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On some reversal-invariant complexity measures of multiary words

(O nekim reverznoinvarijantnim merama složenosti višearnih reči)

-Ph.D. Thesis-

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„Semmit se szeressek félig csinálni,
vagyis a tökély előtt megelégedni.”
(I hate doing things dodgy,
getting satisfied before reaching perfection.)

Bolyai János

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Contents

Abstract

We focus on two complexity measures of words that are invariant under the operation of reversal of a word: the palindromic defect and the MP-ratio.

The *palindromic defect* of a given word w is defined by $|w|+1-|\mathrm{Pal}(w)|$, where $|\mathrm{Pal}(w)|$ denotes the number of palindromic factors of w. We study infinite words, to which this definition can be naturally extended. There are many results in the literature about the so-called *rich words* (words of defect 0), while words of finite positive defect have been studied significantly less; for some time (until recently) it was not known whether there even exist such words that additionally are aperiodic and have their set of factors closed under reversal. Among the first examples that appeared were the so-called *highly potential words.* In this thesis we present a much more general construction, which gives a wider class of words, named *generalized highly potential words,* and analyze their significance within the frames of combinatorics on words.

The *MP-ratio* of a given n-ary word w is defined as the quotient $\frac{|rws|}{|w|}$, where r and s are words such that the word rws is minimal-palindromic and that the length $|r|+|s|$ is minimal possible; here, an n-ary word is called *minimal-palindromic* if it does not contain palindromic subwords of length greater than $\lceil \frac{|w|}{n} \rceil$. In the binary case, it was proved that the MP-ratio is well-defined and that it is bounded from above by 4, which is the best possible upper bound. The question of well-definedness of the MP-ratio for larger alphabets was left open. In this thesis we solve that question in the ternary case: we show that the MP-ratio is indeed well-defined in the ternary case, that it is bounded from above by the constant 6 and that this is the best possible upper bound.

Izvod (in Serbian)

Izučavamo dve mere složenosti reči koje su invarijantne u odnosu na operaciju preokretanja reči: palindromski defekt i MP-razmeru date reči.

Palindromski defekt reči w definiše se kao $|w|+1-|\mathrm{Pal}(w)|$, gde $|\mathrm{Pal}(w)|$ predstavlja broj palindromskih faktora reči w. Mi izučavamo beskonačne reči, na koje se ova definicija može prirodno proširiti. Postoje mnogobrojni rezultati u vezi sa tzv. *bogatim rečima* (reči čije je defekt 0), dok se o rečima sa konačnim pozitivnim defektom relativno malo zna; tokom jednog perioda (donedavno) nije bilo poznato ni da li uopšte postoje takve reči koje su, dodatno, aperiodične i imaju skup faktora zatvoren za preokretanje. Među prvim primerima koji su se pojavili u literaturi su bile tzv. *visokopotencijalne reči*. U disertaciji ćemo predstaviti znatno opštiju konstrukciju, kojom se dobija značajno šira klasa reči, nazvanih *uopštene visokopotencijalne reči*, i analiziraćemo njihov značaj u okvirima kombinatorike na rečima.

MP-razmera date n-arne reči w definiše se kao količnik $\frac{|rws|}{|w|}$, gde su r i s takve da je reč rws minimalno-palindromična, i dužina $|r|+|s|$ je najmanja moguća; ovde, za n-arnu reč kažemo da je *minimalno-palindromična* ako ne sadrži palindromsku podreč dužine veće od $\left\lceil \frac{|w|}{n} \right\rceil$. U binarnom slučaju dokazano je da je MP-razmera dobro definisana i da je ograničena odozgo konstantom 4, što je i najbolja moguća granica. Dobra definisanost MP-razmere za veće alfabete je ostavljena kao otvoren problem. U ovoj tezi rešavamo taj problem u ternarnom slučaju: pokazaćemo da MP-razmera jeste dobro definisana u ternarnom slučaju, da je ograničena odozgo sa 6, i da se ta granica ne može poboljšati.

Preface

In the literature on combinatorics on words, many functions are introduced and studied that can be thought of as measures of various kinds of complexity of given words. In this thesis we focus on certain such measures that are invariant under the operation of reversal of a word. As will be seen (and as is expected), there is a strong connection between such functions and the notion of palindromes. We choose two actual research directions on this topic and answer many questions about them.

One research direction is based on the result of Droubay, Justin and Pirillo [30], who noted that a word of length n can have at most $n+1$ different palindromic factors. The difference between this upper bound and the actual number of palindromic factors of a given word is called the *palindromic defect* (or only *defect*) of a given word [19] (by definition, the defect is always non-negative). Though the definition of defect fundamentally relies on finiteness of a given word, it turns out that it can be naturally extended to infinite words (the defect of an infinite word is defined as the supremum of defects of all of its finite factors). Words of defect 0 are called *full* [19] or *rich* [34], and there are many results about them in the literature [47, 21, 50, 59, 36, 60, 53, 51].

However, infinite words of finite positive defect have been studied significantly less. One of the reasons for that is the fact that explicit constructions of such words (maybe with some additional constraints, such as aperiodicity, since periodic words are more-or-less straightforward to analyze) are somewhat deficient in the literature. For example, aperiodic words of finite positive defect, having the set of factors closed under reversal, had been deemed interesting from the point of view of some (then open) conjectures [17, 20], but examples of such words were missing. In the article [15], an infinite family of infinite words is constructed, called *highly potential words,* which are all aperiodic, have the set of factors closed under reversal, and are of finite positive defect (in fact, the presented construction shows a method to obtain such a word from *any* finite nonpalindromic word). As one can see in that article, those words seem to be a useful supply of examples and counterexamples for various problems of words (which explains their name). We should also say that chronologically the first example of an aperiodic infinite word of finite positive defect, whose set of factors is closed under reversal, had been given somewhat earlier: see [9, Example 3.4], where such a word has been constructed, one that is uniformly recurrent. In the article [14] an example that is not uniformly recurrent has been constructed, which was used to demonstrate a flaw in a proof from the article [10]; this word, although it has much in common with the family of highly potential words, does not belong to that family.

In this thesis we construct a new family of infinite words whose defect is finite, and in many cases positive (with fully characterized cases when the defect is 0). The constructed family contains, as two special cases, both the family of highly potential words (because of this, we dub them *generalized highly potential words*), as well as the mentioned word from [14]. Further, in [34, Proposition 2.10] the authors show the existence of rich infinite words that are recurrent but not uniformly recurrent, by providing three examples; it turns out that all these three words also belong to the class of generalized highly potential words. We believe that all this suggests that our construction extends the class of highly potential words in a fairly noteworthy way. All the words from our family have the set of factors closed under reversal, and each of them is either periodic (which is a less interesting case, and explicitly characterized), or recurrent but not uniformly recurrent. The fact that they are not uniformly recurrent (unless they are periodic) is of a particular significance since: first, there are some results and examples here and there featuring uniformly recurrent words of finite defect (see, e.g., [34, Proposition 4.8], or the article [9], or the counterexample to the so-called Zero defect conjecture from [23], which is defined as a fixed point of a primitive morphism, and it is known [6, Theorem 10.9.5] that fixed points of primitive morphisms are always uniformly recurrent), while next to nothing is known about aperiodic words that are not uniformly recurrent; second, it is shown in [49, Theorem 2] that any uniformly recurrent word of finite defect is a morphic image of some word of zero defect (while the result that the authors obtain without assuming uniform recurrence is weaker, and in the last section they discuss the significance of uniform recurrence and leave as an open question whether the stronger result is valid without it), everything of which suggests that uniformly recurrent words are somewhat easier to work with, and that those that are not uniformly recurrent are less explored territory that deserves a closer look.

Holub and Saari [39] introduced yet another way to measure how "rich" in palindromes a given word is, the so-called *MP-ratio*. MP-ratio is a rational number greater than or equal to 1 such that, the greater MP-ratio is, the given word is "richer" in palindromes (the authors of [39] say that such words are "highly palindromic"); those words whose MP-ratio equals 1 are called *minimal-palindromic*. It turns out that some properties of MP-ratio are not so easy to grasp, since, as shown in [13], it can behave in a quite unpredictable way. The concept of MP-ratio is based on palindromic *subwords* (and not factors) of a given word, which have been noticeably less considered in the literature. They, however, have some interesting properties. As shown in [39], a binary word can be reconstructed, up to reversal, from the set of its palindromic subwords. Also in [39], a property of a word being *abelian bordered* is defined, and it is shown that each binary minimal-palindromic word is abelian unbordered (which is a strong form of unborderedness); abelian (un)borderedness of words has attracted a growing attention in recent times [28, 35, 26, 7, 16]. However, the main drawback of the notion of MP-ratio is the fact that it is defined only for binary alphabet. Though there is a natural analogous way to extend the definition of MP-ratio to a larger alphabet, it is not clear whether in that case the notion is well-defined at all. For that reason, the authors of [39] left the question of well-definedness of MP-ratio for larger alphabets as an open problem. In this thesis we solve that question for ternary alphabet. We show that the MP-ratio is well-defined in the ternary case, that it is bounded from above by the constant 6, and that this bound is the best possible.

The thesis is organized as follows.

In Chapter 1 we give the necessary background. In Section 1.1 we recall the basic notions and theorems about words in general. In Sections 1.2 and 1.3 we present relevant results about the defect and the MP-ratio, respectively. Everything in this chapter is already known in the literature and given with a reference.

Chapters 2 and 3 contain fully original work, mostly included in the articles [4], respectively [2] and [3].

Chapter 2 is devoted to generalized highly potential words. Their definition and some technical preliminary results are given in Sections 2.1 and 2.2. In Section 2.3 we give a necessary and sufficient condition for periodicity of generalized highly potential words, we show that their set of factors is closed under reversal (which implies that they are recurrent), and we further show that the ones that are not periodic are not uniformly recurrent. In Section 2.4 we prove that their defect is always finite, and give a necessary and sufficient condition for the defect to be positive. Separately, in Section 2.5 at the end, we analyze periodic generalized highly potential words (which is a less interesting case).

Chapter 3 deals with the MP-ratio. In Section 3.1 we show that there always exists an MP-extension (r, s) of any ternary word w; in fact, since for our construction holds $|rws| = 6|w|$, we get that the MP-ratio is bounded from above by 6. During the course of the proof, two technical results are needed, and they are given as appendices in Sections 3.3 and 3.4 (where Section 3.3 is self-contained, and Section 3.4 relies only on Section 3.3; thus we believe that this will not cause confusion to the reader); further, those two results are essentially results on *binary* words (and there might be a slim chance that they could be also useful somewhere else), which again makes it natural to give them separated from the proof from Section 3.1. In Section 3.2 we show that the MP-ratio can be arbitrarily close to the constant 6, which gives that 6 is the best possible upper bound on the MP-ratio in the ternary case.

1 Introduction

1.1 On words

In this section we recall basic definitions and properties that will be needed through the thesis. All these notions are mainly standard and can be found, for example, in [6].

A *word* (respectively *infinite word*) is a finite (respectively infinite) sequence of symbols taken from a nonempty finite set Σ, which is called the *alphabet*, and its elements are called *letters*. (We shall sometimes abuse the terminology and say only "word" when it is clear from the context that it must be infinite, or additionally emphasize "finite word" when we feel that this is appropriate.) Let Σ^* denote the set of all finite words and by Σ^∞ the set of all finite or infinite words. In the case $|\Sigma| = 2$ we speak about *binary* words, in the case $|\Sigma| = 3$ we speak about *ternary* words and, generally, in the case $|\Sigma| = n$ we speak about *n-ary* words. If $w = a_1a_2...a_n$ with $a_1, a_2, \ldots, a_n \in \Sigma$, we say that the *length* of w is n, and write $|w| = n$. The unique word of length 0, called the *empty word*, is denoted by ε.

The *concatenation* (or *product*) of words u and v, $u = a_1a_2\ldots a_n$ and $v = b_1b_2\ldots b_m$, is the word $a_1a_2\ldots a_nb_1b_2\ldots b_m$, denoted by uv. The product uv for $u \in \Sigma^*$ and $v \in \Sigma^\infty\backslash\Sigma^*$ can be similarly defined. For a word w and a positive integer k we write w^k for the word $\underbrace{ww\ldots w}_{k}$ and w^∞ for the infinite word $wwww\ldots$; it is also convenient to define $w^0 = \varepsilon$ for any word w. A word $w \in \Sigma^*$ is *primitive* if and only if it is not of the form z^k for $z \in \Sigma^*\backslash\{\varepsilon\}$ and an integer k, $k \geqslant 2$.

For $A \subseteq \Sigma$, we write A^* for the set

$$\{a_1a_2\dots a_k : k \geqslant 0 \text{ and } a_i \in A \text{ for each } i\},$$

and we write $A^+ = A^*\backslash\{\varepsilon\}$. If the set A has only one element, say $A = \{a\}$, we write a^* and a^+ instead of $\{a\}^*$ and $\{a\}^+$. If A and B are two sets of words, we write $AB = \{uv : u \in A, v \in B\}$. Since concatenation of words is an associative operation, the product of more than two sets of words is also well-defined.

A word $u \in \Sigma^*$ is called a *factor* (respectively *prefix*, *suffix*) of a word $w \in \Sigma^\infty$ if and only if there exist words $x \in \Sigma^*$ and $y \in \Sigma^\infty$ such that $w = xuy$ (respectively $w = uy$, $w = xu$). A word $u \in \Sigma^*$ is a *subword* of $w \in \Sigma^*$ if and only if there exist words $x_1, x_2, \dots, x_n, x_{n+1} \in \Sigma^*$ and $y_1, y_2, \dots, y_n \in \Sigma^*$ such that $u = y_1y_2\dots y_n$ and $w = x_1y_1x_2y_2\dots x_ny_nx_{n+1}$ (or, equivalently, u is a subword of w if u is its subsequence). The set of all factors (respectively prefixes, suffixes, subwords) of a word w is denoted by $\mathrm{Fact}(w)$ (respectively $\mathrm{Pref}(w)$, $\mathrm{Suff}(w)$, $\mathrm{Subw}(w)$).

We write $w[i]$ for the i^{th} letter of the word w, and for any pair (i, j) of integers such that $1 \leqslant i \leqslant j \leqslant |w|$ we write $w[i, j]$ for the factor $w[i]w[i + 1]\dots w[j]$ (obviously, $w[i, i] = w[i]$). In the case $i > j$, as well as $i > |w|$ or $j < 1$, we define $w[i, j] = \varepsilon$. By convention, this operation has precedence over concatenation; in other words, $uv[i]$ (and similarly $uv[i, j]$) will always denote $u(v[i])$, not $(uv)[i]$.

If i and j are positive integers and $i \leqslant j$, $[i, j]_{\mathbb{N}}$ denotes the set $\{i, i + 1, i + 2, \dots, j\}$. (Note that $\mathbb{N}$ denotes the set of positive integers, while $\mathbb{N}_0$ denotes the set of nonnegative integers.)

For words u and v, let $|u|_v$ denote the number of distinct occurrences of v in u, that is:

$$|u|_v = |\{i : 1 \leqslant i \leqslant |u| - |v| + 1, u[i, i + |v| - 1] = v\}|.$$

We say that a letter c is *prevalent* in a word w if and only if $|w|_c = \max\{|w|_a : a \in \Sigma\}$. (Note that a prevalent letter is not necessarily unique.) We say that a factor v of a word $w \in \Sigma^\infty$ is *unioccurrent* in w if and only if $|w|_v = 1$.

