Estimating the cost of deposit insurance for a commercial bank following an optimal investment strategy

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Declaration

I declare that

Estimating the cost of deposit insurance for a commercial bank following an optimal investment strategy

is my own work, that it has not been submitted for any degree or examination in any other university, and that all the sources I have used or quoted have been indicated and acknowledged by complete references.

Itani Matamba March, 2020

Signed: ....

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Abstract

Commercial banks play a dominant role in facilitating the economic growth of a country by acting as an intermediary between the deficit spending unit (borrowers) and the surplus spending unit (lenders). In particular, they transform short-term deposits into medium and long-term loans. Due to their important role in the economy and the financial system as a whole, commercial banks are subject to high regulation standards in most countries. According to an international set of capital standards known as the Basel Accords, banks are required to hold a minimum level of capital as a buffer to protect their depositors and the financial market in an event of severe unexpected losses caused by financial risk. Moreover, government regulators aim to maintain public confidence and trust in the banking system through the use of a deposit insurance scheme (DIS). Deposit insurance (DI) has the effect of eliminating mass withdrawals of deposits in an event of a bank failure. However, DI comes at a cost. The insuring agent is tasked with estimating a fairly priced premium that the bank should be charged for DI.

In this thesis, we model a commercial bank holding an asset portfolio of riskless and risky assets in a constant interest rate-financial market. Firstly, we study an optimal control problem that involves maximization of bank capital. In particular, we employ the stochastic optimal control approach to derive optimal investment strategies in the bank's assets that maximize an expected utility of the bank's capital at future date $T > 0$. Secondly, we study a deposit insurance (DI) pricing problem based on the aforementioned bank. The latter problem entails employing a Monte Carlo simulation method to estimate the cost of DI for a coverage horizon of T years. The period of DI coverage, of duration T years, coincides with the interval on which the optimal investment strategy is followed. This enables us to estimate the price for DI under the optimal investment strategy. We present numerical simulations based on the optimal control and DI pricing problems. This includes studying the behaviours of the optimal investment strategy and optimized capital numerically. By means of numerical simulations, we also study the effect of changes in various model parameters on the estimate for the DI premium. Our results suggest that the optimal investment strategy is to diversify the bank's asset portfolio away from the risky asset and towards the riskless asset. Under the optimal investment strategy our results pertaining to the DI pricing problem suggest that for a fixed intial leverage ratio (deposit-to-asset ratio) the cost of the DI premium will increase as either the volatility in the asset portfolio of the bank or the coverage horizon increases. Similarly, for an increase in the initial leverage level the cost of the DI premium will increase as either the volatility in the asset portfolio of the bank or the coverage horizon increases.

Keywords

Asset portfolio

Bank capital

Basel Accord

Commercial bank

Deposit insurance

Deposit-to-asset ratio

Expected utility

Monte Carlo simulation

Stochastic optimal control

Optimal investment strategy

List of Acronyms

Basel Committe on Banking Supervision (BCBS)

Capital adequacy ratio (CAR)

Defined contribution (DC)

Deposit insurance (DI)

Deposit insurance fund (DIF)

Deposit insurance scheme (DIS)

Explicit deposit insurance (EDI)

Federal Deposit Insurance Corporation (FDIC)

Hamilton Jacobi Bellman (HJB)

Implicit deposit insurance (IDI)

Ordinary differential equation (ODE)

Partial differential equation (PDE)

Regulatory bank capital (RBC)

Risk weighted assets (RWAs)

Stochastic differential equation (SDE)

List of Notations

R_1 - The price of a riskless bank asset

R_2 - The price of a risky bank asset

A - The value of the total assets of the bank

D - The total deposits of the bank

B - The total borrowings of the bank

C - The total capital of the bank

$\hat{P}$ - The fairly priced deposit insurance premium of the bank

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Chapter 1

Introduction and scope

A commercial bank is a financial institution that provides services such as accepting demand and time deposits, checking accounts services, making loans to individuals and organizations, and offering basic financial products like certificates of deposit and saving accounts to individuals and small businesses [41]. Commercial banks play an important role in the economy and the financial system as a whole. Banks use deposits and borrowed funds (the liabilities of the bank) to make loans and/or to purchase securities (the assets of the bank). According to Petersen and Mukuddem-Petersen [33], banks try to manage their assets in the following ways. Firstly, they endeavour to grant loans to creditors who are likely to pay high interest rates and are unlikely to default on their loans. Secondly, they try to purchase securities with high returns and low risk. Lastly, in managing their assets, banks attempt to lower risk by diversifying their investment portfolio. The main categories of assets held by banks are loans, treasuries and reserves.

According to Saeed and Zahid [36], banks face many serious problems due to unsuccessful credit risk management, but credit lending remains the chief activity of the banking sector throughout the world. If many of a bank's borrowers default on their loans when due, the bank's creditors, including its depositors, risk loss. If a large number of customers of a bank withdraw their deposits simultaneously due to uncertainty about the bank's solvency, the bank might experience what is known as a bank run. Many bank runs can lead to the failure of the banking system. This will result in heightened interest rates, which will have devastating effects on the economy. Government regulators therefore aim to maintain stability by encouraging a certain level of public confidence and trust in the banking system through the use of a deposit insurance scheme (DIS). Deposit insurance (DI) is based on the idea that if depositors know that their deposits are covered in the event of a bank failure, then they will not feel the need to withdraw all their deposited funds. DI thus protect depositors and give them confidence that their funds are not at risk, hence minimising, if not preventing, the likelihood of bank runs.

A DI scheme could be either explicit (EDI) or implicit (IDI). EDI differs from IDI due to its reliance on formal regulation through central bank law, banking law or the constitution. EDI supposedly sets the rules of the game regarding coverage, participants, and funding. The aforementioned come at a cost. When countries elect not to introduce EDI, then by default their insurance is IDI [21]. Under IDI on the other hand, there is no formal law or regulation relating to the compensation of depositors in the event of a bank failure. If the bank is facing a financial crisis, the government can intervene and make direct payments to the depositors. According to the moral hazard theory, EDI can encourage banks to be less careful about risk behaviour since the deposit insurer will cover a large part of the bank's debts in case of default. The moral hazard problem associated with DI is usually interpreted in terms of an incentive for a bank to increase risk in search for higher profits [13]. In this thesis we will be studying a DI pricing problem based on EDI.

In 1977, Merton [24] suggested an analogy between DI and a put option to value DI contracts. In this paper a formula is derived to evaluate the cost of DI coverage. The author of [24] suggested that the strike price of the option equals the value of the insured deposits, and that the underlying asset in the contract is the bank assets. The maturity of the DI contract, under the model of [24], is equal to the length of time until the next bank audit. If the value of the bank's assets is below the value of the insured deposits at the time of the bank audit, the bank has the right to sell the assets at the value of the insured deposits, otherwise, the option is not exercised. Since Merton's [24] analogy, there has been a tradition of modelling DI as a one-period European put option. Examples of research papers from the literature of modelling DI in this way, which we discuss in detail in Chaper 2, are for instance that of Marcus and Shaked [23], Ronn and Verma [34], Lee *et al.* [22] and Duan [10].

Allen and Saunders [1] were the first to depart from the tradition of modelling DI as a one-period European put option, as they instead modelled DI as a callable perpetual American put option with consideration of both regulatory closure policy and self-closure policy. Hwang *et al.* [17] extended the model of Allen and Saunders [1] by introducing bankruptcy costs as an additional risk factor. In the paper [11], Duan and Yu proposed an alternative way of interpreting DI in a multiperiod framework. The defaulting banks in the model of Duan and Yu [11] are assumed to have their assets reset to the level of the outstanding deposits plus accrued interests when an insolvency resolution takes place. Based on the framework of Duan and Yu [11] , Muller [30] developed a DI pricing model that incorporates the explicit solution of an optimal capital control problem in conjunction with an asset value reset rule comparable to the typical practice of insolvency resolution by insuring agencies.

In order to promote the soundness and stability of the international banking industry, the Basel Committee on Banking Supervision (BCBS) regulates the international banking industry on behalf of the government [14, 37, 35, 32, 31]. In this regard, the BCBS introduced the Basel Capital Accord which stipulates minimum capital requirements for internationally active commercial banks so as to reduce the risk in the international banking system. The Basel Capital Accord set out capital requirements that required banks to hold a minimum level of capital as a buffer to protect their depositors and the financial market in the event of severe unexpected losses caused by financial risks. This minimum capital requirement is known as the capital adequacy of the bank, expressed as a ratio of a bank's capital base and its risk weighted assets (RWAs). From a shareholder's perspective, utilizing more capital increases asset earnings and lead to higher return on equity. From the regulator's perspective, increasing buffer capital reduces risk by cushioning the volatility of earnings [31]. However, decreasing capital increases risk by increasing the bank's financial leverage and, hence, a high probability of failure.

The Basel Capital Accord was finalized by the BCBS in July 1988 in Basel. The aim of the 1988 Capital Accord, also known as the Basel I Accord, was to set up regulatory minimum capital requirements in order to ensure that banks , at all times, are able to return depositors' funds. The 1988 Capital Accord called for a minimum ratio of capital to RWAs of 8% to be implemented by the end of 1992. In other words, the bank's capital should be greater than or equal to

8% of their RWAs. However, the 1988 Capital Accord was based on simplified calculations and classifications, which have simultaneously called for its disappearance. In June 1999, the Basel Committee issued a proposal for a new capital adequacy framework to address the shortcomings of the 1988 Capital Accord. The proposal lead to the 2004 revised capital framework known as the Basel II Accord. The Basel II Accord consisted of three key pillars: Pillar 1 covered the minimal capital requirement, Pillar 2 covered the supervisory review process, and Pillar 3 covered market discipline and disclosure [2, 40]. In response to the 2007-2008 financial crisis, the BCBS introduced a comprehensive set of reform measures known as the Basel III Accord. The Basel III Accord builds upon the Basel II Accord, but aims to further strengthen global capital standards. The Basel III Accord contains changes in the following areas: (i) augmentation in the level and quality of capital; (ii) introduction of liquidity standards; (iii) modifications in provisioning norms; (iv) introduction of a leverage ratio [18, 32, 31].

The stochastic optimal control method is a popular optimization technique for solving optimization problems in finance. The stochastic optimal control method involves solving the Hamilton-Jacobi-Bellman (HJB) partial differential equation (PDE), derived under the principle of dynamic programming. The aforementioned technique originated in the seminal paper of Merton [25]. In the paper [25], the investor wishes to allocate his/her wealth between a risk-free bond and a risky stock so as to maximize the expected utility of his/ her terminal wealth. The author of [25] explicity solved the HJB PDE under a constant volatility of the risky stock. Another important optimization technique used in finance is the so called Martingale method. This method, which relies on the theory of Lagrange multipliers, was developed by Cox and Huang [8] in a setting of complete markets. The Martingale approach incorporates a risk-neutral measure and generally involves solving a PDE [38]. The Martingale method was employed in banking by Witbooi *et al.* [38] who studied a portfolio optimization problem. In this study we shall follow the stochastic optimal control approach.

In this thesis we study an optimal control problem that involves maximizing bank capital. In particular, we employ the stochastic optimal control approach to derive optimal investment strategies in the bank's assets that will maximize an expected utility of the bank's capital at the future date $T > 0$. The constant interest rate financial market that the bank operates in consists of risky and riskless assets. Furthermore, we employ a Monte Carlo simulation method to obtain estimates for the premium the bank should be charged for entering into a DI contract. The Monte Carlo method incorporates an asset value reset rule comparable to what insuring agencies typically employ. By embedding the optimal investment strategies from the control problem into the Monte Carlo simulation method, we are able to see how the investment strategies affect the DI premium.

