Robust covariance estimators for mean-variance portfolio optimization with transaction lots

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ABSTRACT

This study presents an improvement to the mean-variance portfolio optimization model, by considering both the integer transaction lots and a robust estimator of the covariance matrices. Four robust estimators were tested, namely the Minimum Covariance Determinant, the S, the MM, and the Orthogonalized Gnanadesikan–Kettenring estimator. These integer optimization problems were solved using genetic algorithms. We introduce the lot turnover measure, a modified portfolio turnover, and the Robust Sharpe Ratio as the measure of portfolio performance. Based on the simulation studies and the empirical results, this study shows that the robust estimators outperform the classical MLE when data contain outliers and when the lots have moderate sizes, e.g. 500 shares or less per lot.

1. Introduction

For several decades, trading activities in several stock exchanges have increased, since more investors buy assets and sell them again or keep them until the dividend is shared. However, the investors are always facing the risk that the asset price will move downward. To reduce this type of risk, the investor can diversify by buying different assets. Therefore, a loss in an asset can be covered by profits of other assets.

To determine the capital allocation in a portfolio of stocks, Markowitz [24] introduced the mean-variance model. In this model, the variance was used as a risk measure, and the goal was to minimize the variance of the portfolio. As has been noted, e.g. in Britten-Jones [5], the Markowitz model became important, popular, and widely used in practice.

In the Markowitz model, the classical mean and covariance matrix estimators are used. These estimators rely on the assumption of multivariate normal distributed returns, which is rarely fulfilled in real applications. Lauprete et al. [19] showed that many empirical portfolio returns have the sample skewness and the sample kurtosis which exhibit fat tails, follow a non-symmetric distribution and have multivariate tail dependence.

Robust estimators can be applied to handle data that contain outliers and deviate from the assumption of multivariate normality. Huber and Ronchetti [14] characterized the robustness of estimators in terms

of their sensitivity to small deviations from the assumptions of the underlying distribution of the data. The properties of robust estimators have been studied extensively and described in the literature, for example, see Hampel et al. [12], Huber and Ronchetti [14], and Marrona et al. [25].

The usage of robust estimators in the portfolio optimization problem dates back to Lauprete et al. [19], who examined how the Gaussian assumption in the Markowitz mean-variance model is influenced by marginal heavy tail. Lauprete et al. [19] also prove that robust alternatives to the classical variance estimator have lower risk of loss than the non-robust ones. Perret-Gentil and Victoria-Feser [27] use robust S-estimators to estimate the mean and variance in Markowitz's portfolio optimization model. Welsch and Zhou [37] consider several robust covariance estimators in the mean-variance portfolio optimization, such as FAST-MCD, Iterated Bivariate Winsorization, and Fast 2D-Winsorization. DeMiguel and Nogales [9] prove that certain robust estimators produce more stable and less sensitive portfolios than the traditional portfolio. Kaszuba [18] compares several robust estimators that are used in the mean-variance portfolio optimization model and the traditional mean-variance portfolio. He finds that the risk of M-Portfolios and LAD-Portfolios is significantly different from the risk of a classical portfolio. Recently, Supandi et al. [34] show that portfolios that use constrained M-estimators are better than the classical portfolio. It must be noted that the term 'Robust Portfolio Optimization' does not

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only refer to the usage of robust estimators in portfolio optimization. The usage of robust optimization techniques for solving the portfolio optimization problem (see e.g. Goldfarb and Iyengar [11] is also known as robust portfolio optimization.

On the other hand, all stock markets around the world have determined a minimum number of shares or assets that can be traded regularly, known as *transaction lot*. For example, in the Indonesia Stock Exchange, one lot of stocks consists of 100 shares, so the investor can trade 100, 200, etc. shares regularly. Since the portfolio weights obtained by the Markowitz model were presented as fractions or percentages to the total capital, they could not be applied directly by investment managers. The portfolio optimization problem by the Markowitz model with the transaction lot constraint has been discussed in the literature, for example, by [20,23], and Buchheim et al. [6], Mencarelli and D. Ambrosio [26], Soleimani et al. [32], including the review paper in Mansini et al. [22]. Other studies with the transaction lot constraint, but using different risk measures are available, for instance, Setiawan and Rosadi [30] which uses the Value at Risk; Barati et al. [4] which uses the semi variance, etc. However, all of these studies only used the classical variance estimator with additional integer lots constraint and did not consider the robustness of their estimator.

In this paper, we present an improvement to the mean-variance portfolio optimization model with the integer transaction lots constraint, by considering robust estimators of the covariance matrices to deal with the presence of outliers in the data. Four robust estimators are tested, namely the Minimum Covariance Determinant (MCD) estimator, the Tukey's Biweight S-estimator, the Orthogonalized Gnanadesikan-Kettenring (OGK) estimator, and the MM-estimator. To obtain the integer solution to the portfolio optimization problem, we use a genetic algorithm (GA), see, e.g., Arnone et al. [1], Soleimani et al. [32] and Chang et al. [8]. Several nature-inspired metaheuristics alternatives or the (mixed) integer programming methods can also be considered to solve this problem (which is a subject of future research), see, e.g., Bacanin and Tuba [2], [3], Lubis et al. [21], Sitopul et al. [31], Strumberger et al. [33], Tuba and Bacanin [36] and the references therein. Therefore, this study can be viewed as a mixture of robust statistics based and transaction lots based portfolio optimization. We also introduce the lot turnover measure, a modified portfolio turnover, and the Robust Sharpe Ratio as the measure of portfolio performance. Based on the simulation studies and the empirical results, this study shows that the robust estimators outperform the classical MLE when data contain outliers and when the lots have moderate sizes.

This paper is organized as follows. In Section 2, we discuss the theoretical formulation of the mean-variance portfolio optimization and present various robust estimators. In Section 3, we present the mean-variance portfolio optimization problem with transaction lots constraint and the robust GA approach solving the problem. In the next section, Section 4, we present simulation studies whereas in Section 5 we provide the empirical results using stocks from Indonesia Stocks Exchange (IDX). The last section is devoted to the conclusion and some remarks on directions of future research.

2. Robust statistics in the portfolio optimization model

In recent decades, robust statistical methods have been used in the portfolio optimization model, i.e.

- Robust risk measures have been used to substitute the variance in portfolio optimization. Instead of the variance, the portfolio's risk is measured by the Least Absolute Deviation (LAD) or by other types of M-estimators, as presented e.g. in Kaszuba [18] and Fabozzi et al. [10]. According to Kaszuba [17], this method is known as a one-step approach.
- Robust methods have been used to estimate the covariance matrix in the Markowitz mean-variance portfolio optimization model. This model has been discussed in several papers, for example, Kaszuba

[18], Welsch and Zhou [37], and Supandi et al. [34]. There are many robust estimators of the covariance matrix, for example, the Minimum Covariance Determinant (MCD) estimator, the Constrained M-estimator, etc. According to Kaszuba [17], this method is also known as a two-steps approach.

In the following subsection, we discuss the Markowitz portfolio optimization model and some robust statistics that can be used to estimate the covariance matrix in our modified Markowitz model.

