma W_t dB_t \tag{2.11}$$

Milevsky (2001) performs 25000 simulations of the wealth process at different time intervals. For simulating interest rates, he uses continuous time model of interest rate behavior introduced by Cox et al. (1985), as in equation (2.12). The parameters include $\widetilde{r}$, the long run average level of interest rate, γ, which denotes the speed of adjustment in the mean reverting process, and σ_r, which is the volatility of interest rates. The long term rate assumed in Milevsky (2001) is really high, this is because if the current rates are below the long term rate $\widetilde{r}$, interest rates are expected to increase, and vice versa. Note: rate r_t is used for pricing annuities in stochastic case, see results section.

$$dr_t = \gamma(\widetilde{r} - r_t)dt + \sigma_r \sqrt{r_t} dB_t \tag{2.12}$$

A vector of 25000 random interest rates were produced to obtain the simultaneous $\widetilde{a}_{x+T}$ future random annuity prices. This along with the wealth process were used to determine the density function of $\widetilde{c}_T = \widetilde{W_T}/\widetilde{a}_{x+T}$. The author calculates whether the individual has enough consumption attainable at time T in the future to replicate the original consumption c. If the simulated future consumption is lower than the original consumption, it is considered as a failure in deferring annuitization to time T.

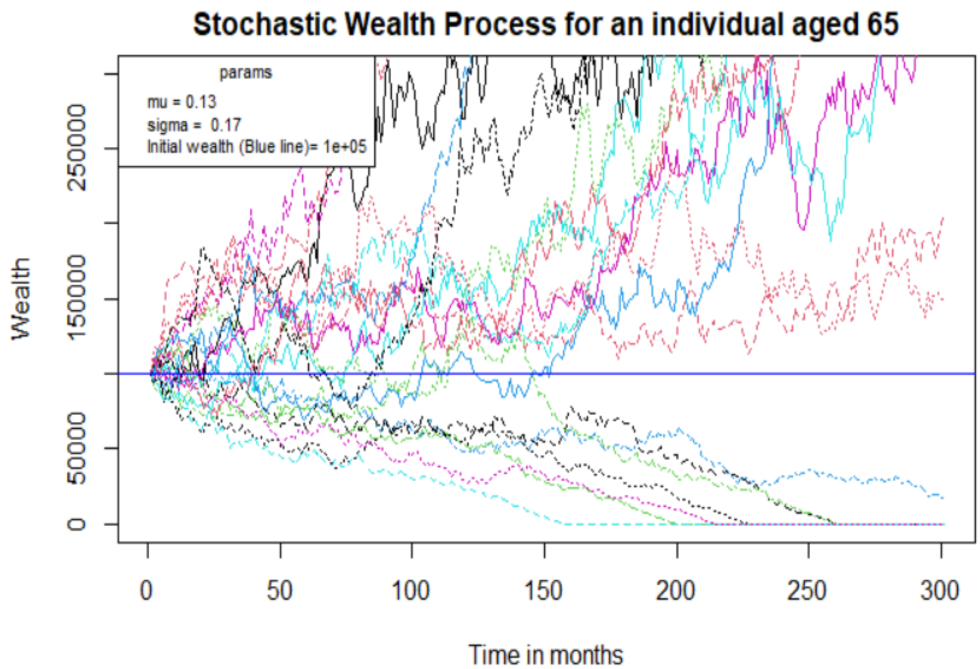

Figure 2.4: The stochastic wealth process for a retiree aged 65 for time period of 25 years, showing 20 simulated paths out of 25000, $w = \$100{,}000$, $\mu = 0.13$, $\sigma = 0.17$

In this paper, we first simulate the model with the parameters and procedure used in Milevsky (2001) to test if we are consistent with its results. Ignoring sampling error, we obtain results close to but not statistically identical with those in Milevsky (2001). We further used the same model with the updated market values of μ and σ at 6% and 20%. Also based on the current long term annuity interest rates, the rate (r) used for pricing annuities is set around 2-4% for simulations. We produce Figure 2.4 as a glimpse of the simulated paths with the parameters from Milevsky (2001), we use time in months and wealth starting at \$100,000. The results of the Monte Carlo simulations are discussed later in Section 2.3.

2.3 Results and Review

In this Section, we discuss the implications of Milevsky's strategy as revealed by the analysis of Monte Carlo simulations. We walk-through the numerical results and their interpretation.

First of all, Milevsky runs the model to project wealth movement only up to certain time points in retirement. That is 5,10,15 and 20 years from the time of retirement. We use the same final time points in our simulations and fixed interest rates of 5%, 7% and 9%

Table 2.2: Simulation results: Probability of beating a life annuity for a retiree aged 65, using Milevsky (2001) parameters, $\mu = 13\%$, $\sigma = 17\%$, $l = 10\%$

Time interval	$r = 5\%$		$r = 7\%$		$r = 9\%$	
	Male	Female	Male	Female	Male	Female
5	78.6%	80.6%	71.7%	74.0%	63.6%	66.3%
10	84.1%	87.1%	77.3%	80.6%	66.3%	70.5%
15	86.1%	89.8%	78.6%	83.0%	66.6%	71.9%
20	86.7%	91.1%	78.6%	83.6%	65.8%	71.8%

The results obtained support the argument made about interest spread. The higher the interest rate on the annuity market, lower the annuity prices, which allows the retiree to consume at a faster rate relative to the fund movement in the market investment.

For a sample scenario, Milvesky's implication of the results states that, if the interest rate on annuities is 7%, a 65 year old retiree female (male) has as a 83% (79%) chance of being able to beat the life annuity rate by delaying annuitization to age 80. These probabilities are conditional on the retiree being alive at age 80.

The major concern to many retirees may be the downside of not being able to beat the original life annuity rate. In our approach, we use the model to calculate the risk of adopting Milevsky's strategy and subsequently being worse off than immediate annuitization. Even more critically, what is the probability that a retiree adopts the strategy but runs out of money in doing so? We use the same parameters from Milevsky (2001) and review the worst 5% of cases. The results in Table 2.3 show that the risk of ruin occurring is most severe for time periods of 10 years and above.

From the results in Table 2.3 we see that the same retiree aged 65 female (male) who had an 83% (79%) chance of beating life annuity rate has a 5% (8%) chance of running out of money before they even reach age 80. In many other cases they would have to purchase an annuity that pays them lower income than desired. Figure 2.5 shows the consumption levels of a 65 year old female along with the fixed annuity benchmark (in blue) level at retirement. The upside gain contrasts with the risk from those paths that fail to even achieve the desired income level. It is also evident from the figure on how quickly the ruin occurs in certain paths. This can hurt a retiree choosing the strategy at such market parameters. Had she opted for a life annuity, she would be guaranteed an income of \$8293.25 /yr for life. Likewise, in case of 9% annuity rate and 20 years of deferral the risk of ruin is as high as 21% for a female and 26% for a male retiring at 65.

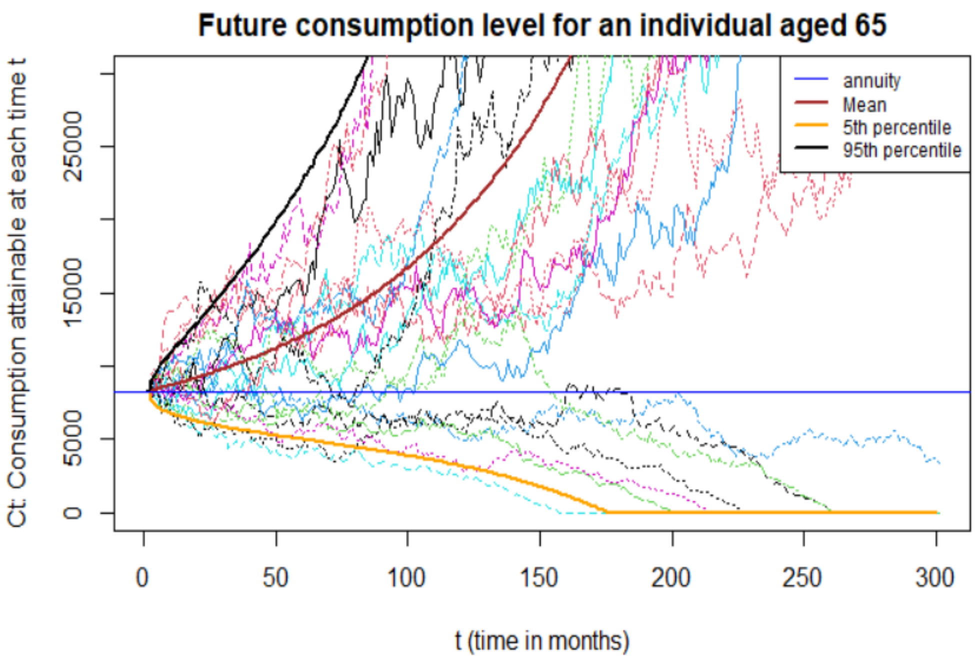

Figure 2.5: Quantile plot for the future consumption levels attainable by a female retiree aged 65 assuming she annuitizes at a discrete time T. Using Milevsky's parameters: $\mu = 13\%$, $\sigma = 17\%$, $r = 7\%$, $l = 10\%$ and total expected lifetime = 25 years. The figure shows 20 simulated paths out of 25000. Here, $c_T = \frac{W_T}{\bar{a}_{x+T}}$.

We now use the model with the current parameters and run a similar set of simulations. Table 2.4 provides results with μ at 6%, σ being 20% and $l = 10\%$. The current long term annuity interest rate is around 2%-4%. Here is an example, using these parameters a retiree aged 65 female (male) with initial wealth $w =$ \$ 100,000, contemplating on buying a life annuity quoted at $r = 2\%$ and $\bar{a}_{65} = 20.34$ (17.75) per dollar of lifetime consumption, will earn an income of $\frac{w}{\bar{a}_{65}} =$ \$4916.42 (\$5635.39) for the rest of their life through immediate annuitization. The retiree may decide to defer annuitization by investing in the market and consume the same amount, with a hope to replicate the same exact annuity on a future date. The simulations show that there is 58.9 % (52.8 %) chance of beating the life annuity if they wait for another 15 years, conditional on being alive. On the other hand, the chance of ruining themselves before they reach 80 is 11% (17%). The probability is quite high and the retiree is going to face an adverse downside trusting the strategy.

Table 2.3: Risk of ruin occurring before annuitizing for a retiree aged 65 under Milevsky's parameters, $\mu = 13\%$, $\sigma = 17\%$, $l = 10\%$

Time interval	$r = 5\%$		$r = 7\%$		$r = 9\%$	
	Male	Female	Male	Female	Male	Female
5	0%	0%	0%	0%	0%	0%
10	1%	1%	2%	1%	5%	3%
15	4%	2%	8%	5%	17%	12%
20	8%	4%	15%	10%	26%	21%

The results in Table 2.3 and 2.4 are quite counter-intuitive. The higher the risk free rate r the higher is the risk of getting ruined. This is because higher r implies that annuity prices are low, which leads to higher consumption c to be consumed by the retiree initially. Hence the probability of the wealth approaching ruin is quicker, as a constant amount of wealth is consumed at each time period. So as we go along the columns of higher interest rates we see a spike in the ruin probability simultaneously.

Table 2.4: Risk of ruin occurring before annuitizing for a retiree aged 65 using current world parameters, $\mu = 6\%$, $\sigma = 20\%$, $l = 10\%$

Time interval	$r = 2\%$		$r = 3\%$		$r = 4\%$	
	Male	Female	Male	Female	Male	Female
5	0%	0%	0%	0%	0%	0%
10	2%	1%	5%	2%	7%	4%
15	17%	11%	23%	15%	30%	22%
20	33%	23%	39%	30%	47%	38%

In conclusion, the take-away points from this Chapter are:

- Milevsky's advice to his readers fails to communicate the fact that there is substantial risk involved. His advice is purely based on the upside gains in the outcomes and it completely ignores the risk of severely adverse outcomes. A real life retiree can have a severe impact on his/her financial status if they are independent retirees (or no bequest motives) and trust this strategy in the scope of earning better in retirement years. The strategy can be quite overwhelming if we only look at the chance of upside gain, while current markets are unpredictable and are constantly adapting.

- Factors like uncertainty in the interest rates (r), market returns (μ) or market volatility (σ) are not evidently taken into account. These factors probably have the most significant impact in the way the results are produced. Given these rates are kept constant and/or are not close to the current world parameters can have adverse affect had the strategy been adopted by a real life retiree.

  In order to test for the uncertainty around future interest rates we use the CIR model mentioned in Milevsky (2001) with parameters ($\widetilde{r}$, γ, σ_r). We updated the values of probability of ruin occurring in Table 2.5 and probability of successful deferral from Table 2.2. The values of c_T are obtained by simulating vector of W_T and vector of $\overline{a}_{x+T}$. An element-by-element procedure was adopted on these two vectors. Figure 2.6 shows the simulated paths of interest rates starting at 9%.

- In Table 2.5, the probability of beating the life annuity has increased for lower interest rates when compared to Table 2.2. The reason behind this is that the annuities are priced on a long term risk free rate, so fluctuations in the interest rate cause a hike

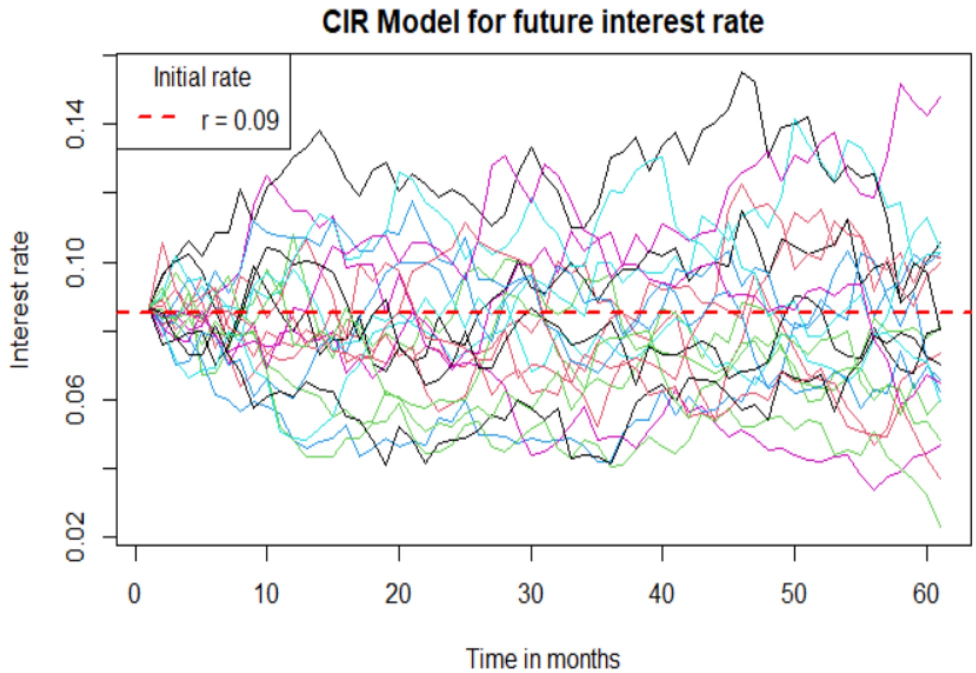

Figure 2.6: Future interest rate simulation model. Parameters are r = 0.09, $\widetilde{r}$ = 0.085, γ = 0.25 and σ_r = 0.08. The 20 simulation paths of interest rate model are based on the parameters provided in Milevsky (2001) for a 5 year period out of the sample of 25000 paths. The paths follow the equation given in (2.12).

in the annuity prices. This allows for lower consumption rates and hence a higher chance of successful deferral is seen, it is not quite as high when compared to the values in Table 2.2. On the other hand, the risk of being ruined is more stable in comparison to Table 2.3, because wealth movement is independent of future annuity prices and there is zero investment in the risk free asset. Hence the chance of ruin is similar in both cases.

- A density plot of the consumption attainable at time T= 10 years for a female individual aged 65 is shown in Figure 2.8. The plot is highly right skewed with majority of values lying between \$0-\$20,000. The original consumption rate at retirement is shown in blue and the 5th and 95th quantiles are shown in dotted red and green lines. From the figure, we can interpret that the mode is below the original c. About 5% of values lie close to 0 i.e. to say ruin almost occurs in the worst 0-5% cases if the retiree chooses to wait for 10 years. The value of ruin from our tables for this

Table 2.5: Probability of beat life annuity (risk of ruin occurring before annuitizing) for a retiree aged 65 using Milevsky's market parameters, $\mu = 13\%$, $\sigma = 17\%$, $l = 10\%$ and CIR model parameters, $\widetilde{r} = 0.085$, $\gamma = 0.25$ and $\sigma_r = 0.25$.

Time interval	$r = 5\%$		$r = 7\%$		$r = 9\%$	
	Male	Female	Male	Female	Male	Female
5	85.6 (0%)	87.5 (0%)	74.1 (0%)	76.3 (0%)	60.3 (0%)	62.6 (0%)
10	88.1 (0.4%)	90.8 (0.1%)	77.7 (2%)	81.6 (1%)	63.9 (5%)	68.3 (3%)
15	88.3 (3%)	92.0 (2%)	78.0 (9%)	83.1 (5%)	64.5 (18%)	70.2 (12%)
20	87.7 (9%)	92.1 (4%)	77.1 (16%)	83.2 (10%)	63.4 (28%)	70.3 (21%)

case shows that it is close to 1%.