The scope of this thesis takes the following form. The current chapter introduces briefly some optimization problems studied in finance, the techniques used to solve them, as well as the concept of deposit insurance pricing. Here we also discuss the importance of the regulation of the international banking system. In Chapter 2 we discuss some of the papers that applied optimization theory in finance and some papers on DI pricing. In Chapter 3 we introduce some basic concepts and properties from finance and probability and measure theory that are used throughout this study. In Chapter 4 we provide an explanation of the balance sheet of the general commercial banking model and then introduce formulae describing the balance sheet items pertaining to our study. More specifically, we describe the bank's asset, liabilities, and capital by means of differential equations. In Chapter 5 we present the optimal control problem and obtain its solution. Here we also present a simulation study to illustrate graphically the behaviour of the optimal proportions of the capital invested in the risky and riskless assets, as well as the behaviour of the optimized bank capital. In Chapter 6 we derive the multiperiod DI pricing model for the bank in question. This chapter also includes graphic illustrations of the insured deposits and the bank's asset portfolio under the asset value reset rule. Towards the end of this chapter, we present our main results of the DI pricing problem. The thesis is concluded with Chapter 7.

Chapter 2

Literature review

We now discuss the works of some of the authors who studied optimization problems in finance. In particular, we summarize the works of Merton [26], Devolder *et al.* [9], Mukuddem-Petersen *et al.* [28], Mukuddem-Petersen and Petersen [33], Mulaudzi *et al.* [29], Witbooi *et al.* [38], Muller and Witbooi [31] and Muller [30]. In addition, we will also discuss the works of some of the authors who contributed to the development of deposit insurance pricing models. These include the contributions of Merton [24], Marcus and Shaked [23], Ronn and Verma [34], Allen and Saunders [1], Duan [10], Duan and Yu [11], Hwang *et al.* [17], Lee *et al.* [22] and Muller [30].

The seminal work of Merton [26] is considered as a pioneering point for the stochastic control and dynamic programming method for continuous-time portfolio optimization. Merton [26] used the Hamilton-Jacobi-Bellman (HJB) equation of the dynamic programming technique to explicitly solve the question of optimal portfolio allocation in a market with a riskless bond and a risky stock as an investment alternative. The stock price process in [26] is assumed to be driven by a geometric Brownian motion. In the paper [26] it is assumed that an investor wishes to maximize his/her terminal wealth under a power utility function. Since Merton's [26] seminal paper, numerous authors have applied optimization theory in order to find similar optimal investment strategies.

Devolder *et al.* [9] applied stochastic optimal control theory to obtain an optimal investment policy, before and after retirement, for a defined contribution (DC) pension plan. The benefits of the plan are paid under the form of annuities which are guaranteed during a certain fixed period of time. During the activity period of the contract, the contributions in [9] are invested in the financial market with one riskless asset and one risky asset. The reserve obtained at retirement age is the amount accumulated without any special guarantee given by the insurer. At retirement age, the insurer uses the reserve to purchase a paid-up annuity. After retirement the insurer has to pay the guaranteed annuity and also decide on how much of the remaining mathematical reserve should be invested in the financial market in question. In view of the fact that the liability is present after retirement, the authors of [9] split the problem into two periods. For the first period before retirement, i.e., the period without liability, they optimized the utility function of the final wealth at retirement. For the second period after retirement they optimized the utility function of the final surplus. For each period they used both the power law and the exponential utility function.

In their paper [33], Mukuddem-Petersen and Petersen applied stochastic optimization theory to asset and capital adequacy management in banking. Their study was motivated by banking regulation under Basel II that emphasized risk minimization practices associated with assets and capital of a bank. The analysis of the paper [33] depend on the dynamics of the capital adequacy ratio (CAR) which they computed in a stochastic setting by dividing regulatory bank capital (RBC) by risk weighted assets (RWAs). The aforementioned authors demonstrated how the CAR can be optimized in terms of the bank equity allocation and the rate at which additional debt and equity is raised. To verify their results, the authors of [33] employed the dynamic programming algorithm for stochastic optimization.

Mukuddem-Petersen *et al.* [28] solved a stochastic maximization problem related to consumption and banking profit on a random time interval. The authors considered a bank balance sheet that consists of items such as assets (loans, treasuries and reserves) and liabilities (deposits) that are balanced by bank capital (shareholder equity and subordinate debt). Here the bank aims to (i) optimize its expected utility of discounted depository consumption during a random time interval and (ii), optimize its profit at terminal random time. The term depository consumption refers to the consumption of the bank's profits by the taking and holding of deposits. In particular, Mukuddem-Petersen *et al.* [28] determined an analytic solution for the associated HJB equation in the case where the utility function are either of power, logarithmic, or exponential type. In the aforementioned study, the control variables are the depository consumption, the value of the depository financial institution's investment in loans and provisions for loan losses. Furthermore, the authors of [28] analyzed certain aspects of the banking model and optimization against the regulatory backdrop offered by the then latest banking regulation in the form of the Basel II Capital Accord. Mukuddem-Petersen *et al.* [28] showed that it is possible for a bank to maximize its expected utility of discounted depository consumption on a random time interval, $[t, \tau]$, and its final profit at time τ.

In the paper [29], Mulaudzi *et al.* investigated the investment of bank funds in loans and treasuries with the aim of generating an optimal final fund level. The results of [29] took behavioural aspects such as risk and regret into account. More specifically, the authors applied a branch of optimization theory that enable them to consider a regret attribute alongside a risk component as an integral part of the utility function. In this case, regret-aversion corresponds to the convexity of the regret function and the bank's preference is assumed to be represented by optimization subject to the utility. Moreover, Mulaudzi *et al.* [29] provided a comparison between risk- and regret-averse banks in terms of optimal asset allocation between loans and treasuries. The authors of [29] reached an analytical solution with regard to the optimal securitization problem with control variable being the portfolio of mortgage-backed securities.

Witbooi *et al.* [38] applied stochastic optimization theory to asset and capital adequacy management in banking. Under the assumption of a complete and fictionless financial market which allows at least two types of financial assets that can be bought and sold without incurring any transaction cost or restriction on short sale. In the aforementioned paper, the authors addressed the problem of obtaining an optimal equity allocation strategy that will optimize the terminal utility of a banking portfolio consisting of three assets, namely a treasury, security and loan under the Cox-Huang [8] methodology. At the same time Witbooi *et al.* [38] constructed a continuous-time model of the Basel II CAR computed from the bank's RWAs and capital in a stochastic setting. A simulation of the optimal equity investment strategy in the paper [38] indicates that the optimal porportion invested in the treasuries increases with respect to time. On the other hand, the optimal proportion invested in the loans progressively decreases with respect to time, while the proportion invested in securities remains constant.

In the paper [31], Muller and Witbooi investigated the investment strategy that maximizes an expected utility of a commercial bank's asset portfolio at a future date. The bank modelled by the aforementioned authors operates in a stochastic interest rate financial market consisting of a treasury security, a marketable security, and a loan. The aforementioned investigation entails obtaining formulas for the optimal amount of bank capital invested in different assets. Based on the optimal investment strategy, a model for the Basel III CAR is derived. Moreover, the authors of [31] considered the optimal investment strategy subject to a constant CAR at the minimum prescribed level set by the BCBS, then derived a formula for the bank's asset portfolio at a constant (minimum) CAR value. Muller and Witbooi [31] presented numerical simulations based on different scenarios. Their results indicate that the asset portfolio at constant (minimum) CAR value grows considerably slower than the asset portfolio of the original investment problem.

Muller [30] employed the stochastic optimal control method to derive an optimal investment strategy in a bank's assets that maximizes an expected exponential utility function of its capital at a future date. The paper considered a bank that trades in a financial market where the interest rate is constant and where it is possible to invest in a treasury, a marketable security and a loan. The aforementioned author provided a simulation study pertaining to the optimal proportions of the capital in the treasury, marketable security and loans. The simulation results reveal that the optimal investment strategy is to diversify the asset portfolio of the bank away from the risky assets and towards the riskless treasury.

Merton [24] suggested that the premium for deposit insurance can be modelled as a put option, with the strike price of the option equal to the value of the bank's deposits, and with the underlying asset being the bank assets. In [24] the asset portfolio of the bank is assumed to be driven by geometric Brownian motion, and the maturity of the DI contract is equal to the length of time until the next bank audit. Thus, using the Black-Scholes formula [7], Merton [24] showed that it is possible to find the value of the option, which is considered to be the insurance premium. The option is to be exercised if the bank is found insolvent during the audit. The practical application of the model [24] requires some important variables which include the value of the bank's assets, the volatility of the return on the assets and the impact of stochastic interest rate on the total bank assets.

For countries that have adopted or are adopting DI, pricing DI as accurately as possible is vital. In countries with explicit deposit insurance (EDI), DI is under-priced (over-priced) if the deposit insurer actually charges less (more) for its services than the estimated opportunity cost value of these services [21]. Marcus and Shaked [23] estimated the fair value of Federal Deposit Insurance Corporation (FDIC) insurance using the model [24] of Merton calibrated with data from 40 large banks that accounted for 25% of the US demand deposit in 1980. In their analysis, Marcus and Shaked [23] encountered two practical challenges that often arise when valuing government guarantees to firms using the option pricing approach. These are the limitations of bank assets and the volatility of the return on the bank assets not being observed directly. The latter authors concluded that the fair value of FDIC insurance is over-priced.

Ronn and Verma [34] found that FDIC insurance is under-priced. They took the assumption of Merton [24] that the time until maturity of the debt is equal to the time until the next bank audit. Ronn and Verma [34] further took the assumption that the strike price of the put option is equal to the total debt of the bank, rather than just the total deposits. Their model relies on two variables, i.e., the bank's asset value and the equity volatility. The bank's asset value can be observed, but the equity volatility must be estimated. According to [34] the sample standard deviation of equity returns should thus be used as the equity volatility.

Duan [10] suggested that the methodology of Ronn and Verma [34] is flawed, because they incorrectly treated the equity volatility as a constant. Duan [10] further suggested that if a bank's assets are assumed to follow a process with constant variance as in [24], and if bank equity is a call option on bank assets, then the bank equity must have a non-constant variance. Duan [10] offered an alternative methodology which overcomes some of the shortcomings of the model [34]. Duan [10] proposed the maximum likelihood estimation method to estimate theoretically correct values for the mean return and equity volatility from a sample of a bank's equity values.

Allen and Saunders [1] modelled DI as a callable perpetual American put option with consideration of both regulatory closure policy and self-closure policy. The authors of [1] argued that DI can be described as a callable put in the sense that DI is a perpetual put option with the insuring agent holding the right to terminate the put option prematurely. The aforementioned authors assumed that the FDIC's closure rule is strictly observed and that there is no additional forbearance. More specifically, banks cannot be granted permission to continue operating with capital levels below the regulatory standards, except in the case of the largest banks. When it comes to the right to exercise, the DI is actually not a standard put option. If the option expires in the money, bank shareholders may choose not to exercise because this would imply voluntary bank closure. In [1], the closure decision is used to control the timing to exercise.