The *reversal* of $w = a_1a_2\dots a_n$, where $a_1, a_2, \dots, a_n \in \Sigma$, is defined by $\widetilde{w} = a_na_{n-1}\dots a_1$. We say that the set of factors of w is *closed under reversal* if and only if for any $v \in \mathrm{Fact}(w)$ holds $\widetilde{v} \in \mathrm{Fact}(w)$. A word w is a *palindrome* (or *palindromic*) if and only if $w = \widetilde{w}$. (The empty word is also a palindrome.) A palindromic subword of a given word will be called a *subpalindrome*. Let $\mathrm{Pal}(w) = \{u \in \mathrm{Fact}(w) : u = \widetilde{u}\}$.

A function $\varphi : \Sigma^* \to \Sigma^*$ is called a *morphism* if and only if, for all $u, v \in \Sigma^*$, we have $\varphi(uv) = \varphi(u)\varphi(v)$.

Before the next theorem [45, Proposition 1.3.2], we need the following notion: a word w' is a *conjugate* of a word w if and only if there exist words x and y such that $w = xy$ and $w' = yx$.

Theorem 1.1. *Let $x, y \in \Sigma^* \backslash \{\varepsilon\}$. Then $xy = yx$ if and only if there exist $t \in \Sigma^*$ and positive integers p, q such that $x = t^p$, $y = t^q$. In other words, if a word is equal to one of its conjugates (different from itself), then it must be a power of exponent at least 2.*

Proof. ($\Leftarrow$): If such a word t exists, then clearly holds $xy = yx = t^{p+q}$.

($\Rightarrow$): We proceed by induction on $|x| + |y|$. The base for $|x| = |y| = 1$ is obvious. Let $|x| + |y| > 2$, and assume, without loss of generality, $|x| \geqslant |y|$. We distinguish two cases.

- $|x| = |y|$:

  Since $xy = yx$ implies $x = y$, we can take $t = x = y$ and $p = q = 1$.

- $|x| > |y|$:

  In this case y is a prefix of x; therefore, we can write $x = yy_1$. Now, $xy = yx$ implies
  $$yy_1y = yyy_1,$$
  that is,
  $$y_1y = yy_1.$$
  As $|y| + |y_1| < |x| + |y|$, by the inductive assumption there exist a word $t \in \Sigma^*$ and positive integers p and q such that $y = t^p$, $y_1 = t^q$, which gives $x = t^{p+q}$ and $y = t^p$.

The proof is completed. ∎

An infinite word w is *periodic* if and only if it is of the form $w = u^\infty$ for some $u \in \Sigma^*$, it is *eventually periodic* if and only if it is of the form vu^∞ for some u, $v \in \Sigma^*$, and it is *aperiodic* if and only if it is not eventually periodic. A positive integer p is a *period* of w if and only if $w[i] = w[i + p]$ for each $i \geqslant 1$. (A period is not unique.) An infinite word w is *recurrent* if and only if each of its factors occurs infinitely many times in w, and it is *uniformly recurrent* if and only if for every finite factor u of w there exists an integer n such that $u \in \mathrm{Fact}(v)$ for every $v \in \mathrm{Fact}(w)$ such that $|v| = n$.

The following three theorems are well-known and can be found in [6], Exercise 10.50a), Example 10.9.1 and Exercise 10.37, respectively.

Theorem 1.2. *For an infinite word w, if* $\mathrm{Fact}(w)$ *is closed under reversal, then w is recurrent.*

Proof. Let $u \in \mathrm{Fact}(w)$. It is enough to prove that for any occurrence of the factor u in w there exists one more occurrence of it, which is to the right of the originally considered occurrence. Write $u = w[i, i + |u| - 1]$ for an integer i. Let v be a prefix of w such that $|v| = 2i + |u| - 1$. Since $\mathrm{Fact}(w)$ is closed under reversal, $\widetilde{v} \in \mathrm{Fact}(w)$ and $\widetilde{u} \in \mathrm{Fact}(\widetilde{v})$. By the choice of v, we have an occurrence of $\widetilde{u}$ in w that begins at least at the $(i+1)^{\mathrm{st}}$ letter of w, that is, this occurrence of $\widetilde{u}$ is to the right of the initially considered occurrence of u. Now in a similar manner we find an occurrence of $\widetilde{\widetilde{u}}(= u)$ which is to the right of the considered occurrence of $\widetilde{u}$, and therefore to the right of the initially considered occurrence of u. This completes the proof. ■

Theorem 1.3. *Every periodic word is uniformly recurrent.*

Proof. Let w be a periodic word with a period k, and let $u \in \mathrm{Fact}(w)$, $|u| = n$. Note that $u \in \mathrm{Fact}(v)$ for each $v \in \mathrm{Fact}(w)$ such that $|v| = k + n - 1$; by definition, w is uniformly recurrent. ■

Theorem 1.4. *Every recurrent, eventually periodic word is periodic.*

Proof. Let $w = uv^{\infty}$ be an eventually periodic word for some $u, v \in \Sigma^*$. We may assume that $|v|$ is the smallest period of v^{∞}. Note that this assumption implies that v is not a power. If $u = \varepsilon$, there is nothing to prove, so we assume $u \neq \varepsilon$. Since w is recurrent, its factor uv occurs infinitely many times in w, and thus it must occur in v^{∞}. We can write $uv = v_2 v^k v_1$, where $v_2 \in \mathrm{Suff}(v)$ and $v_1 \in \mathrm{Pref}(v)$, $|v_1| < |v|$. If $v_1 \neq \varepsilon$, we would have that v is equal to one of its conjugates different from itself, and thus, by Theorem 1.1, it would be a power, contradicting the assumption. Therefore, $uv = v_2 v^k$. Write $v = v' v_2$. We then have

$$w = uv^{\infty} = uvv^{\infty} = v_2(v'v_2)^k(v'v_2)^{\infty} = (v_2 v')^{\infty},$$

which shows that w is periodic. ■

1.2 Palindromic defect

The following inequality was noted by Droubay, Justin and Pirillo [30, Proposition 2].

Theorem 1.5. *Let w be a finite word. Then:*

$$|\mathrm{Pal}(w)| \leqslant |w| + 1.$$

Proof. First, it is not hard to see that, if $u \in \Sigma^*$ and $a \in \Sigma$, then

$$|\mathrm{Pal}(ua)| = \begin{cases} |\mathrm{Pal}(u)| + 1, & \text{if the longest palindromic suffix of } ua \\ & \text{is unioccurrent in } ua; \\ |\mathrm{Pal}(u)|, & \text{otherwise.} \end{cases} \tag{1.1}$$

Similarly,

$$|\mathrm{Pal}(au)| = \begin{cases} |\mathrm{Pal}(u)| + 1, & \text{if the longest palindromic prefix of } au \\ & \text{is unioccurrent in } au; \\ |\mathrm{Pal}(u)|, & \text{otherwise.} \end{cases} \tag{1.2}$$

Indeed, the only possible palindromic factor of ua that is not a palindromic factor of u is the longest palindromic suffix of ua. Namely, if x is a palindromic suffix of ua that is not the longest one, then x is also a prefix of the longest palindromic suffix of ua, and thus x is a factor of u. Now, clearly the longest palindromic suffix of ua is a palindromic factor of ua that is not a palindromic factor of u if and only if it is unioccurrent in ua. This gives (1.1). The explanation of (1.2) is analogous.

Let $w = a_1 a_2 \ldots a_n$. Then, in view of (1.1), we have

$$\begin{aligned} |\mathrm{Pal}(w)| &= |\mathrm{Pal}(a_1 a_2 \ldots a_n)| \leqslant |\mathrm{Pal}(a_1 a_2 \ldots a_{n-1})| + 1 \\ &\leqslant \cdots \leqslant |\mathrm{Pal}(a_1)| + n - 1 \leqslant |\mathrm{Pal}(\varepsilon)| + n \\ &= n + 1, \end{aligned}$$

which was to be proved. ■

Inspired by this inequality, Brlek et al. [19] introduced the notion of *palindromic defect* (or only *defect*) of a word w, denoted by $D(w)$, and defined as:

$$D(w) = |w| + 1 - |\mathrm{Pal}(w)|.$$

They noticed that the defect of a word w is no smaller than the defect of any of its factors; in other words:

Theorem 1.6. *Let w be a finite word and $v \in \mathrm{Fact}(w)$. Then*

$$D(v) \leqslant D(w).$$

Proof. For $u \in \Sigma^*$ and $a \in \Sigma$, by the definition of defect, and by (1.1) and (1.2), we have the following equalities:

$$D(ua) = \begin{cases} D(u), & \text{if the longest palindromic suffix of } ua \\ & \text{is unioccurrent in } ua; \\ D(u) + 1, & \text{otherwise} \end{cases} \tag{1.3}$$

and

$$D(au) = \begin{cases} D(u), & \text{if the longest palindromic prefix of } au \\ & \text{is unioccurrent in } au; \\ D(u) + 1, & \text{otherwise.} \end{cases} \tag{1.4}$$

Now, let $w = a_1 \dots a_n v b_1 \dots b_k$. Then by (1.3) and (1.4) we have

$$\begin{aligned} D(v) &\leqslant D(vb_1) \leqslant \cdots \leqslant D(vb_1 \dots b_k) \\ &\leqslant D(a_n v b_1 \dots b_k) \leqslant \cdots \leqslant D(a_1 \dots a_n v b_1 \dots b_k) \\ &= D(w), \end{aligned}$$

which was to be proved. ■

This motivates the following extension of the definition of defect to infinite words: for $w \in \Sigma^\infty \backslash \Sigma^*$, we define

$$D(w) = \sup_{v \in \mathrm{Fact}(w)} D(v).$$

(Of course, this equality also holds for finite words.) Note that the defect of any finite or infinite word is always nonnegative or infinite.

1.2.1 Defect of some periodic words

In this subsection we shall see a special case of periodic words when the defect is always finite, and easily calculable. Theorems 1.7 (see [19, Lemma 5] and [45, Proposition 1.3.4]) and 1.8 (see [19, Theorem 6]) are somewhat technical results, aimed toward the main point of this section, Theorem 1.9 (from [19, Corollary 8]).

Theorem 1.7. *Let $w = xy = yz$. If w is a palindrome, then there exist palindromes u and v such that $x = uv$, $z = vu$ and $y = (uv)^{i-1}u$ for a positive integer i. Furthermore, then xyz also is a palindrome.*

Proof. If $xy = yz$, then for each positive integer i we have

$$x^i y = yz^i. \tag{1.5}$$

We choose i such that $i|x| \geqslant |y| > (i-1)|x|$. Then (1.5) implies $y = x^{i-1}u$, $x = uv$ and $z^i = vy$ for some words u and v. We then have $z^i = vy = vx^{i-1}u = v(uv)^{i-1}u = (vu)^i$, which implies $z = vu$. Finally, since $w = xy = (uv)^i u$ and w is a palindrome, we have that u and v also are palindromes, and then, since $xyz = (uv)^{i+1}u$, it follows that xyz also is a palindrome. The proof is now completed. ■

Theorem 1.8. *Let p be a primitive word that is a product of two palindromes u and v, with $|u| \geqslant |v|$. Then*

$$D(p^\infty) = D\left(p^\infty\left[1, |uv| + \left\lfloor \frac{|u|-|v|}{3} \right\rfloor\right]\right).$$

Proof. By the proof of Theorem 1.6, we see that the defect of an arbitrary (possibly infinite) word equals the number of its prefixes whose longest palindromic suffix is not unioccurrent. Therefore, we need to show that, for each prefix f (of p^∞) longer than $|uv| + \lfloor \frac{|u|-|v|}{3} \rfloor$, we have that the longest palindromic suffix of f is unioccurent in f. We distinguish three cases depending on the length of f.

- $|f| > |uvuv|$: In this case, there exists k, $k \geqslant 1$, such that we have either

  $$f = u(vu)^k vy, \quad \text{where } y \in \mathrm{Pref}(u) \setminus \{\varepsilon\},$$

  or

  $$f = uv(uv)^k uy, \quad \text{where } y \in \mathrm{Pref}(v) \setminus \{\varepsilon\}.$$

  Let g be the longest palindromic suffix of f. Then, depending on the form of f, one palindromic suffix of f is $\widetilde{y}(vu)^k vy$, respectively $\widetilde{y}(uv)^k uy$, and thus g is at least that long. Therefore, we may write $f = xg$, where $|x| < |uv| \leqslant |g|$. If we suppose that g has another occurrence in f, then the two occurrences must overlap, and then, in view of Theorem 1.7 (where we take g for w, and the overlapping part of the two occurrences of g for y), we get a palindromic suffix of f longer than g; a contradiction with the choice of g.

- $|uvuv| \geqslant |f| > |uvu|$: We write $f = uvuy$, where $y \in \mathrm{Pref}(v) \setminus \{\varepsilon\}$. Let g again be the longest palindromic suffix of f and suppose that it has another (earlier) occurrence in f. Then the two occurrences cannot overlap (since otherwise we would get the same contradiction as in the previous case). Since $\widetilde{y}uy$ is a palindromic suffix of f, we can write $g = suy$, where $s \in \mathrm{Suff}(uv) \setminus \{\varepsilon\}$. Since there is an occurrence of g in f that does not overlap with the rightmost one, we have $g \in \mathrm{Fact}(uv)$, and since $|u| \geqslant |v|$, it follows that there are two overlapping occurrences of u in uv, where the nonoverlapping part has length at least $|s|$. Then, by Theorem 1.7, the overlapping part is of the form $(tr)^{i-1}t$ for some palindromes t and r, and $u = (tr)^i t$ (where $|tr| \geqslant |s|$). This implies that v begins with rt (since the overlapping part in uv ends where v begins), and there is y after rt (since there is y in g after u, and the observed rt is the ending of u that is within g); in other words, $rty \in \mathrm{Pref}(v)$ (indeed, it cannot extend beyond v, since rty is contained in g that is a part of uv). This implies $\widetilde{y}tr \in \mathrm{Suff}(v)$, but then $\widetilde{y}truy$ is a suffix of vuy and thus also a suffix of f, and $|\widetilde{y}truy| > |suy| = |g|$; a contradiction with the choice of g again.

- $|uvu| \geqslant |f| > |uv| + \left\lfloor \frac{|u|-|v|}{3} \right\rfloor$: In this case we write $f = uvy$, where y is a prefix of u longer than $\left\lfloor \frac{|u|-|v|}{3} \right\rfloor$. Let g be the longest palindromic suffix of f. Then
$$|g| \geqslant |\widetilde{y}vy| = |v| + 2|y|.$$
We again suppose that g has another occurrence in f, and we may assume (as in the previous cases) that the two occurrences do not overlap. This gives
$$2|g| \leqslant |f| = |u| + |v| + |y|.$$
The previous two inequalities give $2|v| + 4|y| \leqslant |u| + |v| + |y|$, from which we get $|y| \leqslant \frac{|u|-|v|}{3}$, and thus also $|y| \leqslant \left\lfloor \frac{|u|-|v|}{3} \right\rfloor$, a contradiction.