2.1. Markowitz's portfolio optimization model

Suppose that there are n risky assets traded in the stock market. Define $\mathbf{w} = [w_i]$, $i = 1, 2, ..., n$ as a column vector representing the portfolio weights, and $\boldsymbol{\mu} = [\mu_i]$, $i = 1, 2, ..., n$, a column vector representing the expected returns of each asset. The expected returns for this portfolio are Fabozzi et al. [10]

$$\boldsymbol{\mu}_p = \mathbf{w}^T \boldsymbol{\mu} = \sum_{i=1}^{n} w_i \mu_i$$

To calculate the portfolio's variance, one should calculate the variance-covariance matrix

$$\boldsymbol{\Sigma} = \begin{pmatrix} \sigma_{11} & \sigma_{12} & ... & \sigma_{1n} \\ \sigma_{21} & \sigma_{22} & ... & \sigma_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{n1} & \sigma_{n2} & ... & \sigma_{nn} \end{pmatrix}. \tag{1}$$

The portfolio's variance is defined as

$$\sigma_p^2 = \mathbf{w}^T \boldsymbol{\Sigma} \mathbf{w} = \sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j \sigma_{ij} \tag{2}$$

By assuming that short-selling is prohibited, the Markowitz portfolio optimization problem can be stated as

$$\min_{w} \sigma_p^2 = \mathbf{w}^T \boldsymbol{\Sigma} \mathbf{w}$$

subject to

1. $\mathbf{w}^T \boldsymbol{\mu} = \mu_0$
2. $\mathbf{w}^T \mathbf{1}_p = 1$
3. $w_i \geq 0$, for $i = 1, 2, ..., n$.

Here, $\mu_0 \in R$ denotes a constant.

To solve this optimization problem in practice, one needs to know the covariance matrix $\boldsymbol{\Sigma}$ and the mean vector $\boldsymbol{\mu}$, which are usually estimated from the historical data of the asset returns.

Let $\mathbf{y}_{t.}$ represent an n-dimensional column vector containing the data of the sample returns of all n assets at time t, namely $t = 1, 2, ..., T$. Suppose the historical data follow a multivariate normal distribution, then the maximum likelihood estimation for each parameter is

$$\hat{\boldsymbol{\mu}} = \frac{1}{T} \sum_{t=1}^{T} \mathbf{y}_{t.} \tag{3}$$

and

$$\hat{\boldsymbol{\Sigma}} = \frac{1}{T} \sum_{t=1}^{T} (\mathbf{y}_{t.} - \hat{\boldsymbol{\mu}})(\mathbf{y}_{t.} - \hat{\boldsymbol{\mu}})^T. \tag{4}$$

As noted by Perret-Gentil and Victoria-Feser [27], both equations above are the most efficient estimators, under the assumption of multivariate normality of the data.

2.2. Robust covariance estimators

Statistical procedures often depend on one or several assumptions about the (probability) distribution of the data. Fabozzi et al. [10] state

that robust statistics addresses the problem of making estimates that are insensitive to small deviations from the basic assumption of the statistical models employed. In this section, we review some well-known robust estimators of the mean and the covariance matrix. We describe several robust estimators that can withstand a high fraction (up to 50%) of outliers, such as the Minimum Covariance Determinant (MCD) estimator, S-estimators and MM-estimators. We also discuss a method suitable for high-dimensional data, namely the Orthogonalized Gnanadesikan-Kettenring (OGK) estimator.

2.2.1. Orthogonalized Gnanadesikan-Kettenring estimator

The OGK estimator is a robust estimator of covariance that is symmetric but not necessarily positive definite, nor affine equivariant. According to Marrona et al. [25], the computation of the OGK estimator is as follows. As in the previous sub-section, let $\mathbf{Y} = [y_{ti}]$ be a $T \times n$ data matrix, where the rows $\mathbf{y}_{t \cdot}$ denote the returns of each stock at time t, $t = 1, 2, ..., T$. Hence, the columns $\mathbf{y}_{\cdot i}$ denote the changes of the returns for each stock, $i = 1, 2, ..., n$.

1. Compute a normalized data matrix $\mathbf{X}$ with columns

$$\mathbf{x}_{\cdot i} = \frac{\mathbf{y}_{\cdot i}}{\hat{\sigma}(\mathbf{y}_{\cdot i})},$$

where $\hat{\sigma}$ is a univariate robust estimator of variance. Consequently, the rows will be:

$$\mathbf{x}_{t \cdot} = \mathbf{A}^{-1} \mathbf{y}_{t \cdot}, \quad t = 1, 2, ..., T,$$

where

$$\mathbf{A} = \text{diag}(\hat{\sigma}(\mathbf{y}_{\cdot 1}), ..., \hat{\sigma}(\mathbf{y}_{\cdot n})).$$

2. Compute a robust correlation matrix $\mathbf{U}$, with elements $U_{ii} = 1$ and

$$U_{ij} = \frac{1}{4}[\hat{\sigma}(\mathbf{x}_{\cdot i} + \mathbf{x}_{\cdot j})^2 - \hat{\sigma}(\mathbf{x}_{\cdot i} - \mathbf{x}_{\cdot j})^2],$$

for $i \neq j$.

3. Compute the eigenvalues of $\mathbf{U}$ and the corresponding eigenvectors. Let $\mathbf{E}$ be the matrix where the columns are the eigenvectors of $\mathbf{U}$. Then, compute the matrix $\mathbf{Z}$ with the rows

$$\mathbf{z}_{t \cdot} = \mathbf{E}^T \mathbf{x}_{t \cdot} = \mathbf{E}^T \mathbf{A}^{-1} \mathbf{y}_{t \cdot}, \quad t = 1, 2, ..., T.$$

4. For each $i = 1, 2, ..., n$, compute $\hat{\mu}(\mathbf{z}_{\cdot i})$ and $\hat{\sigma}(\mathbf{z}_{\cdot i})$, where $\hat{\mu}$ is a robust univariate location estimator. Then, set

$$\Gamma = \text{diag}(\hat{\sigma}(\mathbf{z}_{\cdot 1})^2, ..., \hat{\sigma}(\mathbf{z}_{\cdot n})^2)$$

and

$$\boldsymbol{\nu} = (\hat{\mu}(\mathbf{z}_{\cdot 1}), ..., \hat{\mu}(\mathbf{z}_{\cdot n}))^T$$

5. We obtain the Orthogonalized Gnanadesikan-Kettenring estimator for the (multivariate) mean as

$$\hat{\boldsymbol{\mu}}(\mathbf{Y}) = (\mathbf{AE})\boldsymbol{\nu}$$

and for the covariance matrix as,

$$\hat{\boldsymbol{\Sigma}}(\mathbf{Y}) = (\mathbf{AE})\Gamma(\mathbf{AE})^T.$$

2.2.2. Minimum covariance determinant estimator

The Minimum Covariance Determinant (MCD) estimator is a robust estimator of location and covariance, introduced by [28]. This estimator uses both mean and covariance matrix of h data points, for $\frac{T}{2} \leq h < T$, with the smallest determinant of the population covariance matrix. The mean is estimated by

$$\hat{\boldsymbol{\mu}}_{MCD} = \frac{1}{h} \sum_{t=1}^{h} \mathbf{y}_{t \cdot}$$

and the estimated covariance is

$$\hat{\boldsymbol{\Sigma}}_{MCD} = \frac{1}{h} \sum_{t=1}^{h} (\mathbf{y}_{t \cdot} - \hat{\boldsymbol{\mu}}_{MCD})(\mathbf{y}_{t \cdot} - \hat{\boldsymbol{\mu}}_{MCD})^T$$

It is known that the breakdown value of this estimator equals $\frac{T-h}{T}$.

The exact MCD estimator is very hard to compute, as it requires the evaluation of all $\binom{n}{h}$ subsets of size h. To simplify and to speed up the computation, one can implement the FAST-MCD algorithm, as proposed by [29]. The FAST-MCD algorithm is as follows:

1. Randomly draw h observations from T available data, where $\frac{T}{2} \leq h < T$. With this subset, compute the empirical mean and covariance matrix, namely $\hat{\boldsymbol{\mu}}_0$ and $\hat{\boldsymbol{\Sigma}}_0$. Note that if $\det(\hat{\boldsymbol{\Sigma}}_0) = 0$, add one or more observation until $\det(\hat{\boldsymbol{\Sigma}}_0) \neq 0$.
2. For each observation $t = 1, 2, ..., T$, compute the distance as

$$d_0^2(t) = (\mathbf{y}_{t \cdot} - \boldsymbol{\mu}_0)^T \hat{\boldsymbol{\Sigma}}_0^{-1} (\mathbf{y}_{t \cdot} - \boldsymbol{\mu}_0)$$

and set $\hat{\boldsymbol{\Sigma}}_{old} := \hat{\boldsymbol{\Sigma}}_0$.