Hwang *et al.* [17] extended the callable perpetual American put option model of Allen and Saunders [1]. In particular, their model incorporates explicit consideration of bankruptcy costs and more realistic closure rules considering possible forbearance can be accounted for. Bankruptcy cost in [17] plays an important role and it is set as a function of asset return volatility. Applying the isomorphic relationship between DI and a put option, Hwang *et al.* [17] first obtained a closed-form solution for the pricing model with bankruptcy costs and closure policies. Thereafter, they modified the barrier option approach to price the DI. In their model, Hwang *et al.* [17] assume that at the time of bank solvency, deposit holders are entitled to a prorated fraction of the asset value with all debt holders. Hence, the model [17] assume that all debts are of equal liquidation. According to Hwang *et al.* [17], the big challenge in fair pricing of DI is how to make the premium properly reflect the risk of the insured bank.

Based on the framework of Merton [24], Lee *et al.* [22] developed a DI pricing model that incorporates assets correlation, i.e., a measurement for the systemic risk of a bank to account for the risk of joint bank failures. According to [22] the joint bank failure risk is a systematic risk representing the joint loss distribution of dependence among bank's assets. The authors of the latter paper introduced a systematic risk factor to capture the risk of joint bank failures, with the expected cost of DI being the value of the put option. In [22] banks with higher risk of joint bank failures and higher asset correlations are required to pay higher DI premiums.

Duan and Yu [11] proposed an alternative way of interpreting DI in a multiperiod framework. The defaulting banks in [11] are assumed to have their assets reset to the level of the outstanding deposits plus accrued interest when an insolvency resolution takes place. According to the deposit insurance contract, the amount required to reset the assets is the legal liability of the insuring agent. The setup of paper [11] is supported by the United State's historical experience of deposit insurance. Through the use of either purchase-and-assumption or the government-assisted merger method, the majority of defaulting depository institutions were resolved. According to the data reported in the paper of Bartholomew [3], 1730 thrifts were resolved from 1980 to 1990, of which 85.4% thrifts were resolved through this form of reorganization. According to Table 125 of the 1990 FDIC's annual report, 1813 banks closed during the period 1945 through 1990. Among these banks, 69.6% were resolved through this form of reorganization. According to Duan and Yu [11], even after reorganization, the majority of banks continue defaulting. At the point of solvency resolution such banks can be regarded as receiving an at-the money put option. Hence, the DI in [11] can be viewed as a stream of one-period Merton-type put options with occasional asset value resets. The fairly-priced premium rate of Duan and Yu's [11] model is found to be substantially different from that of Merton [24]. The former authors incorparated capital forbearance and moral hazard into their model. Their results suggest that fairly-priced premium is not neutral to forbearance policy even in the absence of moral hazard.

Muller [30] derived a multiperiod DI pricing model that incorporates the explicit solution of his optimal control problem (discussed earlier). The DI premium in [30] is estimated by incorparating the optimal investment strategies for different bank assets in the sense that the expressions describing these strategies are embedded in the bank's asset portfolio formula. The author of [30] employed Monte Carlo simulation method to estimate the value of the DI premium for a coverage horizon of T years. Moreover, the author of [30] assumed that the bank does not pay any dividends to its shareholders through the life of the contract. Similar to the paper by Duan and Yu [11], the pricing model of [30] incorporates an asset value reset rule comparable to the typical practice of insolvency resolution by insuring agencies. Muller [30] assumed that the insuring agent adopts a purchase-and-assumption or government assisted merger as a means to conduct insolvency resolution, in which case the insuring agent provides a lump sum transfer to the acquirer of the insolvent bank at the time of bank audit. The lump sum amount is sufficient to cover the face value of the insured deposits plus accrued interest. The author of [30] found that, under the optimal investment strategy, for a fixed initial leverage level the DI premium increases when either the risk in the bank asset portfolio or the DI coverage horizon is increased. On the other hand, for the rising initial leverage levels it was found that the DI premium rises as the risk in the bank asset portfolio is raised. However, as the coverage horizon is increased, the DI premium drops.

Chapter 3

Preliminaries

In this chapter we introduce mathematical concepts from finance as well as probability and measure theory that will be used throughout this thesis. In particular, we will present relevant definitions, lemmas and theorems here. Our main references for these items are the books by Bhattacharya and Waymire [6], Etheridge [12], Baz and Chacko [5] and Hull [16].

3.1 Concepts from probability and measure theory

Definition 3.1. **(σ-algebra)** *Let Ω be any non-empty set. A σ-algebra or σ-field on Ω is a class $\mathcal{F}$ of subsets of Ω with the following three properties:*

1. $\Omega \in \mathcal{F}$;
2. *if* $\{A(t)\}$ *is a finite or infinite sequence of sets in* $\mathcal{F}$*, then* $\cup A(t) \in \mathcal{F}$;
3. *if* $A \in \mathcal{F}$ *then* $A^c \in \mathcal{F}$.

Definition 3.2. **(Filtration)** *A filtration is a family* $\{\mathcal{F}\}_{t\in J}$ *of σ-algebras* $\mathcal{F}(t) \subset \mathcal{F}$ *which is increasing in the sense that whenever* $s,t \in J$ *and* $s \leq t$*, then* $\mathcal{F}(s) \subset \mathcal{F}(t)$.

Definition 3.3. **(Probability triple)** *A probability triple* $(\Omega, \mathcal{F}, \mathbb{P})$*, consists of a set* Ω *(sample space), a collection of subsets* $\mathcal{F}$ *of* Ω *(events) and a probability measure* $\mathbb{P}$*, which specifies the probability of each event* $A \in \mathcal{F}$*. The collection* $\mathcal{F}$ *is assumed closed under the operations of countable union and taking complements (σ-field). The probability measure* $\mathbb{P}$ *must of course satisfy the following axioms:*

1. $0 \leq \mathbb{P}[A] \leq 1$ *for all* $A \in \mathcal{F}$*;*

2. $\mathbb{P}[\Omega] = 1$*;*

3. $\mathbb{P}[A \cup B] = \mathbb{P}[A] + \mathbb{P}[B]$ *for any disjoint* A *and* B *in* $\mathcal{F}$*;*

4. *if* $A(n) \in \mathcal{F}$ *for all* $n \in N$ *and* $A(1) \subseteq A(2) \subseteq \ldots$, *then* $\mathbb{P}[A(n)] \uparrow \mathbb{P}[\cup_n A(n)]$ *as* $n \uparrow \infty$.

Definition 3.4. (Stochastic process) *Given an indexed set* I*, a stochastic process indexed by* I *is a collection of random variables* $\{B_\lambda : \lambda \in I\}$ *on a probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *taking values in a set* S*. The set* S *is called the state space of the process.*

Definition 3.5. (Simple random walk) *A stochastic process* $\{S(n)\}_{n\geqslant 0}$ *is a simple random walk under the probability measure* $\mathbb{P}$ *if* $S(n) = \sum_{i=1}^{n} \xi(i)$ *where the* $\xi(i)$ *can only take the values* $\{+1, -1\}$ *and are independent and identically distributed under* $\mathbb{P}$.

Definition 3.6. (Brownian motion) *A real-valued stochastic process* $\{B(t)\}_{t\geq 0}$ *is a* $\mathbb{P}$*-Brownian motion (or a* $\mathbb{P}$*-Wiener process) if for some real constant* σ*, under* $\mathbb{P}$*,*

1. *for each* $s \geq 0$ *and* $t > 0$ *the random variable* $B(t+s) - B(s)$ *has the normal distribution with mean zero and variance* $\sigma^2 t$*;*

2. *for each* $n \geq 1$ *and any times* $0 \leq t(0) \leq t(1) \leq t(2) \ldots \leq t(n)$*, the random variables* $\{B(t(r)) - B(t(r-1))\}$ *are independent;*

3. $B(0) = 0$*;*

4. $B(t)$ *is continuous in a variable* t.

Theorem 3.7. *The Hamilton-Jacobi-Bellman (or HJB) equation of optimal control for Ito's process for the optimization problem*

$$J(0, X) = \max_{y} \mathbb{E}\left[\int_0^T f(t, X, y)dt + B(T, X(T)) \Big| F(0)\right]$$

subject to the constraints

$$dX = \mu(t, X, y)dt + \sigma(t, X, y)dW$$

and with $X(0)$ fixed, is of the form

$$-\frac{\partial J(t,X)}{\partial t} = \max_y \left[f(t,X,y) + \frac{\partial J(t,X)}{\partial X(t)}\mu(t,X,y) + \frac{1}{2}\frac{\partial^2 J(t,X)}{\partial X^2(t)}\sigma^2(t,X,y) \right].$$

The HJB equation is a PDE with boundary condition

$$J(T, X(T)) = B(T, X(T)). \qquad \big(\text{see Baz and Chacko [5]}\big).$$

The variable y is called the decision/ control variable, whereas the variable X is called the state variable.

Alternative notation for the HJB equation, which we will be using in Chapter 5, is:

$$-J_t = \max_y \left[f(t,X,y) + J_x\mu(t,X,y) + \frac{1}{2}J_{xx}\sigma^2(t,X,y) \right]$$

Example 3.8. We now present the optimal control problem addressed by Devolder *et al.* [9]. In particular, we show how the aforementioned authors derived the "best" investment policy for the assets backing a defined contribution pension plan's liabilities before the retirement of the participant. During this period, the participant's contributions can be invested in a riskless asset, X_1, or a risky asset, X_2; the reserve obtained at retirement age is the amount accumulated without any special guarantee given by the insurer. At retirement of the participant, this reserve is used to purchase a paid up annuity for the participant.

If $t \in [0, N]$, where N is the retirement date of the participant, then the problem above involves optimizing the utility of the final wealth at retirement. The state variable for the optimization problem is the assets of the pension plan, $F(t)$, where $t \in [0, N]$. The decision variable is chosen as the proportion of the contribution invested in the risky asset.

The financial market considered by Devolder *et al.* [9] is assumed to be described by the two assets given by

$$dX_1(t) = rX_1(t)dt \tag{3.2}$$

and

$$dX_2(t) = \alpha X_2(t)dt + \sigma X_2(t)dw(t), \tag{3.3}$$

where r, α and σ are positive constants and $w(t)$ is a standard Brownian motion. Suppose that the proportion of the contribution invested in the risky asset at time t is $u(t)$, then the proportion invested in the riskless asset is $1 - u(t)$. The optimization problem of Devolder *et al.* [9] is to find the optimal process for $u(t)$.