This completes the proof. ∎

Theorem 1.9. *If p is a primitive word that is a product of two palindromes (one of which can be empty), then there exists a conjugate q of p such that*
$$D(p^\infty) = D(q).$$

Proof. Let $p = uv$, where u and v are palindromes. We claim that there exists a conjugate q of p such that $q = u'v'$ where u' and v' are palindromes and

$|u'| - |v'| \in \{0, 1, 2\}$. If $|u| \geqslant |v|$, let

$$u'' = u\left[1 + \left\lfloor \frac{|u| - |v|}{4} \right\rfloor, |u| - \left\lfloor \frac{|u| - |v|}{4} \right\rfloor\right]$$

and

$$v'' = u\left[|u| - \left\lfloor \frac{|u| - |v|}{4} \right\rfloor + 1, |u|\right] vu\left[1, \left\lfloor \frac{|u| - |v|}{4} \right\rfloor\right],$$

while if $|u| < |v|$, let

$$u'' = v\left[|v| - \left\lceil \frac{|v| - |u|}{4} \right\rceil + 1, |v|\right] uv\left[1, \left\lceil \frac{|v| - |u|}{4} \right\rceil\right]$$

and

$$v'' = v\left[1 + \left\lceil \frac{|v| - |u|}{4} \right\rceil, |v| - \left\lceil \frac{|v| - |u|}{4} \right\rceil\right]$$

(clearly, $u''v''$ is a conjugate of p). In the first case we have $|u''| - |v''| = (|u| - 2\lfloor \frac{|u|-|v|}{4} \rfloor) - (|v| + 2\lfloor \frac{|u|-|v|}{4} \rfloor) = |u| - |v| - 4\lfloor \frac{|u|-|v|}{4} \rfloor = (|u| - |v|) \bmod 4$, and we get the same equality in the second case. Since $(|u| - |v|) \bmod 4 \in \{0, 1, 2, 3\}$, it follows that we can take $u' = u''$ and $v' = v''$, unless $|u| - |v| \equiv 3 \pmod 4$. However, if the latter case happens, we can do the same procedure on the word vu instead of p; since vu is a conjugate of p, any conjugate of vu is also a conjugate of p, and since $|v| - |u| \equiv 1 \pmod 4$, the conjugate that we obtain that way satisfies the requirements.

Now, by the previous theorem we have

$$\begin{aligned} D(q^\infty) = D((u'v')^\infty) &= D\left((u'v')^\infty \left[1, |u'v'| + \left\lfloor \frac{|u'| - |v'|}{3} \right\rfloor \right] \right) \\ &= D((u'v')^\infty[1, |u'v'|]) = D(q). \end{aligned} \tag{1.6}$$

Further, since $\mathrm{Fact}(p^\infty) = \mathrm{Fact}(q^\infty)$, we have

$$D(p^\infty) = D(q^\infty). \tag{1.7}$$

Now (1.6) and (1.7) give $D(p^\infty) = D(q)$, which was to be proved. ■

Note. In particular, given the definition of defect for infinite words, for q from the previous theorem we may (and have to) choose that one conjugate for which $D(q)$ is maximal (or any such, if there are more of them).

1.2.2 Highly potential words

A class of infinite words called *highly potential words* has been introduced in [15]. Given a word w that is not a palindrome, let c denote any letter that does not appear in w, and let:

$$w_0 = w;$$

$$w_i = w_{i-1} c^i \widetilde{w_{i-1}},\ i \in \mathbb{N};$$

$$\mathrm{hpw}(w) = \lim_{i \to \infty} w_i.$$

The infinite word $\mathrm{hpw}(w)$ is called *highly potential word generated by* w. (The limit is well-defined since each w_i is a prefix of w_{i+1}.)

The main properties of highly potential words are given in the following two theorems.

Theorem 1.10. *Let* $\mathrm{hpw}(w)$ *be a highly potential word. Then:*

- $\mathrm{hpw}(w)$ *is aperiodic;*
- $\mathrm{Fact}(\mathrm{hpw}(w))$ *is closed under reversal;*
- $\mathrm{hpw}(w)$ *is recurrent;*
- $\mathrm{hpw}(w)$ *is not uniformly recurrent.*

Proof. In view of Remark 2.2 (see page 37), the proof follows from the respective properties of generalized highly potential words (that will be introduced and thoroughly analyzed in Chapter 2), in particular from Section 2.3. ∎

Theorem 1.11. *Let* $\mathrm{hpw}(w)$ *be a highly potential word. Then:*

$$D(\mathrm{hpw}(w)) = D(w) + 1.$$

Proof. We first prove $D(w_1) = D(w)$. Since $|w_1| - |w| = |w| + 1$, in order to show $D(w_1) = D(w)$ we need to find $|w| + 1$ new palindromes in w_1 (by "new palindromes" we mean those palindromes that appear in w_1 but do not appear in w). It is not hard to see that the set of new palindromes is exactly the set $\{\widetilde{x} c x : x \in \mathrm{Pref}(\widetilde{w})\}$, whose cardinality is $|w| + 1$. Therefore,

$$D(w_1) = D(w). \tag{1.8}$$

Let us calculate $D(w_2)$. Again, it can be easily checked that that the set of new palindromes is the set $\{\widetilde{x}ccx : x \in \mathrm{Pref}(w_1)\}$, whose cardinality is $|w_1| + 1$. Since $|w_2| - |w_1| = |w_1| + 2$, we conclude that

$$D(w_2) = D(w_1) + 1. \tag{1.9}$$

We show now that, for $i \geqslant 2$,

$$D(w_{i+1}) = D(w_i). \tag{1.10}$$

In order to prove this, since $|w_{i+1}| - |w_i| = |w_i| + i + 1$, we need to find $|w_i| + i + 1$ new palindromes in w_{i+1}. We distinguish two classes of palindromes.

- We first enumerate new palindromes that have the factor c^{i+1} in the center; they can be obtained by "expanding" (to the left and the right side) the boxed part below:

  $$w_i \boxed{c^{i+1}} w_i.$$

  All of the words $\{\widetilde{x}c^{i+1}x : x \in \mathrm{Pref}(w_i)\}$ are palindromes that appear for the first time in that step, because none of the factors of w_i contains the factor c^{i+1}. Therefore, we have $|w_i|+1$ new palindromes of this type.

- We now enumerate new palindromes that have the factor w_{i-1} in the center; they can be obtained by "expanding" the boxed part below:

  $$w_i c^{i+1} \boxed{w_{i-1}} c^i w_{i-1}.$$

  Again, all of the words $\{c^k w_{i-1} c^k : 1 \leqslant k \leqslant i\}$ are palindromes that appear for the first time in that step. Indeed, w_{i-1} appears as a factor of w_i only two times, at the beginning and at the end, and thus the observed polynomials are not factors of w_i. Therefore, we have i new palindromes of this type.

It can be easily checked that the list above contains all the new palindromes, and that the types are disjoint. Therefore, we have found in total

$$|w_i| + 1 + i$$

new palindromes, which proves (1.10).

Equalities (1.8), (1.9) and (1.10) now give

$$\sup_{i \in \mathbb{N}_0} D(w_i) = D(w) + 1.$$

By the definition of $\text{hpw}(w)$ and by Theorem 1.6 we have

$$\sup_{z \in \text{Fact}(\text{hpw}(w))} D(z) = \sup_{i \in \mathbb{N}_0} D(w_i),$$

and thus finally

$$D(\text{hpw}(w)) = D(w) + 1.$$

■

1.2.3 A few more aperiodic words of finite defect

The following word was defined in [14].

Theorem 1.12. *Let f be the morphism defined by $f(1) = 1213$, $f(2) = \varepsilon$ and $f(3) = 23$. Let $f^\infty(1) = \lim_{i\to\infty} f^i(1)$. The infinite word $f^\infty(1)$ has the following properties:*

- $f^\infty(1)$ *is aperiodic;*
- $\text{Fact}(f^\infty(1))$ *is closed under reversal;*
- $f^\infty(1)$ *is recurrent but not uniformly recurrent;*
- $D(f^\infty(1))$ *is finite and positive.*

Proof. In view of Remark 2.2 (see page 37), the proof follows from the respective properties of generalized highly potential words, in particular, Section 2.3, as well as Theorem 2.13 and Corollary 2.14 ■

This was the first example in the literature of an infinite word whose set of factors is closed under reversal, which is not uniformly recurrent and which has finite and positive defect. Previously, the following examples of words that are not uniformly recurrent and whose defect is 0 had been seen in [34]: 1) $\varphi_1^\infty(a)$ where $\varphi_1 : a \mapsto aba$, $b \mapsto bb$ (an example taken from [25], where it was considered for another purpose); 2) the Cantor word (also known as the Sierpiński word), that is, $\varphi_2^\infty(a)$ where $\varphi_2 : a \mapsto aba$, $b \mapsto bbb$ (a well-known word; see, for example, [52], which the authors cite); 3) $\varphi_3^\infty(a)$ where $\varphi_3 : a \mapsto abab$, $b \mapsto b$ (the authors' own example); the proof that they have the mentioned properties will also be a special case of some results from this thesis.

1.3 MP-ratio

Clearly, each binary word w contains a subpalindrome of length at least $\lceil \frac{|w|}{2} \rceil$ (e.g., a subpalindrome consisting only of a prevalent letter of w). We say that a binary word w is *minimal-palindromic* if and only if it does not contain a subpalindrome longer than $\lceil \frac{|w|}{2} \rceil$. For $w \in \{0,1\}^*$, a pair (r,s), where $r,s \in \{0,1\}^*$, such that rws is minimal-palindromic, is called an *MP-extension* of w, and if the length $|r|+|s|$ is the least possible, then the pair (r,s) is called a *shortest MP-extension*, or *SMP-extension* of w. The rational number $\frac{|rws|}{|w|}$, where (r,s) is an SMP-extension of w, is called the *MP-ratio* of w. As shown in [39], each binary word possesses an MP-extension (and thus also an SMP-extension, that is, the MP-ratio is well-defined); further, the MP-ratio of any binary word is bounded from above by 4, and this is the best possible upper bound. We first prove the upper bound.

Theorem 1.13. *The MP-ratio of any binary word is at most* 4.

Proof. Let $w \in \{0,1\}^*$. We shall prove that $(r,s) = (0^{2|w|-|w|_0}, 1^{2|w|-|w|_1})$ represents an MP-extension of w, that is, the word

$$0^{2|w|-|w|_0} \;\; w \;\; 1^{2|w|-|w|_1}$$

is always minimal-palindromic. Because of $|rws| = 4|w|$, we need to prove that rws does not contain a subpalindrome whose length exceeds $2|w|$. Let us consider a subpalindome of the form $0p0$. If $0p0 \in 0^*$, then clearly

$$|0p0| \leqslant |rws|_0 = 2|w|.$$

If $|0p0|_1 \neq 0$, we can write $0p0 = 0p'1q'0$, where the highlighted 1 is the first appearance of the letter 1 in the word $0p'1q'0$. Then clearly $|0p'| \leqslant |q'0|$. Also, $1q'0 \in \mathrm{Subw}(w)$, which implies $|1q'0| \leqslant |w|$. Therefore, we have

$$\begin{aligned} |0p0| = |0p'1q'0| = |0p'| + |1q'0| &\leqslant |q'0| + |1q'0| \\ &\leqslant |w| - 1 + |w| = 2|w| - 1 < 2|w|, \end{aligned}$$

which was to be proved. For subpalindromes of the form $1p1$ the proof is analogous. ∎

We shall now prove that 4 is the best possible upper bound on the MP-ratio in binary case. Before proceeding to the main theorem, we need to introduce the concept of k-economic and economic words and prove some auxiliary lemmas.

We say that a word $w \in \{0,1\}^*$ is k-*economic* (with respect to the letter 1) if and only if w is a palindrome and the word $w1^k$ contains a subpalindrome of length at least $|w|_1 + k + 2$. Each such subpalindrome can be written in the form $1^m q 1^m$ where $0 \leqslant m \leqslant k$ and $1^m q \in \mathrm{Subw}(w)$; the pair (q, m) is then called a k-*witness* of w. We say that w is *economic* if and only if it is k-economic for every k, $k = 0, 1, \ldots, |w|_1$.

Lemma 1.14. *Let $w \in \{0,1\}^*$ and let (r, s) be an MP-extension of w. If w is economic, then $|rs|_1 > |w|_1$.*

Proof. Suppose the contrary, that the required inequality does not hold, that is, $|rs|_1 \leqslant |w|_1$. Let $|r|_1 = i$ and $|s|_1 = j$ and suppose that $i \leqslant j$ (in the case $j \leqslant i$ the proof is analogous because w is a palindrome). Since w is economic and $j - i \leqslant |s|_1 \leqslant |rs|_1 \leqslant |w|_1$, we conclude that w is $(j - i)$-economic. Therefore, $w1^{j-i}$ contains a subpalindrome of length at least $|w|_1 + j - i + 2$, and that subpalindrome can be written in the form $1^m q 1^m$ for $m \leqslant j - i$ and $1^m q \in \mathrm{Subw}(w)$. This implies that $1^{m+i} w 1^{m+i}$ is a subpalindrome of rws and for its length we have:

$$
\begin{aligned}
|1^{m+i} q 1^{m+i}| &= 2i + |1^m q 1^m| \geqslant 2i + |w|_1 + j - i + 2 = |w|_1 + i + j + 2 \\
&= |rws|_1 + 2 > |rws|_1 + 1 \geqslant \left\lceil \frac{|rws|}{2} \right\rceil .
\end{aligned}
$$

Contradiction, since the word rws is minimal-palindromic. Thus, the proof is completed. ■

Lemma 1.15. *Let $w \in \{0,1\}^*$ and let (r, s) be an MP-extension of w. If w is economic, then $|rws| > 4|w|_1$.*

Proof. First, it is easy to see that in a minimal-palindromic word the number of letters 0 and 1 differ by at most 1. Since rws is a minimal-palindromic word, we have:

$$
\begin{aligned}
|rws| &= |rws|_0 + |rws|_1 \geqslant 2|rws|_1 - 1 = 2|w|_1 + 2|rs|_1 - 1 \\
&\geqslant 2|w|_1 + 2(|w|_1 + 1) - 1 > 4|w|_1,
\end{aligned}
$$

where we used the previous lemma in the penultimate step. ■

Lemma 1.16. *Let w_0 be an economic word. We define the sequence $(w_i)_{i\geqslant 0}$ recursively by*

$$w_{i+1} = w_i 1^{t_i} w_i, \tag{1.11}$$

where $(t_i)_{i\geqslant 0}$ is a given sequence of positive integers. If for each nonnegative integer i we have $t_i < |w_i|_0$, then all the words w_i are economic.

Proof. We prove the lemma by induction on i. For $i = 0$ there is nothing to prove. Let us assume that w_i is economic and prove that then w_{i+1} also must be. We do it by proving that w_{i+1} is k-economic for each k, $k = 0, 1, \dots, |w_{i+1}|_1$. We distinguish three cases depending on the value of k, and in each case find a subpalindrome p in $w_{i+1}1^k$ whose length is at least $|w_{i+1}|_1 + k + 2$.