3. Sort the distances from the smallest to the largest one, i.e.

$$d_0(\pi(1)) \leq d_0(\pi(2)) \leq ... \leq d_0(\pi(T)),$$

where $\pi(t)$ indicates the permutation to obtain the ordering.

4. Get the new subset of size h as the observations corresponding to the smallest h distances, namely the index set $H_1 = \{\pi(1), \pi(2). ...\pi(h)\}$.
5. In the new subset, compute the new mean and covariance matrix: $\hat{\boldsymbol{\mu}}_0$ and $\hat{\boldsymbol{\Sigma}}_0$.
6. Since $\det(\hat{\boldsymbol{\Sigma}}_0) \leq \det(\hat{\boldsymbol{\Sigma}}_{old})$, repeat steps 2–5 until $\det(\hat{\boldsymbol{\Sigma}}_0) = \det(\hat{\boldsymbol{\Sigma}}_{old})$.
7. Repeat the step 1–6 for several initializations.
8. The last $\hat{\boldsymbol{\mu}}_0$ and $\hat{\boldsymbol{\Sigma}}_0$ are the MCD estimates.

In step 7, the algorithm will stop and obtain an approximate MCD solution by taking several initial choices of h, applying steps 1–7 to each random choice of h observation, and keeping the solution with the lowest determinant. Since there is only a finite number of h subsets, the sequence of determinants calculated by this algorithm must converge within a finite number of steps. Note that there is no guarantee that the final iteration will give the global minimum of the covariance determinant. Another algorithm to calculate the MCD estimator for the mean and the covariance matrix has been suggested by Hubert et al. [16], known as the deterministic MCD algorithm.

2.2.3. S-estimator

The S-estimator could be recognized as a generalization of the Minimum Volume Ellipsoid (MVE) estimator, to increase the efficiency of the MVE multivariate location and scatter estimator. According to Marrona et al. [25], the S-estimators of location $\hat{\boldsymbol{\mu}}_S$ and covariance $\hat{\boldsymbol{\Sigma}}_S$ are defined such that the determinant of the matrix $\hat{\boldsymbol{\Sigma}}_S$ is minimized under the constraint

$$\frac{1}{T} \sum_{t=1}^{T} \rho\left(\sqrt{(\mathbf{y}_{t \cdot} - \hat{\boldsymbol{\mu}}_S)^T \hat{\boldsymbol{\Sigma}}_S^{-1} (\mathbf{y}_{t \cdot} - \hat{\boldsymbol{\mu}}_S)} \right) = b,$$

where b is a constant, $\hat{\boldsymbol{\mu}}_S \in \mathbb{R}^n$, and $\hat{\boldsymbol{\Sigma}}_S$ is a positive definite symmetric matrix. Here, the loss function $\rho(.)$ must be carefully selected from the continuously differentiable function class, such that the S-estimator will have a high breakdown and is asymptotically normal. A popular choice of continuously differentiable loss functions (which is also used in this paper) is Tukey's biweight function [15], given by

$$\rho(u) = \begin{cases} \frac{m}{6}\left(1 - \left[1 - \left(\frac{u}{m}\right)^2\right]^3\right), & \text{for} |u| \le m \\ \frac{m^2}{6}, & \text{for} |u| > m. \end{cases}$$

The breakdown value of Tukey's biweight S-estimator depends on the constant m, i.e. $\epsilon* = \frac{6b}{m}$. Other ρ functions could be used as well, for example, Hampel's function or Huber's function Hubert and Rousseeuw [15].

2.2.4. MM-estimator

The MM-estimator Tatsuoka and Tyler [35] is a type of high breakdown value estimator, an extension of the S-estimator, that has high efficiency under multivariate normality. The computation of the MM-estimator for the returns data $\mathbf{y}_{t\cdot}$, $t = 1, 2, ..., T$, consists the following four steps:

1. First, choose a loss function ρ to compute the S-estimators of location and covariance, namely $(\breve{\boldsymbol{\mu}}, \breve{\boldsymbol{\Sigma}})$.
2. Compute $\hat{\hat{\sigma}} = |\breve{\boldsymbol{\Sigma}}|^{\frac{1}{2n}}$.
3. Find the MM-estimator of the location and the shape parameter, $(\hat{\hat{\boldsymbol{\mu}}}_{MM}, \hat{\hat{\boldsymbol{\Gamma}}})$, that minimize

$$\frac{1}{T}\sum_{t=1}^{T}\rho_1\left(\frac{[(\mathbf{y}_{t\cdot} - \boldsymbol{\mu})^T\boldsymbol{\Gamma}^{-1}(\mathbf{y}_{t\cdot} - \boldsymbol{\mu})]^{\frac{1}{2}}}{\hat{\hat{\sigma}}}\right)$$

for all $\boldsymbol{\mu} \in \mathrm{R}^n$ and $|\boldsymbol{\Gamma}| = 1$, $\boldsymbol{\Gamma} \in PDS(n)$, the class of positive definite symmetric matrices. Here ρ_1 does not need to be the same as ρ.
4. Compute the MM-estimator of covariance matrix, $\hat{\hat{\boldsymbol{\Sigma}}}_{MM} = \hat{\hat{\sigma}}^2\hat{\hat{\boldsymbol{\Gamma}}}$.

The breakdown point of the MM-estimator is inherited by that of the S-estimator used for its computation Hubert and Rousseeuw [15].

3. Minimum transaction lots in the Markowitz model

Most of the financial asset transactions take place at the stocks market, where companies offer their stocks, bonds, or derivative shares. A minimum number of shares that can be regularly traded in the stock markets is called a transaction lot. As an example, for regular transactions at the Indonesia Stocks Exchange (IDX), one transaction lot for stocks and options contract is 100 units and 1 unit, respectively. The number of asset lots that can be traded by an investor is represented as an integer.

3.1. Optimal portfolio with transaction lots and robust covariance estimator

Suppose that b is the amount of capital that will be invested into n available shares at time t. Suppose that the price per share of the ith shares is c_i, and the number of shares per lot is k. If Σ denotes the variance-covariance matrix (estimated using previous data), we can determine the weight of the ith asset, w_i as well as the number of lots of the ith asset, x_i, by solving the following optimization problem (assuming that short sale is prohibited):

$\min \mathbf{w}^T\boldsymbol{\Sigma}\mathbf{w}$

subject to

1. $\sum_{i=1}^{n} kx_i c_i \le b$
2. $w_j = \frac{c_j x_j}{\sum_{i=1}^{n} c_i x_i}$, $j = 1, 2, ..., n$.
3. x_j is an integer.

It must be noted that the sample mean would contribute a high estimation error to the out-of-sample performance DeMiguel and Nogales [9], Kaszuba [18]. Therefore, following Kaszuba [18], our model only considers the minimum-variance portfolios, which are obtained by removing the constraint of the sample mean. Hence, the

optimal portfolio weights only depend on the covariance matrix. The first constraint in the optimization above ensures that the total money spent to buy the stocks does not exceed the available capital (b), while the last constraint confirms that the lot numbers are integers.

In the practical computation, the covariance matrix is replaced by its estimates. Besides the classical maximum likelihood estimation (MLE), in this study, we apply four different robust estimators to estimate the Σ, namely the MCD estimator, the OGK estimator, the S-estimator, and the MM-estimator.