Devolder *et al.* [9] assumed that a lump sum is paid to the pension plan at time $t = 0$ and that there are no other future contributions. Taking into account that the riskless asset's evolution is described by Eq.(3.2), while that of the risky asset is described by Eq.(3.3), the process F is a solution of the stochastic differential equation (SDE).

$$dF(t) = F(t)\big[u(t)\alpha + (1 - u(t))r\big]dt + F(t)u(t)\sigma dw(t),$$

with

$$F(0) = P > 0.$$

Now the optimization problem of Devolder *et al.* [9] which recall, is to optimize the utility of the final wealth at retirement, can be written as

$$\max_u \mathbb{E}U\Big[F(N)\Big]$$

with

$$dF(t) = F(t)\big[u(t)\alpha + (1 - u(t)r)\big]dt + F(t)u(t)\sigma dw(t)$$

and

$$F(0) = P > 0.$$

Devolder *et al.* [9] introduced the value function of the problem as:

$$W(t, F) = \max_u \mathbb{E}\big[U(F(N)|F(t) = F)\big].$$

The HJB equation of the problem, according to Theorem 3.7, can be written as

$$-\frac{\partial W(t,F)}{\partial t} = \max_u \left[\big[u(t)(\alpha - r) + r\big]F\frac{\partial W(t,F)}{\partial F(t)} + \frac{1}{2}u^2(t)\sigma^2F^2(t)\frac{\partial^2 W(t,F)}{\partial F^2(t)}\right]$$

or

$$0 = \max_u \left[\frac{\partial W}{\partial t} + \big[u(t)(\alpha - r) + r\big] F \frac{\partial W}{\partial F} + \frac{1}{2} u^2(t) \sigma^2 F^2 \frac{\partial^2 W}{\partial F^2} \right]. \tag{3.4}$$

The latter equation can be written as

$$0 = \max_u \{H\}.$$

This leads to two equations and a second-order condition given by

$$H(u^*) \quad = \quad 0; \tag{3.5}$$

$$\frac{\partial H}{\partial u}(u^*) \quad = \quad 0; \tag{3.6}$$

$$\frac{\partial^2 H}{\partial u^2}(u^*) \quad < \quad 0.$$

Eq.(3.6) gives

$$0 = (\alpha - r) F \frac{\partial W}{\partial F} + u^*(t) F^2 \sigma^2 \frac{\partial^2 W}{\partial F^2},$$

from which a first explicit form for the optimal investment proportion u^* in the risky asset emerges:

$$u^*(t) \quad = \quad -\frac{\frac{\partial W}{\partial F}}{F\big(\frac{\partial^2 W}{\partial F^2}\big)} \frac{\alpha - r}{\sigma^2} \tag{3.7}$$

Substituting this into Eq.(3.5) yields a PDE for the value function

$$\frac{\partial W}{\partial t} + rF \frac{\partial W}{\partial F} - \frac{1}{2} \frac{(\alpha - r)^2}{\sigma^2} \frac{\big(\frac{\partial W}{\partial F}\big)^2}{\frac{\partial^2 W}{\partial F^2}} = 0 \tag{3.8}$$

with limit condition

$$W(N, F) = U(F).$$

The problem is now to solve Eq.(3.8) for the value function W and replacing it in Eq.(3.7) to obtain the optimal policy.

By using the power law utility

$$U(F) = \frac{F^\gamma}{\gamma},$$

with $\gamma < 1$ and $\gamma \neq 0$, and the structure

$$W(t, F) = b(t)\frac{F^\gamma}{\gamma}$$

where $b(N) = 1$, Devolder *et al.* [9] derived the explicit optimal policy as

$$\begin{aligned} u^*(t) &= -\frac{b(t)}{Fb(t)(\gamma - 1)F^{\gamma-2}}\frac{\alpha - r}{\sigma^2} \\ &= \frac{\alpha - r}{\sigma^2}\frac{1}{1-\gamma}. \end{aligned} \tag{3.9}$$

This is a constant proportion depending on the risk premium $\alpha - r$, the volatility σ^2 and risk aversion γ.

The optimal amount of the participant's wealth to invest in the risky asset is thus

$$F(t)u^*(t) = F(t)\frac{\alpha - r}{\sigma^2}\frac{1}{1-\gamma}. \tag{3.10}$$

We now present a simulation of Eq.(3.10) based on the parameters

$N = 10,\ P = 1,\ \alpha = 0.12,\ r = 0.11,\ \sigma = 0.16$ and $\gamma = 0.9$.

t	0	1	2	3
$F(t)u^*(t)$	3.9062	16.9415	12.5199	6.3658
t	4	5	6	7
$F(t)u^*(t)$	7.1090	40.3987	54.3738	50.6230
t	8	9	10	
$F(t)u^*(t)$	99.7404	271.9812	609.8181	

Table 3.1: The optimal amount of the defined contribution pension plan member's wealth to invest in the risky asset

In Table 3.1 we present the optimal amounts of the participant's wealth $F(t)u^*(t)$ that should be invested in the risky asset at intervals of 1 year over a 10 year period. We observe an upward trend in the value of $F(t)u^*(t)$ over the 10 year period. We plot the evolution of $F(t)u^*(t)$ over a period of 10 years in Figure 3.1 below. We observe that the range of $F(t)u^*(t)$ is interval form [3.9062, 609.8181].

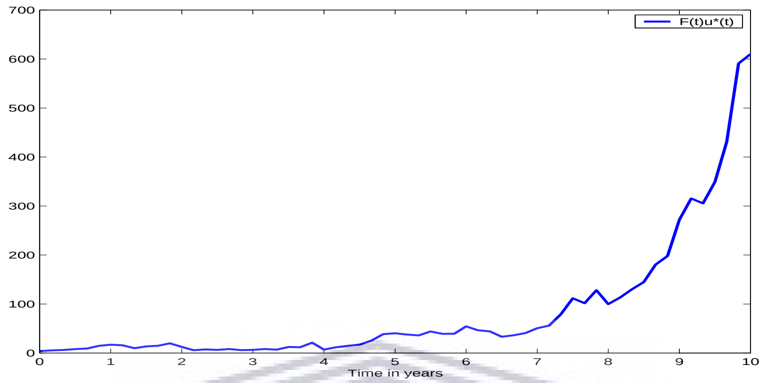

Figure 3.1: A simulation of the evolution of the optimal amount of wealth of the defined contribution pension plan participant over a 10 year period.

3.2 Stock price evolution over time

Lemma 3.9. (Itô's Lemma) *Suppose that $S(t)$ denotes the price of a stock at time t and that it follows the Itô process*

$$dS(t) = \mu S(t)dt + \sigma S(t)dB(t), \tag{3.11}$$

where μ and σ are finite positive constants denoting, respectively, the mean and standard deviation of the random variable S. The stochastic process $B(t)$ is a standard Brownian motion under the risk-neutral probability. In this case $S(t)$ is a geometric Brownian motion.

Itô's lemma states that for a suitable function $G(t, S(t))$ of t and $S(t)$, the differential $dG(t, S(t))$ can be expressed as

$$\begin{aligned} dG(t,S(t)) &= \left[\mu S(t)\frac{\partial G(t,S(t))}{\partial S(t)} + \frac{\partial G(t,S(t))}{\partial t} + \frac{\sigma^2 S^2(t)}{2}\frac{\partial^2 G(t,S(t))}{\partial S^2(t)}\right]dt \\ &+ \sigma S(t)\frac{\partial G(t,S(t))}{\partial S(t)}dB(t), \end{aligned}$$

where $B(t)$ is the standard Brownian motion. Therefore, if $G(t, S(t)) = \ln(S(t))$, then using Itô's formula, $G(t, S(t))$ follows the process

$$dG(t, S(t)) = \left(\mu S(t)\frac{1}{S(t)} - \frac{\sigma^2 S^2(t)}{2}\frac{1}{S^2(t)}\right)dt + \sigma S(t)\frac{1}{S(t)}dB(t),$$

or

$$dG(t, S(t)) = \left(\mu - \frac{\sigma^2}{2}\right)dt + \sigma dB(t).$$

3.3 Concepts from finance

Definition 3.10. (Option) *An option is a contract which gives the holder the right, but not the obligation, to buy (call option) or sell (put option) the underlying asset by a certain date for a certain price.*

Definition 3.11. (European option) *A European option is an option which gives the buyer or seller the right to exercise the option only at the maturity date.*

Definition 3.12. (Payoff of a European option) *The payoffs (both long and short position) of European options with strike price K, expiration date T and final price of the underlying asset $S(T)$, are as follow:*

1. *The payoff from a long position in a European call option is* $\max(S(T) - K, 0)$;
2. *The payoff from a short position in the European call option is* $-\max(S(T) - K, 0) = \min(K - S(T), 0)$;
3. *The payoff from a long position in a European put option is* $\max(K - S(T), 0)$;
4. *The payoff from a short position in a European put option is* $-\max(K - S(T), 0) = \min(S(T) - K, 0)$.

We now consider a European put option written on a risky asset whose price dynamic evolves according to the geometric Brownian motion process

$$\frac{d\hat{S}(t)}{\hat{S}(t)} = \hat{r}dt + \hat{\sigma}d\hat{B}(t), \quad t \geq 0.$$

Here $\hat{r}$ is the risk-free rate of interest per annum and $\hat{\sigma}$ the volatility of the asset price. If $\hat{K}$ denotes the exercise price of the option at the expiration date T, then using the Black-Scholes formula the price $\tilde{P}$ of the option can be expressed as

$$\tilde{P} = \hat{K}e^{-\hat{r}T}N(-d_2) - \hat{S}(0)N(-d_1).$$

In the above formular,

$$d_1 = \frac{ln\big(\frac{\hat{S}(0)}{\hat{K}}\big) + \big(\hat{r} + \frac{\hat{\sigma}^2}{2}\big)T}{\hat{\sigma}\sqrt{T}}$$

and $d_2 = d_1 - \hat{\sigma}\sqrt{T}$. The function $N(\cdot)$ is the cumulative probability distribution function for a standardized normal distrbution.

In the example below we employ the Black-Scholes formula to compute the price of a European put option written on some risky asset.

Example 3.13. The price of a European put option with expiration date $T = 1$ years and strike price $\hat{K}$=85, written on a risky asset with price $\hat{S}$ and $\hat{r} = 0.05$, $\hat{\sigma} = 0.08$ and $\hat{S}(0) = 100$, is:

$$\begin{aligned} \tilde{P} &= 85e^{-0.05*1}N\Bigg(-\frac{ln(\frac{100}{85}) + \big(0.05 + \frac{(0.08)^2}{2} * 1\big)}{0.08 * \sqrt{1}} - 0.08 * \sqrt{1}\Bigg) \\ &- 100N\Bigg(-\frac{ln(\frac{100}{85}) + \big(0.05 + \frac{(0.08)^2}{2} * 1\big)}{0.08 * \sqrt{1}}\Bigg) \\ &= 0.0088 \end{aligned}$$

Chapter 4

The general commercial banking model

In this chapter we present some theory on the general commercial banking model. More specifically, we explain the stylized balance sheet variables of commercial banks, which are needed to formulate the banking problems studied in this thesis.

We consider a commercial bank that is assumed to trade in a financial market that is complete and frictionless and continuously open over a fixed time interval $[0, T]$. We assume throughout this thesis that we are working with a probability space $(\Omega, \mathcal{F}, \{\mathcal{F}(t)\}_{t\geq 0}, \mathbb{P})$, where $\mathbb{P}$ is the real world probability measure. The Brownian motions W, W_D and W_B appearing in the dynamics of the balance sheet variables to be introduced later in this chapter are assumed to be defined on the probability space $(\Omega, \mathcal{F}, \{\mathcal{F}(t)\}_{t\geq 0}, \mathbb{P})$. The filtration $\{\mathcal{F}(t)\}_{t\geq 0}$ is generated by the Brownian motions and satisfies the usual conditions (see Definition 3.6 of the pleliminary chapter).

4.1 Stylized balance sheet

We now discuss the stylized balance sheet of commercial banks, which records the assets and liabilities of banks. Assets are items banks own, while liabilities are the banks' debts. Commercial banks use liabilities to finance their assets. According to references [27, 38, 31, 30], the balance sheet of a commercial bank, at time $t \geq 0$, can be described by the equation

$$R(t) + S(t) + L(t) = D(t) + B(t) + C(t). \tag{4.1}$$

In the above equation, the variables R , S , L , D , B and C are regarded as stochastic processes representing the values of reserves, securities, loans, deposits, borrowings and capital, respectively. The sum of the reserves, securities and loans of the bank make up the total assets of the commercial bank, while the total liabilities is the sum of the deposits and borrowings of the bank.