Assume first $0 \leqslant k \leqslant |w_i|_1$. Then w_i is k-economic by the inductive assumption. Let (q, m) be a k-witness of w_i. Consider the word

$$p = 1^m q 1^{t_i+m} q 1^m.$$

We have $1^m q \in \mathrm{Subw}(w_i)$, since (q, m) is a k-witness of w_i. Also, since $m \leqslant k$, we have $1^m \in \mathrm{Subw}(1^k)$. Altogether, $p \in \mathrm{Subw}(w_i 1^{t_i} w_i 1^k) = \mathrm{Subw}(w_{i+1}1^k)$, and we have

$$\begin{aligned}|p| &= 3m + 2|q| + t_i = 2(2m + |q|) - m + t_i \geqslant 2(|w_i|_1 + k + 2) - k + t_i \\ &= |w_{i+1}|_1 + k + 4 > |w_{i+1}|_1 + k + 2\end{aligned}$$

(where $|2m| + q = |1^m q 1^m| \geqslant |w_i|_1 + k + 2$ follows from the fact that w_i is k-economic). Therefore, w_{i+1} is also k-economic.

Assume now that $|w_i|_1 < k \leqslant |w_i|_1 + t_i$. For the word

$$p = 1^k w_i 1^k$$

we have $p \in \mathrm{Subw}(w_{i+1}1^k)$, because of $1^k \in \mathrm{Subw}(w_i 1^{t_i})$. Furthermore, since $t_i + 1 \leqslant |w_i|_0$ and $|w_i|_1 + 1 \leqslant k$, we have

$$|p| = 2k + |w_i|_1 + |w_i|_0 \geqslant k + 2|w_i|_1 + 1 + t_i + 1 = |w_{i+1}|_1 + k + 2,$$

which means that w_{i+1} is k-economic.

Finally, suppose that $|w_i|_1 + t_i < k \leqslant |w_{i+1}|_1$. Let $j = |w_i|_1 + t_i$ and $l = k - j$. Since $l < |w_i|_1$, the word w_i is l-economic. Let (q, m) be an l-witness of w_i. Consider the word

$$p = 1^{m+j} q 1^{m+j}.$$

By its construction, p is a subword of $w_{i+1}1^k$ (indeed, $1^j \in \mathrm{Subw}(w_i 1^{t_i})$, $1^m q \in \mathrm{Subw}(w_i)$ and $1^{m+j} \in \mathrm{Subw}(1^k)$), and

$$|p| = 2j + |1^m q 1^m| \geqslant 2j + |w_i|_1 + l + 2 = |w_{i+1}|_1 + k + 2.$$

Therefore, w_{i+1} is economic also in this case, which completes the proof. ∎

In the following lemma we show that there exist arbitrarily long economic words.

Lemma 1.17. *For a sequence $(t_i)_{i\geqslant 0}$, let $w(t_0, t_1, \ldots, t_{j-1})$ denote the word w_j defined by (1.11) with the initial term $w_0 = 0000$. For each k, $k \geqslant 448$, there exists an economic word v_k of length k such that $v_k = w(t_0, t_1, \ldots, t_{n-1})$ for some n, $n \geqslant 6$, and some integers $t_0, t_1, \ldots, t_{n-1}$ satisfying $2^i \leqslant t_i < 2^{i+2}$ for each i, $i = 0, 1, \ldots, n-1$.*

Proof. First of all, it is easily checked that w_0 is an economic word. Further,

$$|w_j|_0 = |w(t_0, t_1, \ldots, t_{j-1})|_0 = 2^{j+2}$$

independently of the sequence $(t_i)_{i\geqslant 0}$.

Now, if $2^i \leqslant t_i < 2^{i+2}$ for all i, $0 \leqslant i \leqslant j-1$, we then have $t_i < |w_i|_0$ for each such i, so by Lemma 1.16 the word $w(t_0, t_1 \ldots, t_{j-1})$ is economic. We calculate its length:

$$\begin{aligned}|w(t_0, t_1 \ldots, t_{j-1})| &= |w(t_0, t_1 \ldots, t_{j-1})|_0 + |w(t_0, t_1 \ldots, t_{j-1})|_1 \\ &= 2^{j+2} + t_{j-1} + 2t_{j-2} + 2^2 t_{j-3} + \cdots + 2^{j-1} t_0 \\ &= 2^{j+2} + \sum_{i=0}^{j-1} 2^{j-1-i} t_i.\end{aligned}$$

Let $(\alpha_i)_{i\geqslant 0}$ be the sequence defined by $\alpha_i = 2^i$ and $(\beta_i)_{i\geqslant 0}$ the sequence defined by $\beta_i = 2^{i+2} - 1$. Then

$$\begin{aligned}|w(\alpha_0, \alpha_1 \ldots, \alpha_{j-1})| &= 2^{j+2} + \sum_{i=0}^{j-1} 2^{j-1-i} 2^i = 2^{j+2} + \sum_{i=0}^{j-1} 2^{j-1} \\ &= 2^{j+2} + j 2^{j-1} = 2^{j-1}(8 + j),\end{aligned}$$

while

$$
\begin{aligned}
|w(\beta_0, \beta_1 \dots, \beta_{j-1})| &= 2^{j+2} + \sum_{i=0}^{j-1} 2^{j-1-i}(2^{i+2} - 1) \\
&= 2^{j+2} + \sum_{i=0}^{j-1} 2^{j+1} - \sum_{i=0}^{j-1} 2^{j-1-i} \\
&= 2^{j+2} + j2^{j+1} - (2^j - 1) \\
&= 2^j(3 + 2j) + 1.
\end{aligned}
$$

Note that the inequality

$$|w(\alpha_0, \alpha_1, \dots, \alpha_j)| < |w(\beta_0, \beta_1 \dots, \beta_{j-1})| \tag{1.12}$$

is equivalent to

$$2^j(9 + j) < 2^j(3 + 2j) + 1,$$

which is true for $j \geqslant 6$. Also, notice that

$$\{[|w(\alpha_0, \alpha_1, \dots, \alpha_{j-1})|, |w(\alpha_0, \alpha_1, \dots, \alpha_j)| - 1]_{\mathbb{N}} : j \in \mathbb{N}\}$$

represents a partition of the set of integers greater than 8, which means that for each k large enough there exists an integer j such that

$$|w(\alpha_0, \alpha_1, \dots, \alpha_{j-1})| \leqslant k < |w(\alpha_0, \alpha_1, \dots, \alpha_j)|. \tag{1.13}$$

Now (1.12) and (1.13) imply that for each k, $k \geqslant 448$ (where $k \geqslant 448$ comes from $448 = |w(\alpha_0, \alpha_1, \dots, \alpha_5)|$), there exists an integer n, $n \geqslant 6$, such that

$$|w(\alpha_0, \alpha_1, \dots, \alpha_{n-1})| \leqslant k < |w(\beta_0, \beta_1 \dots, \beta_{n-1})|.$$

We shall show now that for each integer k from the interval

$$[|w(\alpha_0, \alpha_1, \dots, \alpha_{n-1})|, |w(\beta_0, \beta_1 \dots, \beta_{n-1})| - 1]_{\mathbb{N}}$$

there exists a word $v_k = w(t_0, t_1, \dots, t_{n-1})$, $|v_k| = k$, where the elements of the sequence $(t_i)_{i=0}^{n-1}$ satisfy $\alpha_i \leqslant t_i \leqslant \beta_i$. We prove it by induction. Assume that the assertion holds for a given k. Let

$$k = |w(t_0, t_1 \dots, t_s, \beta_{s+1}, \dots, \beta_{n-1})|,$$

where s denotes the largest index such that $t_s < \beta_s$. We claim that then

$$k+1 = |w(t_0, t_1 \ldots, t_{s-1}, t_s+1, \beta_{s+1}-1, \ldots, \beta_{n-1}-1)|.$$

Indeed,

$$\begin{aligned}
|w(t_0,&t_1 \ldots, t_{s-1}, t_s+1, \beta_{s+1}-1, \ldots, \beta_{n-1}-1)| \\
&= 2^{n+2} + \sum_{i=0}^{s-1} 2^{n-1-i} t_i + 2^{n-1-s}(t_s+1) + \sum_{i=s+1}^{n-1} 2^{n-1-i}(\beta_i - 1) \\
&= 2^{n+2} + \sum_{i=0}^{s} 2^{n-1-i} t_i + 2^{n-1-s} + \sum_{i=s+1}^{n-1} 2^{n-1-i}\beta_i - \sum_{i=s+1}^{n-1} 2^{n-1-i} \\
&= 2^{n+2} + \sum_{i=0}^{s} 2^{n-1-i} t_i + \sum_{i=s+1}^{n-1} 2^{n-1-i}\beta_i + 2^{n-1-s} - (2^{n-1-s}-1) \\
&= 2^{n+2} + \sum_{i=0}^{s} 2^{n-1-i} t_i + \sum_{i=s+1}^{n-1} 2^{n-1-j}\beta_i + 1 \\
&= |w(t_0, t_1 \ldots, t_s, \beta_{s+1}, \ldots, \beta_{n-1})| + 1 = k+1.
\end{aligned}$$

The proof is now completed. ∎

We are now ready to prove that the obtained upper bound is optimal [39, Theorem 5].

Theorem 1.18. *Let $R_2(n)$ denote the maximal MP-ratio over all the words $w \in \{0,1\}^*$, $|w| = n$. We have*

$$\lim_{n\to\infty} R_2(n) = 4.$$

Proof. We shall use the notation from the previous lemmas. Notice that for each $j \geqslant 1$ we have

$$|w(t_0, t_1, \ldots, t_{j-1})|_1 \geqslant |w(\alpha_0, \alpha_1, \ldots, \alpha_{j-1})|_1,$$

while

$$|w(t_0, t_1, \ldots, t_{j-1})|_0 = |w(\alpha_0, \alpha_1, \ldots, \alpha_{j-1})|_0,$$

from which we can get

$$1 \geqslant \frac{|w(t_0, t_1, \ldots, t_{j-1})|_1}{|w(t_0, t_1, \ldots, t_{j-1})|} \geqslant \frac{|w(\alpha_0, \alpha_1, \ldots, \alpha_{j-1})|_1}{|w(\alpha_0, \alpha_1, \ldots, \alpha_{j-1})|}.$$

For the right-hand side, we know that

$$\frac{|w(\alpha_0, \alpha_1, \ldots, \alpha_{j-1})|_1}{|w(\alpha_0, \alpha_1, \ldots, \alpha_{j-1})|} = \frac{2^{j-1}j}{2^{j-1}(j+8)} = \frac{j}{j+8}.$$

This is enough to conclude that the words v_k satisfy

$$\lim_{k\to\infty} \frac{|v_k|_1}{|v_k|} = 1. \tag{1.14}$$

In order to prove the theorem, it is enough to show that for any positive real number η there exists a positive integer k_0 such that, for each $k \geqslant k_0$, MP-ratio of the word v_k is greater than $4 - \eta$. Let η be given. Choose an integer k_0 such that, for each $k \geqslant k_0$, we have

$$\frac{|v_k|_1}{|v_k|} > 1 - \frac{\eta}{4}$$

(such k_0 exists because of (1.14)). Let a pair (r, s) be an MP-extension of v_k, $k \geqslant k_0$. By Lemma 1.15, due to the fact that the word v_k is economic, we have

$$\frac{|rv_ks|}{|v_k|} > \frac{4|v_k|_1}{|v_k|} > 4 - \eta;$$

therefore, the MP-ratio of v_k is greater than $4 - \eta$. This completes the proof. ■

2

Generalized highly potential words

2.1 Construction

Definition 2.1. Let $w, u, v \in \Sigma^*$, where $wuv \neq \varepsilon$ and u and v are palindromes, and let $A = (a_i)_{i\in\mathbb{N}}$ be a strictly increasing sequence of positive integers. We recursively define:

$$w_0 = w;$$

$$w_i = w_{i-1}(uv)^{a_i}u\widetilde{w_{i-1}}, \;\; i \in \mathbb{N};$$

and then:

$$\mathrm{ghpw}(w, u, v, A) = \lim_{i\to\infty} w_i. \tag{2.1}$$

(The limit is well-defined since each w_i is a prefix of w_{i+1}.) The infinite word $\mathrm{ghpw}(w, u, v, A)$ is called *generalized highly potential word generated by* w*,* u*,* v *and* A*.*

Remark 2.2. We first note that generalized highly potential words are indeed a generalization of highly potential words: if w is a nonpalindromic word, c a letter that does not appear in w, and I the sequence $(i)_{i\in\mathbb{N}}$, then we clearly have

$$\mathrm{hpw}(w) = \mathrm{ghpw}(w, \varepsilon, c, I).$$

For the word mentioned in Theorem 1.12 in Subsection 1.2.3 holds

$$f^\infty(1) = \mathrm{ghpw}(1213121, 3, 2, I).$$

Also, it is easy to see that the other three words in the same subsection can be represented respectively as

$$\begin{aligned}
\varphi_1^\infty(a) &= \mathrm{ghpw}(a, \varepsilon, b, (2^{i-1})_{i\in\mathbb{N}}),\\
\varphi_2^\infty(a) &= \mathrm{ghpw}(a, \varepsilon, b, (3^{i-1})_{i\in\mathbb{N}}),\\
\varphi_3^\infty(a) &= \mathrm{ghpw}(a, \varepsilon, b, I).
\end{aligned}$$

2.2 Standard form

Different quadruples of parameters (w, u, v, A) can lead to the same generalized highly potential word. In the following lemma we shall prove that for each generalized highly potential word there can be chosen a quadruple with some particular properties that will be very useful.

Lemma 2.3. *Let* $\mathrm{ghpw}(w, u, v, A)$ *be a generalized highly potential word. Then there are words* w^S*,* u^S*,* v^S *and a sequence* A^S *such that* w^S *is a palindrome,* $u^S v^S$ *is primitive, and*

$$\mathrm{ghpw}(w, u, v, A) = \mathrm{ghpw}(w^S, u^S, v^S, A^S).$$

Proof. We first show that a quadruple can be chosen such that w is a palindrome. Suppose that $w \neq \widetilde{w}$. Since $w_1 = w(uv)^{a_1}u\widetilde{w}$, we see that w_1 is always a palindrome. It is not hard to see that

$$\mathrm{ghpw}(w, u, v, A) = \mathrm{ghpw}(w_1, u, v, B),$$

where $B = (b_i)_{i\in\mathbb{N}}$, $b_i = a_{i+1}$.

Suppose now that uv is not primitive, that is, $uv = t^n$ for a word t and an integer n, $n \geqslant 2$. We can assume that t is primitive. Note that we can write $t = u'v'$ where $u' \in \mathrm{Suff}(u)$ and $v' \in \mathrm{Pref}(v)$ (one of u' and v' can be ε). Then we have:

$$\begin{aligned}
u &= (u'v')^k u',\\
v &= v'(u'v')^l
\end{aligned}$$

for some integers k and l that satisfy $k + l = n - 1$. Also, since u' is both the prefix and the suffix of the palindrome u, we conclude that u' is also a palindrome; in a similar manner, v' is a palindrome, too. We prove that

$$\mathrm{ghpw}(w, u, v, A) = \mathrm{ghpw}(w, u', v', C),$$

where C is an increasing sequence defined by $C = (c_i)_{i\in\mathbb{N}}$,

$$c_i = na_i + k.$$

This follows by induction, noting that

$$\begin{aligned} w_{i+1} &= w_i(uv)^{a_{i+1}}uw_i = w_i((u'v')^n)^{a_{i+1}}(u'v')^k u'w_i \\ &= w_i(u'v')^{na_{i+1}+k}u'w_i = w_i(u'v')^{c_{i+1}}u'w_i. \end{aligned}$$

The proof is completed. ■

If a quadruple (w, u, v, A) is such that w is a palindrome and uv is primitive, we shall say that $\mathrm{ghpw}(w, u, v, A)$ is in *standard form*. The previous lemma shows that each generalized highly potential word can be presented in standard form.