- The author's simulations do not account for uncertainty in mortality rates. This tends to be one of the weakest points in his simulation analysis. The ever changing demographical, life span, and health statuses account for future adjustments in mortality. In line with the current trend, future mortality patterns may vary, thus ultimately affecting the annuity prices even for a fixed interest rate. It is quite evident in the comparison between Milevsky's survival probabilities and the updated probabilities provided in Table 2.1.

- Taking modern economic parameters into consideration, Figure 2.7 shows the simulated paths of the self-annuitizing strategy using current parameters. The paths when compared to Figure 2.5 majorly fall below the desired annuity benchmark. It implies that with current market scenarios, this strategy isn't really doing a good job at replicating a life annuity.

- Milevsky's reason for investing all of the wealth into risky assets is the long term propensity of equity based investments to outperform fixed income products, independent of how the annuity markets perform. One of the key factors in the probability of successful deferral is the level of interest rates in the market. The mean equity risk premium, which is $\mu - r$, shows the gain in income from the difference in rates.
  - For instance, when μ is 13% and r is 5% (Milevsky's parameters) the expected equity premium is 8%.
  - For more current parameters of $\mu = 6\%$ and r of 2% the difference is about 4%, less than assumed by Milevsky. The lower the risk premium, the quicker the wealth goes to zero.

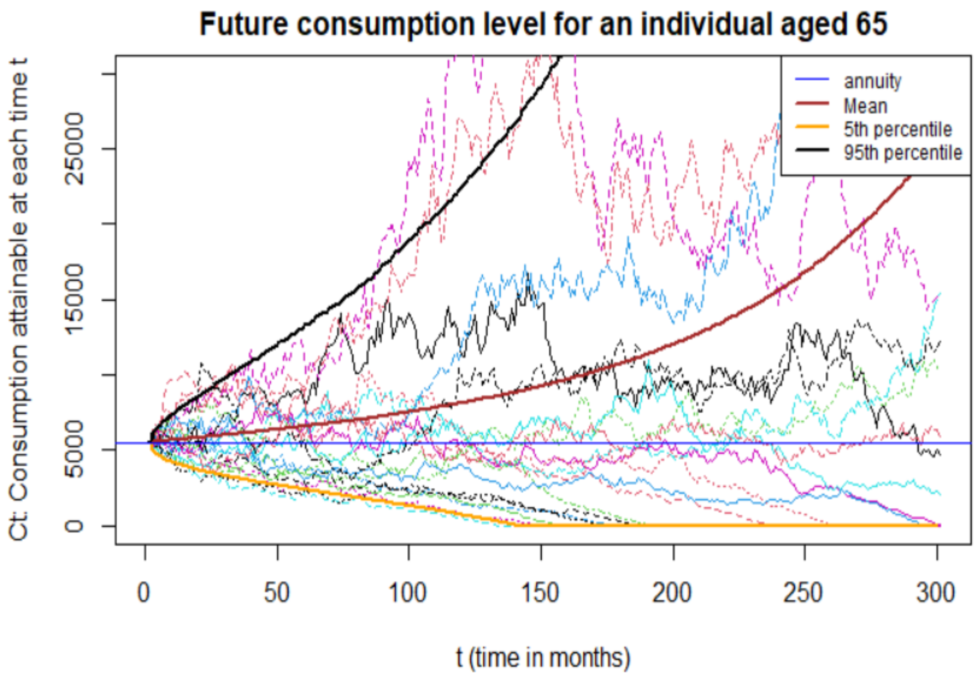

Figure 2.7: Quantile plot for the future consumption level attainable by a female retiree aged 65 assuming she annuitizes at a discrete time T. Using updated parameters, $\mu = 6\%$, $\sigma = 20\%$, $r = 3\%$, $l = 10\%$ and total future lifetime = 25 years. The figure shows 20 simulated paths out of 25000. Here $c_T = \frac{W_T}{\bar{a}_{x+t}}$

The median equity risk premium accounts for the moving volatility. It is given by $(\mu - \frac{\sigma^2}{2}) - r$, and is a measure of both volatility and drift coefficients.

- When μ is 13%, r is 5% and $\sigma = 17\%$ the median risk premium is 6.55%,
- $\mu = 6\%$, r is 2% and $\sigma = 20\%$, the median risk premium is 2%.

Given the risk involved, it is very questionable to advise a retiree to go "all or nothing" in risky assets in their portfolio choice.

- Unlike the discrete returns case, the strategy under the stochastic setting does not give the latest time at which one has to annuitize. Rather the author leaves it open to the reader to choose among the 5-year time periods, based on the probability of beating the starting income.

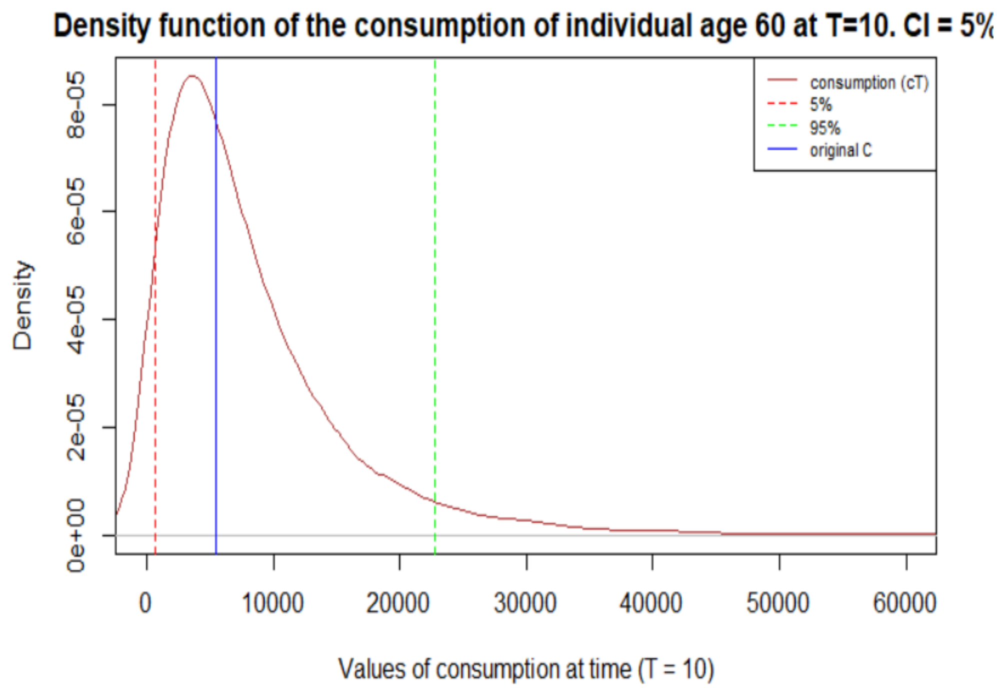

Figure 2.8: Density distribution of future consumption outcomes at time $T = 10$. $\widetilde{c}_T = \frac{\widetilde{W}_T}{\widetilde{a}_{x+T}}$, Parameters used here are $r = 5\%$, $\mu = 13\%$, $\sigma = 17\%$ and $l = 10\%$.

- Overall, the normative advice from the paper does not fit well with the descriptive behavior of the retirees. Nor does it seem highly acceptable. Retirees willing to defer annuitization are likely to hold both stocks and bonds. This normative advice will ruin people with quite high probability.

Chapter 3

Annuitization and Asset allocation by Milevsky and Young (2007)

3.1 Highlights

Milevsky and Young (2007) examines optimal annuitization, investment and consumption strategies for a utility maximizing retiree facing a stochastic time of death under various institutional restrictions.

One of the frameworks introduced in the paper is termed the *"all or nothing arrangement"*. Under this arrangement, the retiree invests in financial markets initially and makes periodic withdrawals for consumption, but may annuitize all of her remaining wealth at *one point in time*. Retirees are assumed to invest based on their risk preferences alongside their own subjective assessment of their mortality.

Milevsky and Young (2007) assume that the decision maker seeks to maximize their expected utility of discounted consumption over admissible consumption and investment parameters. The value function derives the optimal age to annuitize, which occurs when the option to delay has zero time value. This optimum depends on the retiree's risk aversion, as well as their subjective mortality beliefs. Considering these criteria, Milevsky and Young (2007) locate an optimal age to annuitize and simultaneously develop a metric to capture the loss from annuitizing prematurely. The normative scope of the paper suggests that a retiree who postpones annuitization, roughly has a 70% chance of consuming more throughout their remaining life than if they annuitize at retirement. Altogether, the results show that under the *all or nothing arrangement*, using plausible risk aversion values, the optimal annuitization time does not occur before age 70.

The objective of this paper is similar to Milevsky (2001). We list a few similarities and differences to give a sense of what to expect from this chapter. The similarities are:

- The retiree is expected to invest all of his/her wealth (W_t) into the investments. Likewise, they are expected to withdraw all of their wealth from the portfolio once they reach annuitization time. Other external sources of income are not considered to be a part of the strategy.
- The authors in both the papers recommend the strategy to a retiree seeking to gain some flexibility in wealth. Retirees are advised to invest and consume in the first few years of retirement when the equity risk premium pays more than the mortality credits, and annuitize when the mortality credits becomes larger.
- Ultimately both the papers are providing normative advice.

The differences to notice are:

- Milevsky and Young (2007) is built around expected utility of discounted lifetime consumption in advising a retiree maximizing their utility, whereas Milevsky (2001) uses a probability based approach to advise the retirees.
- The model in Milevsky and Young (2007) allows for mixed investments to be made in both risky and risk-free assets, unlike Milevsky (2001), which restricted investments to risky assets only.
- The optimal consumption (c_t) is a fixed percentage of current wealth in Milevsky and Young (2007). In Milevsky (2001), it is a constant, c.
- The optimal annuitization time is computed in Milevsky and Young (2007) through the use of an objective function. Hence $T \in \mathbb{R}^+$ is a fixed time point in future, whereas Milevsky (2001) focuses on different discrete time points in future.
- There is no separate loading factor used in this paper.

In this Chapter, we are interested in the dynamics of the *all or nothing framework*. We illustrate the strategy proposed under this arrangement through Monte Carlo simulations. Later we incorporate current real world parameters to test for the up-to-date risk in adopting such a strategy.

3.2 Strategy: All or nothing framework

The choice of portfolio is crucial. An individual would like to invest in risky assets to gain higher returns, but balance out with some risk-free assets.

We describe the model from Milevsky and Young (2007) of financial and annuity markets. An individual can choose to buy stocks whose price at time t is S_t, which follows a Geometric Brownian Motion, and also risk-free assets, whose price at time t is X_t, where

$$\begin{aligned} dS_t &= \mu S_t dt + \sigma S_t dB_t \,,\; S_0 = S > 0 \\ dX_t &= r X_t dt \,,\; r \geq 0 \end{aligned} \tag{3.1}$$

Here $\mu > 0$ is the drift coefficient , $\sigma > 0$ is the volatility coefficient of the risky asset and B_t is a standard Brownian motion with respect to filtration $\mathcal{F}_t$ of the probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

The wealth process W_t assumes the retiree invests π_t in stocks and $1 - \pi_t$ in bonds at time t. This leads us to the following stochastic differential equation for the wealth function:

$$dW_t = \Big(\big[r + \pi_t(\mu - r)\big]W_t - c_t\Big)dt + \sigma\pi_t W_t dB_t \tag{3.2}$$

where c_t is the consumption at time t for the retiree and $W_0 = w$.

Moving onto the insurance and actuarial assumptions. The expected present value of a life annuity function that pays \$1 per year continuously to an individual aged x is written as $\overline{a}_x$, defined by:

$$\overline{a}_x = \int_0^\infty e^{-rt} \, {}_t p_x \, dt \tag{3.3}$$

The risk free rate r used here is equal to the risk-free government yield bond rate. Milevsky and Young (2007) do not specify the use of any loading factor separate from loading adjusted to mortality.

In Milevsky and Young (2007), the individual values their expected utility via his or her own subjective mortality, while the annuity is priced using objective mortality, which may or may not be the same. However for numerical illustrations we avoid complexity by working only with objective mortality rates. In other words, both the mortalities are equal.

The individual's t-year survival probability is denoted by ${}_tp_x$. It is defined via the subjective force of mortality μ_{x+t}, by the formula:

$$ {}_tp_x = \exp\left(- \int_0^t \mu_{x+s} ds \right) \tag{3.4} $$

An individual's risk preference can be measured through the utility based value function associated with her choices. It is assumed that an individual seeks to maximize her expected utility. In Milevsky and Young (2007), $V(W_t, t)$, from equation (3.5), denotes the value function associated with an individual maximizing her expected utility of discounted lifetime consumption over admissible $\{c_t, \pi_t\}$, where the expectation is conditional on current size of the fund W_t.

$$ V(W_t,t) = \sup_{c_t,\pi_t,T} \mathbf{E}_t \left[\int_t^T {}_{s-t}p_{x+t} e^{-\rho(s-t)} u(c_s)\, ds + \int_T^\infty {}_{s-t}p_{x+t} e^{-\rho(s-t)} u\left(\frac{W_T}{\bar{a}_{x+T}} \right) ds \right] \tag{3.5} $$

At time $t = 0$, $W_0 = w$, and the value function is:

$$ V(w,0) = \sup_{c_0,\pi_0,T} \mathbf{E}_0 \left[\int_0^T {}_sp_x e^{-rs} u(c_s)\, ds + {}_T\, p_x e^{-rT} u\left(\frac{W_T}{\bar{a}_{x+T}} \right) \bar{a}_{x+T} \right] \tag{3.6} $$

In equation (3.6), the value function is discounted to time $t = 0$, when an individual had just retired. The first term in the expression discounts for utility gained from consuming in the interval $(0, T)$, given the retiree is alive. The second term is the discounted utility gained from annuitizing at time T. The individual annuitizes all her wealth at time T, and subsequently consumes at a constant rate of $\frac{W_T}{\bar{a}_{x+T}}$. So for each time t the retiree has an option of stopping her investment process and annuitizing for a certain utility gain. Or she could continue to invest in the hope that stopping at a later date will provide her a bigger reward. So among all possible stopping times there exists an optimal stopping time, which is some fixed future time T. One can note the use of t is explicitly for generalized equations, since the strategy is mainly based on comparison between annuitizing at retirement or at later date; t would thus be the starting time 0.

• ρ is the subjective discount rate representing the individual's degree of impatience relative to their consumption preference over time. Milevsky and Young (2007) assume this rate to be equal to r. The authors reason that, the discount rate for perceived risk and other subjective factors is adjusted through the use of force of mortality μ_{x+t} effectively.

This adjustment is thus made by including the term $\overline{a}_{x+T}$ to the second term. The use of r to discount utility is based on a flawed assumption, this is discussed later in the chapter.

• u is an increasing concave utility function, written as equation (3.7), that exhibits a constant relative risk aversion (CRRA) co-efficient γ, which is assumed to be a constant.

$$u(c) = \frac{c^{1-\gamma}}{1-\gamma}, \gamma > 0, \gamma \neq 1 \tag{3.7}$$

If the value V is negative for all values of $T > 0$, then it is optimal to annuitize immediately, written as equation (3.8), if and only if $\gamma > 1$.

$$V(w, 0 \mid T = 0) \geq V(w, 0 \mid \forall\ T > 0) \tag{3.8}$$

Using CRRA utility, the authors show that the stopping time T is not random, but rather deterministic. This is because the CRRA utility factors out wealth from the value function. The optimal stopping time follows a smooth solution that satisfies the value function in (3.5). The Hamilton-Jacobi-Bellman equation is introduced with variational inequality as in equation (3.9). The hypothesis that supports this form of solution is that the continuation region lies from $(0, T)$. In this region, the reward for self-annuitization is higher than the reward for purchasing an annuity. Therefore, one does not annuitize their wealth until time T. Here the equality holds when $t = T$, i.e. it falls in the stopping region, so the reward for purchasing an annuity is higher and it coincides with the value function.

$$V(W_t, t) \geq \overline{a}_{x+t} u\left(\frac{W_t}{a_{x+t}}\right) = \frac{u(c_t)}{\rho + \mu_{x+t}} \tag{3.9}$$

The authors compute optimal values of investment and consumption by maximizing the HJB equation with respect to π_t and c_t, to derive the following results:

$$C_t^* = c^*(W_t^*, t) = \frac{W_t^*}{\psi(t)} \tag{3.10}$$

and

$$\Pi_t^* = \pi^*(W_t^*, t) = \frac{\mu - r}{\sigma^2 \gamma} W_t^* \tag{3.11}$$

where $\psi^\gamma(t) = \overline{a}^\gamma_{x+t}$. The optimal proportion of wealth to be invested in risky assets before annuitization is given by equation (3.11) and the optimal rate of consumption before annuitization by equation (3.10), in which W^*_t is the optimally controlled wealth before annuitization.

The authors define a function $\widetilde{V}(W_t, t; T) = \frac{1}{1-\gamma} W_t^{1-\gamma} \psi^\gamma(t; T)$. This function, when it hits the stopping region ($t = T$), obtains the unique value T^* such that $V(W_t, t) = \max_{T \geq 0} \widetilde{V}(W_t, t; T)$. The key feature to note is that wealth and time are multiplicatively separable in $\widetilde{V}$, this is to imply that optimal time to annuitize is independent of wealth and is therefore deterministic.