On the other hand, the above mentioned portfolio minimum variance selection with transaction lots is a multi-objective decision problem with integer feasible regions. This problem can be solved using the genetic algorithm method. Other alternatives are available, such as the (mixed) integer programming method or other heuristic methods, see e.g. Bacanin and Tuba [2], Lubis et al. [21] and the references therein.

3.2. Solving the optimization using the genetic algorithm

One of the most popular heuristic technique that is widely used in portfolio optimization is the genetic algorithm (GA). Proposed by Holland [13], the GA had been developed from the concept of natural selection and genetic theory. In the GA, each solution candidate must be represented as chromosomes. To initialize the algorithm, an initial population that consists of several chromosomes is generated. Afterwards, several pairs of chromosomes as 'parents' are selected from the population, which produce new chromosomes (as the next generation) by crossover and/or mutation operator with predefined probability. A fitness function, which is defined based on the objective function of the problem, is used to evaluate the performance of each solution represented by each chromosome. These processes have to be repeated for several generations until the final (termination) condition has been reached. As explained in Chang et al. [7], the basic steps of the genetic algorithm are as follows.

1. Generate an initial population consisting of several chromosomes.
2. Evaluate the fitness of each chromosome in the population.
3. Select 'parents' from the population.
4. Recombine parents to produce the next generation, using crossover and mutation.
5. Evaluate the fitness of the next generation.
6. Replace some or all populations by the next generation.
7. Repeat steps 3–6 until a satisfactory solution has been found.

The application of the genetic algorithm to solve the portfolio optimization problem was introduced by Arnone et al. [1]. Several studies, for example Chang et al. [7], Soleimani et al. [32], and Chang et al. [8], showed that the genetic algorithm could efficiently find a near optimal or even the optimal solution of a portfolio optimization problem.

Following Lin and Liu [20], in this paper chromosomes with n elements or genes were used, where n is equal to the number of stocks in the portfolio. Each gene is a real number between 0 and 1. Let u_j, $j = 1, 2, ..., n$ be the jth genes in a chromosome. Since the genes should represent the portfolio weights, we must divide each number by the total sum of genes in the chromosome,

$$v_j = \frac{u_j}{\sum_{j=1}^{n} u_j}, \quad j = 1, 2, ..., n.$$

Suppose that the price of the jth asset is c_j and the total investment capital is b, while the number of lots for each asset could be calculated as

$$x_j = \left\lfloor b\frac{v_j}{c_j} \right\rfloor, \quad j = 1, 2, ..., n$$

The calculation of the lots of each asset would cause the change of

the portfolio weight. Hence we define the modified weight after lot calculation, that is,

$$w_j = \frac{x_j c_j}{\sum_{j=1}^{n} x_j c_j}, \quad j = 1, 2, \dots, n.$$

The modified weight, as part of the weight vector **w**, is used in the fitness function:

$$F = \mathbf{w}^T \Sigma \mathbf{w} - \epsilon \left| b - \sum_{j=1}^{n} x_j c_j \right|$$

In this function, the second term is a penalty to ensure that the amount of money is not far from the total available capital b. By adjusting the ϵ value, the algorithm can find the portfolio with total price as near as possible to b. In other words, the investor may spend most of all available capital.

3.3. Measuring the portfolio performance: rolling horizon and the lot turnover measure

Following DeMiguel and Nogales [9] and Supandi et al. [34], to evaluate the performance of a robust estimation method in portfolio optimization, we use the rolling horizon method. This method consists of the following steps:

1. Obtain the data of n stock for the period T. Determine the length of the estimation window η, where $\eta < T$.
2. Compute the portfolio assets weights **w** and/or the number of asset lots **x** based on the data in the estimation window. To implement this step, estimate the new covariance matrix for each estimation window using both the classical and the robust estimation method.
3. Hold the portfolio (obtained in the second step) and determine the out-of-sample returns data r_{t+1}.
4. Change the estimation window by including one more recent data point, and discard one the earliest data point.
5. Repeat steps 2, 3, and 4 until the end of the data set is reached.

The out-of-sample returns data $\hat{r}_{t+1}$ obtained in the third step could be used to evaluate the portfolio performance, that is, by calculating the out-of-sample mean returns

$$\mu_s = \frac{1}{T - \eta} \sum_{t=\eta+1}^{T} \hat{r}_t,$$

and the standard deviation

$$\sigma_s = \sqrt{\frac{1}{T - \eta - 1} \sum_{t=\eta+1}^{T} (\hat{r}_t - \mu_s)^2}$$

We also consider the Sharpe Ratio (SR) or excess returns to risk ratio, that is,

$$SR_s = \frac{\mu_s - r_f}{\sigma_s}$$

where r_f represents the risk-free investment returns. Since the μ_s and σ_s used in the above formula are not robust, we introduce a new robust ratio measure, called Robustified Sharpe Ratio (RSR), as

$$RSR_s = \frac{\mu_{\alpha,s} - r_f}{\sigma_{\alpha,s}}$$

while $\mu_{\alpha,s}$ and $\sigma_{\alpha,s}$ denote the trimmed mean and the trimmed standard deviation, respectively. For these ratios, a higher Sharpe Ratio or Robust Sharpe Ratio indicates better portfolios, since it means that more returns are obtained with the same risk, or, the portfolio has a smaller risk for the same returns.

In addition, to measure the portfolio's stability, DeMiguel and Nogales [9] defines the portfolio turnover as

Table 1
The parameterization of the genetic algorithm.

Method (Parameters)	Value
Chromosome length	6 (equal to the number of assets in the portfolio)
Representation of genes	real number between 0 and 1
Number of population	150
Crossover method	Uniform crossover
Crossover probability	0.7
Mutation method	Random mutation
Mutation probability	0.1
Selection method	Tournament (roulette-wheel) selection
Elitism	3
Number of generation	2000

$$PT = \frac{1}{T - \eta - 1} \sum_{i=1}^{n} \sum_{t=\eta}^{T-1} |w_{t,i} - w_{t+1,i}|,$$

where $w_{t,i}$ and $w_{t+1,i}$ denote the ith asset weight at time t and time $t + 1$, respectively.

In our research, since the result of portfolio optimization is presented as the number of lots, we modify the above mentioned formula to represent the average change in the number of lots, that is,

$$LT = \frac{1}{T - \eta - 1} \sum_{i=1}^{n} \sum_{t=\eta}^{T-1} |x_{t,i} - x_{t+1,i}|$$

where $x_{t,i}$ and $x_{t+1,i}$ denote the number of lots of the i-th asset at time t and $t + 1$, respectively.

Denote p_{ti} as the price of the assets i at time t. By defining

$$\tilde{w}_{t,i} = \frac{x_{ti} p_{ti}}{\sum_{i=1}^{n} x_{ti} p_{ti}}$$

and using the result $\tilde{w}_{t,i}$ to replace the $w_{t,i}$ in the formula of the portfolio turnover, we can obtain the modified portfolio turnover (MPT) that could be compared with the portfolio turnover (PT). The portfolio turnover, lot turnover, as well as the modified lot turnover, represent the rebalancing of assets in the portfolio that must be done to maintain the portfolio at optimal conditions. Rebalancing assets need some money to buy the assets and pay the transaction cost, hence the better portfolio is the one that exhibits a smaller turnover.