Remark 2.4. The assumption that uv is primitive will be used very much, most of the times in the form of the following consequence: in that case, by Theorem 1.1, uv appears as a factor of $uvuvuv\ldots$ only at the "obvious positions" (in other words, $|uvuv|_{uv} = 2$; to be more precise, by this term we shall onward refer to the appearances of uv within $uvuvuv\ldots$ that begin at a position i where $i \equiv 1 \pmod{|uv|}$); furthermore, the same also holds for each conjugate of uv (each conjugate of a primitive word is again primitive, which is also easily seen by Theorem 1.1).

Another (technical) consequence of the assumption that uv is primitive (that will also be useful) is given in the following lemma.

Lemma 2.5. *Assume that u and v are palindromes, $uv \neq \varepsilon$, such that the word uv is primitive. Let x be a palindrome such that $|x| \geqslant 2|uv| - 1$ and $x\big[1, \lfloor\frac{|x|}{2}\rfloor + |uv|\big] = (vu)^\infty\big[1, \lfloor\frac{|x|}{2}\rfloor + |uv|\big]$. Then there exists a positive integer m such that $x = (vu)^m v$.*

Proof. Let $y = x\big[\lfloor\frac{|x|}{2}\rfloor + 1, \lfloor\frac{|x|}{2}\rfloor + |uv|\big]$. By the lemma's assumption, y is a conjugate of vu, and thus we may write $y = (vuvu)[i, j]$ for some i and j, where $1 \leqslant i, j \leqslant |vuvu|$ and $j - i + 1 = |uv|$. Since x matches $(vu)^\infty$ for the first $\lfloor\frac{|x|}{2}\rfloor + |uv|$ letters, Remark 2.4 leads to

$$\left\lfloor \frac{|x|}{2} \right\rfloor + 1 \equiv i \pmod{|uv|}.$$

We also have $\widetilde{y} = (\widetilde{vuvu})[2|uv| - j + 1, 2|uv| - i + 1] = (uvuv)[2|uv| - j + 1, 2|uv| - i + 1]$ and (since x is palindromic) $\widetilde{y} = x\big[\lceil \frac{|x|}{2} \rceil - |uv| + 1, \lceil \frac{|x|}{2} \rceil\big]$; this implies (by again appealing to Remark 2.4 in a similar manner)

$$\left\lceil \frac{|x|}{2} \right\rceil \equiv |v| + (2|uv| - i + 1) \equiv |v| + 1 - i \pmod{|uv|}.$$

Adding the two congruences together gives

$$\left\lfloor \frac{|x|}{2} \right\rfloor + 1 + \left\lceil \frac{|x|}{2} \right\rceil \equiv |v| + 1 \pmod{|uv|},$$

that is,

$$|x| \equiv |v| \pmod{|uv|}.$$

Together with the lemma's assumption and the fact that x is palindromic, this gives the required conclusion. ∎

2.3 Basic properties

We first present a necessary and sufficient condition for a generalized highly potential word to be periodic.

Theorem 2.6. *Let the word* $\mathrm{ghpw}(w, u, v, A)$ *be given in standard form. Then* $\mathrm{ghpw}(w, u, v, A)$ *is periodic if and only if either* $w = (vu)^m v$ *for a nonnegative integer* m*, or exactly one of the words* w*,* u *and* v *is nonempty.*

Proof. We first assume $w = (vu)^m v$. Then

$$\mathrm{ghpw}((vu)^m v, u, v, A) = (vu)^m v (uv)^{a_1} u (vu)^m v (uv)^{a_2} u \ldots = (vu)^\infty.$$

Also, if exactly one of w, u, v is nonempty, we have

$$\begin{aligned} \mathrm{ghpw}(w, \varepsilon, \varepsilon, A) &= w^\infty; \\ \mathrm{ghpw}(\varepsilon, u, \varepsilon, A) &= u^\infty; \\ \mathrm{ghpw}(\varepsilon, \varepsilon, v, A) &= v^\infty. \end{aligned}$$

We conclude that in all these cases the constructed word is periodic, which completes the ($\Leftarrow$) part.

To prove the converse, we assume that $\mathrm{ghpw}(w,u,v,A)$ is periodic. Then we can write

$$\mathrm{ghpw}(w,u,v,A) = w(uv)^{a_1-1}uvuwuv(uv)^{a_2-1}uw(uv)^{a_1}uw\cdots = s^{\infty},$$

where s can be chosen (long enough) such that $vuwuv \in \mathrm{Fact}(s)$. Assume that at least one of u and v is nonempty (there is nothing to prove if $u = v = \varepsilon$). Then we can choose i large enough such that $s \in \mathrm{Fact}((uv)^{a_i})$, which implies $vuwuv \in \mathrm{Fact}((uv)^{a_i})$. Recall (by Remark 2.4) that vu and uv appear in $(uv)^{a_i}$ only at the obvious positions. If $u \neq \varepsilon$, then the above gives

$$w = (vu)^m v \quad \text{for a nonnegative integer } m, \tag{2.2}$$

which was to be proved (note that $w = v = \varepsilon$ is a special case of this); if $u = \varepsilon$, then we conclude $w = v^l$ for a nonnegative integer l, that is, w is again of the form (2.2), or $w = u = \varepsilon$. This completes the proof. ■

We shall now prove that each generalized highly potential word is either periodic, or recurrent but not uniformly recurrent. We first need the following assertion.

Proposition 2.7. $\mathrm{Fact}(\mathrm{ghpw}(w,u,v,A))$ *is closed under reversal.*

Proof. Let $x \in \mathrm{Fact}(\mathrm{ghpw}(w,u,v,A))$. Choose a large enough integer i such that $x \in \mathrm{Fact}(w_i)$. Since

$$w_{i+1} = w_i(uv)^{a_{i+1}}u\widetilde{w}_i,$$

we have

$$\widetilde{x} \in \mathrm{Fact}(\widetilde{w}_i) \subseteq \mathrm{Fact}(w_{i+1}) \subseteq \mathrm{Fact}(\mathrm{ghpw}(w,u,v,A)),$$

which was to be proved. ■

Now, in view of Theorem 1.2, we have the following corollary.

Corollary 2.8. *Each generalized highly potential word is recurrent.*

Concerning uniform recurrence, we have:

Proposition 2.9. *A generalized highly potential word is uniformly recurrent if and only if it is periodic.*

Proof. The part ($\Leftarrow$) is clear by Theorem 1.3. Let us prove the other direction. Suppose the contrary: $\mathrm{ghpw}(w, u, v, A)$ is uniformly recurrent but not periodic. Since $vuwuv$ is a factor of $\mathrm{ghpw}(w, u, v, A)$, there exists a positive integer n such that $vuwuv$ is a factor of any factor of $\mathrm{ghpw}(w, u, v, A)$ of length n. Choose x such that $|x| = n$ and $x \in \mathrm{Fact}((uv)^{a_i})$ for some i. Now in the same manner as in the proof of Theorem 2.6 we get that either $w = (vu)^m v$ for a nonnegative integer m, or that exactly one of the words w, u and v is nonempty; in other words, $\mathrm{ghpw}(w, u, v, A)$ is periodic, which is a contradiction. ■

Finally, we have the following proposition.

Proposition 2.10. *If a generalized highly potential word is not periodic, then it is aperiodic.*

Proof. Suppose the contrary: $\mathrm{ghpw}(w, u, v, A)$ is eventually periodic but not periodic. By Corollary 2.8, it is recurrent, but then Theorem 1.4 implies that it must be periodic; contradiction. ■

2.4 Defect of generalized highly potential words

In this section we prove that the defect of each generalized highly potential word is always finite. Before proceeding to the main theorem, we need two technical lemmas.

Lemma 2.11. *Let a nonperiodic* $\mathrm{ghpw}(w, u, v, A)$ *be given in standard form, where* $vu \notin \mathrm{Pref}(wuv)$. *Assume that there exists an integer* i *such that* $i \geqslant 3$ *and*

$$|w_i|_{(uv)^{a_i}u} = 1. \tag{2.3}$$

Then

$$|w_{i+1}|_{w_i} = 2 \tag{2.4}$$

and

$$|w_{i+1}|_{(uv)^{a_{i+1}-1}u} = 2 + 2|w_i|_{(uv)^{a_{i+1}-1}u}.$$

Proof. Let us first prove (2.4). Write

$$w_{i+1} = w_i(uv)^{a_{i+1}}uw_i.$$

Clearly, $|w_{i+1}|_{w_i} \geqslant 2$. Suppose that there is a third copy of w_i in w_{i+1}. Note that $(uv)^{a_i}u$ occurs in the center of the considered copy of w_i, and we now conclude that this copy of $(uv)^{a_i}u$ must (partly) overlap the central copy of $(uv)^{a_{i+1}}u$ in w_{i+1} (otherwise we would have $|w_i|_{(uv)^{a_i}u} \geqslant 2$, which is impossible). Suppose that the length of the overlapping part is greater than or equal to $|uv|$. Then the overlapping part contains the factor vu or uv, and by Remark 2.4 this factor is positioned within the central copy of $(uv)^{a_{i+1}}u$ at one of the obvious positions. But this means that the central copy of $(uv)^{a_i}u$ in w_i must be preceded by uv or followed by vu, and since it is preceded by vuw and followed by wuv, we have a contradiction with $vu \notin \mathrm{Pref}(wuv)$ (or, which is the same, $uv \notin \mathrm{Suff}(vuw)$). Therefore, the overlapping part is shorter than $|uv|$. We may assume, without loss of generality, that the overlapping part and the considered copy of $(uv)^{a_i}u$ have a common endpoint (the other possibility: that they have a common starting point, is analogous). The part of the considered copy of $(uv)^{a_i}u$ that does not overlap presents a suffix of w_i. Suppose that its length is greater than or equal to $|vuw|$. Since $vuw \in \mathrm{Suff}(w_i)$, we have that vuw is a suffix of the considered part of $(uv)^{a_i}u$. But then Remark 2.4 gives that the beginning of that suffix vuw must coincide with an obvious position of vu within the considered copy of $(uv)^{a_i}u$ (the one that is not in the center of w_i); since that suffix vuw is followed by uv, altogether we obtain that wuv begins with vu, which is in contradiction with the lemma's assumption. Finally, we need to check the case when the length of the non-overlapping part of the considered copy of $(uv)^{a_i}u$ is less than $|vuw|$. But then we have $(uv)^{a_i}u \in \mathrm{Fact}(vuwuv)$, and since $vuwuv \in \mathrm{Fact}(w_{i-1})$ (recall that $i \geqslant 3$), we conclude $(uv)^{a_i}u \in \mathrm{Fact}(w_{i-1})$, which contradicts (2.3). This proves (2.4).

Let us now prove

$$|w_{i+1}|_{(uv)^{a_{i+1}-1}u} = 2 + 2|w_i|_{(uv)^{a_{i+1}-1}u}.$$

The inequality ($\geqslant$) is clear (there are two copies of $(uv)^{a_{i+1}-1}u$ in the center of w_{i+1}, plus the copies in the starting and the ending w_i). We are left to show only that there are no copies of $(uv)^{a_{i+1}-1}u$ in w_{i+1} that overlap the central copy of $(uv)^{a_{i+1}}u$ but are not encompassed within it. Suppose the contrary, that there exists such a copy. Suppose first that the overlapping part is of length $|uv|$ or more. Then the overlapping part contains the factor vu or uv, and Remark 2.4 gives that this factor is positioned within the central copies of $(uv)^{a_{i+1}}u$ at one of the obvious positions. But this means that the central copy of $(uv)^{a_{i+1}}u$ in w_{i+1} must be preceded by uv or followed by vu, and since it is preceded by vuw and followed by wuv, we have a contradiction with $vu \notin \mathrm{Pref}(wuv)$ (or $uv \notin \mathrm{Suff}(vuw)$). That leaves only the case when the overlapping part is of length less than $|uv|$, but this case also leads to a contradiction in completely the same manner as in the previous paragraph. This completes the proof of the lemma. ■

Lemma 2.12. *Let a nonperiodic* $\mathrm{ghpw}(w, u, v, A)$ *be given in standard form, where* $vu \notin \mathrm{Pref}(wuv)$*. Then there exists a positive integer* i *such that:*

1) $|w_i|_{(uv)^{a_i}u} = 1$ *(where that one copy of* $(uv)^{a_i}u$ *is in the center of* w_i*);*

2) $|w_{i+1}|_{w_i} = 2$ *(where those two copies of* w_i *are at the beginning and at the end of* w_{i+1}*);*

3) $|w_{i+1}|_{(uv)^{a_{i+1}-1}u} = 2 + 2|w_i|_{(uv)^{a_{i+1}-1}u}$ *(which equals either* 2 *or* 4*, depending on whether* $a_{i+1} - 1$ *is greater than* a_i *or equal to it, respectively).*

Furthermore, if i *is any number that satisfies 1), 2) and 3), then each number* k*,* $k \geqslant i$*, has the same properties.*

Note. Since Lemmas 2.11 and 2.12 seem to somewhat overlap and might confuse the reader, before we proceed to the proof, we shall say a few words on their mutual relationship (including a sketch of the proof of Lemma 2.12, in order to let the reader know what structure of the proof to expect, and what will be the role of Lemma 2.11 there).

The proof of Lemma 2.12 consists of two parts: in the first part we prove that such a number i exists, and then in the second part we prove the last sentence from the lemma's statement.

We do the first part by finding a number i, $i \geqslant 3$, that has the property 1). Lemma 2.11 then automatically implies that the same value i also has the properties 2) and 3), which finishes the first part of the proof.

We then proceed to the second part of the proof, which we do by induction. We assume that a number $k - 1$ is given that has all the properties 1), 2) and 3), and then prove that the number k also has the properties 1), 2) and 3), which we show one by one (the inductive assumption stands for the whole time, assuming that $k - 1$ has all three properties simultaneously). In this part of the proof we have to show all the three properties one by one, that is, we cannot only show the property 1) and then refer to Lemma 2.11 for 2) and 3) (as we do in the first part of the proof), because for that we would need the condition $k \geqslant 3$, which might not hold. However, the proofs for 2) and 3) here are not just repeating the proof of Lemma 2.11 all over (although some of the steps are indeed quite similar), since the assumptions are different: in Lemma 2.11 we proved that, if k had the property 1) and $k \geqslant 3$, then k also had the properties 2) and 3), while in this proof we do not have the inequality $k \geqslant 3$ anymore, but instead have the assumption that $k-1$ has all the properties 1), 2) and 3).

We hope that this additional explanation will clear any eventual confusion of the reader. We also add that Lemma 2.12 is a key result that we shall refer to multiple times in the thesis, while Lemma 2.11 is a technical device that we use in the proof of Lemma 2.12 (only in the first part of the proof), and after that we shall not refer to it anymore.