The next step is to find the optimal annuitization time T. An optimal T that maximizes $\widetilde{V}(W_t, t; T)$ in equation (3.12) is obtained through the first derivative with respect to T, assuming $t < T$. So among all the possible stopping times of T we are looking for the optimal time $\{T^* \in \mathbb{R} \mid 0 \leq T < T^*\}$ which has positive derivative for $T < T^*$, and for all values of $T > T^*$ it becomes negative. That is,

$$\frac{\partial \widetilde{V}}{\partial T}\bigg|_{t<T^*} = 0 \tag{3.12}$$

The derivative obtained up to proportionality constant is given by Milevsky and Young (2007) as:

$$\frac{\partial \widetilde{V}}{\partial T} \propto [\delta - (r + \mu_{x+T})] \tag{3.13}$$

where $\delta = r + (1/2\gamma)((\mu - r)/\sigma)^2$.

The derivative is independent of π_t and c_t or wealth factors. This is counter-intuitive and also supports the previous argument made about wealth getting factored out due to the use of CRRA utility.

The proportionality in this case is due to the changing force of mortality with respect to time t. Either $\delta \leq (r + \mu_x)$, which indicates that it is optimal to annuitize immediately, *or* $\delta > (r + \mu_x)$, from which it follows that there exists a future time T at which it will be optimal to annuitize one's wealth.

Re-arranging the above equation (3.13) in terms of μ_x, using $\delta = r+(1/2\gamma)((\mu-r)/\sigma)^2$, derives a constant as in equation (3.14). It implies that the optimal age to purchase a life annuity is when the force of mortality is greater than a constant similar to Merton's constant.

$$\mu_{x+T} = \frac{1}{2\gamma}\left(\frac{\mu - r}{\sigma}\right)^2 \tag{3.14}$$

Similar to the approach in Section 2.2, the idea is to quantify the risk of the decision maker's choice and calculate the probabilities associated with various consumption outcomes. In order to do so, Milevsky and Young (2007) develop a metric to measure the loss in value from annuitizing prematurely. This is done by computing the additional wealth that needs to be added to the current wealth, W_t, that makes the retiree indifferent between annuitizing now with extra wealth, and annuitizing at time T without the extra wealth. In the case of immediate annuitization, this would mean an additional wealth (h) that is required to be added to wealth at $t = 0$ i.e. w, in order to compensate the utility maximizing retiree for forced immediate annuitization, shown in equation (3.15). The value of option to delay h is expressed as a percentage of w in the equation that follows:

$$\widetilde{V}(w, 0; T) = \widetilde{V}(w(1+h), 0; 0) \tag{3.15}$$

If the value of h determined at $t = 0$ is 0 then it means one should annuitize immediately.

To implement this model we require the force of mortality assumptions and the capital market parameters. The force of mortality (μ_x) is assumed to follow a two parameter Gompertz distribution as in equation (2.2). The stock price follow a Geometric Brownian process with drift $\mu = 0.12$ and volatility $\sigma = 0.20$, using the original parameters from Milevsky and Young (2007). The authors assume a nominal rate of return on risky bonds of 6%. Given all these factors, the optimal time for annuitization is computed.

The proportion of wealth invested in stocks used in Milevsky and Young (2007) given parameters of $\gamma = 2$, $\mu = 12\%$, $\sigma = 20\%$ and $r = 6\%$, is 75%. They also explored CRRA parameter $\gamma = 1$,2 and 5. Before annuitization, the optimal consumption (c_t) is a percentage of current wealth.

In this chapter, we follow most of the aspects covered in Milevsky and Young's strategy. As mentioned earlier, we avoid complexity by working only with objective mortality rates. The subjective mortality constraints though are interesting to look at, but are very specific to each individual's assumption. Our aim is to understand the risk in the strategy when subjective and objective forces of mortality are equal, and using up-to-date parameters. The results of the above strategy along with ours are mentioned in the following section.

3.2.1 Results and review

In this section, we first apply the strategy using the parameters from Milevsky and Young (2007) and subsequently we apply the strategy with updated parameters. The results based on Milevsky and Young's parameters were consistent with those in the paper.

First, the optimal times to annuitization were calculated for CRRA with $\gamma = 1$, 2 and 5, along with the probability of having a lower income at the time of annuitization. Figure 3.1 shows an example for a female (male) retiree aged 60, who attain their optimal age to annuitize at 78 (73).

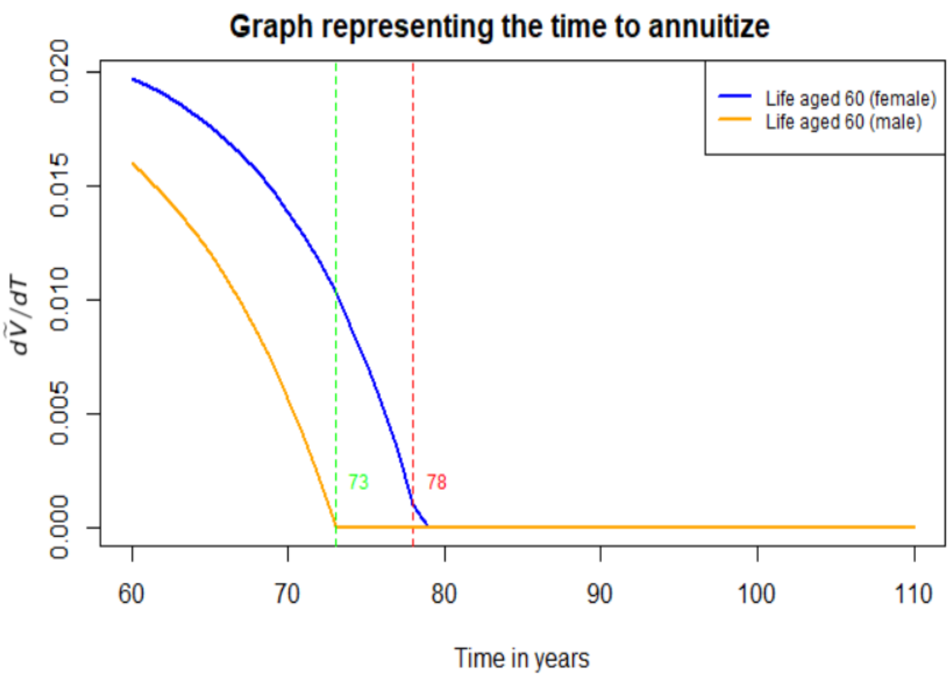

Figure 3.1: Optimal age to annuitize for a male and female retiree aged 60 obtained through the derivative $\frac{\partial \widetilde{V}}{\partial T} = 0$ from equation (3.13). Parameters used here are $\gamma = 2$, $\mu = 12\%$, $\sigma = 20\%$, $r = 6\%$

Table 3.1 matches with the results from Milevsky and Young (2007). Using these optimal time conditions, the wealth function in equation (3.2) is simulated for the expected lifetime of the retiree[1] and a probability of obtaining a life annuity with income lower than the income at retirement is computed. This is shown in Table 3.2.

[1] Refer to appendix for the calculation of the expected lifetime

Table 3.1: All or nothing framework: Optimal time to annuitize for female (male) with risk aversion of $\gamma = 1$, 2 and 5. Using the parameters in Milevsky and Young (2007)

Age	$\gamma = 1$	$\gamma = 2$	$\gamma = 5$
	Female (Male)	Female (Male)	Female (Male)
60	84 (80)	78 (73)	70 (63)
65	84 (80)	78 (73)	70 (Now)
70	84 (80)	78 (73)	Now (Now)
75	84 (80)	78 (Now)	Now (Now)

Table 3.2: All or nothing framework: Probability of lower annuity income and Probability of wealth going to zero before annuitization for risk aversion of $\gamma = 1$ and 2. Using the parameters in Milevsky and Young (2007)

Age	$\gamma = 1$		$\gamma = 2$	
	Female (Male)		Female (Male)	
	Prob. Lower Annuity	Prob. of ruin	Prob. Lower Annuity	Prob. of ruin
60	26.5 % (32.3 %)	10.3 % (5.1 %)	26.8 % (33.2 %)	10.4 % (9.8 %)
65	30.4 % (37.2 %)	5.1 % (0.4 %)	32.1 % (37.6 %)	9.7 % (1.8 %)
70	35.5 % (43.6 %)	0.4 % (0.0 %)	36.3 % (43.2 %)	1.8 % (0.0 %)
75	42.4 % (N/A)	0.0 % (N/A)	42.2 % (N/A)	0.0 % (N/A)

Table 3.2 shows the probability of purchasing a lower annuity along with the probability of wealth hitting zero before the retiree reaches the prescribed annuitization time. The figures clearly show that there is some evident level of the retiree being ruined when he/she is still alive. Such a risk has not been communicated well in Milevsky and Young (2007) to its readers.

For instance, in Figure 3.2 we show results for a female aged 60 investing 75% of her wealth ($W =$ \$100,000) in risky assets and 25% in risk-free assets before annuitization, with $\mu = 12\%$ and $\sigma = 20\%$. She will gain access to life annuity payout of \$7676.95 per year using an interest rate of 6%, which is 7.676% of original wealth if she annuitizes immediately. The blue line indicates the consumption if she chooses to annuitize at retirement. The paths show her consumption given she does not annuitize in her lifetime. The mean consumption path from the figure is seen to be increasing over time, implying that at a future time of annuitization, the rate of consumption would be higher on average.

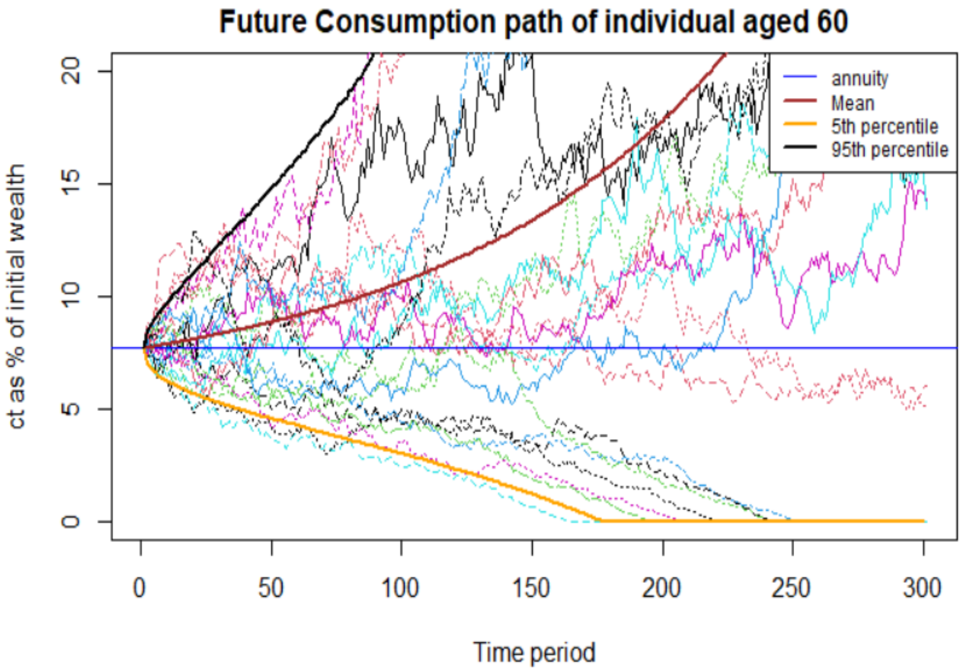

Figure 3.2: Future simulated consumption path of a female retiree age 60. Parameters: $\gamma = 2$, $\mu = 12\%$, $\sigma = 20\%$ and $r = 6\%$. The figure shows 20 simulated paths out of 25000

If she defers her annuitization based on Milevsky and Young's strategy, she would be annuitizing around the age of 78, with a 26% chance of obtaining a lower payout annuity from that age than from age 60. The authors claims that this probability of deferral failure is compensated by the probability of ending up with more than 120% of initial annuity income. In this case, the probability of consuming at least 20% more from the optimal age of annuitization, compared with annuitizing immediately, is 69%.

Figure 3.3 shows the paths of consumption if the retiree (from our example above) chooses to annuitize as prescribed by the strategy. Time here is measured in months, i.e. at 216 months or age 78 we see the consumption becomes constant. Though there is a majority of paths above the blue line, we should also be concerned by the risk of this strategy doing poorly. Thus we look at the probability of the consumption path going to zero before the retiree hits annuitization; indicating ruin.

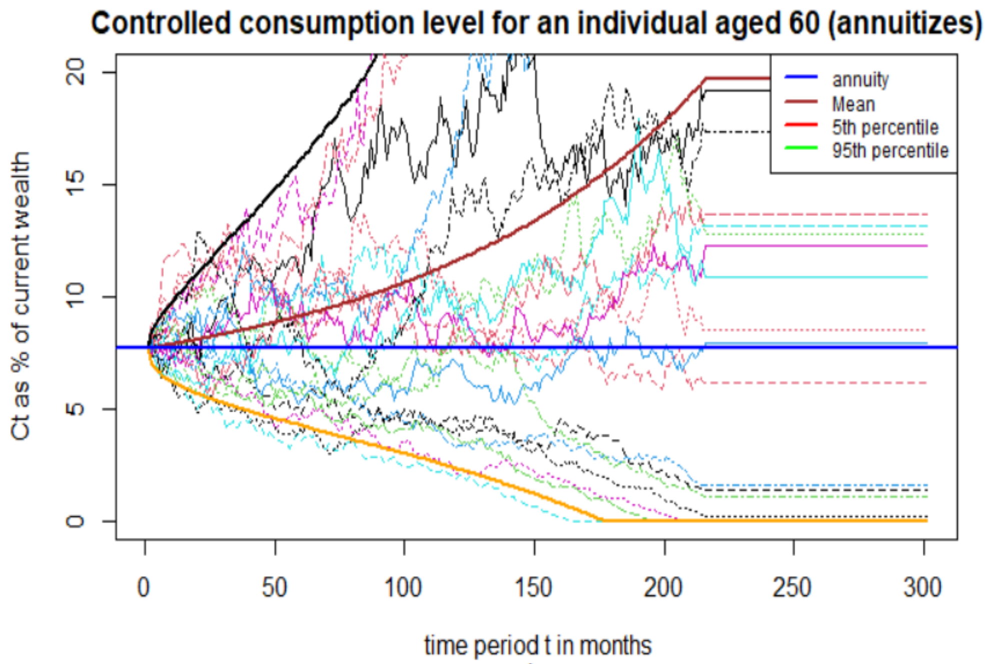

Figure 3.3: Future consumption path and the path followed by the retiree after annuitizing at age 78, for a female aged 60. The figure shows 20 simulated paths out of 25000

We now incorporate the strategy with up-to-date parameters. Using the derivative of utility in equation (3.13) we compute the optimal ages of annuitization for both male and female ages 60 to 75. The optimal ages to annuitize for a male and female age 60 are 64 and 71 respectively, this can be witnessed from Figure 3.4. All values are presented in Table 3.3.

A female retiree aged 60, is able to earn an income of \$4,730.15 by annuitizing at retirement, which is 4.73% of her original wealth. Assume her coefficient of risk aversion (γ) is 1, along with market parameters of $\mu = 6\%$, $\sigma = 20\%$ and $r = 2\%$. Based on the strategy, the optimal time for her to annuitize is at age 77. Figure 3.5 shows the variability of consumption paths of the retiree if she does not annuitize in her expected lifetime, while Figure 3.6 shows the paths of the retiree where she annuitizes. The majority of the paths in both these figures struggle to even meet the benchmark annuity level. The retiree here has a 42.1% chance of failing to meet the annuity income at retirement, and the risk of ruin is 13.5%.

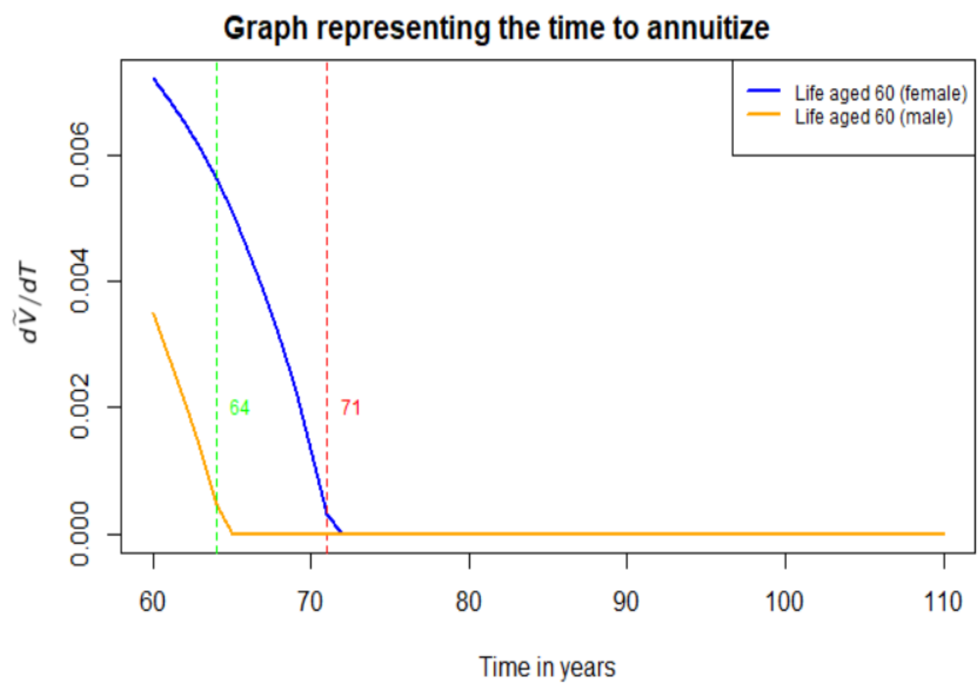

Figure 3.4: Optimal age to annuitize for a male and female retiree aged 60 obtained through the derivative $\frac{\partial \widetilde{V}}{\partial T} = 0$ from equation (3.13). Parameters used here are $\gamma = 2$, $\mu = 6\%$, $\sigma = 20\%$, $r = 2\%$

We list the retiree ages and their corresponding probabilities of obtaining a lower annuity, along with the probability of ruin in Table 3.4. In Figure 3.5, we observe that many paths lie below the blue benchmark and some tend to hit zero at a later date in retirement years.