4. Simulation study

In this section, we study the property of the modified mean-variance portfolio optimization problem with additional transaction lots constraint and robust estimators for the covariance matrix via simulation studies. Here we consider a mixture of a multivariate normal distribution $F_\delta(r)$ with contamination. In this model, $\delta \in (0, 1)$ represents the proportion of contamination to the multivariate normal model. Hence, the actual asset returns follow the following distribution,

$$F_\delta(r) = (1 - \delta) N_p(\mu, \Sigma) + \delta H,$$

where $N_p(\mu, \Sigma)$ denotes the multivariate normal distribution with mean vector μ and covariance matrix Σ.
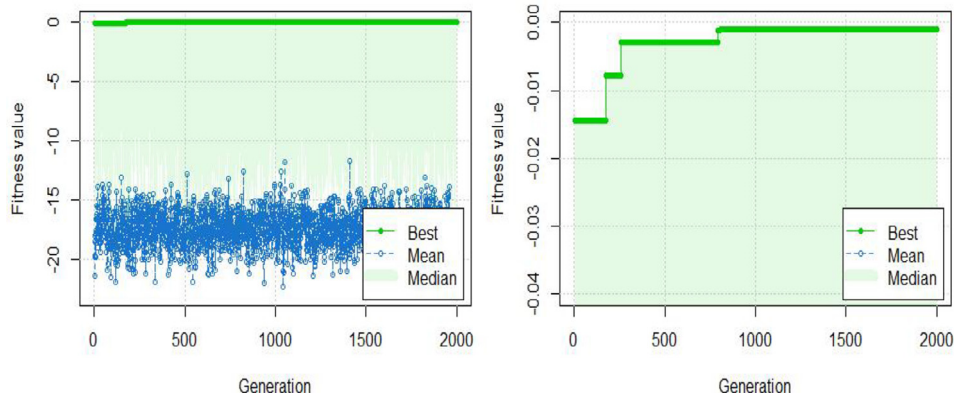
Following the simulation study of Supandi et al. [34], a set of returns data for three independent assets is generated. The generated data are multivariate normally distributed, with mean vector

$$\mu = (0.12, \quad 0.15, \quad 0.13)$$

and covariance matrix Σ, where we consider the case of the independent returns with

$$\Sigma = \begin{pmatrix} 0.012 & 0 & 0 \\ 0 & 0.015 & 0 \\ 0 & 0 & 0.016 \end{pmatrix} \tag{5}$$

and the dependent returns with

Fig. 1. The fitness function values of all populations for each generation of the genetic algorithm (in the case of independent returns with lot constraint). In the right panel, we zoom into the plot by only showing the best values of the fitness function. It clearly shows that after 2000 generations, the results are already stable.
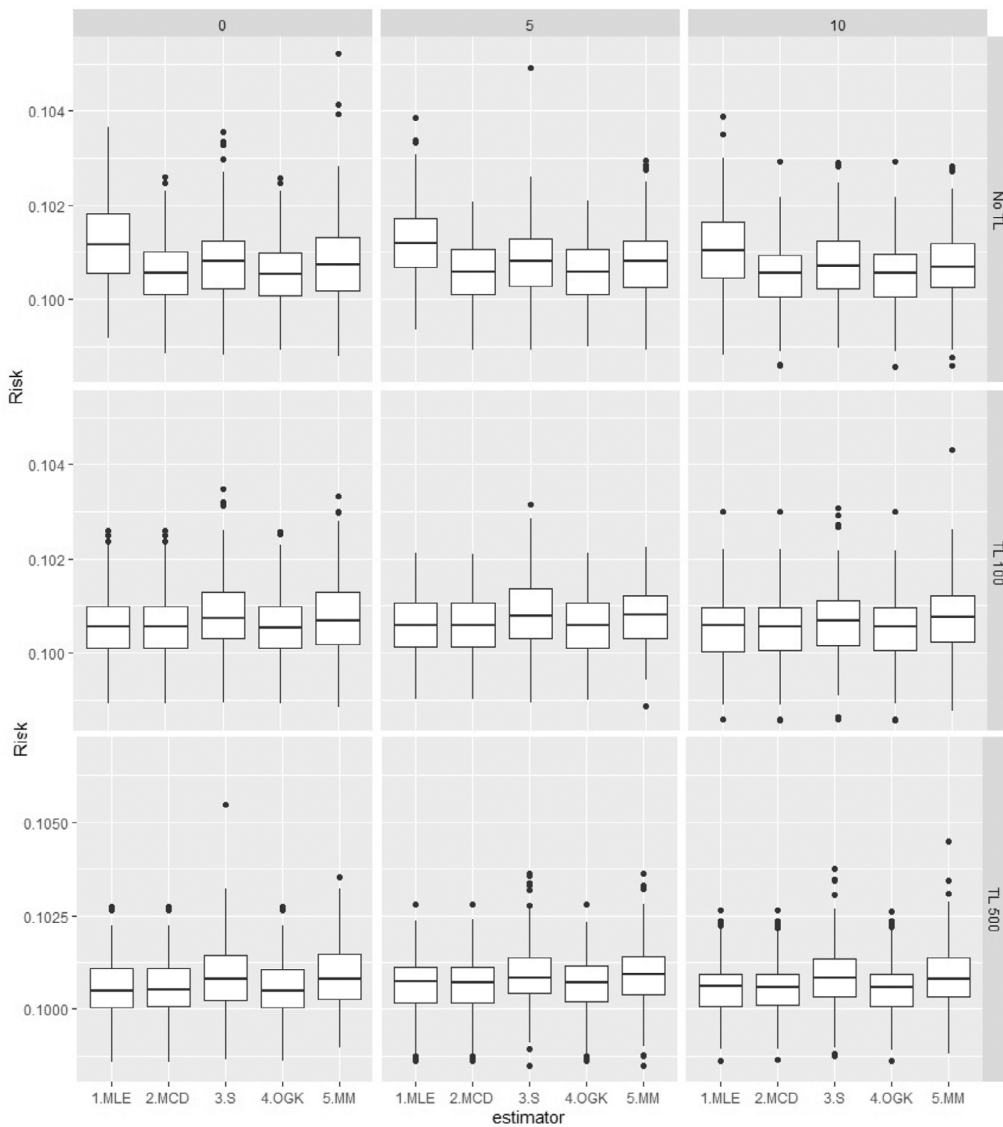
Fig. 2. Risk of portfolios calculated with transaction lots (TL) (with size 100 and 500 shares per lot) and without transaction lots (NTL) based on classical MLE and several robust estimators (independent returns).
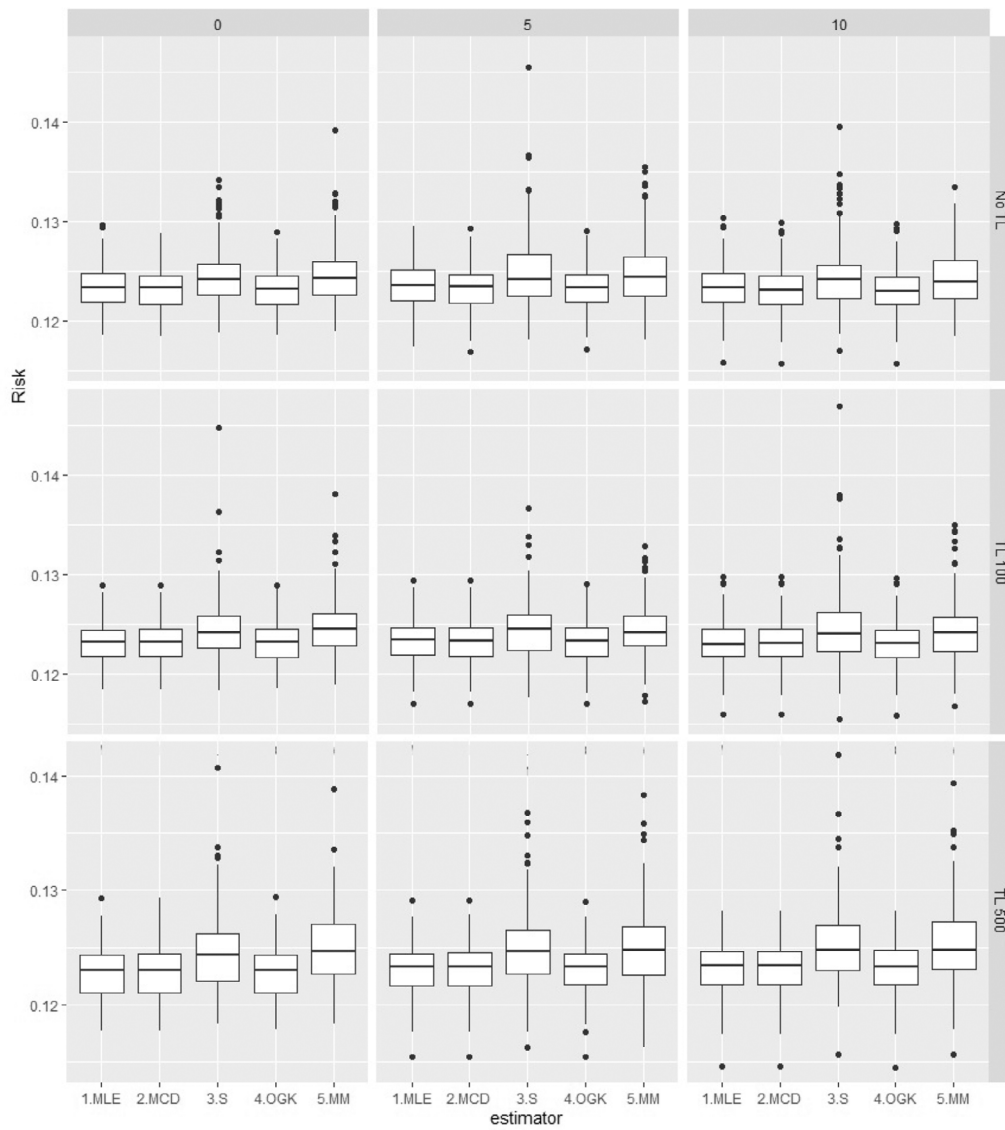
Fig. 3. Risk of portfolios calculated with transaction lots (TL) (with size 100 and 500 shares per lot) and without transaction lots (NTL) based on classical MLE and several robust estimators (dependent returns).