Bank reserves refer to the sum of currency that commercial banks hold in the form of deposits in accounts with the central bank, as well as currency that it physically holds in its vault (vault cash). The vault cash is used to meet the day-to-day currency withdrawals by the banks' customers [31, 30]. The minimum reserve requirements of commercial banks are set by the central bank. However, only a small portion of the total deposits is needed as reseverves since it is uncommon for depositors to withdraw all their funds at the same time [28].

The *primary securities* banks own are treasury securities (treasuries) and marketable securities. *Treasuries* are bonds issued by national treasuries in most countries as a means of borrowing money to meet government expenditures not covered by tax revenues [31]. There are four types of treasuries, namely treasury bills, treasury notes, treasury bonds and savings bonds. All of the treasury securities besides savings bonds are very liquid [28]. On the other hand, *marketable securities* are stocks and bonds that can be sold quickly and easily in the secondary market when a bank is in need of extra cash [30]. They are often referred to as secondary reserves with a readily determined fair market value.

Loans granted by a commercial bank include business loans, mortgage loans (land loans), consumer loans and interbank loans [31]. A *business loan* is a loan specifically intended for business purposes. *Mortgages* are long term loans used to buy property. A *consumer loan* is a loan to individuals for personal or household purposes. An *interbank loan* is a loan between banking institutions, with terms ranging from overnight to one week. There are two basic categories that most loans types fall into. These are secured and unsecured loans. A *secured* loan is a loan in which a lender accepts some asset as collateral for the loan. Secured loans are safer for the lender and more affordable for the borrower, as the lower risk allows for lower interest rates. *Unsecured* loans are loans that do not have collateral associated with them. Unsecured loans carry more risk for the lender, and to compensate for the increased risk, lenders charge higher interest to these types of loans. The bank sets the fixed period over which the loan is provided, as well as the rate of interest and the amount of repayment. Bank loans earn more interest than banks have to pay on deposits. Thus, bank loans are a major source of revenue for a bank.

Bank deposits refer to money that the bank's customers place in the banking institution for safekeeping [31, 30]. Deposits can be classified as demand deposits or time deposits. *A demand deposit* is a deposit account that gives the depositor the right to withdraw their funds from the account without prior notice. An example of a demand account offered by the bank is a checking account. *A time deposit* is an interest-bearing bank deposit account that has a specified date of maturity. A time deposit can only be withdrawn prior to its maturity with advanced notice and/or by paying a penalty. A time deposit refers to a savings account or certificate of deposit offered by a bank.

The term *borrowings* refers to the funds that commercial banks borrow from other banks (via interbank market) and/or the central bank [31, 30]. Banks are required to hold an adequate amount of liquid assets in order to be able to cover any unexpected and large withdrawal request. Commercial banks borrow from the central bank or interbank market in order to meet these reserve requirements when their cash at hand is low before the close of business. Some banks on the other hand, have excess liquid assets above the liquidity requirements. These banks will lend money in the interbank market, receiving interest on the assets.

Bank capital represents the net worth of the bank and is defined as the value of the bank's assets minus the value of its liabilities. The more capital the bank has the better it can absorb losses on its assets before it becomes insolvent. Bank capital is raised by selling new equity, retaining earnings and by issuing debt or building up loan-loss reserves. It is usually the bank's risk management department's responsibility to calculate its capital requirements, which is then approved by the bank's top executive management. The dynamics of bank capital is stochastic in nature as it depends in part on the uncertainty related to debt and shareholder contributions.

In theory, the bank can decide on the rate at which debt and equity is raised [31, 30].

The regulatory bank capital is divided into different tiers based on subordination and the ability to absorb losses, with sharp distinction of capital instrument when a bank is still solvent versus after it goes bankrupt [39, 4]. Under Basel III, bank capital, C, takes the form

$$C(t) = C_{\mathrm{T1}}(t) + C_{\mathrm{T2}}(t),$$

where $C_{\mathrm{T1}}(t)$ and $C_{\mathrm{T2}}(t)$ are Tier 1 and Tier 2 capital respectively [4, 32, 31, 30].

Tier 1 capital consists of shareholders' equity and retained earnings. Tier 1 capital is the core measure of the banks' financial strength from the regulator's perspective. It is always available and acts as a buffer against losses without ceasing business operations. The amount of Tier 1 capital affects returns for shareholders, while a minimum amount of such is required by regulatory authorities. *Tier 2* capital includes revaluation reserves, undisclosed reserves, hybrid securities, subordinated debt and general loan-loss reserves. Tier 2 capital is supplementary capital. From the regulator's perspective, it measures the banks' financial strength with regard to the second most reliable form of financial capital [4, 32, 31, 30, 42].

4.2 Modelling the underlying bank

We now introduce formulae for the assets in which the bank modelled in this thesis invests. We consider a constant interest rate $r > 0$. We assume that the riskless and risky asset dynamics, respectively, follow the differential equations

$$\begin{aligned} \frac{dR_1(t)}{R_1(t)} &= rdt, \\ R_1(0) &> 0 \end{aligned} \tag{4.2}$$

and

$$\begin{aligned} \frac{dR_2(t)}{R_2(t)} &= \mu dt + \sigma dW(t), \\ R_2(0) &> 0. \end{aligned} \tag{4.3}$$

In Eq.(4.3), $\mu = r + m_1$, where m_1 is a positive constant.

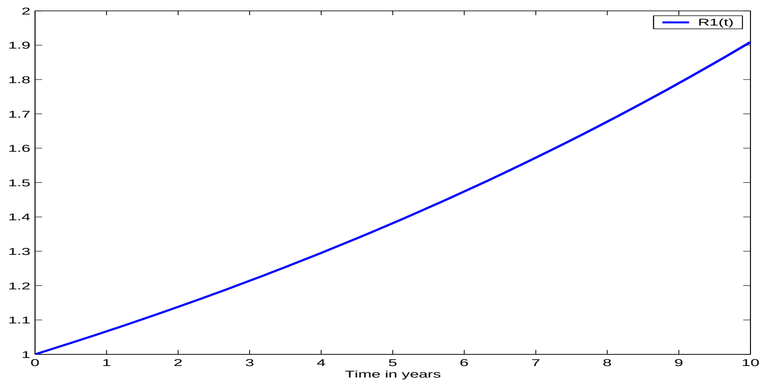

Figure 4.1: A simulation of the evolution of the price of the riskless asset, $R_1(t)$, with $r = 0.065$, $R_1(0) = 1$ and $T = 10$ years.

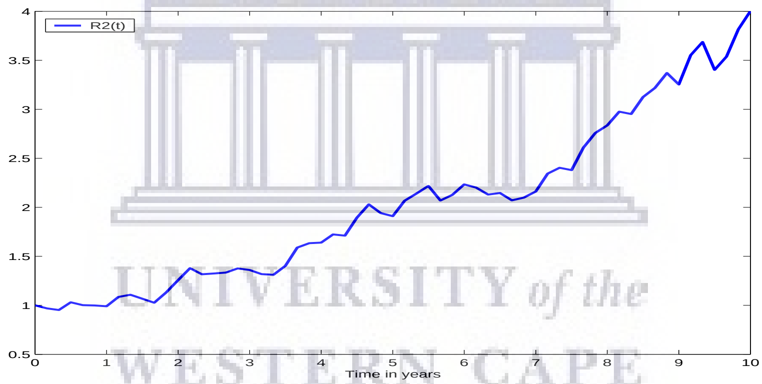

Figure 4.2: A simulation of the evolution of the price of the risky asset, $R_2(t)$, with $\mu = 0.1$, $\sigma = 0.12$, $R_2(0) = 1$ and $T = 10$ years.

In Figures 4.1 and 4.2 we simulate the evolution of the riskless and risky asset dynamics over a period of 10 years. The simulations are based on the parameters $r = 0.065$, $m_1 = 0.035$ and $\sigma = 0.12$. We consider initial conditions $R_1(0) = R_2(0) = 1$. The prices of the assets both exhibit upward behaviour.

The amount of capital invested in the risky asset and the riskless asset at time t are denoted by $\theta(t)$ and $[A(t) - \theta(t)]$, respectively. Thus, the asset portfolio consiting of the risky asset and riskless asset is given by (see [15, 31, 30] for instance)

$$dA(t) = \Big[A(t) - \theta(t)\Big]\frac{dR_1(t)}{R_1(t)} + \theta(t)\frac{dR_2(t)}{R_2(t)} + dK(t),$$

which takes the form

$$\begin{aligned} dA(t) &= \Big[A(t) - \theta(t)\Big] rdt + \theta(t)\Big[\mu dt + \sigma dW(t)\Big] + dK(t) \\ &= A(t)rdt - \theta(t)rdt + \theta(t)\mu dt + \theta(t)\sigma dW(t) + dK(t) \\ &= \Big[A(t)r - \theta(t)r + \theta(t)\mu\Big] dt + \theta(t)\sigma dW(t) + dK(t) \\ &= \Bigg\{\Big[A(t) - \theta(t)\Big] r + \theta(t)\mu\Bigg\} dt + \theta(t)\sigma dW(t) + dK(t) \end{aligned} \quad (4.4)$$

when Eq.(4.2) and Eq.(4.3) are imported into the expression for $dA(t)$.

In Eq.(4.4), $dK(t)$ represents the rate at which shareholders raise capital that is invested in the bank's assets. If $dK(t)$=Mdt, for M a positive constant, then

$$dA(t) = \Bigg\{\Big[A(t) - \theta(t)\Big] r + \theta(t)\mu\Bigg\} dt + \theta(t)\sigma dW(t) + Mdt$$

or

$$dA(t) = \Bigg\{\Big[A(t) - \theta(t)\Big] r + \theta(t)\mu + M\Bigg\} dt + \theta(t)\sigma dW(t). \quad (4.5)$$

We assume that the total liabilities, $L(t)$, of the bank is given by the equation

$$L(t) = B(t) + D(t),$$

where, recall, B and D denote the borrowings and deposits of the bank. Hence, the SDE governing $L(t)$ is

$$dL(t) = dB(t) + dD(t). \quad (4.6)$$

We take the assumption that $B(t)$ and $D(t)$, respectively, evolve according to SDEs

$$dB(t) = \mu_B dt + \sigma_B dW_B(t) \quad (4.7)$$

and

$$dD(t) = \mu_D dt + \sigma_D dW_D(t). \tag{4.8}$$

Here the coefficients μ_B, σ_B, μ_D and σ_D are positive constants.

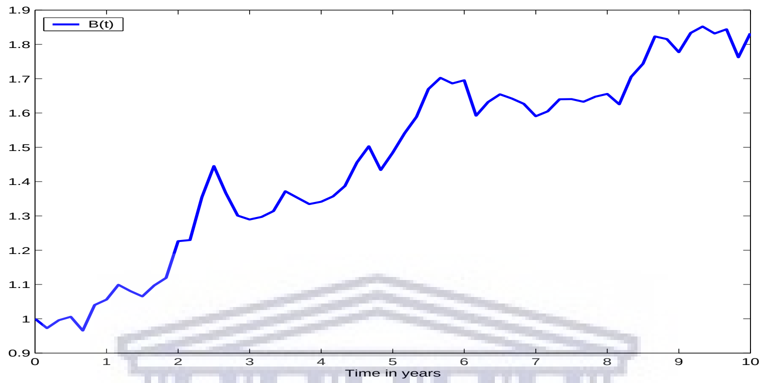

Figure 4.3: A simulation of the evolution of the bank's borrowings, $B(t)$, with $\mu_B = 0.1$, $\sigma_B = 0.12$, $B(0) = 1$ and $T = 10$ years.