Table 3.3: All or nothing framework: Optimal time to annuitize for female (male) with risk aversion of $\gamma = 1$, 2 and 5. Using real world parameters

Age	$\gamma = 1$	$\gamma = 2$	$\gamma = 5$
	Female (Male)	Female (Male)	Female (Male)
60	77 (71)	71 (64)	63 (Now)
65	77 (71)	71 (Now)	Now (Now)
70	77 (71)	71 (Now)	Now (Now)
75	77 (Now)	Now (Now)	Now (Now)

Table 3.4: All or nothing framework: Probability of lower annuity income and Probability of wealth going to zero before annuitization for risk aversion of $\gamma = 1$ and 2. Using current real world parameters μ= 6% , σ = 20% and r = 2%

Age	$\gamma = 1$		$\gamma = 2$	
	Female (Male)		Female (Male)	
	Prob. Lower Annuity	Prob. of ruin	Prob. Lower Annuity	Prob. of ruin
60	42.1 % (45.2 %)	13.5 % (3.3 %)	38.8 % (44.2 %)	0.2 % (3.0 %)
65	44.3 % (47.2 %)	5.5 % (0.0 %)	42.5 % (N/A)	0.0 % (N/A)
70	46.4 % (48.7 %)	0.2 % (0.0 %)	46.7 % (N/A)	0.0 % (N/A)
75	48.4 % (N/A)	0.0 % (N/A)	50.2 % (N/A)	0.0 % (N/A)

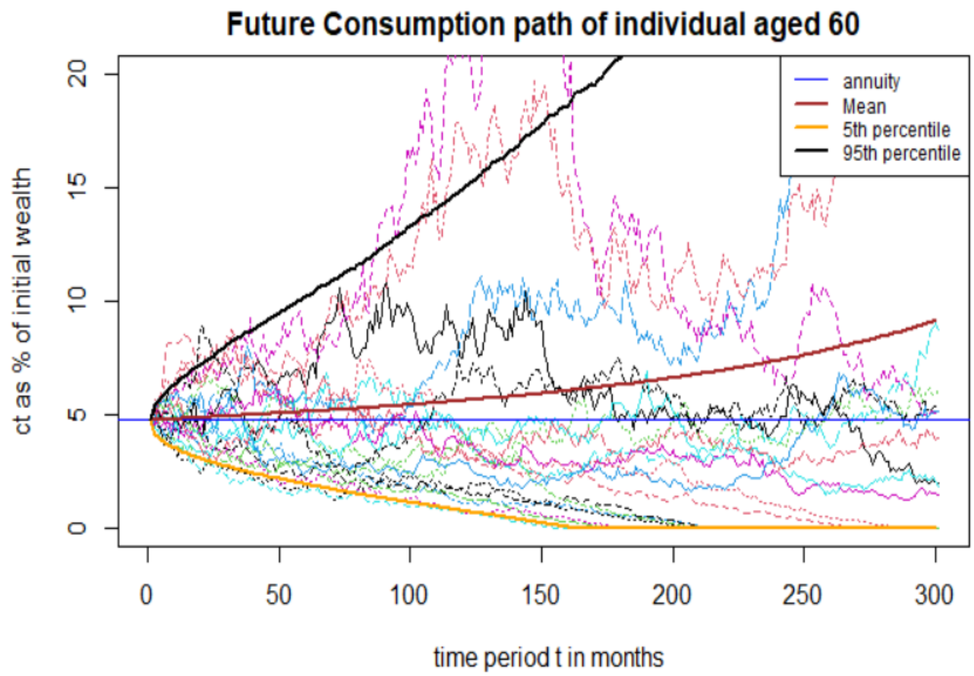

Figure 3.5: Future simulated consumption path of a female retiree age 60. Parameters: $\gamma = 1$, $\mu = 6\%$, $\sigma = 20\%$ and $r = 2\%$. The figure shows 20 simulated paths out of 25000

In conclusion, we highlight a few key points that offer more insight into the strategy in Milevsky and Young (2007).

- Using expected discounted lifetime utility function of consumption explicitly assumes that future gratification can be discounted and aggregated in the fashion that resembles the present value of cash flows. This is governed by the "subjective discount rate" ρ, which reflects the "pure rate of time preference". Whereas it constitutes of human rationality that might be conflicting as opposed to the normative understanding. In Milevsky and Young (2007), this subjective rate is set equal to r. This reflects flawed thinking, that the subjective discount is related to the time value of money, but it is not. Given $r = 6\%$, it implies that a retiree consuming \$100 currently would only need to consume $100e^{-0.06\times 20} = 30.1$ in 20 years (ignoring inflation and assuming survival). In fact, she would likely need at least \$100 in 20 years or perhaps more.
- Another aspect of expected utility is that, implicitly, the less frequent but high magnitude upside scenarios compensate for the downside scenarios. In Milevsky and

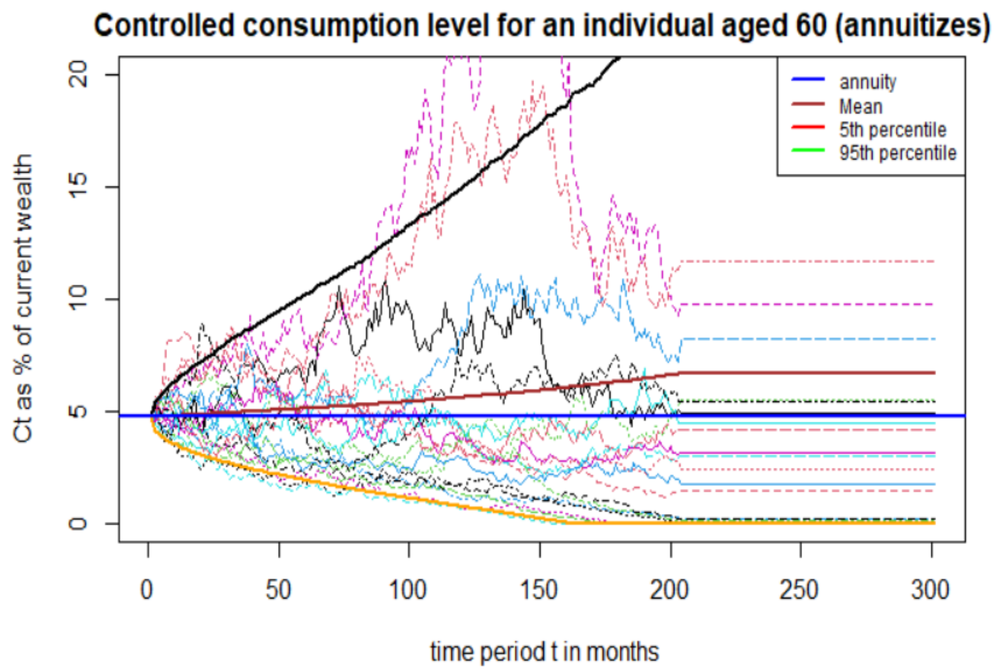

Figure 3.6: Future consumption path and the path followed by the retiree after annuitizing at age 77, for a female aged 60. With risk aversion parameter of $\gamma = 1$, $\mu = 6\%$, $\sigma = 20\%$ and $r = 2\%$. The figure shows 20 simulated paths out of 25000

Young (2007), the upside gain in the annuity income is emphasized. The authors do not communicate the fact that there is greater downside risk that comes along in adopting the strategy. Given that a retiree here is advised to fully invest all of his/her wealth, the prospects of losing all of their wealth in the retirement years can be disastrous.

- Consider Figure 3.7, which represents the consumption path of a female retiree aged 60, with $\gamma = 2$. This person invests her current wealth for a future expectation of higher consumption at older ages. Under Milevsky and Young (2007) parameters, we notice that majority of the time the 5th percentile (red dotted line) is below the benchmark level (blue line). Which is to say that in the worst 5% case of the adopting the strategy, this retiree is never going to be able to earn an income from her future annuity that at least matches the income had she annuitized at retirement (immediately).

- The results in Tables 3.1 and 3.3 show that the possible times of annuitization are

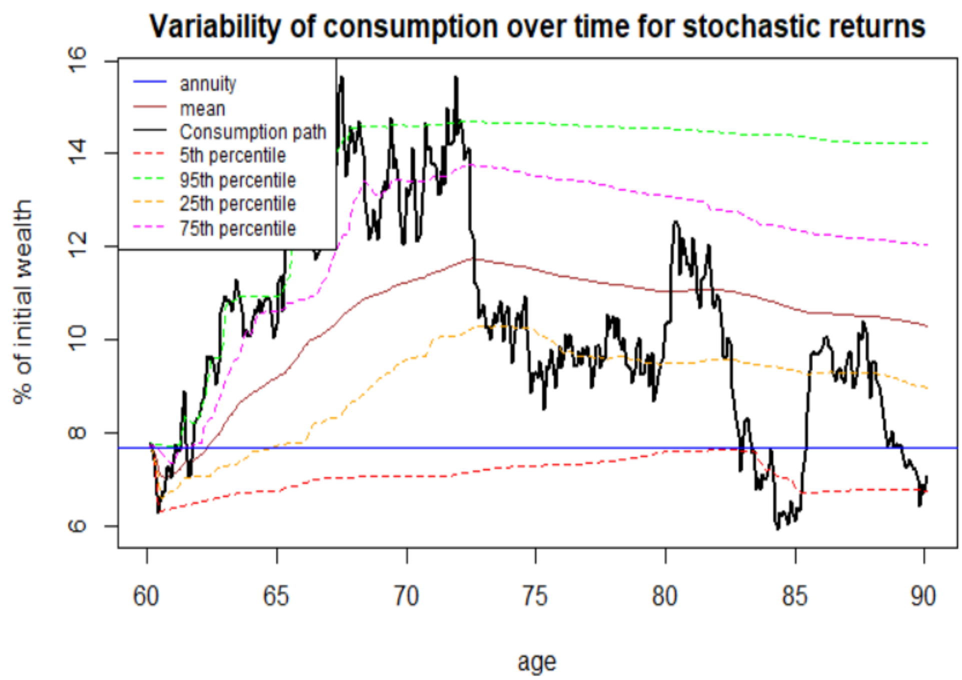

Figure 3.7: Variability of consumption path over time. Range of possible outcomes, If one decided to defer annuitization. The consumption path here is of a female aged 60 with parameters of CRRA $\gamma = 2$, $\mu = 12\%$, $\sigma = 20\%$, $r = 6\%$

the same for all ages ranging from 60 to 75. Such a pattern is observed because we are assuming that the future annuity rates are deterministic and are conditional on the basis of interest rates. If we look back at equation (3.14), with known parameters of μ, r, σ and γ we are able to say when one needs to annuitize, and these factors being constant across all the ages is the reason why the optimal time is similar across all ages. This would imply that if one delays annuitization, one runs the risk that annuity prices will actually increase.

- On related notes, the reason behind lower values of ruin at higher ages is because the retiree is advised to only invest for a few years, because the mortality risk is increasing at a faster pace, and the retiree would be less sure about surviving more than a few years. Thus, investment sounds like a wiser choice. However, if the retiree lives longer than expected they would experience the downfall in the consumption curve, as seen in Figure 3.7, accompanied by higher probabilities of ending up purchasing a lower annuity than desired.

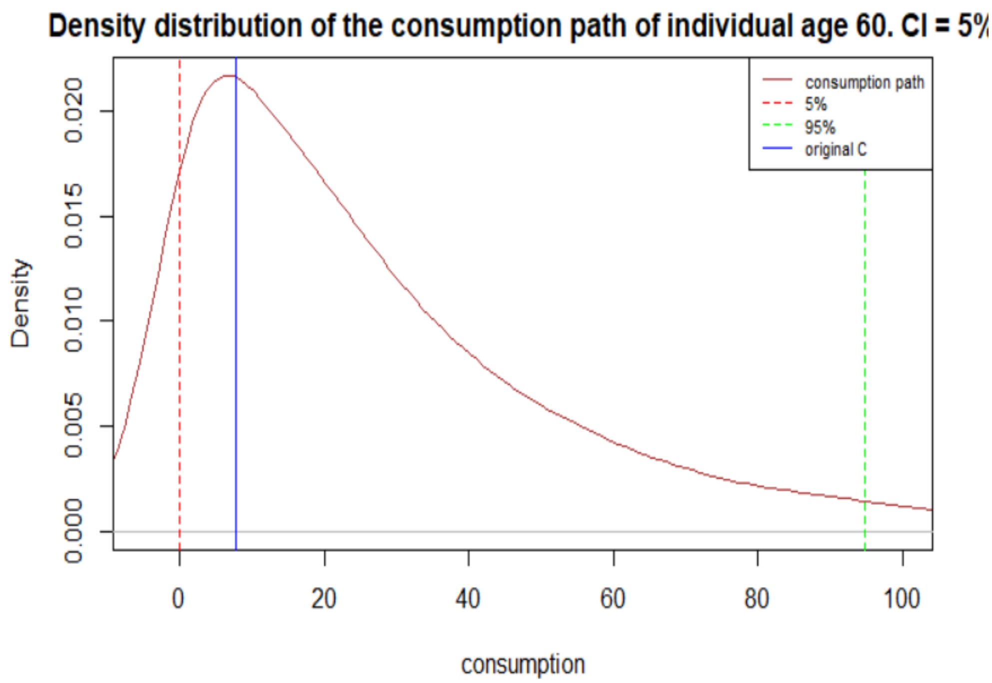

Figure 3.8: Density distribution of the consumption path of a female retiree aged 60 deferring her annuitization until age 78. The skewness of the distribution shows the consumption range at the time of her annuitization, along with 5% and 95% quantiles. The original consumption $c = 7.67\%$, $\gamma = 2$, $\mu = 12\%$, $\sigma = 20\%$ and $r = 6\%$

- Figure 3.8 shows the density distribution of the consumption paths at the time of annuitization for a female retiree aged 60. The retiree has a risk preference of $\gamma = 2$, with market parameters of $\mu = 12\%$, $\sigma = 20\%$ and $r = 6\%$. She is recommended to annuitize at age 78. The distribution is heavily right skewed, implying that more than 50% of the distribution lies below the mean. The median consumption value is $c_t = 20\%$. Fewer values tend to be reach a higher consumption rate at this time point, whereas at least 26% of the values fall below the blue line (original annuity or the mode of the distribution) and around 10.4% of the time ruin happens.