Table 2
The mean of Sharpe Ratio and Robust Sharpe Ratio (in italic) from simulated data (independent case).

Estimator	No Transaction Lots			Lot = 100			Lot = 500		
Type	0 pct.	5 pct.	10 pct.	0 pct.	5 pct.	10 pct.	0 pct.	5 pct.	10 pct.
MLE	4.2232	4.2208	4.2285	4.2420	4.2372	4.2449	4.2434	4.2331	4.2377
	4.5132	*4.5108*	*4.5191*	*4.5309*	*4.5259*	*4.5345*	*4.5319*	*4.5227*	*4.5271*
MCD	4.2531	4.2481	4.2532	4.2441	4.2393	4.2452	4.2425	4.2359	4.2423
	4.5418	*4.5369*	*4.5425*	*4.5337*	*4.5285*	*4.5346*	*4.5312*	*4.5255*	*4.5315*
S	4.2530	4.2480	4.2532	4.2421	4.2394	4.2467	4.2416	4.2348	4.2414
	4.5418	*4.5368*	*4.5424*	*4.5311*	*4.5284*	*4.5359*	*4.5303*	*4.5245*	*4.5312*
OGK	4.2530	4.2481	4.2532	4.2428	4.2387	4.2464	4.2411	4.2351	4.2391
	4.5417	*4.5369*	*4.5425*	*4.5318*	*4.5278*	*4.5357*	*4.5297*	*4.5245*	*4.5285*
MM	4.2531	4.2483	4.2533	4.2466	4.2400	4.2438	4.2375	4.2357	4.2399
	4.5418	*4.5371*	*4.5426*	*4.5354*	*4.5292*	*4.5333*	*4.5262*	*4.5249*	*4.5293*

$$\Sigma = \begin{pmatrix} 0.012 & 0.001 & 0.001 \\ 0.001 & 0.015 & 0.001 \\ 0.001 & 0.001 & 0.016 \end{pmatrix}. \tag{6}$$

For the presence of any outliers, we assume that H is multivariate normal distributed with the same covariance matrix Σ but with different mean vector, that is,

$$\mu_H = (-0.12, \; -0.15, \; -0.13).$$

For each of these covariance matrices Σ and mean vector μ_H, three sets of data were generated: (1) data without outliers, (2) data with 5% outliers, and (3) data with 10 percent outliers. Data from each type

Table 3
The mean of Sharpe Ratio and Robust Sharpe Ratio (in italic) from simulated data (dependent case).

Estimator	No Transaction Lots			Lot = 100			Lot = 500		
Type	0 pct.	5 pct.	10 pct.	0 pct.	5 pct.	10 pct.	0 pct.	5 pct.	10 pct.
MLE	3.4719	3.4669	3.4675	3.4452	3.4376	3.4286	3.4272	3.4229	3.4071
	3.7145	*3.7106*	*3.7096*	*3.6839*	*3.6779*	*3.6673*	*3.6650*	*3.6590*	*3.6458*
MCD	3.4794	3.4762	3.4745	3.4408	3.4331	3.4391	3.4239	3.4194	3.4165
	3.7226	*3.7208*	*3.7172*	*3.6793*	*3.6745*	*3.6798*	*3.6611*	*3.6556*	*3.6548*
S	3.4799	3.4759	3.4744	3.4466	3.4342	3.4434	3.4267	3.4294	3.4101
	3.7232	*3.7204*	*3.7171*	*3.6849*	*3.6750*	*3.6834*	*3.6643*	*3.6663*	*3.6495*
OGK	3.4793	3.4755	3.4739	3.4429	3.4392	3.4347	3.4361	3.4322	3.4185
	3.7225	*3.7200*	*3.7165*	*3.6817*	*3.6800*	*3.6737*	*3.6747*	*3.6691*	*3.6571*
MM	3.4797	3.4762	3.4750	3.4413	3.4439	3.4391	3.4234	3.4222	3.4156
	3.7230	*3.7209*	*3.7178*	*3.6805*	*3.6849*	*3.6782*	*3.6607*	*3.6582*	*3.6552*

Table 4
Price range and descriptive statistics of each asset returns.

Stock	Stock price		Stock returns					
Code	Min	Max	Mean	Median	Std.Dev	MAD	Skewness	Kurtosis
S1	3500	8375	0.046	0.00	0.022	0.019	0.066	3.311
S2	5050	9250	−0.013	0.00	0.021	0.017	0.069	5.226
S3	3450	9950	0.001	0.00	0.020	0.017	0.216	5.331
S4	3175	8250	0.008	0.00	0.019	0.015	0.102	5.769
S5	3100	7500	0.073	0.00	0.019	0.013	0.196	4.185
S6	4680	9200	0.019	0.00	0.021	0.014	0.102	5.049

were generated 200 times, meaning that we simulated returns data for three stocks in 200 observations. By assuming that the price of each stock at the beginning of the period is $P_0 = 1000$, the price at the end of the period can be calculated using the sum of the generated log returns. This simulation was repeated 200 times, yielding 200 different data sets.

For each generated data set, the portfolio optimization problem with transaction lots (we only consider lot size 100 and 500 shares here) and without transaction lots were calculated as described in Section 3. The covariance matrix was estimated using the mentioned methods: the MLE, the MCD estimator, the S-estimator, the OGK estimator, and the MM-estimator. Based on the five covariance estimates, the optimum weights in the portfolios are obtained using the genetic

algorithm approach (see Sections 3.1. and 3.2), assuming that short selling is prohibited. The parametrization for the genetic algorithm is presented in Table 1. We generate three different optimal portfolios and compare the fitness function's value for each combination. Only portfolios with the best (largest) fitness were considered and further analyzed.