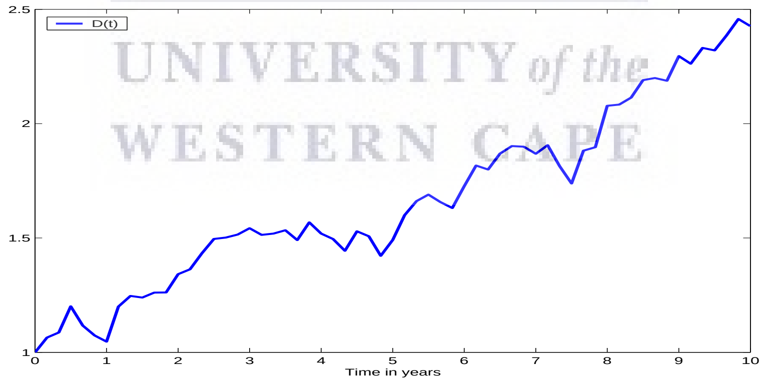

Figure 4.4: A simulation of the evolution of the bank's deposits, $D(t)$, with $\mu_D = 0.12$, $\sigma_D = 0.15$, $D(0) = 1$ and $T = 10$ years.

In the Figures 4.3 and 4.4 we simulate, respectively, the evolution of the borrowings and deposits over a 10-year period. We consider the parameters $\mu_B = 0.1$, $\sigma_B = 0.12$, $\mu_D = 0.12$, $\sigma_D = 0.15$ and the initial conditions $B(0) = D(0) = 1$ in the simulations. We note that both liabilities exhibit upward behaviour.

The bank's capital can then be defined as the bank's assets minus the bank's liabilities, i.e.,

$$C(t) = A(t) - L(t),$$

for which we can write

$$\begin{aligned} dC(t) &= dA(t) - dL(t) \\ &= dA(t) - [dB(t) + dD(t)]. \end{aligned}$$

Thus, the SDE governing $C(t)$ is therefore

$$dC(t) = dA(t) - dB(t) - dD(t). \tag{4.9}$$

Subsituting the right hand sides of Eqs.(4.5), (4.7) and (4.8) for the expressions $dA(t)$, $dB(t)$ and $dD(t)$ in Eq.(4.9) yields:

$$\begin{aligned} dC(t) &= \left\{ \Big[A(t) - \theta(t)\Big] r + \theta(t)\mu + M \right\} dt + \theta(t)\sigma dW(t) - \mu_B dt - \sigma_B dW_B(t) - \mu_D dt - \sigma_D dW_D(t) \\ &= \left\{ \Big[A(t) - \theta(t)\Big] r + \theta(t)\mu + M - \mu_B - \mu_D \right\} dt \\ &+ \theta(t)\sigma dW(t) - \sigma_B dW_B(t) - \sigma_D dW_D(t). \end{aligned} \tag{4.10}$$

We are now ready to formulate the optimal control for the commercial bank that we modelled above. This is done in the next chapter.

Chapter 5

The optimal control problem

We now present the optimal control problem and derive its solution. In particular, we determine the investment strategy that maximizes an expected utility of the bank's capital at time $T > 0$.

The problem we wish to solve is as follows.

Problem: The objective is to maximize the expected utility of the bank's capital at time $T > 0$, i.e.,

$$\max_{\theta} \mathbb{E}\Big[U\Big(C(T)\Big)\Big],$$

with the dynamics of $C(t)$ described by the SDE

$$\begin{aligned} dC(t) &= \Big\{\Big[A(t) - \theta(t)\Big]r + \theta(t)\mu + M - \mu_B - \mu_D\Big\}dt + \theta(t)\sigma dW(t) - \sigma_B dW_B(t) - \sigma_D dW_D(t), \\ C(0) &> 0. \end{aligned}$$

We define the value function of our problem as

$$H(t, C) = \sup_{\theta} \mathbb{E}\Big[U\Big(C(T)\Big)|C(t) = C\Big]$$

for $0 < t < T$. The value function can be considered as a kind of utility function. While the marginal utility of the value function is a constant, the marginal utility of the original utility function $U(\cdot)$ decreases to zero as $C \to \infty$ according to Kramkov and Schachermayer [20].

According to Jonsson and Sircar [19], the value function inherits the convexity of the utility function and is strictly convex for $t < T$ even if $U(\cdot)$ is not.

The maximum principle leads to the Hamilton-Jacobi-Belman equation:

$$\begin{aligned} H_t &+ \max_{\theta}\left\{\Big[(A-\theta)r+\theta\mu+M-\mu_D-\mu_B\Big]H_c\right. \\ &+ \left.\frac{1}{2}\Big[(\theta\sigma)^2+\sigma_D^2+\sigma_B^2\Big]H_{cc}\right\}=0, \end{aligned} \tag{5.1}$$

where the variable t has been suppressed.

Differentiation of Eq.(5.1) with respect to θ yields:

$$(-r+\mu)H_c+\frac{1}{2}(2\theta)\sigma^2 H_{cc} = 0$$

or

$$(-r+\mu)H_c+\theta\sigma^2 H_{cc} = 0,$$

from which we obtain the first-order maximizing condition for the optimal inverstment strategy in the risky asset as

$$\theta = \frac{(r-\mu)H_c}{\sigma^2 H_{cc}}. \tag{5.2}$$

We substitute the RHS of Eq.(5.2) into Eq.(5.1) and get

$$\begin{aligned} H_t &+ \left\{\Big[A-\frac{(r-\mu)H_c}{\sigma^2 H_{cc}}\Big]r+\frac{(r-\mu)H_c}{\sigma^2 H_{cc}}\mu+M-\mu_D-\mu_B\right\}H_c \\ &+ \frac{1}{2}\left\{\Big[\frac{(r-\mu)H_c}{\sigma^2 H_{cc}}\sigma\Big]^2+\sigma_D^2+\sigma_B^2\right\}H_{cc} \\ &= 0. \end{aligned} \tag{5.3}$$

The problem is now to solve the PDE (5.3) for the value function H and placing it in Eq.(5.2) to obtain the optimal investment strategy θ. The PDE (5.3) admits an explicit solution for the utility function of the form

$$U(C) = -\frac{1}{g}e^{-gC},$$

where $g > 0$ is a positive constant

for which

$$-\frac{U''(C)}{U'(C)} = g.$$

(See Devolder *et al.* [9] and Muller [30]). We try to find an explicit solution for the PDE (5.3) with the structure

$$H(t, C) = -\frac{1}{g} e^{-gC+b(t)} \tag{5.4}$$

of Muller [30], for which it was assumed that $b(T) = 1$.

We calculate the partial derivatives appearing in Eq.(5.2) from Eq.(5.4) as:

$$H_t = -\frac{b'(t)}{g} e^{-gC+b(t)}$$

$$H_C = e^{-gC+b(t)}$$

$$H_{CC} = -ge^{-gC+b(t)}$$

Substitution of these derivatives into Eq.(5.3) yields

$$\begin{aligned} & - \frac{b'(t)}{g} e^{-gC+b(t)} + \left\{ \left[A - \frac{(r-\mu)e^{-gC+b(t)}}{\sigma^2(-ge^{-gC+b(t)})} \right] r + \frac{(r-\mu)e^{-gC+b(t)}}{\sigma^2(-ge^{-gC+b(t)})} \mu + M - \mu_D - \mu_B \right\} e^{-gC+b(t)} \\ & + \frac{1}{2} \left\{ \left[\frac{(r-\mu)e^{-gC+b(t)}}{\sigma^2(-ge^{-gC+b(t)})} \sigma \right]^2 + \sigma_D^2 + \sigma_B^2 \right\} (-ge^{-gC+b(t)}) \\ & = 0, \end{aligned} \tag{5.5}$$

which simplifies to

$$\begin{aligned} & - \frac{b'(t)}{g} e^{-gC+b(t)} + \left\{ \left[A + \frac{(r-\mu)}{\sigma^2 g} \right] r - \frac{(r-\mu)}{\sigma^2 g} \mu + M - \mu_D - \mu_B \right\} e^{-gC+b(t)} \\ & - \frac{1}{2} \left\{ \left[-\frac{(r-\mu)}{\sigma g} \right]^2 + \sigma_D^2 + \sigma_B^2 \right\} g e^{-gC+b(t)} \\ & = 0. \end{aligned} \tag{5.6}$$

If, at the same time, we multiply Eq.(5.6) by $-g$ and divide it by $e^{-gC+b(t)}$, we get

$$\begin{aligned} b'(t) &- g\left\{\left[A+\frac{(r-\mu)}{\sigma^2 g}\right]r-\frac{(r-\mu)}{\sigma^2 g}+M-\mu_D-\mu_B\right\} \\ &+ \frac{1}{2}g^2\left\{\left[\frac{(\mu-r)}{\sigma g}\right]^2+\sigma_D^2+\sigma_B^2\right\}=0. \end{aligned} \tag{5.7}$$

Now if we let

$$\lambda = g\left\{\left[A+\frac{(r-\mu)}{\sigma^2 g}\right]r-\frac{(r-\mu)}{\sigma^2 g}+M-\mu_D-\mu_B\right\}-\frac{1}{2}g^2\left\{\left[\frac{(\mu-r)}{\sigma g}\right]^2+\sigma_D^2+\sigma_B^2\right\},$$

then Eq.(5.7) can be written as the ordinary differential equation (ODE)

$$b'(t)-\lambda = 0. \tag{5.8}$$

We proceed to solve Eq.(5.8) for b as follows:

$$\begin{aligned} b'(t)-\lambda=0 \Rightarrow b'(t) &= \lambda \\ \Rightarrow b(t) &= \int \lambda dt \\ &= \lambda t+\kappa \end{aligned}$$

By imposing the condition that $b(T)=1$, we find that $\kappa = 1-\lambda T$,

hence

$$\begin{aligned} b(t) &= \lambda t+1-\lambda T \\ &= \lambda(t-T)+1. \end{aligned}$$

Our value function thus becomes

$$H(t,C) = -\frac{1}{g}e^{-gC+\lambda(t-T)+1}.$$

We note that the second-order condition is also satisfied, as

$$\begin{aligned} \sigma^2 H_{CC} &= \sigma^2(-ge^{-gC+\lambda(t-T)+1}) \\ &= -g\sigma^2 e^{-gC+\lambda(t-T)+1} \\ &< 0. \end{aligned}$$

From Eq.(5.2), we find that:

$$\begin{aligned}\theta = \frac{(r-\mu)H_c}{\sigma^2 H_{cc}} &= \frac{(r-\mu)e^{-gC+\lambda(t-T)+1}}{\sigma^2(-g)e^{-gC+\lambda(t-T)+1}} \\ &= \frac{(r-\mu)}{\sigma^2(-g)} \\ &= \frac{(\mu-r)}{\sigma^2 g}\end{aligned}$$

The amount of capital to invest in the riskless asset is thus

$$\begin{aligned}A-\theta &= A-\frac{(\mu-r)}{\sigma^2 g} \\ &= A+\frac{(r-\mu)}{\sigma^2 g}.\end{aligned}$$

The proportions of capital to invest, respectively, in the risky and riskless assets are thus given by

$$Z_1 = \frac{\theta}{A} = \frac{(\mu-r)}{\sigma^2 g A} \tag{5.9}$$

and

$$Z_2 = \frac{A-\theta}{A} = 1-\frac{\theta}{A} = 1-\frac{(\mu-r)}{\sigma^2 g A} = 1-Z_1. \tag{5.10}$$

We now present a simulation study to characterize the behaviour of the optimal proportions of the capital invested in the risky and riskless assets, as well as the behaviour of the optimized bank capital.