- One of the strongest assumptions in the original paper is the constant risk free rate and risk premium. In comparison, to Section 2.3, where interest rates were modelled using CIR, the impact on the ruin probability was minimal. The strategy was adopted to only invest in risky assets. However, since *"all or nothing arrangement"* in this chapter incorporates the choice of risk free trading we choose to implement

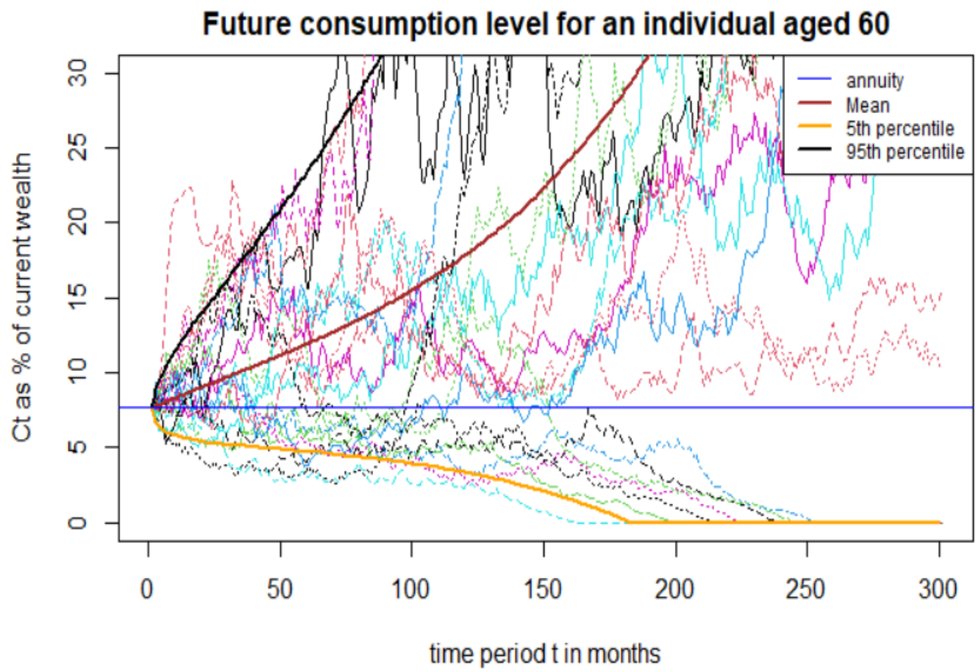

Figure 3.9: The impact of stochastic interest rates on the future consumption paths of a female individual aged 60. Using parameters from Milevsky and Young (2007): $\gamma = 2$, $\mu = 12\%$, $\sigma = 20\%$, $r = 6\%$. The interest rates are modelled using CIR model and corresponding parameters from Chapter 2

the stochastic interest rates model to see if the impact of investing in risk-free assets. We use CIR as an external factor to the given model, to add closeness to real world complexity of interest rate changes. We first simulate the stochastic interest rates through the CIR model, with the parameters $(\widetilde{r}, \sigma_r, \gamma_r)$ of (0.085, 0.25, 0.25) used in Milevsky (2001). The model generates a vector of interest rates for 25000 paths across 30 years. These interest rates are used for X_t in equation (3.1), as well as for the annuity price. We calculate the $P\,[\widetilde{W}_T <= 0]$. The impact of interest rates can be noticed from Figure 3.9 where the paths are more fluctuating in nature than in Figure 3.2. Figure 3.10 uses our parameters along with the CIR model, to be compared with the fixed interest rate model used in Figure 3.6. The ruin probabilities in both the cases are 9.3% and 13.3%. These values are close to the values we see in the Tables 3.2 and 3.4.

The impact of stochastic interest rates is minimal over the long term, as our opti-

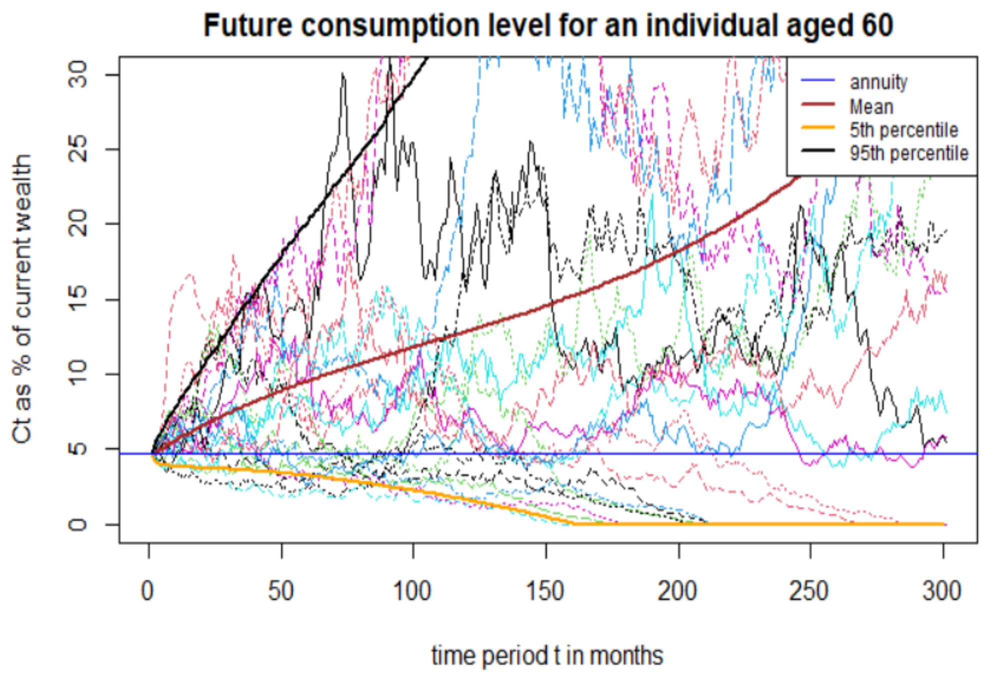

Figure 3.10: The impact of stochastic interest rates on the future consumption paths of a female individual aged 60. Using parameters from Milevsky and Young (2007): $\gamma = 1$, $\mu = 6\%$, $\sigma = 20\%$, $r = 2\%$. The interest rates are modelled using CIR model and it's corresponding parameters from Chapter 2

mal times to annuitization for individual in these cases are 18 and 17 years after retirement. A shorter term would have more impact on the wealth process. Out of curiosity, the process was applied for a female aged 70 and 75 in both the cases of $\gamma = 1$ (our parameters) and $\gamma = 2$ (parameters from Milevsky and Young (2007)). However, there is minimal difference observed in the ruin probabilities of 0.4% and 0.2% (for age 70) and 0 for age 75. The only change in the values is of the probability of obtaining a lower annuity. Figure 3.11 shows the process for a female individual aged 70. We can see by the time this person reaches 36 months or 3 years, the optimal time of annuitization, she is now facing a downside risk of 53% (significantly worse than 36% from table 3.2) while the probability of ruin remains at zero.

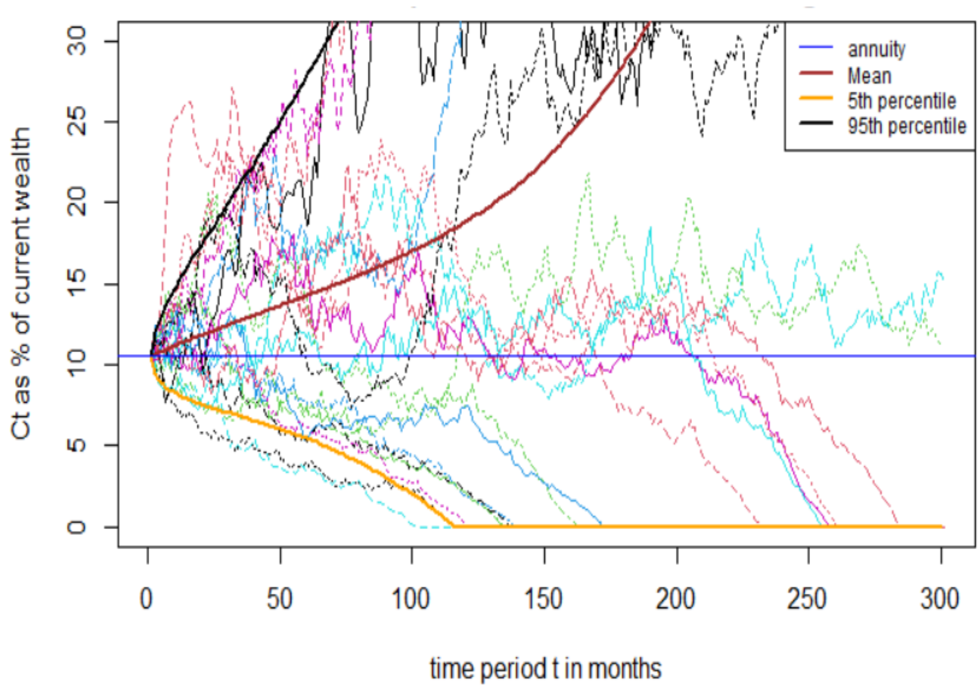

Figure 3.11: The impact of stochastic interest rates on the future consumption paths of a female individual aged 70. Using parameters from Milevsky and Young (2007): $\gamma = 2$, $\mu = 12\%$, $\sigma = 20\%$, $r = 6\%$. The interest rates are modelled using CIR model and it's corresponding parameters from Chapter 2

Chapter 4

On the sub-optimality cost of immediate annuitization in DC pension funds Giacinto and Vigna (2012)

4.1 Highlights

Giacinto and Vigna (2012) formulate and extend the solution given in Gerrard et al. (2012) for an optimal time of annuitization in the context of a member of defined contribution (DC) pension scheme. In this scheme, a member makes contribution, typically as a percentage of salary, during their employment period in order to fund their post-retirement income. The fund's growth depends on the performance of the investment returns during this period. A member has the possibility of taking periodic withdrawals (also popularly known as "income drawdown option" or "phased withdrawals") from the fund after retirement. According to this option, she can choose to defer her annuitization to a future time which can prove to be optimal based on some criteria. Thus it adds flexibility in the choice of pension benefits, which is not available in many countries, where immediate annuitization is compulsory at retirement. In this paper, the focus lies on obtaining the optimal time of annuitization for a retiree who has the option to choose her own investment and consumption strategy. However, the optimal time to annuitize is constrained to occur within a certain time limit, or not at all.

Giacinto and Vigna (2012) supports the availability of programmed withdrawals as an option to retirees of DC pension plans. Such a retiree has three principle degrees of freedom, namely:

- She can decide what investment strategy to adopt in investing the fund at her disposal;
- She can decide how much of the fund to withdraw at any time between retirement and ultimate annuitization (if any).
- She can decide when to annuitize (if ever).

The first two choices represent a classical inter-temporal decision-making problem which can be dealt with using optimal control techniques in the typical Merton framework (Merton (1971)), whereas the third choice is tackled by defining an optimal stopping time problem. Hence, Giacinto and Vigna (2012) follow the footsteps of Gerrard et al. (2012) in formulating a combined stochastic control and optimal stopping problem. They consider a quadratic loss function as a criterion for the optimal annuitization time, and provide a closed form solution. They conclude that the optimal annuitization time depends on financial market parameters, and parameters linked to retirees risk preferences. Retirees in Giacinto and Vigna (2012) can find themselves annuitizing a few years (6-7 years) from retirement or at a later time (10-15 years) based on these typical conditions.

The authors choose to answer the question "Is immediate annuitization optimal? if not, what is the cost of mandating annuitization at retirement?". They give an insight into the extent of loss in wealth suffered by a retiree who lacks the option of programmed withdrawals, but is obliged to annuitize at retirement. The authors introduce sub-optimality cost as a tool to measure the loss of expected present value of consumption from retirement to death. In other words, it indicates the trade-off on how much percentage of a retiree's future wealth is given up when she annuitizes immediately. Under sufficient conditions, it is proved that immediate annuitization is sub-optimal within this model.

We list the similarities and differences between this Chapter and the previous two for coherent purposes. Some similarities to note are:

- Giacinto and Vigna (2012) allows for mixed investment portfolios holding risky and risk-free assets with consumption and investment proportions acting as control variables. This is similar to Milevsky and Young (2007).
- The interim consumption is determined by the level of annuity purchasable at initial time (or at retirement). This is similar to the approach in Milevsky (2001).

Noticeable differences are:

- The optimal annuitization time is optimized through a quadratic loss function, wherein we are minimizing the loss from annuitizing prematurely.
- The authors point out that it is reasonable for a retiree to have a target income for future. It would then be optimal, when they retiree is close enough to the target, to then switch and annuitize for the desired income rate.
- The paper uses an assumption of constant force of mortality. However for consistency, we choose to continue with the use of Gompertz mortality assumptions for our simulations.

In this chapter, we are mainly interested in the strategy proposed to find optimal annuitization time. We run extensive simulations of the model with updated parameters to see if it works flexibly in helping a real world retiree to make retirement decisions. We provide results of quantile measures used to assess the risk in adopting the strategy and interpret them.

4.1.1 A Self-annuitizing strategy using Quadratic loss model

The model in Giacinto and Vigna (2012) considers the position of a member of defined contribution plan who chooses the drawdown option at retirement. The member gets to withdraw a certain income until she achieves the age at which the purchase of annuity is mandatory. In other words, she is able to select the time of eventual annuitization up to a maximum future time.

The retiree starts with a lump sum of $W_0 > 0$ at retirement. This amount is invested in a risk-free asset with a constant instantaneous rate of return r, and a risky asset whose price is assumed to follow Geometric Brownian Motion (GBM) with parameters μ and σ, as in equation (3.1). The model assumes that the remaining lifetime of the retiree aged x is exponentially distributed with a constant force of mortality (δ). However, we continue to use a more realistic force of mortality, through the Gompertz assumption. The values presented in Table 4.1 show the difference in assumption of mortality over the survival probabilities. We know that for a male aged 60 the annuity price at $r = 3\%$ is 16.53 using the Gompertz mortality assumption. Equating this to the annuity price at constant force of mortality, we have $\overline{a}_x = \frac{1}{r+\delta} \Rightarrow \delta = 0.0305$. For comparative purposes, we use this value in Table 4.1.

Table 4.1: Difference between the survival probabilities (${}_tp_x$) of life aged 60 male using constant force of mortality model and Gompertz model.

time	Survival prob. (male) using constant $\delta = 0.0305$	Survival prob. (male) using Gompertz
1	0.9699	0.9932
2	0.9408	0.9858
3	0.9126	0.9776
4	0.8852	0.9688
5	0.8586	0.9592

Until the time of annuitization, the retiree is assumed to have the choice in the proportion of investment to be made in risky assets and the desired level of income to be consumed. Deviations from these targets will lead in a loss for him/her. However, if the amount of money in the fund is ever exhausted, then no further trading or withdrawal is permissible.

We begin with some notation:

- T is the time of annuitization
- T_0 is the time when the fund hits 0.
- W_t is the fund at time t, where $t < \min(T, T_0)$.
- $(1+l)(\overline{a}_x)^{-1}$ is a positive constant, representing the amount of annuity which can be purchased with one unit of money. The size of the annuity purchasable with sum W_t is $\frac{W_t}{(1+l)\overline{a}_x}$, where $(\overline{a}_x)^{-1} > r$. The annuity price under a constant force of mortality of δ would be $(\overline{a}_x)^{-1} = \frac{r+\delta}{1+l}$, where l stands for the loading factor. In this chapter, the loading factor is set to 0.
- π_t is the proportion invested in the risky asset.
- c_t is an instantaneous amount of income withdrawn from the fund.

The stochastic differential equation that describes the growth of the fund is:

$$dW_t = [W_t (\pi_t(\mu - r) + r) - c_t] \, dt + \pi_t W_t \sigma dB_t \tag{4.1}$$

where B_t represents the standard Brownian motion.

The reason a retiree chooses the income drawdown option is the hope of being able to purchase a life annuity in the future that provides higher income than the income provided on immediate annuitization. Hence it seems reasonable to assume that a retiree has a certain target income in mind when they choose to defer annuitization. This leads us to the introduction of quadratic loss (or disutility) function.

The model uses two continuous control variables π_t and c_t, and a stopping time T, chosen to minimize the following quadratic cost function

$$J^{c,\pi,T}(t, W_t) = E_t\left[\int_t^{\tau} e^{-(\rho+\delta)s}(c_0 - c_s)^2 ds + e^{-(\rho+\delta)\tau}\left(c^* - \frac{W_\tau}{\bar{a}_{x+\tau}}\right)^2 \bar{a}_{x+\tau}\right] \tag{4.2}$$

where:

- $J(t, W_t)$ is the expected loss criterion discounted to time t, for a retiree consuming in the interval $t \in [0, \tau]$. The second term relates to purchasing an annuity at optimal time τ.
- $\tau = min(T, T_0)$,
- ρ is the subjective discount factor. This is assumed to be equal to r.
- δ is the constant force of mortality. This notation is similar to the force of mortality we used earlier as μ_x, under this constant assumption: $\mu_{x+t} = \mu$, whereas we use the asset returns variable to be μ. To avoid confusion we prefer using δ.
- c_0 is the benchmark income before purchasing the annuity. We assume that $c_0 = \frac{W_0}{\bar{a}_x}$, which is the size of the annuity the retiree could have purchased if she had annuitized immediately at retirement.
- c^* is the income target after purchasing the annuity.
- The disutility $(c_0 - c_t)^2$, is continuously experienced when the income drawdown from the fund is below or above the ideal level of income before annuitization. $(c_0 - c_t)$ is typically the deviation in the fund from a benchmark level to indicate the loss/gain in income at each time t.
- The disutility $\left(c^* - \frac{W_\tau}{\bar{a}_{x+\tau}}\right)^2$, is experienced when the level of fund is below or above the final target level after annuitization. For $T \geq \tau$, $\frac{W_\tau}{\bar{a}_{x+\tau}} = c_T$.

- If $\tau = T_0$, then $\frac{W_\tau}{\overline{a}_{x+\tau}} = 0$

A terminal disutility is engendered at annuitization time T, when the level of the fund is $W_T \geq 0$, by a discrepancy between the level of annuity actually purchased and the ideal level set by the retiree. The expected present value of the squared loss of income is:

$$K(T, W_T) = \overline{a}_{x+T} \left(c^* - \frac{W_T}{\overline{a}_{x+T}} \right)^2 \tag{4.3}$$

Intuitively, if the fund hits $c^*\overline{a}_{x+T}$, the retiree would be able to purchase a life annuity that can pay her c^*, and the penalty on meeting the desired consumption rate would be zero thereafter. Hence it is optimal to purchase an annuity as soon as the fund level reaches $c^*\overline{a}_{x+T}$. The ratio $\frac{c^*}{c_0}$ infers the risk propensity of the retiree, the higher the ratio; higher is the targeted income, hence lower the risk aversion and vice versa.