Proof. We first show the existence of such an integer i. Let i be such that $i \geqslant 3$ and $|(uv)^{a_i}u| \geqslant |vuwuv|$. Recall

$$w_i = w_{i-1}(uv)^{a_i}uw_{i-1}.$$

Let us show that this choice of i satisfies the properties 1), 2) and 3) from the statement. It is enough to show only the part 1), since then the parts 2) and 3) will follow by Lemma 2.11.

We show that the factor $(uv)^{a_i}u$ occurs exactly once in w_i. We clearly have one copy of it in the center, so we need to prove that there are no other copies. Suppose the contrary, that there is another copy. Assume first that that copy (partly) overlaps the central copy, and that, without loss of generality, it is positioned to the left of the central copy. We again have, as in the proof of Lemma 2.11, that the length of the overlapping part cannot be greater than or equal to $|uv|$; therefore, the overlapping part is shorter than $|uv|$. The part of the considered copy of $(uv)^{a_i}u$ that does not overlap presents a suffix of w_{i-1} and its length is greater than $(uv)^{a_i-1}u$, which is, by the choice of i, at least $|vuw|$. Since $vuw \in \mathrm{Suff}(w_{i-1})$, we have that vuw is a suffix of the considered part of $(uv)^{a_i}u$. But then Remark 2.4 gives that the beginning of that suffix vuw must coincide with an obvious position of vu within the considered copy of $(uv)^{a_i}u$; since that suffix vuw is followed by uv, altogether we obtain that wuv begins with vu, which is in contradiction with the lemma's assumption. Therefore, we are left to analyze only the case when there is no overlap, that is, $(uv)^{a_i}u \in \mathrm{Fact}(w_{i-1})$. But this implies in the same way $(uv)^{a_i}u \in \mathrm{Fact}(w_{i-2})$, then $(uv)^{a_i}u \in \mathrm{Fact}(w_{i-3})$ etc., which is clearly a contradiction. This proves that the chosen value of i indeed satisfies the properties 1), 2) and 3).

Let us now prove the last sentence from the lemma's statement. Assume that i is given as required. We proceed by induction on k. We have the base for $k = i$. Now assume that the assertion holds for $k-1$.

We first prove
$$|w_k|_{(uv)^{a_k}u} = 1. \tag{2.5}$$
Recall
$$w_k = w_{k-1}(uv)^{a_k}uw_{k-1}.$$
Clearly, $|w_k|_{(uv)^{a_k}u} \geqslant 1$. Suppose that there is another copy of $(uv)^{a_k}u$ (besides the one in the center) in w_k. It cannot overlap the central copy of $(uv)^{a_{k+1}}u$ for a length of $|uv|$ nor more (as we have already seen several times); therefore, the length of the overlap is less than $|uv|$. Then there is a copy of $(uv)^{a_k-1}u$ in w_{k-1}. Note that this copy contains the factor $(uv)^{a_{k-1}}u$, and by the inductive assumption, we have that the only copy of this factor in w_{k-1} is the one in the center; however, it is followed by wuv and preceded by vuw (or only w both times, in the special case $k = 2$), and now because of the assumption $vu \notin \mathrm{Pref}(wuv)$ (then also $uv \notin \mathrm{Suff}(vuw)$) we get that this copy of $(uv)^{a_{k-1}}u$ in w_{k-1} cannot be a part of the considered copy of $(uv)^{a_k}u$ in w_k, a contradiction. This proves (2.5).

Let us now prove
$$|w_{k+1}|_{w_k} = 2. \tag{2.6}$$
(Note that we cannot simply use Lemma 2.11 here, since the inequality $k \geqslant 3$ might not hold.) Recall
$$w_{k+1} = w_k(uv)^{a_{k+1}}uw_k.$$
Clearly, $|w_{k+1}|_{w_k} \geqslant 2$. Suppose that there is a third copy of w_k in w_{k+1}. Note that $(uv)^{a_k}u$ occurs in the center of the considered copy of w_k, and we now conclude that this copy of $(uv)^{a_k}u$ must (partly) overlap the central copy of $(uv)^{a_{k+1}}u$ in w_{k+1} (otherwise we would have $|w_k|_{(uv)^{a_k}u} \geqslant 2$, which is impossible), and, by the argument that we have already seen, the length of the overlapping part cannot be greater than or equal to $|uv|$. But then, since the considered copy of $(uv)^{a_k}u$ is both preceded by and followed by w_{k-1}, we get that one of those two copies of w_{k-1} is encompassed inside w_k, neither at its beginning nor at its end. But this implies $|w_k|_{w_{k-1}} \geqslant 3$, which contradicts the inductive assumption. This proves (2.6).

Finally, we need to prove
$$|w_{k+1}|_{(uv)^{a_{k+1}-1}u} = 2 + 2|w_k|_{(uv)^{a_{k+1}-1}u}.$$

We again assume that there is a “superfluous” copy of $(uv)^{a_{k+1}-1}u$ in w_{k+1}. Then (again) it overlaps the central copy of $(uv)^{a_{k+1}}u$ for a length of less than $|uv|$, which means that the non-overlapping part of the considered copy of $(uv)^{a_{k+1}-1}u$ is of length at least $|(uv)^{a_{k+1}-2}u|$, which is at least $|(uv)^{a_k-1}u|$; however, this non-overlapping part is encompassed within w_k and we see that it contains a copy of $(uv)^{a_k-1}u$ that is “superfluous” in w_k, in contrary to the inductive assumption. This completes the proof of the lemma. ■

We are now ready for the main theorem of this section (and arguably of the whole chapter).

Theorem 2.13. *Let* $\mathrm{ghpw}(w,u,v,A)$ *be a generalized highly potential word. We have*

$$D(\mathrm{ghpw}(w,u,v,A)) < \infty.$$

Proof. If $\mathrm{ghpw}(w,u,v,A)$ is periodic, the proof will be given in the next subsection. Therefore, assume that $\mathrm{ghpw}(w,u,v,A)$ is not periodic, and we may further assume, without loss of generality, that it is given in standard form. We shall first work under the assumption $vu \notin \mathrm{Pref}(wuv)$ (and then also $uv \notin \mathrm{Suff}(vuw)$), and then return to the general case at the end of the proof.

Let i be a number whose existence is guaranteed by Lemma 2.12. It is enough to prove that for each k, $k \geqslant i+1$, we have

$$D(w_k) = D(w_{i+1}). \tag{2.7}$$

Indeed, in that case we would have, by Theorem 1.6 and the equality (2.1),

$$\begin{aligned} D(\mathrm{ghpw}(w,u,v,A)) &= \sup_{z\in\mathrm{Fact}(\mathrm{ghpw}(w,u,v,A))} D(z) = \sup_{j\in\mathbb{N}_0} D(w_j) \\ &= D(w_{i+1}), \end{aligned}$$

as needed.

In order to show (2.7), it is enough to prove only

$$D(w_{i+2}) = D(w_{i+1}); \tag{2.8}$$

indeed, in that case, by then choosing $i+1$ instead of i (note that, by the second part of Lemma 2.12, $i+1$ indeed satisfies the same requirements as i does), we would get $D(w_{i+3}) = D(w_{i+2})$ in the same way, then $D(w_{i+4}) = D(w_{i+3})$ etc., which gives (2.7).

Therefore, in order to prove (2.8), our goal is to find $|w_{i+2}| - |w_{i+1}|$ palindromes in w_{i+2} that do not occur in w_{i+1}. Since

$$\begin{aligned}|w_{i+2}| - |w_{i+1}| &= |w_{i+1}| + a_{i+2}|uv| + |u| + |w_{i+1}| - |w_{i+1}| \\ &= |w_{i+1}| + a_{i+2}|uv| + |u|,\end{aligned}$$

we need to find $|w_{i+1}| + a_{i+2}|uv| + |u|$ new palindromes. (Note: by our construction of the required number of palindromes, it will not be obvious that our list contains *all* the new palindromes. But this is not relevant: it is enough to find at least the required number of new palindromes, and Theorem 1.6 then implies that there cannot be more of them.) We distinguish four types of palindromes (after defining each type, we first explain why that type is disjoint from all the types before it; these explanations are marked by the symbol "$\lhd$").

- We first enumerate new palindromes that have the factor $(uv)^{a_{i+2}}u$ in the center; they can be obtained by "expanding" (to the left and the right side) the boxed part below:

  $$w_{i+1}\boxed{(uv)^{a_{i+2}}u}w_{i+1}.$$

  Clearly, there is a total of $|w_{i+1}|$ palindromes of this type (not counting the palindrome $(uv)^{a_{i+2}}u$ itself), and all of them must be new because $(uv)^{a_{i+2}}u$ occurs only once in w_{i+2} (and does not occur in w_{i+1}).

- We now enumerate new palindromes that have the factor w_i in the center; they can be obtained by "expanding" the boxed part below:

  $$w_{i+1}(uv)^{a_{i+2}}u\boxed{w_i}(uv)^{a_{i+1}}uw_i.$$

  $\lhd$ This type is disjoint from the first type since, by the property 1) from Lemma 2.12, there is only one copy of $(uv)^{a_{i+2}}u$ in w_{i+2}, which cannot be in the center of a palindrome of this type.

  Since there are only two occurrences of w_i in w_{i+1} (by the choice of i: Lemma 2.12, property 2)), we get that all the palindromes $\widetilde{x}w_ix$ for $x \in \mathrm{Pref}((uv)^{a_{i+1}}u) \setminus \{\varepsilon\}$ are new. But there may be even more new palindromes. Note that w_i begins with wuv. If

  $$t = \max\{|p| : p \in \mathrm{Pref}(wuv) \cap \mathrm{Pref}(vu)\},$$

then the expanding can continue further for t more new palindromes (and since $vu \notin \mathrm{Pref}(wuv)$, this is the best we can do). In total, we have

$$a_{i+1}|uv| + |u| + t$$

new palindromes of this type.

- Let us enumerate new palindromes that have the factor $(uv)^{a_{i+2}-1}u$ in the center; they can be obtained by "expanding" the boxed part below:

  $$w_{i+1}uv\boxed{(uv)^{a_{i+2}-1}u}w_{i+1}.$$

  ◁ This type is disjoint from both the previous types since, by the property 3) from Lemma 2.12, there are either 2 or 4 copies of $(uv)^{a_{i+2}-1}u$ in w_{i+2}, and none of them can be in the center of a palindrome of one of the previous two types.

  Note: these palindromes are new only if $a_{i+2} > a_{i+1} + 1$, and we shall do the counting under this assumption (otherwise, all of them would be factors of w_{i+1}). Since w_{i+1} begins with wuv, we have a total of t new palindromes here.

- Finally, we enumerate new palindromes that are factors of $(uv)^{a_{i+2}}u$.

  ◁ This type is disjoint from the first type since each palindrome of the first type is longer than each palindrome of this type.

  ◁ Let us prove that this type is disjoint from the second type. Suppose the contrary: there is a palindrome p that is of both the second and the fourth type. We shall soon see that all the palindromes of the fourth type are of length strictly greater than $|(uv)^{a_{i+1}-1}u| + 2t$, and thus strictly greater than $|(uv)^{a_i}u| + 2t$. Since p is of the second type, p has $(uv)^{a_i}u$ in the center (because w_i has $(uv)^{a_i}u$ in the center). The letter at $t+1$ positions right of this copy of $(uv)^{a_i}u$ in p is $(wuv)[t+1]$, which is different from $(vu)[t+1]$ by the definition of t. However, by Remark 2.4 and the fact that $p \in \mathrm{Fact}((uv)^{a_{i+2}}u)$ (because p is of the fourth type also), we have that the observed copy of $(uv)^{a_i}u$ in p has to be followed by vu, a contradiction. This proves the claim.

  ◁ Finally, we show that this type is disjoint from the third type. As we shall see in a moment, this type will be divided into two subtypes ((2.9) and (2.10) below). The first subtype is disjoint from the third type since there are either 2 or 4 copies of $(uv)^{a_{i+2}-1}u$ in w_{i+2}, none of which is in the center of $(uv)^{a_{i+2}}u$, while all the palindromes of the first subtype are in the center of $(uv)^{a_{i+2}}u$, and therefore they cannot have a copy of $(uv)^{a_{i+2}-1}u$ in their center. The second subtype is disjoint from the third type since each palindrome of the third type is longer than each palindrome of the second subtype.

Let us first consider palindromes of the form

$$((uv)^{a_{i+2}}u)[j, |(uv)^{a_{i+2}}u| - j + 1]. \tag{2.9}$$

In other words, they can be obtained by removing one by one letter from both ends of $(uv)^{a_{i+2}}u$ simultaneously. At one moment, we shall arrive to $(uv)^{a_{i+1}}u$ or $(uv)^{a_{i+1}-1}u$ (depending on whether a_{i+2} and a_{i+1} are of the same parity or not, respectively). Assume, e.g., the second case (the first one is similar but even easier). At this moment, the palindrome that we arrive to belongs to $\mathrm{Fact}(w_{i+1})$, so there is no point to continue further. We shall now check how many of all those palindromes exist already in $\mathrm{Fact}(w_{i+1})$. Note that, by the choice of i (in particular, the property 3) from Lemma 2.12), we know exact positions of all the copies of $(uv)^{a_{i+1}-1}u$ within w_{i+1}: there are two copies that are parts of the central $(uv)^{a_{i+1}}u$, and additionally, if they exist (that is, if $a_{i+1}-1 = a_i$), two copies in the centers of the starting and ending w_i. Since each of these copies is either preceded by vuw or followed by wuv (or both), it is easy to see that the number of the considered palindromes that belong to $\mathrm{Fact}(w_{i+1})$ is precisely t. The same conclusion holds if a_{i+2} and a_{i+1} are of the same parity. This means that so far we have enumerated $\lceil \frac{a_{i+2}-a_{i+1}}{2} \rceil |uv| - t$ new palindromes.

We now consider palindromes of the form

$$((uv)^{a_{i+2}-1}u)[j, |(uv)^{a_{i+2}-1}u| - j + 1]. \tag{2.10}$$

We again remove one by one letter from both ends of $(uv)^{a_{i+2}-1}u$ simultaneously until we reach $(uv)^{a_{i+1}}u$ or $(uv)^{a_{i+1}-1}u$. The same argument as in the previous paragraph shows that there are $\lfloor \frac{a_{i+2}-a_{i+1}}{2} \rfloor |uv| - t$ new palindromes here, but there is one exceptional case: namely, if $a_{i+2} = a_{i+1} + 1$, then already the starting palindrome $(uv)^{a_{i+2}-1}u$ belongs to $\mathrm{Fact}(w_{i+1})$, and thus then we get 0 new palindromes (the formula above would give a senseless value of $-t$, the explanation of which is that the subtracted t palindromes in this exceptional case are not factors of $(uv)^{a_{i+2}-1}u$, and thus we do not need to subtract them).

Since $\lceil \frac{a_{i+2}-a_{i+1}}{2} \rceil + \lfloor \frac{a_{i+2}-a_{i+1}}{2} \rfloor = a_{i+2} - a_{i+1}$, we may conclude that, altogether, there is a total of

$$(a_{i+2} - a_{i+1})|uv| - 2t$$

new palindromes of this type if $a_{i+2} > a_{i+1} + 1$, and

$$|uv| - t$$

if $a_{i+2} = a_{i+1} + 1$.