We consider an investment horizon of $T = 10$ years and assume that $M = 0.12$. The simulations are based on the following parameter values and intial conditions:

$r = 0.065$, $m_1 = 0.035$, $\sigma = 0.12$, $\mu_B = 0.10$, $\sigma_B = 0.12$, $\mu_D = 0.12$, $\sigma_D = 0.15$, $g = 2.5$, $A(0) = 1$, $D(0) = 0.5$, $B(0) = 0.2$ and $C(0) = 0.3$.

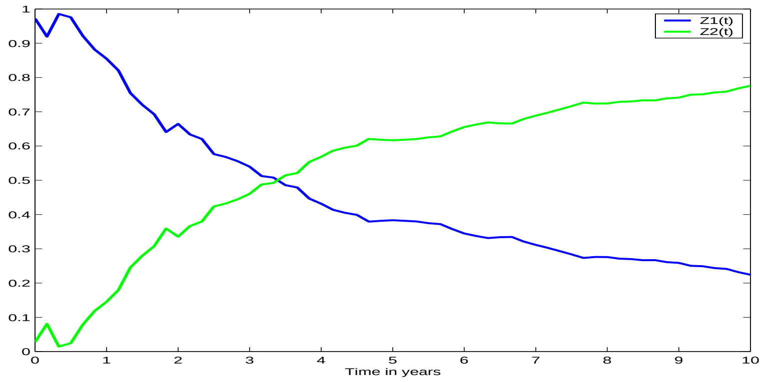

Figure 5.1: A simulation of the optimal proportions $Z_1(t)$ and $Z_2(t)$ of the capital invested, repectively, in the risky and riskless assets.

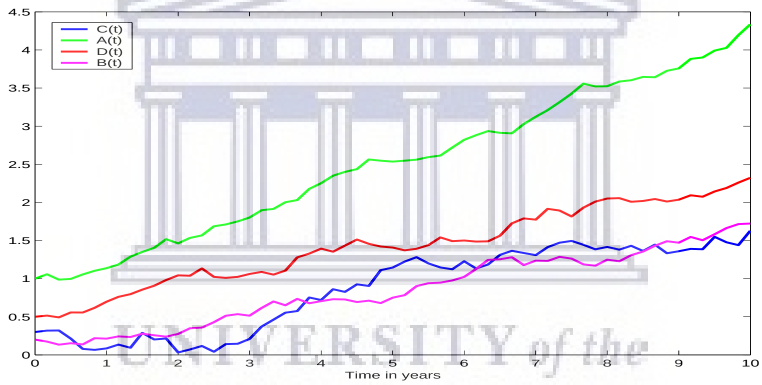

Figure 5.2: A simulation of the optimized bank capital $C(t)$ together with the bank asssets $A(t)$, bank deposits $D(t)$ and bank borrowings $B(t)$.

In Figure 5.1 we simulate the optimal proportions of capital invested in the risky and riskless assets. According to the investment strategy depicted in Figure 5.1, the bank has to initially invest most of its capital in the risky asset. Over time, the amount invested in the risky asset should be reduced while the amount invested in the riskless asset should be increased. In other words, the optimal investment strategy is to diversify the bank's portfolio away from the risky asset and towards the riskless asset. This finding is in accordance with the papers Witbooi *et al.* [38], Muller and Witbooi [31] and Muller [30]. The authors of the aforemantioned papers presented simulations of the optimal proportions of capital invested in the marketable security, loans and treasury. In the paper by Witbooi *et al.* [38], the authors found that the proportion invested in the marketable security remains constant throughout the time horizon of the optimization problem. On the other hand, the optimal proportion invested in the loans progressively decreases with respect to time, while the optimal proportion invested in the treasuries increases. The authors of the papers [31] and [30] both found that, over time, the proportions of capital invested in the risky assets (marketable security and loan) decrease while the proportion invested in the treasury increases.

In Figure 5.2 we simulate the optimized capital of the bank along with the assets, deposits and borrowings. We observe that the optimized bank capital, bank's asset portfolio, the deposits and borrowings exhibit upward trends. A similar observation was made by Muller [30]. The optimal asset portfolio modelled by Muller and Witbooi [31] also exhibits an upward trend. In the paper by Witbooi *et al.* [38], the authors further observed that the Basel II CAR of the commercial bank modelled in their paper, subject to the optimal equity allocation strategy, resembles a mean-reverting process.

We will now derive the DI pricing model that incorporates the solution of the optimal control problem studied in this chapter.

Chapter 6

The multiperiod deposit insurance pricing model

In this chapter we derive the multiperiod DI pricing model for the underlying commercial bank of our study. In deriving the DI pricing model, we follow an options based pricing approach similar to the methodologies of Merton [24], Ronn and Verma [34], Duan and Yu [11] and Muller [30]. The aforementioned authors modelled DI as some form of put option. We assume that the bank does not pay any dividends to its shareholders on the interval $[0, T]$. Furthermore, the bank is assumed to be audited at the times $t(i)$, $i = 1, 2, 3, \ldots, n-1$, n, where $t(i)$ are positive integers such that $0 = t(1) < t(2) < t(3) \cdots < t(n-1) < t(n) = T$.

We let the total insured deposits of the bank at time t be denoted by $\hat{D}(\mathrm{t})$. Following Muller [30], we assume that the total insured deposits are of the form

$$D\hat{(t)} = \rho D(t), \tag{6.1}$$

where $0 \leq \rho \leq 1$, so that the SDE governing $\hat{D}(\mathrm{t})$ is

$$dD\hat{(t)} = \rho dD(t). \tag{6.2}$$

Replacing the $dD(t)$ in Eq.(6.2) by the RHS of Eq.(4.8) we obtain the expression

$$dD\hat{(t)} = \rho[\mu_D dt + \sigma_D dW_D]. \tag{6.3}$$

We further assume, as in the papers [11] and [30], that the bank's total asset value is subject to reset at the time of the audit. More specifically, the total asset value of the bank at time $t(i)$ is determined by the following rule:

- Should $A(t(i)) < e^{rt(i)}\hat{D}(t(i))$, then the bank's total asset value will be reset to the value $e^{rt(i)}\hat{D}(0)$, which is the face value of the total insured deposits plus accrued interest.
- If instead, the bank is found to be solvent, then the bank total asset value will follow the SDE in Eq.(4.5).

The value of the DI at time $t(i)$, for i=1, 2, 3,..., $n-1$, n, can be described as a put option on the assets of the bank $A(t(i))$, with a strike price equal to $e^{rt(i)}\hat{D}(t(i))$. Facing the insuring agent is a stream of put option-like liabilities, each giving rise to payment denoted by $K(t(i))$. Here

$$K(t(i)) = \begin{cases} e^{rt(i)}\hat{D}(t(i)) - A(t(i)), & \text{if } A(t(i)) < e^{rt(i)}\hat{D}(t(i)) \\ 0, & \text{if otherwise.} \end{cases} \tag{6.4}$$

The cash payment, $K(t(i))$, at time $t(i)$ can be thus generalized to

$$K(t(i)) = \left[e^{rt(i)}\hat{D}(t(i)) - A(t(i))\right]^{+}, \tag{6.5}$$

which implies that

$$K(t(i)) = \max\left[0, e^{rt(i)}\hat{D}(t(i)) - A(t(i))\right]. \tag{6.6}$$

The SDE (4.5) describing our bank asset portfolio does not follow a geometric Brownian motion. Hence, we cannot apply the Black-Scholes [7] model to price the option-like liabilities faced by the insuring agent. Instead, we employ a Monte Carlo simulation method for estimating the price of these liabilities.

In our DI pricing problem, as was done in [11] and [30], we take the assumption that the fairly-priced premium for the bank can be determined by the formula

$$\hat{P} = \frac{1}{n\hat{D}(0)} \sum_{j=1}^{n} e^{-rt(i)}\mathbb{E}[K(t(i))], \tag{6.7}$$

which takes the form

$$\hat{P} = \frac{1}{n\hat{D}(0)} \sum_{j=1}^{n} e^{-rt(i)} \mathbb{E}\Big[e^{rt(i)}\hat{D}(t(i)) - A(t(i))\Big]^{+}, \tag{6.8}$$

if we replace $K(t(i))$ in Eq.(6.7) by the RHS of Eq.(6.5).

In Algorithm 1 below, we present the Monte Carlo simulation algorithm of Muller [30] that we apply to compute the fairly-priced premium for our bank model.

An algorithm for the Monte Carlo simulation method used to estimate $\hat{P}$:

While generating 10^5 sets, each consisting of a pair of sample paths for $\hat{D}$ and A on the interval $[0, T]$,

DO

At each $t(i)$, where $i = 1, 2, 3, \ldots, n-1, n$ and $t(1) < t(2) < t(3), \ldots, t(n-1) < t(n) = T$ are positive integers:

Compute the payoff $[e^{rt(i)}\hat{D}(t(i)) - A(t(i))]^{+}$ for each set consisting of the sample paths of $\hat{D}$ and A.

Using all the sets of sample paths of $\hat{D}$ and A, compute the average of the payoffs $[e^{rt(i)}\hat{D}(t(i)) - A(t(i))]^{+}$ as a proxy to $\mathbb{E}[e^{rt(i)}\hat{D}(t(i)) - A(t(i))]^{+}$.

Discount the proxy to time zero by multiplying it by $e^{-rt(i)}$.

END

Sum the values of all the discounted proxies computed at times $t(i)$, where $i = 1, 2, 3, \ldots, n-1, n$.

Divide the sum of the discounted proxies by $n\hat{D}(0)$.

Algorithm 1: Muller's [30] alogrithm for the Monte Carlo simulation used to estimate $\hat{P}$.