It is well known from the theory of optimal stopping time problem that the value function must satisfy the Hamilton-Jacobi-Bellman equation. The objective is to minimize over possible investment and consumption choices the expected discounted future loss from retirement until time τ of equation (4.2). The authors use the results from Gerrard et al. (2012) to show the region where it is optimal to annuitize. The intuition to the variational inequalities of HJB equation is that, when the level of wealth W_t is in the continuation region (U), the loss in value in this case of investment-consumption optimization is lower than that achieved in the case of purchasing an annuity. On the other hand, when W_t is in the stopping region, the value in the case of purchasing an annuity is higher and coincides with the value function, i.e. it is optimal to stop consuming and annuitize at that point of time.

The infimum of the expected loss given by the equation (4.2) is $V(t, W_t)$, where:

$$V(t, W_t) = \inf_{c,\pi,T} J^{c,\pi,T}(t, W_t) \tag{4.4}$$

The authors obtain the optimal proportion of wealth to invest in risky asset as well as the optimal income to withdraw from the fund, when the wealth lies in the continuation region. The intuition from equation (4.5) is that, for optimal proportions of wealth invested, there exists a solution to the problem that minimizes the expected loss for all $t < \tau$. Hence it is possible to analyse the behavior of wealth function under this optimal control.

$$V(t, W_t) = \inf_{\pi_t} J(t, W_t; \pi_t) = J(t, W_t; \pi_t^*) \tag{4.5}$$

A similar problem is expressed in simpler terms in related works, Gerrard et al. (2004) and Gerrard et al. (2006), featuring the behavior of W_t under optimal control.

$$dW_t^* = [W_t^* (\pi_t^*(\mu - r) + r) - c_t]\, dt + \pi_t^* W_t^* \sigma dB_t \tag{4.6}$$

The equation (4.6) is the evolution of fund under optimal control of π_t^*, where

$$\pi^*(t, W_t^*) = \frac{(\mu - r)}{\sigma^2} \left(\frac{F_t - W_t^*}{W_t^*} \right) \tag{4.7}$$

and,

$$F_t = c^* \, \overline{a}_{x+t} \, e^{-r(T-t)} + \frac{c_0}{r}(1 - e^{-r(T-t)}) \tag{4.8}$$

F_t is the running target level of wealth at each time t. The natural interpretation for the choice of F_t is the following. Should the fund reach the value of the target at time t, the pensioner could immediately invest in the riskless asset what would be necessary to reach the final target at time T, and still consume an amount c_0 for the remaining $T - t$ years. Therefore, she would achieve the final target with certainty, meanwhile consuming the amount c_0 that immediate annuitization at retirement would have provided, and thereafter purchase an annuity paying the required amount c^* per unit time. This minimum level of fund that guarantees the fixed consumption by investing entirely in risk free asset is known as the "safety region". However, we notice that, due to the way in which they are constructed, the targets can never be reached. It implies investing in the risky asset a proportion of the positive difference between the amount needed for being in safe region and the fund level. Thus, the optimal amount invested in the risky asset is a proportion of the shortfall $(F_t - W_t^*)$.

Once the fund hits $c^* \, \overline{a}_{x+T}$ the optimal decision is to annuitize. However, the authors obtain a fraction $w^* < c^* \, \overline{a}_{x+T}$ where they show it is still optimal to annuitize. The intuition is that, when the retiree reaches a certain level close enough to the desired target, it is better to annuitize and accept a low penalty, rather than continuing to invest and face the risk that they might drift further from the target. Hence the continuation region where the self-annuitizing strategy works is restricted by the equation:

$$U = [0, w^*) \cup \left(c^* \, \overline{a}_{x+T}, +\infty \right) \tag{4.9}$$

The fraction $w^* = c^*\ \overline{a}_{x+T} - \frac{2rD}{\phi}$ is formulated by minimizing the penalty $K(T, W_T) > 0$ for all $W_t \in [0, c^*\ \overline{a}_{x+T}]$. The intuition along with the parameters in the equation follows:

$$\begin{aligned} D &= \frac{c_0}{r} - c^*\ \overline{a}_{x+T} > 0 \\ \phi &= \beta^2 - 2r + \frac{2}{\overline{a}_{x+T}} \text{ and} \\ \beta &= \frac{\mu - r}{\sigma} \end{aligned} \tag{4.10}$$

In the model outlined above, it is optimal to annuitize immediately at retirement for any initial lump sum $W_t \in [0, c^*\ \overline{a}_{x+T}]$ if and only if

$$\phi \leq \frac{2rD}{c^* \overline{a}_{x+T}} \tag{4.11}$$

The equation (4.11) can be written in various equivalences to interpret the role of each of these factors and their impact on the criterion. For interpretation, the equation in (4.11) can be written as equation (4.12) by plugging in the values of D and ϕ. The inequality is a condition of optimality with Sharpe ratio (β) as the subject variable. The intuitive behavior is that periodic withdrawals only make sense in the hope of being better off than committing to annuitizing immediately. This is possible in the presence of a sufficiently good amount of risky asset. If the Sharpe ratio is too small, it indicates that immediate annuitization is preferable. It means that, at $T^* = 0$ with fund W_t the optimality condition is satisfied.

$$\beta^2 \leq \frac{2}{\overline{a}_{x+T}} \left(\frac{c_0}{c^*} - 1 \right) \tag{4.12}$$

Similarly, substituting for $\overline{a}_{x+T}$ in the equation (4.12) with $\frac{\rho+\delta}{1+l}$, under the assumption of constant force of mortality, with $l = 0$, the condition becomes

$$\beta^2 \leq 2(\rho + \delta) \left(\frac{c_0}{c^*} - 1 \right) \tag{4.13}$$

• The inequality in (4.13) implies that optimal immediate annuitization is not possible if the ratio c^*/c_0 is too high. In other words, if the targeted pension income c^* is more than twice the purchasable annuity income c_0, then the inequality will never hold. The authors also prove that D has to be greater than zero; it can be written as equation (4.14) for better interpretation. Since, $\overline{a}_{x+T}^{-1} > r$ and $\overline{a}_{x+T}^{-1} = \frac{r+\delta}{1+l}$ under their constant force of mortality model, the equation (4.14) can deduce that the risk aversion parameter is bounded and can be chosen such that $1 < c^*/c_0 < (r+\delta)/[r(1+l)]$ is satisfied.

$$D = \frac{c_0}{\overline{a}_{x+T}^{-1}} \left(\frac{\overline{a}_{x+T}^{-1}}{r} - \frac{c^*}{c_0} \right) \tag{4.14}$$

• If the coefficient $\rho + \delta$ in equation (4.13) is too low, it is associated with a high tolerance for the future, and a high expectation of future lifetime. Hence a retiree would likely gain more from self-annuitizing strategy.

All these features are intuitive and desirable in the model considered to verify its optimality. To summarise, For $T^* = 0$ and for every initial wealth, $\frac{w^*}{c^* \, \overline{a}_{x+T}} = 0$ i.e. $c^* \, \overline{a}_{x+T} - \frac{2rD}{\phi} = 0$, is possible when, $\phi = \frac{2rD}{c^* \, \overline{a}_{x+T}}$. The validity of this optimality condition at retirement only holds when:

- sufficiently low value of β,
- sufficiently low values of c^*/c_0,
- sufficiently high value of $\rho + \delta$ is observed.

Moving onto the numerical applications of the strategy, the parameters necessary for the problem are:

$$r, \mu, \sigma, 1/\overline{a}_x, c^*, \rho$$

In Giacinto and Vigna (2012), a male retiree aged 60 is chosen with a time horizon of $t = 30$ years. This is the maximum time available to the retiree to annuitize, so the authors are only looking for an optimal annuitization point T^* that occurs within this time frame, otherwise such a path is not considered. This is an assumption required due to constant force of mortality, where a 90 year old retiree have the same expected future lifetime as a 60 year old. This assumption makes a huge difference in the simulation results.

The annuitization problem for the above mentioned retiree was explored in the original paper under the following assumptions and parameters:

- Wealth $W_0 = 100$
- Risk free rate $r = 3\%$
- δ is constant force of mortality, with values assumed in Table 4.2.

- The conversion factor $\overline{a}_x^{-1}$ used for converting lump sum into annuity is $\overline{a}_x^{-1} = 0.085$.

- c^* is the target level of annuity income. This value is assumed to be based on the risk preference of the retiree. The ratio of $c^*/c_0 > 1$ in all cases.

- c_0 is the size of annuity purchasable at retirement with initial fund W_0; $c_0 = 6.22$ is the initial consumption calculated by the authors using Italian projected mortality table (RG48). We use this value in the simulations for comparative purposes.

- ρ, which is the inter-temporal discount factor, is assumed to be equal to r. The quantity used for measuring the patience of retiree for future events, its value in time is given by $\rho + \delta$. Which again is a flawed assumption, is explained later in Section 4.1.2.

For annuitization to happen at a later date in retirement, the condition in equation (4.11) should be sub-optimal for immediate annuitization. For the continuation region $U = [0, w^*)$ where $w^* < c^*\overline{a}_{x+T}$, the dependence on width of continuation region is checked for the parameters β, c^*/c_0 and $\rho + \delta$. This is done by analysing the ratio $\frac{w^*}{c^*\overline{a}_{x+T}}$.

The four scenarios used in Giacinto and Vigna (2012) are showcased in Table 4.2. The parameter values are chosen in such a way that immediate annuitization is not optimal. The validity of these parameters was also checked by changing one of them while others are constant. This is to make sure that a sub-optimal cost of not being able to defer annuitization can be reported for these scenarios.

Table 4.2: Parameter values used in Giacinto and Vigna (2012) for the four different scenarios A,B,C and D

	A	B	C	D
β	0.25	0.33	0.40	0.50
c^*/c_0	1.50	1.75	2.00	2.25
μ	0.06	0.08	0.102	0.13
σ	0.12	0.15	0.18	0.20
δ	0.06	0.04	0.02	0.005

Each of the scenarios are tested by running 1000 Monte Carlo simulations for the risky asset, the projections were repeated for each value of c^*. The ratios of c^*/c_0 in each scenario is used to simulate c^*. In discretization of the process, the time interval is equal to a week. The optimal π_t^* from equation (4.7) is simulated for all the weeks $t = [0, 1500]$. Across the 4 scenarios, for each trajectory a T^* is the time when the fund first hits w^* and the corresponding $\overline{a}_{x+T^*}$ is calculated. $w^*/\overline{a}_{x+T^*}$ is the amount of annuity purchasable with fund w^* at time T^*. Trajectories where the fund value does not reach w^* within the time frame of 30 years are assigned the value 0. Hence only the relevant cases where $T^* < 30$ are considered.

Statistics of various relevant information from the simulations performed was reported. Important statistics from the simulations include:

- The timing of the optimal annuitization, T^*,
- The probability of achieving the final annuity income at the mean time of optimal annuitization
- The occurrence of negative consumption in the interval of optimal annuitization.
- The mean time to ruin occurring within this time frame.

In this paper, we follow the strategy and use the parameters mentioned in Giacinto and Vigna (2012). We seek to find the optimal annuitization time when the fund reaches $c^*\ \overline{a}_{x+T^*}$ for the first time (T^*), ignoring the fact that the trajectory can go upwards or downwards moving ahead, so our ratio $\frac{w^*}{c^*\ \overline{a}_{x+T^*}} = 1$. At this time, it is possible for a retiree to purchase c^* as desired . In other words, we assume that w^* is the exact value that is desirable by the retiree as their target wealth. At time T^*, $W_{T^*} = w^*$, which is enough to purchase the annuity with an income of c_{T^*}.

Although this may provide results which are slightly sub-optimal, it gives an idea whether choosing c^* as desirable target might help a retiree to achieve their goal income. For consistency, we use the Gompertz mortality in our simulations; we report the value of μ_x at initial age x, for comparative purposes. Annuity prices in our simulations under each scenario are also calculated using the Gompertz mortality assumption. This assumption unlike constant force of mortality can be less optimal according to the strategy, perhaps the idea is to use a more realistic mortality model as an add-on analysis. We use updated parameters in our application along with theirs. The results are presented in the next section.

4.1.2 Results and review

In this section we provide the results of our Monte Carlo simulations, using the Giacinto and Vigna (2012) parameters as well as updated parameters. We implement the simulations for each of the 4 scenarios along with a 5th scenario that includes our set of parameters. We first introduce the scenario with the parameters in Giacinto and Vigna (2012) for a male retiree aged 60 and compare them with ours simultaneously. Scenario A1 uses a constant force of mortality, following Giacinto and Vigna (2012), while scenario A2 uses Gompertz mortality.

Scenario A1: $\mu = 6\%$, $\sigma = 12\%$, $r = 3\%$, $\delta = 6\%$, $\beta = 0.25$, $c^*/c_0 = 1.5$

Scenario A2: $\mu = 6\%$, $\sigma = 12\%$, $r = 3\%$, $\mu_{60} = 0.65\%$, $\beta = 0.25$, $c^*/c_0 = 1.5$

These scenarios model highly risk averse retirees. These retirees seek a target income of 50% above c_0. They do not expose themselves to too much to financial risk and prefer to gain a small value of β on the market. In other words, they are playing safe by investing 75% in risk-free assets and simultaneously earn higher returns by investing 25% in risky assets. The high mortality rate indicates these individuals prefer current income to future income. Some trajectories for this scenario are shown in Figure 4.1.

The authors mention that the starting point of consumption is typically the benchmark level of annuity purchasable at retirement, which is $c_0 = \frac{W_0}{\bar{a}_{60}} = 8.5$, but the value used here is 6.22, i.e. the individual is assumed to be consuming lower than benchmark. Consumption over time in Figure 4.1 is function of the wealth and attainable annuity.

The target income is achieved typically around 110.4 weeks on average i.e. after 2.1 years. However 75% of the time, the fund is lower than the desired fund level at this time. However, the mean time of ruin is 1110 weeks and it falls in the interval of optimal $T^* \in (11, 1305)$. The probability of ruin if the retiree has to wait until the maximum optimal time to meet the desired income is thus 35.9%.

The scenario A2, results are shown in Figure 4.2. The initial consumption level obtained from actuarial calculation of pricing the annuity is 6.05. So the desired future income would be $c^* = 9.075$. Consumption in Figure 4.2 is a function of wealth over annuity achievable along time t. The retiree, being highly risk averse, does not get much out of the market returns. In most of the cases he is seen to fail to meet the basic annuity benchmark c_0. The wealth reaches the desired level at a later stage, i.e. around 399 weeks. The chance of obtaining a lower payout annuity is 75.8% and the risk of getting ruined in the interval $(60, 1289)$ is 34.0%.

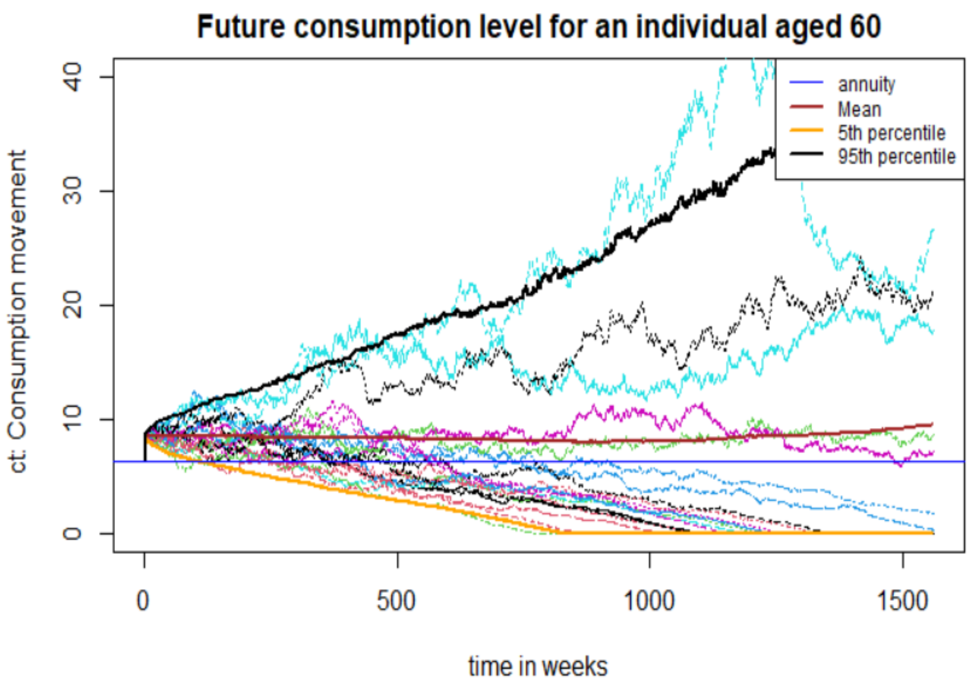

Figure 4.1: Scenario A1: Future consumption path of a male retiree aged 60 along with the 5th and 95th quantiles and mean. Using parameters $\mu = 0.06$, $\sigma = 0.12$, $r = 0.03$, $\delta = 0.06$, $c_0 = 6.22$, $c^*/c_0 = 1.5$, $\beta = 0.25$, $\overline{a}_{x+t}^{-1} = 0.085$

Scenario B1: $\mu = 8\%$, $\sigma = 15\%$, $r = 3\%$, $\delta = 4\%$, $\beta = 0.33$, $c^*/c_0 = 1.75$

Scenario B2: $\mu = 8\%$, $\sigma = 15\%$, $r = 3\%$, $\mu_{60} = 0.65\%$, $\beta = 0.33$, $c^*/c_0 = 1.75$

Again, these scenarios are identical other than the mortality model. The retiree is moderately risk averse. They desire a higher target income than the previous category, along with a β of 0.33 on the market. Once again, the ratio $\rho + \delta$ is higher. The consumption c_0 is set to be 6.22 and the desired target level thus becomes 10.88. Figure 4.3 shows the consumption paths of this retiree.