Fig. 1 shows the fitness function values (the mean, the median and the best) of all populations for each generation in the genetic algorithm (in the case of independent returns with lot constraint). From Fig. 1 (right panel), we clearly can see that the results already stabilize after 2000 generations.

Figs. 2 and 3 show that the portfolio optimization method without transaction lot constraint based on the robust estimators has lower risk compared to the MLE risk. When the transaction lots were applied, the classical MLE and several robust estimators yield portfolios with similar risk.

Tables 2 and 3 summarize the mean of both the Sharpe Ratio (SR) and the Robust Sharpe Ratio (RSR) of the independent and dependent returns, respectively, as defined in Section 3 above. For the optimal portfolio without transaction lots, it can clearly be seen that the robust estimator yields better portfolios than the classical MLE in the presence of outliers, both for the independent and dependent returns. In this case, the MM-method seems to always perform better than the other robust methods. The simulation results obtained for the optimal portfolio with transaction lots can be summarized as follows. In case of
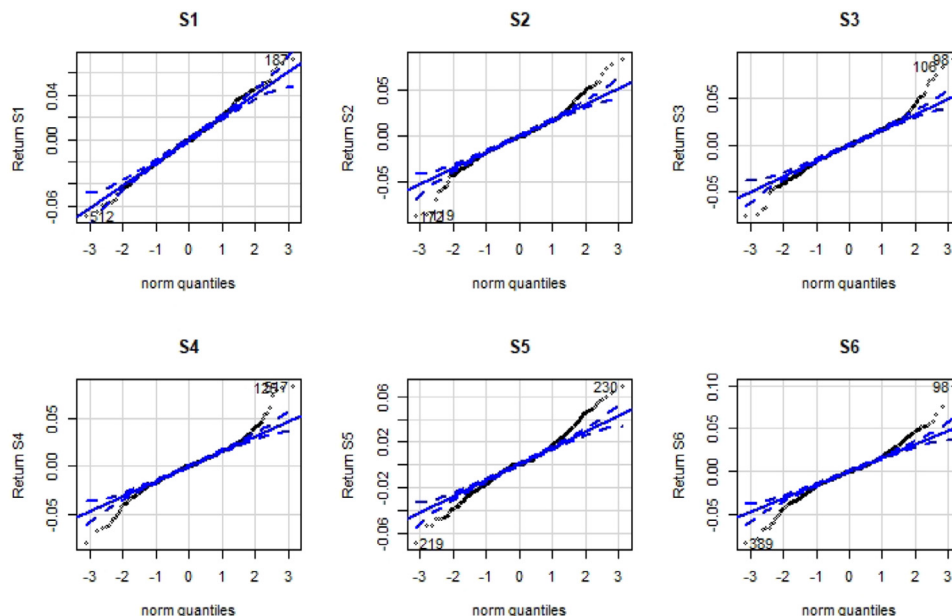

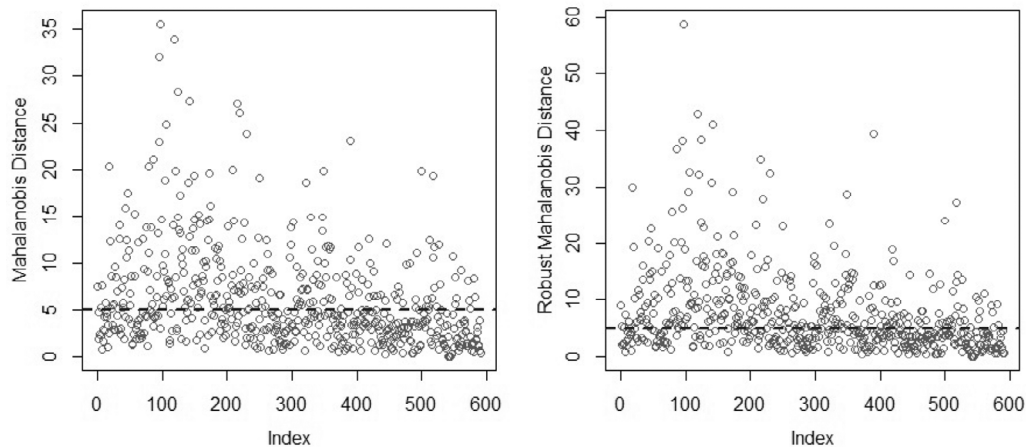**Fig. 4.** Normal QQ-plot of stock S_1 until S_6 returns data.

Fig. 5. Mahalanobis distance of the data. Points above the dashed line represent possible multivariate outliers.

Table 5
Daily (1-day), 3-days, and 5 days average Lot Turnover (LT) for portfolio with two different lot sizes.

Estim-	1-day LT		3-days LT		5-days LT	
ator	100	500	100	500	100	500
MLE	128.34	18.64	152.39	21.46	143.13	20.21
MCD	133.94	14.23	128.85	13.59	146.38	13.96
S	121.52	16.85	124.43	15.14	126.09	15.61
MM	129.27	21.33	135.00	17.71	128.05	19.62
OGK	134.24	19.12	140.74	20.55	145.14	21.66

portfolio optimization with outliers, the robust estimator can yield better portfolios than the classical MLE, both for the independent and dependent returns. There is no single robust method that always performs best in all situations. However, the performance of the MCD method seems to be the best in the independent returns. In the dependent case with moderate lot size (100), we obtain that MM will give the best portfolio, whereas for larger lot size (500), the method OGK will be superior over the other robust methods. We also further study the dependent case with stronger covariance, and the results are very similar, namely, in the presence of outliers, the robust estimators can give a better portfolio than the classical one. However, to save space, we omit the details.

In the simulation study, we do not consider the rolling horizon procedure, therefore the portfolio turnover and the lot turnover cannot be reported here. We report these two measures for the empirical example in the following section.

5. Empirical example

In this section, we apply the proposed approach discussed in Section 3 above to real stock market data.

5.1. Data description

We choose six stocks in the Indonesia Stocks Exchange (IDX) and obtain the data for 20 months. All stocks exhibit similar prices, ranging from IDR 3100 to IDR 9,950. The stock prices as well as the descriptive statistics of the calculated returns data for each asset are shown in Table 4.

From Table 4 it can be seen that most assets have non-negative returns. The skewness shows us that some of the assets have near symmetric distribution. From the sample kurtosis, we know that the asset returns do not follow normal distribution, as confirmed by Fig. 4. The plot of Mahalanobis distances as well as robust Mahalanobis distances (Fig. 5) confirms that the stock returns data might contain outliers.

5.2. Procedures

In this empirical study, we consider the estimation window or rolling horizon procedure. The number of outliers for each stock varies among the estimation windows. We estimate the covariance matrix using various methods, namely the maximum likelihood estimator (MLE), MCD, the S-estimator and the MM-estimator. The optimum weights in the portfolios are obtained using the genetic algorithm approach, assuming that short selling is prohibited. The parametrization for the genetic algorithm is the same as in the simulation study, presented in Table 1 (see Section 4).