Finally, let us sum all the numbers. If $a_{i+2} = a_{i+1} + 1$, then we have found

$$\begin{aligned}|w_{i+1}| + (a_{i+1}|uv| + |u| + t) + (|uv| - t) &= |w_{i+1}| + (a_{i+1} + 1)|uv| + |u| \\ &= |w_{i+1}| + a_{i+2}|uv| + |u|\end{aligned}$$

new palindromes (recall that we ignore the third bullet here); if $a_{i+2} > a_{i+1} + 1$, then we have found

$$\begin{aligned}&|w_{i+1}| + (a_{i+1}|uv| + |u| + t) + t + ((a_{i+2}-a_{i+1})|uv| - 2t) \\ &\qquad = |w_{i+1}| + |u| + a_{i+2}|uv|\end{aligned}$$

new palindromes. In both cases, we get what was to be proved.

We now need to address the case $vu \in \mathrm{Pref}(wuv)$. Let

$$s = \min\{j : (wuv)[j] \neq (vu)^\infty[j]\} - |uv|$$

(such a number s must exist since otherwise Remark 2.4 would imply that w is of the form $vuvu \dots vuv$ and thus $\mathrm{ghpw}(w, u, v, A)$ would be periodic). Note that the assumption $vu \in \mathrm{Pref}(wuv)$ implies that s is positive. We also show that $s \leqslant \lfloor \frac{|w|}{2} \rfloor$: indeed, if this were not the case, then the word $vuwuv$ would be a palindromic word that would match $(vu)^\infty$ for the first $|vu| + \lfloor \frac{|w|}{2} \rfloor + |uv|$ (which is $\lfloor \frac{|vuwuv|}{2} \rfloor + |uv|$) letters, and then Lemma 2.5 would imply $vuwuv = (vu)^m v$; therefore, w would also be of the form $vuvu \dots vuv$, which would contradict the fact that $\mathrm{ghpw}(w, u, v, A)$ is not periodic. Now, let:

$$\begin{aligned} e &= (|u| + 2s) \bmod |uv|;\\ l &= (-s) \bmod |uv|;\\ w' &= w[s+1, |w|-s];\\ u' &= (uvuv)[l+1, l+e];\\ v' &= (uvuv)[l+e+1, l+|uv|];\\ A' &= \left(a_i + \frac{|u|+2s-e}{|uv|}\right)_{i=1}^{\infty} = (a'_i)_{i=1}^{\infty}. \end{aligned} \tag{2.11}$$

Notice that $u'v'$ is a conjugate of uv, and we have

$$\begin{aligned} (u'v')^\infty &= (uv)[l+1, |uv|](uv)^\infty = (u'v')^\infty[1, |uv|-l](uv)^\infty\\ &= (u'v')^\infty[1, s](uv)^\infty, \end{aligned} \tag{2.12}$$

and also

$$\begin{aligned} \widetilde{(u'v')^\infty[1,s]} &= \widetilde{(uv)[l+1, |uv|](uv)^{\lfloor \frac{s}{|uv|} \rfloor}} = (vu)^{\lfloor \frac{s}{|uv|} \rfloor}(vu)[1, |uv|-l]\\ &= (vu)^\infty[1, s]. \end{aligned}$$

Further, by the definition of s, we have

$$w[1,s] = (vu)^\infty[1,s] = \widetilde{(u'v')^\infty[1,s]}.$$

We claim:

$$\mathrm{ghpw}(w, u, v, A) = w[1,s]\,\mathrm{ghpw}(w', u', v', A'). \tag{2.13}$$

It is enough to prove that for each i we have

$$w[1,s]w'_i = w_i[1, |w_i| - s]. \tag{2.14}$$

Before we proceed, we shall first check what do we get of the word $(u'v')^{a'_i}u'$ when we erase the prefix and the suffix of the length s. By (2.12), we notice that, after erasing the prefix, there remains a word of the form $uvuvuv\ldots$. Therefore, since $(u'v')^{a'_i}u'$ is a palidrome, erasing both the prefix and the suffix leaves a word of the form $(uv)^k u$ for a nonnegative integer k. We have $k|uv| + |u| + 2s = a'_i|uv| + |u'|$, which reduces to

$$k = a'_i + \frac{|u'|-|u|-2s}{|uv|} = a_i + \frac{|u|+2s-e}{|uv|} + \frac{e-|u|-2s}{|uv|} = a_i.$$

The proof of (2.14) is now a straightforward induction: the base (for $i = 0$) is clear, and for the induction step we have:

$$\begin{aligned} w[1,s]w'_{i+1} &= w[1,s]w'_i(u'v')^{a'_{i+1}}u'w'_i \\ &= w_i[1,|w_i|-s](u'v')^{\infty}[1,s](uv)^{a_{i+1}}u\widetilde{(u'v')^{\infty}[1,s]}w'_i \\ &= w_i(uv)^{a_{i+1}}uw[1,s]w'_i \\ &= w_i(uv)^{a_{i+1}}uw_i[1,|w_i|-s] \\ &= w_{i+1}[1,|w_{i+1}|-s], \end{aligned}$$

which was to be proved.

Now notice the following: w' is a palindrome (by its definition), $u'v'$ is primitive (since it is a conjugate of uv, which is primitive), and $v'u' \notin \mathrm{Pref}(w'u'v')$ (because of $(v'u')[|v'u'|] = (u'v')[1] = (uv)[l+1] = (vu)^{\infty}[s] = (vu)^{\infty}[s+|uv|]$ and $(w'u'v')[|v'u'|] = (wuv)[s+|uv|]$, and these two letters are different by the choice of s). Therefore, the word $\mathrm{ghpw}(w', u', v', A')$ satisfies all the assumptions of the first part of the proof, and we conclude that its defect is finite. Now, since $\mathrm{ghpw}(w, u, v, A)$ is recurrent, by (2.13) we conclude that each its factor is also a factor of $\mathrm{ghpw}(w', u', v', A')$ (and the other direction obviously holds, too), which finally implies:

$$D(\mathrm{ghpw}(w,u,v,A)) = D(\mathrm{ghpw}(w',u',v',A')) < \infty.$$

The proof is completed. ■

Since, as mentioned in the Preface, infinite words of defect 0 have been studied significantly more than infinite words of finite nonzero defect, it makes sense to give a characterization of generalized highly potential words of (non)zero defect. Such a characterization can easily be inferred from the proof of Theorem 2.13. We give it in the following corollary (the corollary assumes that a word is given in standard form, but if it is not, we can always rechoose the defining parameters as in the proof of Lemma 2.3 and make it in standard form).

Corollary 2.14. *Given a nonperiodic* $\mathrm{ghpw}(w, u, v, A)$ *in standard form, we have:*

1° *If* $vu \notin \mathrm{Pref}(wuv)$*, choose the smallest integer* i *that satisfies 1), 2) and 3) from the statement of Lemma 2.12, and then* $D(\mathrm{ghpw}(w, u, v, A)) = D(w_{i+1})$*.*

2° *If* $vu \in \mathrm{Pref}(wuv)$*, choose* w'*,* u'*,* v' *and* A' *as in (2.11), and then* $D(\mathrm{ghpw}(w,u,v,A)) = D(\mathrm{ghpw}(w',u',v',A'))$*, which is evaluated as in* 1° *above.*

In particular, in this way we can determine whether $D(\mathrm{ghpw}(w,u,v,A))$ *is* 0 *or positive, which gives a characterization of generalized highly potential words of (non)zero defect.*

Also note, we can easily produce many examples of generalized highly potential words of nonzero defect. The simplest way is just to take any of the words w, u or v to have nonzero defect, and then $\mathrm{ghpw}(w,u,v,A)$ also has nonzero defect. This is a sufficient, but not necessary condition: for example, if w, u and v are all rich words, but such that one of the words wu or uv has positive defect, then $\mathrm{ghpw}(w,u,v,A)$ again has positive defect.

In fact, if any two of the words w, u and v are such that they cannot be factors of the same rich word, then $\mathrm{ghpw}(w,u,v,A)$ has positive defect. It was an open question posed in [51] if it is decidable whether two rich words can be factors of the same rich word; this question has been settled (in the affirmative) very recently [54], though the deciding algorithm is not really practical. An elegant necessary condition for two rich words to be factors of the same rich words is as follows [22, Theorem 6]: no two different factors of a rich word can have the same longest palindromic prefix as well as the same longest palindromic suffix (therefore, if we want $\mathrm{ghpw}(w,u,v,A)$ to have positive defect, it is sufficient to have the stated condition violated for any two of the words w, u, v). It has been asked in [59, Open problem 6.2] whether the stated condition is also sufficient (that is, whether any two rich words that have different longest palindromic prefix or longest palindromic suffix must be factors of the same rich word); if true, this would greatly simplify the mentioned algorithm from [54], but up to the present author's knowledge, this problem is still open.

2.5 Periodic case

Finally, we show that periodic generalized highly potential words also have finite defect.

Theorem 2.15. *The defect of a periodic generalized highly potential word is finite.*

Proof. Let $\mathrm{ghpw}(w, u, v, A)$ be given in standard form, and let

$$\mathrm{ghpw}(w, u, v, A) = p^{\infty}.$$

We claim that we may assume that p is a primitive word that is a product of two palindromes (where one of them is possibly ε). Indeed, Theorem 2.6 implies that we may assume either $p = vu$ or $p \in \{w, u, v\}$, but in the latter case, if p is not primitive but, say, $p = t^n$, we may take t in place of p (t is then a palindrome since it is both a prefix and a suffix of a palindrome). Now Theorem 1.9 gives

$$D(\mathrm{ghpw}(w, u, v, A)) < \infty.$$

■

3

MP-ratio in the ternary case

Consider the n-ary alphabet $\Sigma = \{0, 1, \ldots, n-1\}$. Clearly, each $w \in \Sigma^*$ contains a subpalindrome of length at least $\lceil \frac{|w|}{n} \rceil$. Therefore, it is natural to say that a word $w \in \Sigma^*$ is *minimal-palindromic* if and only if it does not contain a subpalindrome longer than $\lceil \frac{|w|}{n} \rceil$. For a word $w \in \Sigma^*$, a pair (r, s), where $r, s \in \Sigma^*$, such that rws is minimal-palindromic, is called an *MP-extension* of w, and we define an SMP-extension and the MP-ratio in the same way as in the binary case. However, as mentioned earlier, in case of an arity greater than 2, it is not clear whether an MP-extension always exists, and thus whether the MP-ratio is well-defined. In this chapter we prove that this is true for ternary alphabet.

We first show an easy proposition that will be useful later.

Proposition 3.1. *Let $w \in \{0, 1, 2\}^*$, and let (r, s) be an SMP-extension of w and $|rs| \geqslant 2$. Then $|rws| = 3k - 2$ for some positive integer k, and the values $|rws|_0, |rws|_1, |rws|_2$ are (in some permutation) either $k-1, k-1, k$ or $k-2, k, k$.*

Proof. Suppose the contrary: (r, s) is an SMP-extension of w, $|rs| \geqslant 2$ and $|rws| = 3k - 1$ (respectively $|rws| = 3k$). Let $r's'$ denote the word obtained by erasing any letter (respectively any two letters) from rs (where r' is a subword of r and s' of s). Clearly, the length of a longest subpalindrome of $r'ws'$ is not greater than the length of a longest subpalindrome of rws, which is at most $\lceil \frac{|rws|}{3} \rceil$. Since $\lceil \frac{|r'ws'|}{3} \rceil = \lceil \frac{3k-2}{3} \rceil = k = \lceil \frac{|rws|}{3} \rceil$, we conclude that (r', s') is an MP-extension, and $|r'| + |s'| < |r| + |s|$, a contradiction.

Therefore, we now know that $|rws| = 3k-2$. Let us show the second part of the statement. Let c be a prevalent letter in rws. Since $\lceil \frac{|rws|}{3} \rceil = \lceil \frac{3k-2}{3} \rceil = k$

and rws is minimal-palindromic, we have $|rws|_c \leqslant k$. If $|rws|_c < k$, then $|rws| \leqslant 3(k-1) < 3k-2$ would follow, which is a contradiction. Therefore, the only possibility is $|rws|_c = k$. If a prevalent letter is unique, then we see that each of the other two letters has to occur exactly $k-1$ times, while if there are two prevalent letters (both occurring k times), then the third letter has to occur $k-2$ times. ∎

3.1 An upper bound on the MP-ratio

Our aim is in this section to show that the MP-ratio of any ternary word w is at most 6. We fix the alphabet $\Sigma = \{0, 1, 2\}$.

The following functions will be needed. For $w \in \Sigma^*$ and $a, b \in \Sigma$, let

$$\gamma(w, a, b) = \min\left\{2\big|w[i, |w|]\big|_a - \big|w[i, |w|]\big|_b : i = 1, 2, \ldots, |w|+1\right\},$$

and let

$$g(w, a, b) = \max\left\{2\big|w[i, |w|]\big|_a - \big|w[i, |w|]\big|_b : i = 1, 2, \ldots, |w|+1\right\}.$$

Further, let $j(a, w)$ denote the position of the last occurrence of a in w (that is, $w[j(a, w)] = a$ and $w[k] \neq a$ for each k, $k > j(a, w)$), and $j(a, w) = 0$ if a does not occur in w. We define

$$g'(w, a, b) = \max\left(\left\{2\big|w[i, |w|]\big|_a - \big|w[i, |w|]\big|_b : i = 1, 2, \ldots, j(a, w)\right\} \cup \{0\}\right).$$

We first show two easy properties of these functions.

Lemma 3.2. *Let w be a finite word and let a and b be two distinct letters. Then:*

a) $g'(w, a, b) \leqslant g(w, a, b)$;

b) $\gamma(w, a, b) + g(\widetilde{w}, a, b) = g(w, a, b) + \gamma(\widetilde{w}, a, b) = 2|w|_a - |w|_b$.

Proof. a) Follows from the definitions of g and g'.

b) We first show that for each i, $1 \leqslant i \leqslant |w|+1$, we have the equality

$$\left(2\big|w[i, |w|]\big|_a - \big|w[i, |w|]\big|_b\right) + \left(2\big|\widetilde{w}[|w|-i+2, |w|]\big|_a - \big|\widetilde{w}[|w|-i+2, |w|]\big|_b\right) = 2|w|_a - |w|_b.$$

The equality follows by observing that each occurrence of the letter a is counted in exactly one of the parenthesis, and the same holds for each occurrence of the letter b. Note that, because of this equality, the first parenthesis reaches its minimum exactly when the second parenthesis reaches its maximum, and vice versa. When the first parenthesis reaches its minimum (and the second one its maximum), the expression on the left-hand side becomes $\gamma(w,a,b)+g(\widetilde{w},a,b)$ (by the definition of γ and g); when the first parenthesis reaches its maximum (and the second one its minimum), the expression on the left-hand side becomes $g(w,a,b)+\gamma(\widetilde{w},a,b)$. This proves the lemma. ■

The following property of the function g is less obvious, but will also be very useful.