The desired target level is achieved around 250 weeks from retirement, i.e. 4.8 years on average. As usual the paths where $T^* > 30$ are ignored. Figure 4.3 shows that ruin does not happen around the time of 250 weeks, it occurs around 866 weeks which is 21 years and occurs at a faster pace when compared to previous case, where the chance of ruin had the person not annuitized would occur around 21.5 years. The risk of earning lower than the target income is 72%. The reason behind mentioning this risk is because

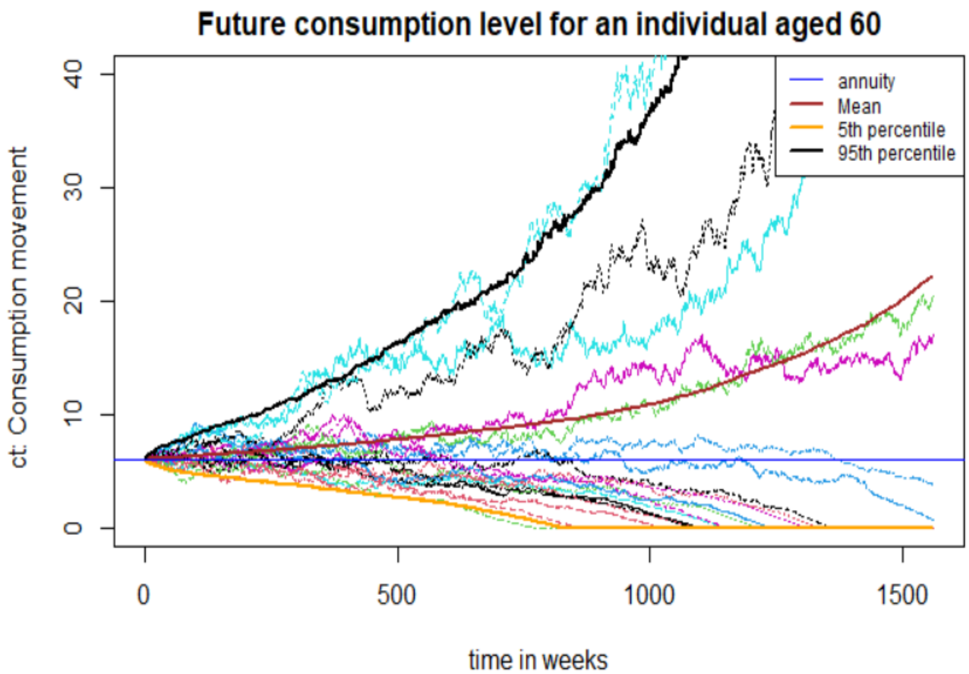

Figure 4.2: Scenario A2: Future consumption path of a male retiree aged 60 along with the 5th and 95th quantiles and mean. Using parameters $\mu = 0.06$, $\sigma = 0.12$, $r = 0.03$, $\mu_{60} = 0.0065$, $c_0 = 6.05$, $c^*/c_0 = 1.5$, $\beta = 0.25$

one does not always achieve their desired target level in 250 weeks. The range of values is $T^* \in (15, 1547)$ within the time frame, Thus, in the event of retiree waiting for next 1547 weeks to meet his target income, will eventually face risk of ruin of 30.1%.

Figure 4.4 shows the case for scenario B2. The desired target level is achieved in 430 weeks i.e. 5 years on average. The chance of ruin here is 0.4% and the risk of consumption being lower than the desired target at this time point is 68.9%. On average, ruin occurs at a similar time point in future i.e. around 1110 weeks, and the risk of being ruined is lower, at 28.8%.

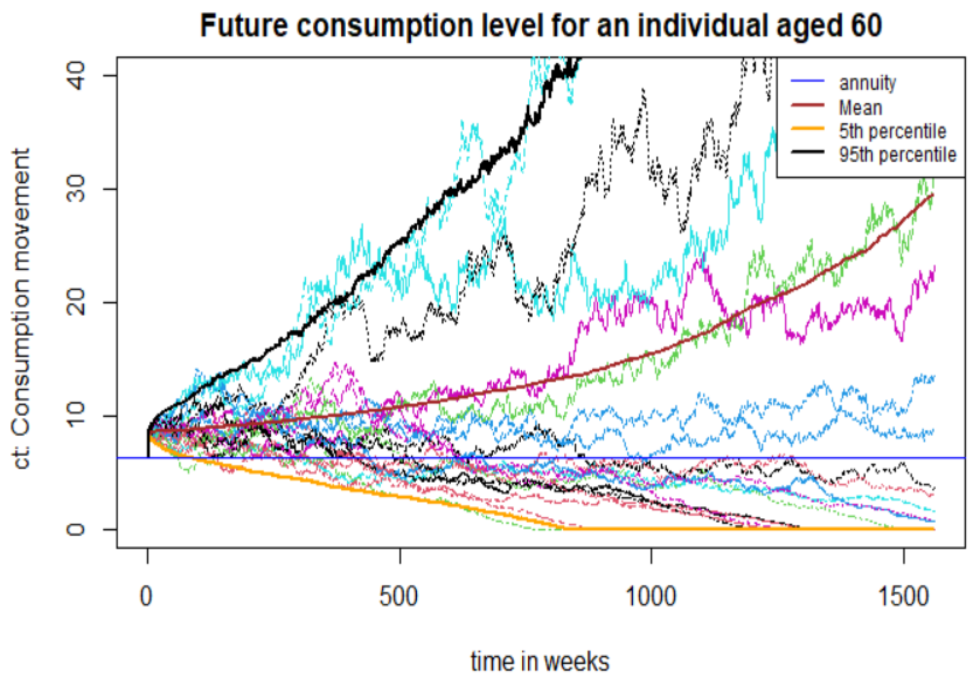

Figure 4.3: Scenario B1: Future consumption path of a male retiree aged 60 along with the 5th and 95th quantiles and mean. Using parameters μ = 0.08, σ = 0.15, r = 0.03, δ = 0.04, c_0 = 6.22, c^*/c_0 = 1.75, β = 0.33

Scenario C1: μ = 10.2%, σ = 18%, r = 3%, δ = 2%, β = 0.40, c^*/c_0 = 2

Scenario C2: μ = 10.2%, σ = 18%, r = 3%, μ_{60} = 0.65%, β = 0.40, c^*/c_0 = 2

The retirees in this category aim to double their future income. Some simulated consumption paths for scenario C1 are shown in Figure 4.5. The retiree is able to earn β of 0.4 on the market; $\rho + \delta$ is lower than previous cases. The optimal time to earn the desired target income is 322 weeks. The risk of obtaining lower than desired target is 65.4%. The paths tend to zero around 20 years after retirement and thus a retiree who is yet to achieve a desired target level between $T^* \in (10, 1545)$ fails to do so by a chance of 20%. The volatility and returns of the risky asset influence the consumption paths to gain quicker returns in shorter time period and also ruin at a quicker pace.

On similar lines, Figure 4.6 shows some consumption paths for scenario C2. The optimal time for earning the desired target income in this model is 440 weeks with a 1% chance of ruin happening at this time point. The returns and volatility of the market enhance the

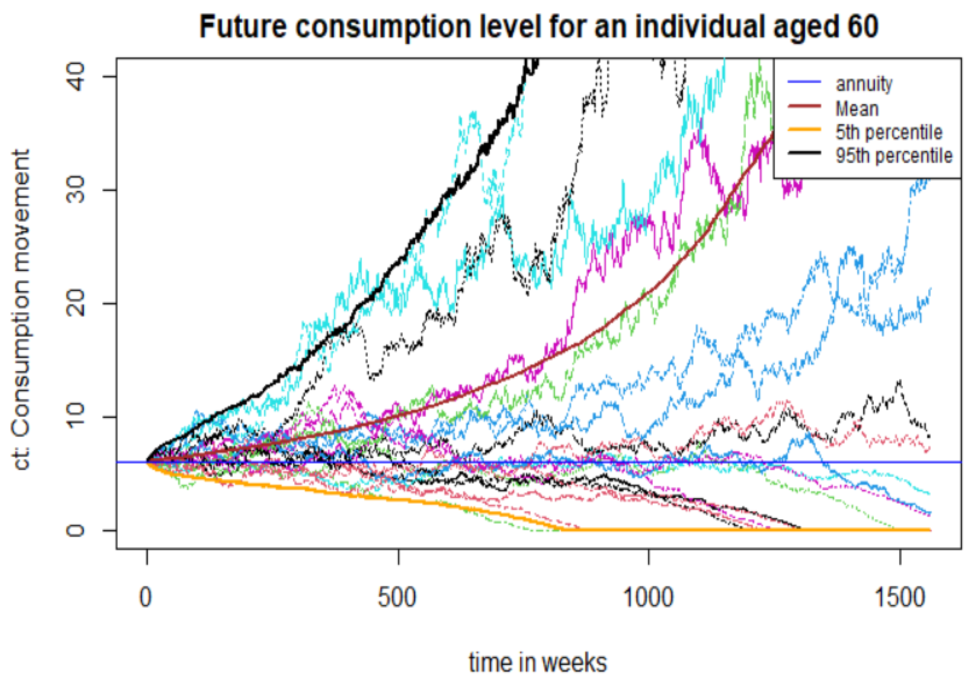

Figure 4.4: Scenario B2: Future consumption path of a male retiree aged 60 along with the 5th and 95th quantiles and mean. Using parameters $\mu = 0.08$, $\sigma = 0.15$, $r = 0.03$, $\mu_{60} = 0.0065$, $c_0 = 6.22$, $c^*/c_0 = 1.75$, $\beta = 0.33$

achievable rate of future target income. Hence the probability of obtaining a lower income annuity at this time point is 64%. The range of annuitization times is $T^* \in (0, 1494)$, with a 19.6% chance of running out of money if the retiree follows the strategy.

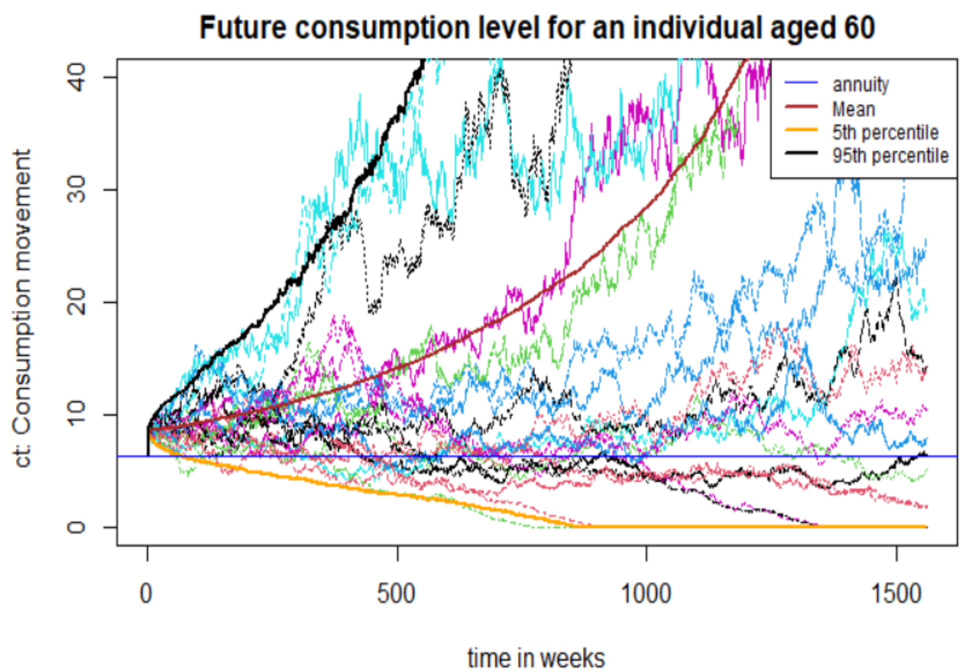

Figure 4.5: Scenario C1: Future consumption path of a male retiree aged 60 along with the 5th and 95th quantiles and mean. Using parameters $\mu = 0.102$, $\sigma = 0.18$, $r = 0.03$, $\delta = 0.02$, $c_0 = 6.22$, $c^*/c_0 = 2$, $\beta = 0.4$

Scenario D1: $\mu = 13\%$, $\sigma = 20\%$, $r = 3\%$, $\delta = 0.5\%$, $\beta = 0.50$, $c^*/c_0 = 2.25$

Scenario D2: $\mu = 13\%$, $\sigma = 20\%$, $r = 3\%$, $\mu_{60} = 0.65\%$, $\beta = 0.50$, $c^*/c_0 = 2.25$

These retirees are risk tolerant. They aim to more than double their future income through the purchase of an annuity at some future optimal time. They get a very high β from the financial market. However the ratio seems unrealistically high and does not obey the argument in terms of equation (4.14), where, for D to be greater than zero, the ratio needs to follow $1 < c^*/c_0 < (r+\delta)/[r(1+l)]$, which in this case is, $1 < 2.25 < 1.17$. We see that the inequality thus does not hold for such high ratios of target income. The optimal time computed for the simulations in Figure 4.7 is 655 weeks. At this time point, the chance of ruin is 0.2% and the chance of obtaining a lower payout annuity is 71.0%. For the scenario, the range of optimal times $T^* \in (123, 1510)$ is quite varied and the risk of retiree falling at a higher range of this bracket is 12.0%.

Figure 4.8 shows the paths along the time frame for a male retiree aged 60. The optimal

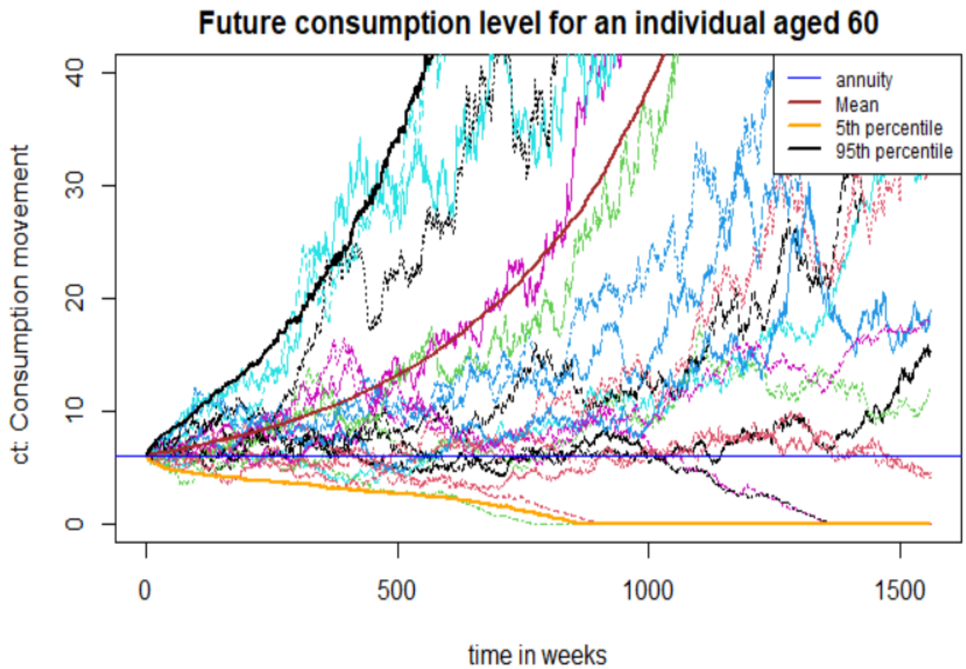

Figure 4.6: Scenario C2: Future consumption path of a male retiree aged 60 along with the 5th and 95th quantiles and mean. Using parameters $\mu = 0.102$, $\sigma = 0.18$, $r = 0.03$, $\mu_{60} = 0.0065$, $c_0 = 6.05$, $c^*/c_0 = 2$, $\beta = 0.4$

time to annuitize is 720 weeks. Doing so will lead to a chance of ruin of 0.2%, while the chance of purchasing a lower income annuity is 61.0%. This retiree lies in the optimal annuitizing range of $T^* \in (186, 1552)$, which brings the risk of 11.2% of ruin occurring if they annuitize towards the higher end of this range.