We generate three different optimal portfolios and compare the fitness function's value for each combination along the rolling horizon. Only portfolios with the best (largest) fitness were considered and further analyzed. To examine the lot size effect, we consider two different sizes of lots, namely 100 shares and 500 shares.
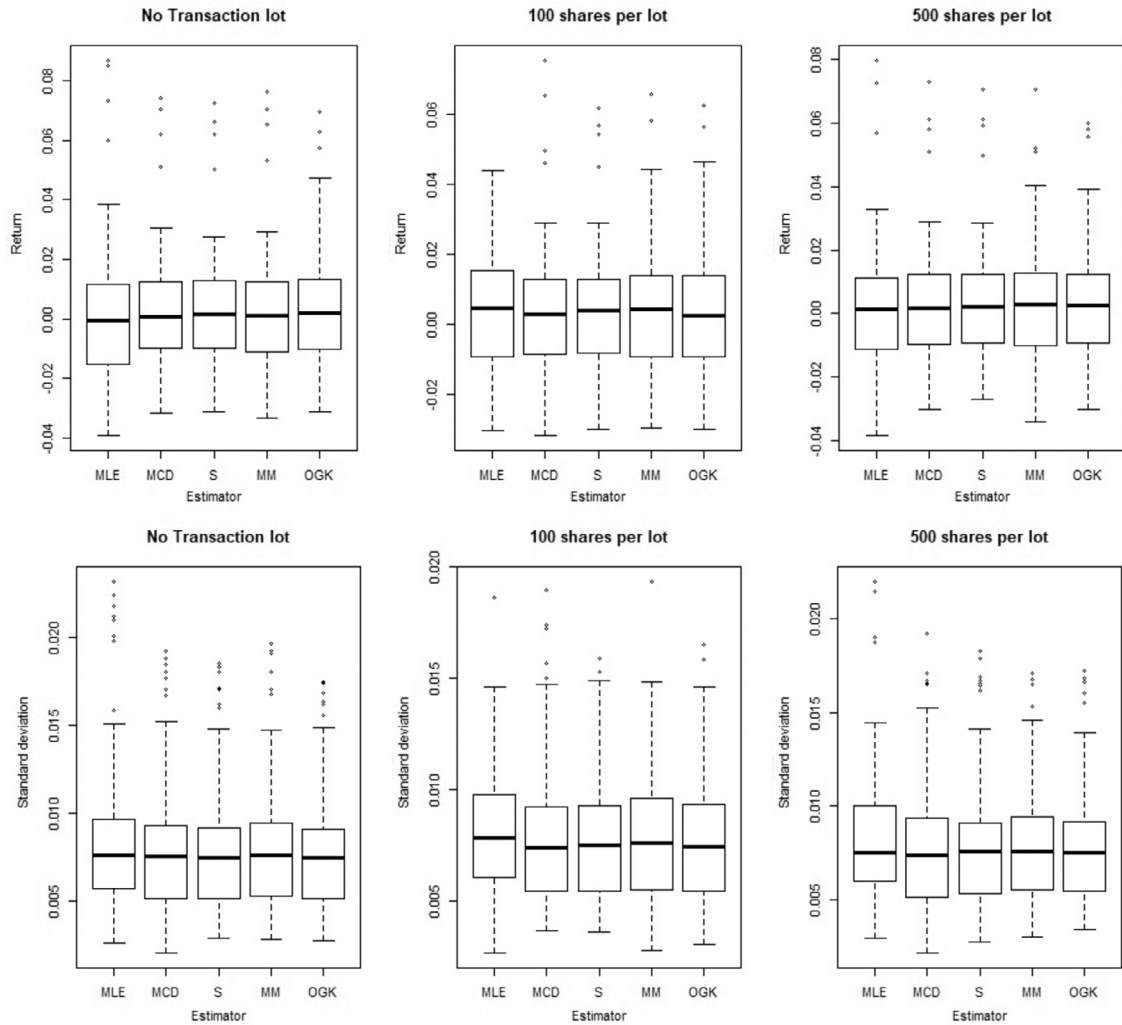
Table 6
The daily (1-day), 3-days, and 5-days Modified Portfolio Turnover (MPT) for each portfolio without transaction lots, with transaction lots equal to 100 unit stocks, and with transaction lots equal to 500 unit stocks.

Estim-	1-day PT			3-days PT			5-days PT		
ator	No.TL	100	500	No.TL	100	500	No.TL	100	500
MLE	0.0010	0.0454	0.0470	0.0022	0.0521	0.0482	0.0031	0.0506	0.0479
MCD	0.0048	0.0239	0.0222	0.0065	0.0224	0.0210	0.0071	0.0244	0.0212
S	0.0011	0.0187	0.0162	0.0022	0.0196	0.0147	0.0029	0.0196	0.0157
MM	0.0010	0.0248	0.0222	0.0021	0.0264	0.0216	0.0028	0.0257	0.0209
OGK	0.0046	0.0200	0.0173	0.0053	0.0210	0.0186	0.0060	0.0202	0.0182

Fig. 6. Total returns of the optimal portfolio calculated based on classical and several robust estimators of the covariance matrix. This figure is obtained when a 1-day rolling horizon procedure is applied to the data.

Table 7

The average of Sharpe Ratio and Robust Sharpe Ratio (in italic) for the portfolio without transaction lot and with transaction lots equal to 100 and 500 shares. This table is obtained when a 1-day rolling horizon procedure is applied to the data.

Estim-ator	Mean		
	No.TL	100	500
MLE	−0.424	−0.09	0.444
	−0.95	*−0.542*	*−0.02*
MCD	−0.051	0.123	0.245
	−0.741	*−0.491*	*−0.236*
S	0.026	0.126	0.272
	−0.618	*−0.419*	*−0.242*
MM	−0.096	0.071	0.256
	−0.717	*−0.505*	*−0.189*
OGK	0.064	0.233	0.279
	−0.467	*−0.231*	*−0.233*

5.3. Results

The result of the portfolio optimization can be seen from two points of view: (1) stability of its weights, and (2) portfolio performance.

Table 5 demonstrates that a higher number of shares per lot implies smaller lot turnover. We notice that using all methods, by increasing the size of the lot from 100 to 500 shares (five times larger) reduces the average 1-day, the average 3-days and the average 5-days lot turnover by more than eighty percent. This result shows that when the lot size is increased, the portfolio weight becomes more stable.

Another important result is that the various robust statistics exhibit different lot turnover (LT). The S-estimator consistently produces lower portfolio turnover than the classical MLE. The OGK estimator yields portfolios that can be with a higher lot turnover than the MLE, where the LT of MCD is similar to the MLE. The LT of MM is similar to the MLE for a daily LT, and significantly lower than MLE for higher numbers of days.

Table 6 presents the values of the modified portfolio turnover (MPT) (see Section 3.3 for the definition of MPT). For portfolios without transaction lot constraint, the MLE gives lower turnover compared to the MCD and the OGK estimator. However, it has almost identical turnover with the S-estimator and the MM-estimator. When the transaction lot constraint was applied in the portfolio selection model, all four robust estimators would give lower MPT compared to the classical MLE. Hence, the portfolio weights become more stable using robust estimators.

The performance of the portfolio could be examined based on the total returns as well and its standard deviation, which are presented in Fig. 6. In general, the MM-estimator gives the highest total returns in any case of lot constraints. For the risk in terms of the standard deviation, the usage of a robust covariance estimator can yield portfolios

with lower risk.

In general, the portfolio performance can be compared using the Sharpe Ratio (SR) and Robust Sharpe Ratio (RSR) as presented in Table 7. In the case of portfolios with integer lot transactions containing outliers, the value of the mean SR (and RSR) show that the robust portfolio can diminish the chance of obtaining higher loss, especially when the number of lots is equal to 100 or less. The same results also apply in the case of no lot constraint used in the optimization. Different results were obtained for the portfolio with 500 shares per lot, which shows that the classical MLE can produce portfolios better than the robust estimators.

6. Conclusion and future research

In this paper, we have extended the study of the mean-variance portfolio optimization with integer transaction lots by considering robust estimators of the covariance matrix. Based on simulation studies and empirical results, it turned out that the robust estimators outperform the classical MLE when the data contain outliers and when the lots have moderate sizes. Further research needs to be done using other robust estimators as well as adding more constraints to represent the real condition of the stock markets.

Acknowledgment

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