Table 6.1: Estimations for the fairly-priced deposit insurance premium

$\mathbf{D(0)/A(0) = 0.80}$	$\sigma = \mathbf{0.08}$	$\sigma = \mathbf{0.10}$	$\sigma = \mathbf{0.12}$	$\sigma = \mathbf{0.14}$	$\sigma = \mathbf{0.16}$
$T = 2$	0.0055	0.0173	0.0319	0.0461	0.0582
$T = 4$	0.0099	0.0418	0.0893	0.1357	0.1753
$T = 6$	0.0162	0.0835	0.1901	0.2940	0.3758
$T = 8$	0.0264	0.1609	0.3736	0.5662	0.7054
$T = 10$	0.0442	0.3069	0.6952	1.0041	1.2078
$\mathbf{D(0)/A(0) = 0.85}$	$\sigma = \mathbf{0.08}$	$\sigma = \mathbf{0.10}$	$\sigma = \mathbf{0.12}$	$\sigma = \mathbf{0.14}$	$\sigma = \mathbf{0.16}$
$T = 2$	0.0082	0.0246	0.0439	0.0620	0.0770
$T = 4$	0.0144	0.0552	0.1106	0.1622	0.2038
$T = 6$	0.0226	0.1055	0.2243	0.3316	0.4119
$T = 8$	0.0356	0.1922	0.4177	0.6063	0.7378
$T = 10$	0.0586	0.3536	0.7470	1.0411	1.2272
$\mathbf{D(0)/A(0) = 0.90}$	$\sigma = \mathbf{0.08}$	$\sigma = \mathbf{0.10}$	$\sigma = \mathbf{0.12}$	$\sigma = \mathbf{0.14}$	$\sigma = \mathbf{0.16}$
$T = 2$	0.0121	0.0344	0.0591	0.0804	0.0975
$T = 4$	0.0200	0.0717	0.1355	0.1906	0.2334
$T = 6$	0.0310	0.1304	0.2594	0.3666	0.4446
$T = 8$	0.0478	0.2295	0.4647	0.6467	0.7677
$T = 10$	0.0759	0.4044	0.8012	1.0765	1.2418
$\mathbf{D(0)/A(0) = 0.95}$	$\sigma = \mathbf{0.08}$	$\sigma = \mathbf{0.10}$	$\sigma = \mathbf{0.12}$	$\sigma = \mathbf{0.14}$	$\sigma = \mathbf{0.16}$
$T = 2$	0.0173	0.0459	0.0755	0.0997	0.1182
$T = 4$	0.0272	0.0899	0.1605	0.2176	0.2590
$T = 6$	0.0415	0.1589	0.2962	0.4022	0.4735
$T = 8$	0.0629	0.2697	0.5101	0.6833	0.7905
$T = 10$	0.0979	0.4606	0.8537	1.1065	1.2501
$\mathbf{D(0)/A(0) = 1.00}$	$\sigma = \mathbf{0.08}$	$\sigma = \mathbf{0.10}$	$\sigma = \mathbf{0.12}$	$\sigma = \mathbf{0.14}$	$\sigma = \mathbf{0.16}$
$T = 2$	0.0237	0.0590	0.0928	0.1186	0.1370
$T = 4$	0.0362	0.1105	0.1862	0.2428	0.2809
$T = 6$	0.0538	0.1880	0.3289	0.4303	0.4939
$T = 8$	0.0803	0.3120	0.5536	0.7132	0.8063
$T = 10$	0.1229	0.5148	0.8981	1.1256	1.2461

In Table 6.1 we employ Algorithm 1 to estimate the fairly-priced premium our bank should be charged for DI. Similar to [30], the premiums obtained in Table 6.1 are computed for different values of σ and T ranging from 0.08 to 0.16 and from 2 to 10 years, respectively. Also, we consider different intial leverage levels (deposit-to-asset ratios) by varying the intial condition $D(0)$, while keeping intial condition $A(0)$ fixed at 1.00. We set the values for $D(0)$ to 0.80, 0.85, 0.90, 0.95 and 1.00. However, for the parameters r, m_1, g, μ_D, σ_D and M we use the same values as for the simulation study of Chapter 5.

For a fixed intial leverage level, we observe that the estimated price of the DI contract increases as either the volatility (σ) or the covarage horizon (T) increases. This means that the bank will pay higher premiums for the DI contract as the volatility in the asset portfolio or as the covarage horizon increases. Also, we notice that when the intial leverage level of the bank increases the estimated price of the DI contract increases as either the volatility in the asset portfolio or the covarage horizon increases. This implies that the bank with higher intial leverage level, compared to the one with a low intial leverage level, will pay higher premiums when either the volatility of the asset portfolio or the covarage horizon increases.

Our first observation is consistent with the findings of Duan and Yu [11], for the case where there is no capital forbearance and moral hazard, and that of Muller [30]. Our second observation is also consistent with the findings of Duan and Yu [11]. However, in the paper of Muller [30], the author found that increasing the intial leverage level increases the DI premium as the volatility in the bank's asset portfoilo is increased, but decreases as the coverage horizon is increased.

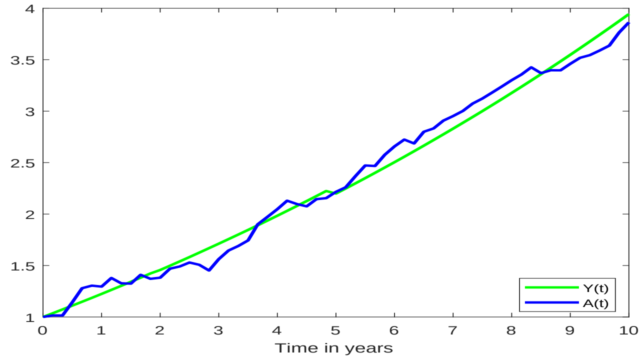

Figure 6.1: A simulation of the average, $Y(t)$, of 10^5 sample paths of $A(t)$ under the asset value reset rule for $D(0)/A(0) = 1$ and $\sigma = 0.08$.

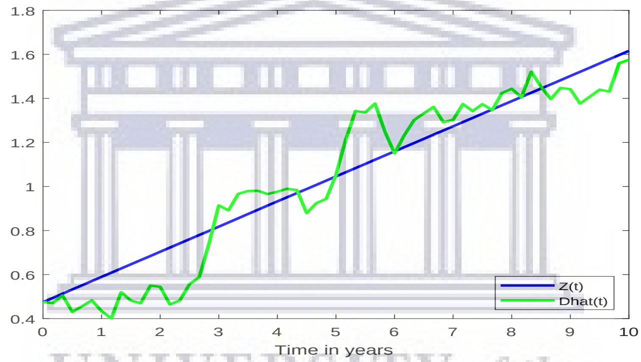

Figure 6.2: A simulation of the average, $Z(t)$, of 10^5 sample paths of $\hat{D}(t)$ for $\rho = 0.95$, $\mu_D = 0.12$ and $\sigma_D = 0.15$.

In Figures 6.1 and 6.2 we observe that the sample path of $A(t)$ and the sample path of $\hat{D}(t)$ revolve around the average $Y(t)$ and $Z(t)$, respectively. In Figures 6.3 and 6.4 we present simulations of 10^5 sample path of $A(t)$ and $\hat{D}(t)$, based on the quantities $\mu = 0.1$, $\sigma = 0.08$, $D(0)/A(0) = 1$, $\rho = 0.95$, $\mu_D = 0.12$, $\sigma_D = 0.15$ and $\hat{D}(0) = 1$.

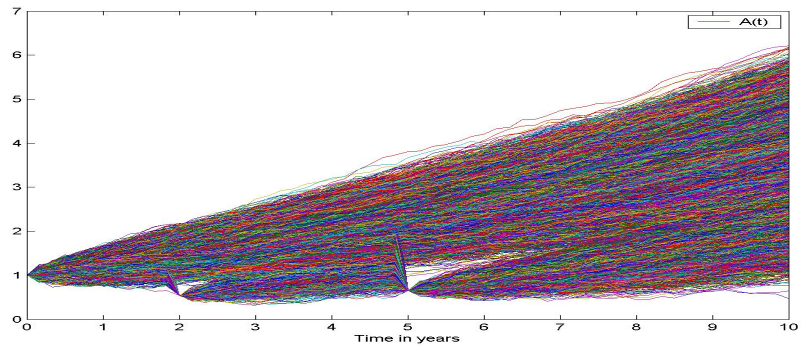

Figure 6.3: A simulation of 10^5 sample paths of $A(t)$ under the asset value reset rule for an initial leverage level of $D(0)/A(0) = 1$ and $\sigma = 0.08$.

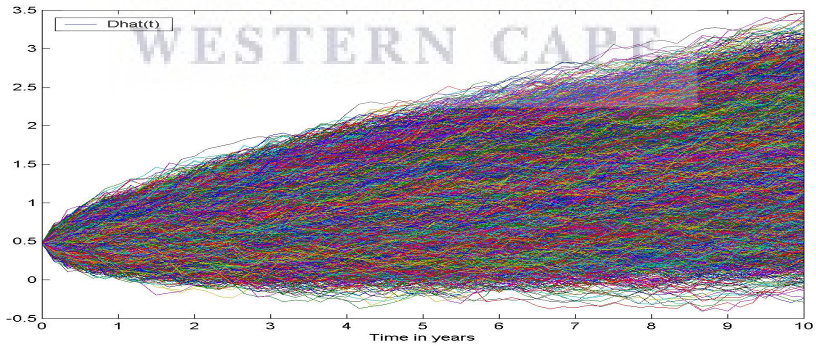

Figure 6.4: A simulation of 10^5 sample paths of $\hat{D}(t)$ for $\rho = 0.95$, $\mu_D = 0.12$, $\sigma_D = 0.15$ and $\hat{D}(0) = 1$.

Chapter 7

Conclusion

To solve optimization problems in finance, one can either employ the Martingale method or the stochastic optimal control technique. In this study we follow the latter approach by showing how to optimize the capital of a commercial bank in terms of utility maximization. We assume that bank capital, raised by the shareholders of the bank, is invested in the bank's assets. Bank capital, by definition, is the difference between the values of a bank's asset portfolio and liabilities. Our bank's liabilities are assumed to come in the form of deposits and borrowings. The financial market in which the bank trades consists of risky and riskless assets and the interest rate in the market is assumed constant. To formulate the optimal control problem, we first introduce SDEs satisfied by the dynamics of the bank's asset portfolio and liabilities and then derive the SDE for the bank's capital. We employ the stochastic optimal control approach to derive optimal investment strategies in the bank's assets that maximize an expected utility of the bank's capital at future date $T > 0$. We find an explicit solution to the optimal control problem by using an exponential utility function similar to the one considered by Devolder *et al.* [9] and Muller [30]. In addition, we study the behaviours of the optimal investment strategies and optimized capital numerically. We find that the optimal investment strategy is to diversify the bank's asset portfolio away from the risky asset and towards the riskless asset over time. That is, the bank should initially invest more of its capital in the risky asset than the riskless asset. Over time, less capital should be invested in the risky asset and more in the riskless one. Our findings are similar to that of the paper by Muller and Witbooi [31] and Muller [30] that both found that, over time, the proportions of capital invested in the risky assets (marketable security and loan) should be decreased while the proportion invested in the treasury should increase.

While banks are important to the economy, they are vulnerable to illiquidity and insolvency. For these reasons, many countries have implemented DI schemes. For the bank considered in the optimal control problem we study a DI pricing problem by following the options based approach followed by Merton [24], Ronn and Verma [34], Duan and Yu [11] and Muller [30]. We employ a Monte Carlo simulation method to estimate the cost of DI for a coverage horizon of T years, where T is the date at which the bank wishes to maximize the utility of its capital. We assume that the bank in question does not pay any dividends to its shareholders on the interval [0,T] and that the audits of the bank is conducted periodically at times $t(i), i = 1, 2, ..., T$. Our Monte Carlo method incorporates an asset value reset rule similar to those of Duan and Yu [11] and Muller [30]. In particular, should the bank be insolvent at the auditing times, then its assets are set to the level of the face value of the total insured deposits plus accrued interest, but follows the SDE describing the asset portfolio otherwise. By embedding the optimal investment strategy from the control problem into the Monte Carlo simulation method, we are able to see how the investment strategy affects the DI premium. By means of numerical simulations, we also study the effect of changes in various model parameters on the estimate for the DI premium. Under the optimal investment strategy our results suggest that, for a fixed initial leverage ratio (deposit-to-asset ratio), the cost of the DI premium must increase as either the volatility in the asset portfolio of the bank or the coverage horizon increases. Similarly, for an increasing intial leverage level the cost of the DI premium must increase as either the volatility in the asset portfolio of the bank or the coverage horizon increases. Our first observation is consistent with the findings of Duan and Yu [11], for the case where there is no capital forbearance and moral hazard, and that of Muller [30]. Our second observation is also consistent with the findings of Duan and Yu [11]. However, in the paper of Muller [30], the author found that increasing the intial leverage level increases the DI premium as the volatility in the bank's asset portfoilo is increased, but decreases as the coverage horizon is increased.

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