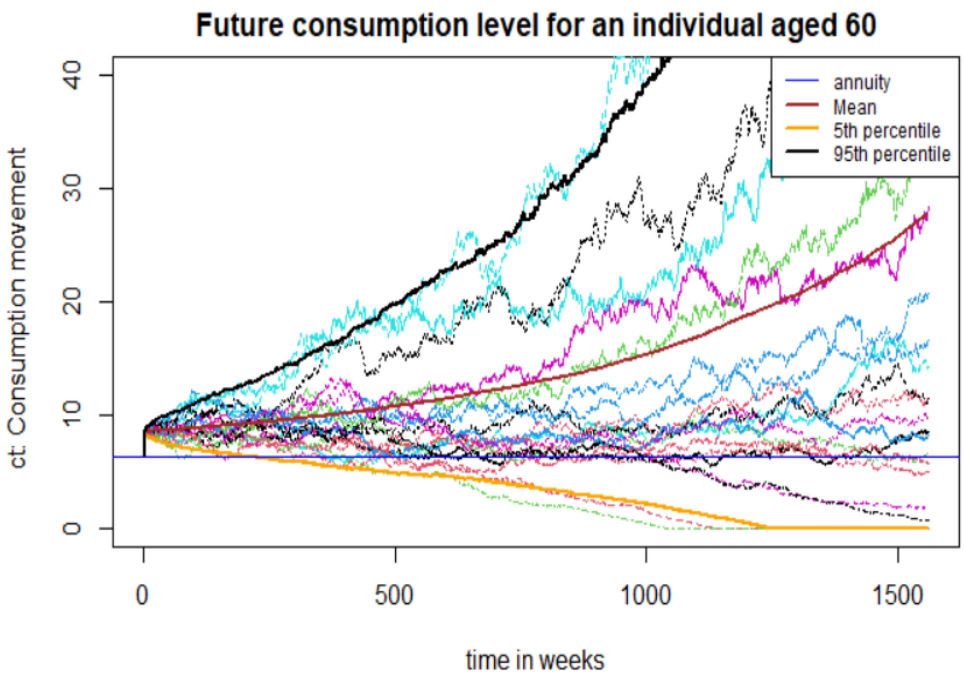

Figure 4.7: Scenario D1: Future consumption path of a male retiree aged 60 along with the 5th and 95th quantiles and mean. Using parameters $\mu = 0.13$, $\sigma = 0.2$, $r = 0.03$, $\delta = 0.005$, $c_0 = 6.22$, $c^*/c_0 = 2.25$, $\beta = 0.5$

Scenario E: $\mu = 6\%$**,** $\sigma = 20\%$**,** $r = 2\%$**,** $\mu_{60} = 0.65\%$**,** $\beta = 0.20$**,** $c^*/c_0 = 1.25$

We choose this retiree with current real world parameters, who is willing to earn a realistic future income of 1.25 times the initial annuity income. The initial consumption of this retiree would be 5.35 and therefore target income is 6.7. Figure 4.9 shows the future consumption path of this retiree for 30 year period. The optimal time for the retiree to achieve his target annuity income is 288 weeks. The chance of this person losing on the ability to earn his desired target income at this time point is 92.8%, which is very high. The minimum time T^* needed for obtaining a desired target income is 62 weeks. The reason being the lower returns and lower interest rate slow down the consumption movement. However, the optimal $T^* \in (62, 804)$ does not meet the mean time of ruin of 1115 weeks, hence a risk of ruin as low as 3% is noticed in the optimal interval.

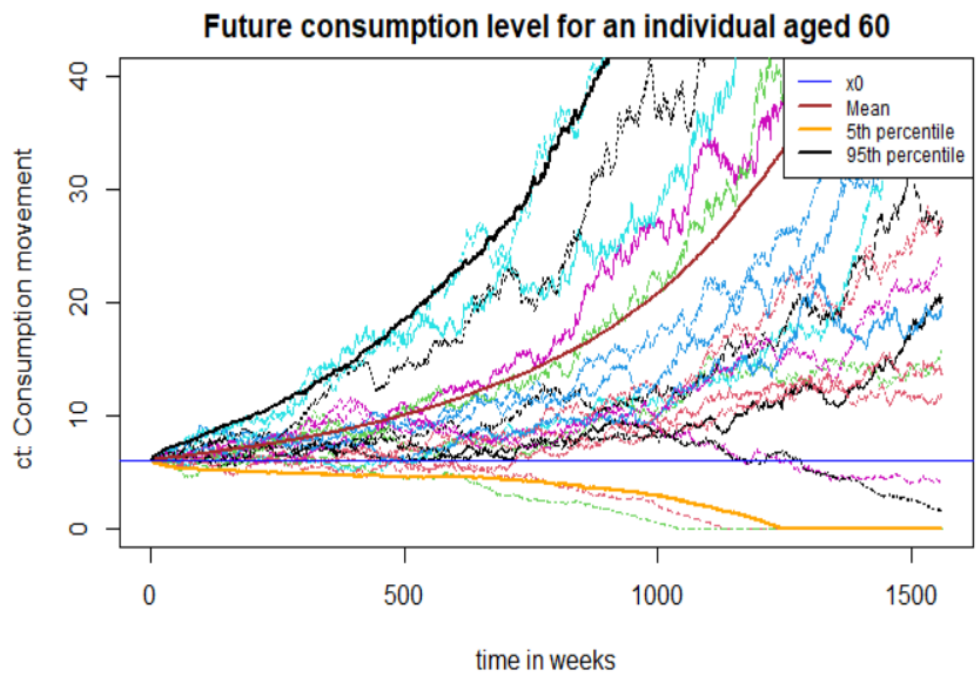

Figure 4.8: Scenario D2: Future consumption path of a male retiree aged 60 along with the 5th and 95th quantiles and mean. Using parameters $\mu = 0.13$, $\sigma = 0.2$, $r = 0.03$, $\mu_{60} = 0.0065$, $c_0 = 6.05$, $c^*/c_0 = 2.25$, $\beta = 0.5$

In this Chapter, we have seen the different scenarios introduced, an attempt was made to slightly alter the approach by updating the strategy in terms of its parameters as well as the risk measures.

- The probability of being ruined in all of the scenarios are very large. This was not well communicated in Giacinto and Vigna (2012). Though it appears to be worth waiting to obtain a higher payout annuity, the risk in the process of attaining this desired income is not communicated.

- The estimate of ruin in Giacinto and Vigna (2012) is lower, as the assumption of constant δ means that annuity prices are constant at all ages. The retiree is assumed to consume less ($c_0 \neq W_0/\overline{a}_x$) than in the case of non-constant force of mortality, and this alters the time by which he gets close to his desired target income. This assumption of a constant force of mortality is flawed, and has a major impact on the annuitization results.

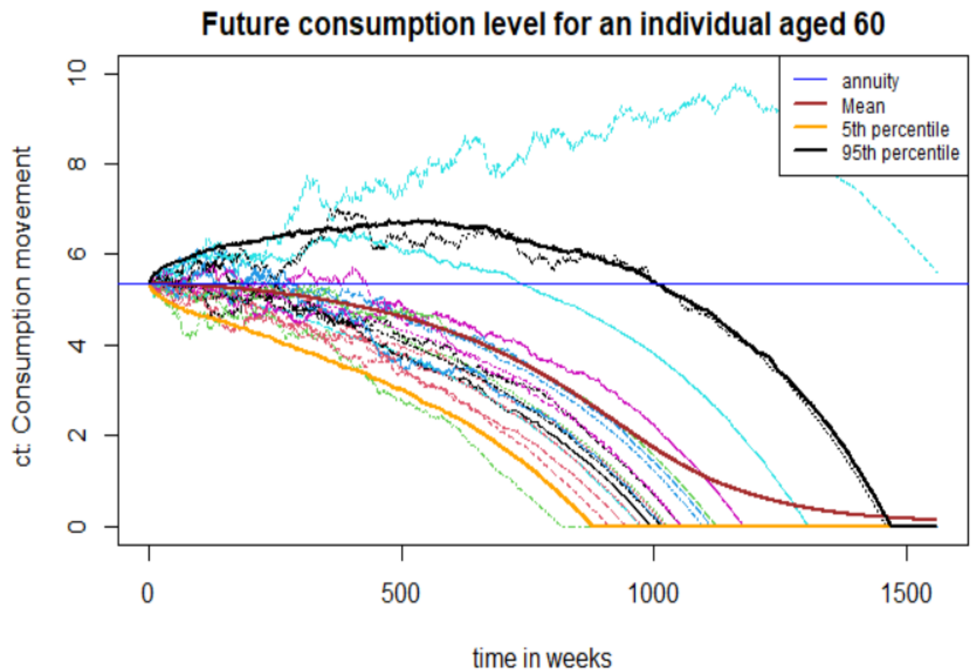

Figure 4.9: Scenario E: Future consumption path of a male retiree aged 60 along with the 5th and 95th quantiles and mean. Using parameters $\mu = 0.06$, $\sigma = 0.2$, $r = 0.02$, $\mu_{60} = 0.0065$, $c_0 = 5.36$, $c^*/c_0 = 1.25$, $\beta = 0.2$

- It is also evident that there is a high chance of a lower annuity attainable at future dates, even though the retiree is advised to optimize at that time. This is intuitive, as the ability to beat the mortality credits for lower values of μ earned from financial market is lower. This trend is visible in Scenario A (low μ) and gradually increases as we move to Scenario D (high μ).

- The mean time to ruin with parameters from Giacinto and Vigna (2012) is observed to occur quicker than when Gompertz mortality is used. This is because the constant consumption level at initial time is higher compared to what we obtain using Gompertz mortality assumption. Hence the retiree is subject to early ruin consuming at a higher rate.

- The mean time to optimal annuitization is seen to be occurring at early time points in Table 4.4. The reason being the change in amount of annuity purchasable is stable, as $\overline{a}_x^{-1} = \frac{r+\delta}{1+l}$ does not depend on x. Whereas, under the Gompertz assumption $\overline{a}_x$

Table 4.3: Risk measures for different Sharpe ratios and other parameters of r, μ_{60}, c^*/c_0 using Gompertz mortality

	Scenario A	Scenario B	Scenario C	Scenario D	Scenario E
T^* (in weeks)	384	430	440	720	288
$P(c_{T^*} <= c^*)$	75.8%	68.9%	64.0%	61.0%	92.8%
$P(c_{T^*} <= 0 \mid T^* = \max T^*)$	34.0%	28.8%	19.6%	11.2%	3%
Mean time of ruin (in weeks)	1112	1110	1045	1240	1115

Table 4.4: Risk measures for different Sharpe ratios and other parameters of r, δ, c^*/c_0 in Giacinto and Vigna (2012)

	Scenario A	Scenario B	Scenario C	Scenario D
T^* (in weeks)	110	250	322	655
$P(c_{T^*} <= c^*)$	75.0%	72.0%	65.4%	71.0%
$P(c_{T^*} <= 0 \mid T^* = \max T^*)$	35.9%	30.1%	20.0%	12.0%
Mean time of ruin (in weeks)	1110	1090	1040	1237

varies with x. Although this point sounds similar to the first one, the impact here is observed on the ratio c^*/c_0. The time needed to reach c^* is lower as the amount of annuity purchasable becomes higher for higher values of δ. Hence, a retiree who would have achieved the same level of annuity price under the exponential mortality distribution now has access to the payout earlier.

- Under the updated parameters in Scenario E, the lower market returns and interest rate delay the mean optimal time of annuitization. The risk of obtaining a lower payout annuity is very high in this scenario. The intuition behind this is that, by the time one hits optimal annuitization, the annuity prices are higher. This in turn lowers the income obtained from purchasing such an annuity.

- Across the columns of Tables 4.3 and 4.4 we observe the increase in the mean time of optimization, even though we increase the values of financial markets for better returns. This implies that as the retiree's desire to earn higher income at a later date is chosen, the time required to reach the necessary target is also higher and even more difficult. In Scenario D, we observe that, in most cases, the horizon $t = 30$ years is not enough for the retiree to reach the target income level.

- The subjective discount rate (ρ) and force of mortality (δ) are often observed to be interpreted as one coefficient. The authors mention that this coefficient is a global discount factor of an individual, that takes into account subjective tolerance towards future time and the expected remaining lifetime. In other words, it is measuring the tolerance towards future income. Using $\rho = r$ links subjective discounting to the time value of money, but this is flawed. Discounting what a retiree might be needing in the future would likely de-value the expectation towards future income. A future feeling is always less influential than a present one.

Chapter 5

Conclusion

In this thesis we have considered three different strategies for optimal investment-consumption-annuitization for an individual at retirement. Decision making in retirement planning is expressed as normative advice in each of the papers studied. Our main findings indicate that the downside risk to a retiree adopting any of these strategies is significant, and under-reported. This is a strong takeaway from the paper.

The "do it yourself and then switch" retirement strategy described in Chapter 2 initially gives an individual full control of their assets, and guides them towards participating in long term upward performance of equity markets. The strategy fails to communicate the downside risk, in the case where equity market performs poorly. In current conditions, with long term interest rates as low as 2%, and market volatility of 20% and above, the chances of successfully adopting this strategy are low.

The "all or nothing arrangement" in Chapter 3 puts forward a normative solution under no bequest motives and fewer constraints when compared to Chapter 2. This strategy is based on a utility based objective function. Expected utility allows upside outcomes to offset downside outcomes, which observes the true underlying risk. Also the assumption of subjective discount rate equal to the risk free rate; is flawed. This assumes significant devaluation of consumption at older ages. The resulting optimal strategy is not an adequate retirement planning advice to be used under a realistic financial framework.

In Chapter 4 we used a quadratic loss function to minimize the loss in annuitizing prematurely. The main drawback in this setting is the exponential future lifetime distribution assumption. The force of mortality plays a major role in determining future annuity prices and survival probabilities, and using a constant rate is indeed a flawed assumption. The original paper, again, emphasizes the successful deferrals that achieve the target income.

However, the time span is varied for different people based on their set of risk preferences and investment choices. This impacts the optimal annuitization time for a retiree, and additionally, the downside risk is not well communicated. This is an important omission.

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APPENDICES

Appendix A

Derivations for Milevsky (2001)

A.1 ODE for wealth function for deterministic returns case

Solving the ordinary differential equation in 2.3 we get:

$$\frac{dW_t}{dt} - kW_t = -c$$

$$e^{-kt}\frac{dW_t}{dt} - e^{-kt}kW_t = -ce^{-kt}$$

$$(W_t e^{-kt})' = -ce^{-kt}$$

Integrating both sides with respect to t:

$$\int (W_t e^{-kt})' dt = \int -ce^{-kt} dt$$

$$W_t e^{-kt} + d = \frac{c}{k} e^{-kt} + e$$

d and e are unknown constants, we can club them as one. And subject W_t as follows:

$$W_t = \frac{c}{k} + de^{kt}$$

We know $W_0 = w$ so:

$$w = \frac{c}{r} + d$$

$$d = w - \frac{c}{k}$$

$$\therefore W_t = \frac{c}{k} + (w - \frac{c}{k})e^{kt} \; for\, all \; t < t^* \tag{A.1}$$

A.2 Computing time of ruin for deterministic returns case

Computing for the time at which the individual is ruined subjecting the equation (2.4) to 0.

$$0 = \frac{c}{k} + (w - \frac{c}{k})e^{kt}$$

$$e^{kt} = \frac{\frac{-c}{k}}{w - \frac{c}{k}}$$

$$e^{kt} = \frac{-c}{kw - c}$$

we know $c = w/a_x$

$$kt = \ln\left(\frac{\frac{-w}{a_x}}{kw - \frac{w}{a_x}}\right)$$

$$kt = \ln\left(\frac{-w}{kw\, a_x - w}\right)$$

$$t = \frac{-\ln(1 - k\, a_x)}{k}$$

$$t^* = \begin{cases} \frac{-\ln(1-k\, a_x)}{k} & k < (a_x)^{-1} \\ \infty & k \geq (a_x)^{-1} \end{cases}$$

A.3 Computing the annuitization time point in deterministic returns case

Solving for T which is the optimal time at which the user needs to switch to annuity before ruin t^*.

This time occurs when marketable Wealth at future time T would match the price of continued lifetime consumption stream c.

$$max\{T\} \quad s.t. \frac{W_T}{a_{x+T}} \geq c$$

$$(w - \frac{c}{k})e^{kT} + \frac{c}{k} \geq c \times a_{x+T}$$

$$e^{kT} \geq \frac{kca_{x+T} - c}{wk - c}$$

We know $c = w/a_x$;

$$e^{kT} \geq \frac{kwa_{x+T} - w}{wka_x - w}$$

$$e^{kT} \geq \frac{la_{x+T} - 1}{ka_x - 1}$$

$$T = \frac{1}{k} \ln \left[\frac{1/k - a_{x+T}}{1/k - a_x} \right]$$

The above equation were s can be infinity is constrained to follow:

$$T* = \begin{cases} \frac{1}{k} \ln \left[\frac{1/k - a_{x+T}}{1/k - a_x} \right] & k < (a_x)^{-1} \\ \infty & k \geq (a_x)^{-1} \end{cases}$$

Similar logic is used in obtaining c_τ as in equation (2.9).

Appendix B

Derivations for Milevsky and Young (2007)

B.1 SDE for wealth function for stochastic returns case

$$dW_t = ((\mu - r)W_t - c)dt + \sigma W_t dB_t$$
$$dW_t = ((1 - \pi_t)rW_t + \pi_t \mu W_t - c_t)dt + \sigma \pi_t W_t dB_t$$
$$dW_t = (rW_t + (\mu - r)W_t - c_t)dt + \sigma \pi_t W_t dB_t$$

Therefore;

$$dW_t = ([r + (\mu - r)]W_t - c_t)dt + \sigma \pi_t W_t dB_t$$