Lemma 3.3. *Let $w \in \Sigma^*$, let b be a prevalent letter in w, and let a be a letter distinct from b. We have:*

$$g(w,a,b) + g(\widetilde{w},a,b) \leqslant 3|w|_a.$$

Proof. First, we have the following sequence of equivalences (where Lemma 3.2b) is used in the first step):

$$\begin{aligned} g(w,a,b) + g(\widetilde{w},a,b) &\leqslant 3|w|_a & \text{if and only if}\\ g(w,a,b) - \gamma(w,a,b) + 2|w|_a - |w|_b &\leqslant 3|w|_a & \text{if and only if}\\ g(w,a,b) - \gamma(w,a,b) &\leqslant |w|_a + |w|_b. \end{aligned}$$

Therefore, it is enough to show that $g(w,a,b) - \gamma(w,a,b) \leqslant |w|_a + |w|_b$.

Now, let K, respectively k, where $1 \leqslant K, k \leqslant |w|+1$, denote the value of i for which the expression

$$2\big|w[i,|w|]\big|_a - \big|w[i,|w|]\big|_b$$

reaches its maximal, respectively minimal, value. In other words,

$$g(w,a,b) = 2\big|w[K,|w|]\big|_a - \big|w[K,|w|]\big|_b$$

and

$$\gamma(w,a,b) = 2\big|w[k,|w|]\big|_a - \big|w[k,|w|]\big|_b.$$

We distinguish two cases depending on which one of k and K is greater, and show that in both cases the expected inequality holds.

Let first $K \leqslant k$. Now, let i transition gradually from K to k, and we monitor changes in the value $2\big|w[i,|w|]\big|_a - \big|w[i,|w|]\big|_b$. If $w[i] = a$, then the value of the expression $2\big|w[i,|w|]\big|_a - \big|w[i,|w|]\big|_b$ in the next step will decrease by 2 (in comparison to the current value); if $w[i] = b$ then the considered value will increase by 1; if $w[i] \notin \{a,b\}$, then the considered value will not change. Since $g(w,a,b) \geqslant \gamma(w,a,b)$, we conclude that the difference between them is at most twice the number of letters a in the factor $w[K,k-1]$ (that is, the maximum is reached when the considered value constantly decreases during the described process). Now we have:

$$g(w,a,b) - \gamma(w,a,b) \leqslant 2|w[K,k-1]|_a \leqslant 2|w|_a \leqslant |w|_a + |w|_b$$

(where the last inequality holds because of the assumption that b is a prevalent letter in w).

Let now $k \leqslant K$. In a similar manner as in the previous paragraph, we get that in this case the difference between $g(w,a,b)$ and $\gamma(w,a,b)$ is at most the number of letters b in the factor $w[k,K-1]$. Therefore, in this case we have:

$$g(w,a,b) - \gamma(w,a,b) \leqslant |w[k,K-1]|_b \leqslant |w|_b \leqslant |w|_a + |w|_b.$$

This completes the proof. ■

Now we are ready to construct an MP-extension of a given word w. For the rest of this section, without loss of generality, we assume $|w|_0 \leqslant |w|_1 \leqslant |w|_2$. We shall describe two extensions of the word w, denoted by $f(w)$ and $f'(w)$, and show that at least one of them is an MP-extension. Those two extensions are:

$$\begin{aligned} f(w) &= 0^{2|w|-|w|_0} 2^{2|w|-|w|_2-g'(w,0,2)} \;\; w \;\; 2^{g'(w,0,2)} 1^{2|w|-|w|_1}; \\ f'(w) &= 1^{2|w|-|w|_1} 2^{g'(\widetilde{w},0,2)} \;\; w \;\; 2^{2|w|-|w|_2-g'(\widetilde{w},0,2)} 0^{2|w|-|w|_0}. \end{aligned}$$

Note that $f'(w) = \widetilde{f(\widetilde{w})}$. By r and s, respectively r' and s', we shall refer to the prefix and the suffix attached to w in $f(w)$, respectively $f'(w)$.

In other words, the letters 1 and 0 are piled up at the ends, and the letter 2 is arranged around w in an asymmetric way. We shall later need a more precise "quantification" of this asymmetry, so let us show that

$$(2|w| - |w|_2 - g'(w,0,2)) - g'(w,0,2) \geqslant |w|_2 \qquad (3.1)$$

(and the same holds with $\widetilde{w}$ in place of w), which reduces to

$$g'(w,0,2) + |w|_2 \leqslant |w|.$$

And indeed:

$$g'(w,0,2)+|w|_2 \leqslant 2|w|_0+|w|_2 \leqslant |w|_0+|w|_1+|w|_2=|w|,$$

which was to be proved.

Note. The presented construction is not the only one possible. Another possibility is to use the function g in place of g' (or any intermediate value), and the proof in that case is completely the same. We decided to present the version with g' because that version is exactly a "borderline" case in the sense that the letters 2 are arranged in the "mostly asymmetric" way possible; in other words, by transferring only one letter 2 from the "smaller pile" to the "larger pile" we would not have an MP-extension anymore.

As already announced, we claim that at least one of the pairs (r,s) and (r',s') represents an MP-extension of w; that is, at least one of the words $f(w)$ and $f'(w)$ does not have subpalindromes whose length exceeds $2|w|$ (having in mind that $|f(w)|=|f'(w)|=6|w|$). The proof consists of a number of intermediate assertions.

Lemma 3.4. *The length of an arbitrary subpalindrome of the form* $0p0$ *in each of the words* $f(w)$ *and* $f'(w)$ *is less than or equal to* $2|w|$*.*

Proof. Without loss of generality, we prove the assertion for the word $f(w)$. (This indeed does not affect the generality: if we prove the claim for $f(w)$ for each w, then it also holds for each $f(\widetilde{w})$, and now we only need to recall the equality $f'(w)=\widetilde{f(\widetilde{w})}$ and the fact that the claimed property remains true for $\widetilde{f(\widetilde{w})}$ if it is true for $f(\widetilde{w})$.) Each subword of $f(w)$ of the form $0p0$ must be a subword of

$$rw=0^{2|w|-|w|_0}2^{2|w|-|w|_2-g'(w,0,2)}w,$$

because s obviously does not contain the letter 0.

If at least $\frac{|0p0|}{2}$ letters from w participate in the palindrome $0p0$ (which means: $0p0$ is a subword of rw obtained by selecting at least $\frac{|0p0|}{2}$ letters from w, while the rest of the letters are selected from r), then, clearly, $|0p0|\leqslant 2|w|$, which was to be proved. Assume now that more than $\frac{|0p0|}{2}$ letters from r participate in the palindrome $0p0$ (it must be so if the assumption from the previous sentence is not true). Then, clearly, $0p0\in 0^*2^*0^*$.

If $0p0\in 0^*$, then we immediately have

$$|0p0|\leqslant |rw|_0=(2|w|-|w|_0)+|w|_0=2|w|,$$

which was to be proved. Therefore, it remains to check the case $0p0 \in 0^*2^+0^*$. Note that then there exists a position i in the word w such that among the letters at the positions $1, 2, \ldots, i-1$, respectively $i, i+1, \ldots, |w|$, only the letters 2, respectively the letters 0, can participate in the palindrome $0p0$. Hence, there can be at most $\big|w[i,|w|]\big|_0$ zeros at the end of $0p0$, and therefore also at the beginning. Altogether, we conclude $|0p0| \leqslant |r|_2 + \big(|w|_2 - \big|w[i,|w|]\big|_2\big) + 2\big|w[i,|w|]\big|_0$. Since $0p0$ ends with 0, we have that i is at most the position of the rightmost letter 0 in w; this gives that the expression from the previous sentence is bounded from above by $|r|_2 + |w|_2 + g'(w,0,2)$ (by the definition of g'). In other words, we again have

$$\begin{aligned}|0p0| &\leqslant |r|_2 + |w|_2 + g'(w,0,2)\\ &= (2|w| - |w|_2 - g'(w,0,2)) + |w|_2 + g'(w,0,2) = 2|w|,\end{aligned}$$

which completes the proof. ■

Lemma 3.5. *The length of an arbitrary subpalindrome of the form $1p1$ in each of the words $f(w)$ and $f'(w)$ is less than or equal to $2|w|$.*

Proof. We again prove the assertion only for the word $f(w)$. We may assume $1p1 \in 1^*2^+1^*$ (everything else can be dealt with in a completely analogous way like in Lemma 3.4). Then we can write the palindrome $1p1$ in the form $1p_wp_2p_11$, where $1p_w \in \mathrm{Subw}(w)$, $p_2 \in 2^*$ and $p_11 \in 1^*$. Since there are at most $|w|_1$ letters 1 to the left of p_2, we conclude $|p_11| \leqslant |w|_1$. Now we have

$$\begin{aligned}|1p1| &= |1p_wp_2p_11| = |1p_w| + |p_2| + |p_11| \leqslant |w| + g'(w,0,2) + |w|_1\\ &\leqslant |w| + 2|w|_0 + |w|_1 \leqslant |w| + |w|_0 + |w|_2 + |w|_1\\ &= 2|w|,\end{aligned}$$

which completes the proof. ■

Lemma 3.6. *Let p and q be two nonempty subpalindromes of w. Let w_p, v, w_q and t be such that $w = w_pv = tw_q$, p is a subword of w_p, and q is a subword of w_q. Then*

$$|p| + 2|v|_2 + |q| + 2|t|_2 \leqslant 4|w|_2 + |w|_1 + |w|_0. \tag{3.2}$$

Proof. Define the word w', $|w'| = |w|$, in the following way:

$$w'[i] = \begin{cases} 1, & \text{if } w[i] = 0 \text{ or } w[i] = 1;\\ 2, & \text{if } w[i] = 2.\end{cases}$$

We obviously have $|w'|_2 = |w|_2$ and $|w'|_1 = |w|_1 + |w|_0$. By the assumption $|w|_2 \geqslant |w|_1 \geqslant |w|_0$ we get $2|w'|_2 \geqslant |w'|_1$. Similarly, let p', v', q', t', be the words obtained from p, v, q, t, respectively, by replacing all 0s by 1s. Then p' and q' are subpalindromes of the word w', and by applying Theorem 3.19 (formulated and proved later in Section 3.4) we get

$$|p'| + 2|v'|_2 + |q'| + 2|t'|_2 \leqslant 4|w'|_2 + |w'|_1.$$

Note that the left-hand side is the left-hand side of (3.2), and the right-hand side is the right-hand side of (3.2), which proves the lemma. ■

Lemma 3.7. *At least one among the words $f(w)$ and $f'(w)$ does not contain a subpalindrome of the form $2p2$ longer than $2|w|$.*

Proof. Suppose the contrary: in both the words $f(w)$ and $f'(w)$ the length of a longest subpalindrome of the form $2p2$ is greater than $2|w|$. Consider the word $f(w)$. Since $|f(w)|_2 = 2|w|$, such a longest subpalindrome of $f(w)$ contains a letter different from 2, and thus, by (3.1), we can write it as $2^{l+|s|_2} p_w 2^{l+|s|_2}$ where $p_w = \widetilde{p_w}$ and $p_w 2^l \in \mathrm{Subw}(w)$. This palindrome has length $|p_w| + 2l + 2|s|_2$, which equals $|p_w| + 2l + 2g'(w,0,2)$. Therefore, the assumption from the beginning reduces to:

$$|p_w| + 2l + 2g'(w,0,2) > 2|w|.$$

In a similar manner, considering $f'(w)$, we get:

$$|q_w| + 2l' + 2g'(\widetilde{w},0,2) > 2|w|$$

(where q_w and l' are defined analogously).

Summing the last two inequalities yields:

$$|p_w| + 2l + |q_w| + 2l' + 2g'(w,0,2) + 2g'(\widetilde{w},0,2) > 4|w|,$$

which is equivalent to:

$$|p_w| + 2l + |q_w| + 2l' > 4|w| - 2g'(w,0,2) - 2g'(\widetilde{w},0,2). \tag{3.3}$$

Note that the left-hand side of (3.3) equals the left-hand side of (3.2), which is, by Lemma 3.6, less than or equal to $4|w|_2 + |w|_1 + |w|_0$. On the other hand, for the right-hand side of (3.3) we have:

$$\begin{aligned}
&4|w| - 2g'(w,0,2) - 2g'(\widetilde{w},0,2) \\
&\qquad \geqslant 4|w| - 2g(w,0,2) - 2g(\widetilde{w},0,2) \geqslant 4|w| - 6|w|_0 \\
&\qquad = 4|w|_2 + 4|w|_1 - 2|w|_0 \geqslant 4|w|_2 + |w|_1 + 3|w|_0 - 2|w|_0 \\
&\qquad = 4|w|_2 + |w|_1 + |w|_0,
\end{aligned}$$

where we used Lemma 3.2a) and Lemma 3.3. This gives a contradiction, and the lemma is thus proved. ■

We are now ready for the main theorem of this section.

Theorem 3.8. *The MP-ratio of any ternary word is at most* 6.

Proof. The assertion follows directly from Lemmas 3.4, 3.5 and 3.7. ■

Note. We make no claim that the considered extension is an SMP-extension. In fact, having in mind Proposition 3.1, we see that this is certainly *not* the case; by erasing any two letters from r and s, we would get a shorter MP-extension, which at the same time shows that the MP-ratio of any ternary word is *strictly* less than 6. However, because of the following section, this does not make any crucial difference. We chose to write the proof in the presented way since we felt that it was a little bit easier (from a technical point of view) if each letter in rws had the same number of occurrences. In any case, an MP-extension obtained by erasing two letters from our extension still does not have to be an SMP-extension. The question of constructing an SMP-extension of a given word is much harder, and seems to be far out of reach even in the binary case [13].

3.2 Optimality of the upper bound

We shall now show that the constant 6 from the previous section is optimal.

In Section 1.3 we introduced the properties of a binary word being economic and k-economic. We slightly modify this definition to make an appropriate adaptation for the ternary case. We say that a word $w \in \{0,1,2\}^*$ is *k-economic* (with respect to the letter 1) if and only if w is a palindrome and the word $w1^k$ contains a subpalindrome of length at least $|w|_1 + k + 3$. Each such subpalindrome can be written in the form $1^m q 1^m$ where $0 \leqslant m \leqslant k$ and $1^m q \in \mathrm{Subw}(w)$; the pair (q, m) is then called a *k-witness* of w. We say that w is *economic* if and only if it is k-economic for every k, $k = 0, 1, \ldots, |w|_1$.

The following lemmas are (more or less) direct adaptations of Lemma 1.14, Lemma 1.15 and Lemma 1.16.

Lemma 3.9. *Let $w \in \{0,1,2\}^*$, and let (r, s) be an MP-extension of w. If w is economic, then $|rs|_1 > |w|_1$.*

Proof. Suppose the contrary: $|rs|_1 \leqslant |w|_1$. Let $|r|_1 = i$ and $|s|_1 = j$, and assume, without loss of generality, $i \leqslant j$. Since w is economic and $j-i \leqslant |s|_1 \leqslant |rs|_1 \leqslant |w|_1$, it follows that w is $(j-i)$-economic. Therefore, $w1^{j-i}$ contains a subpalindrome of length at least $|w|_1 + j - i + 3$, and that subpalindrome can be written in the form $1^m q 1^m$ for $m \leqslant j - i$ and $1^m q \in \mathrm{Subw}(w)$. But we now have that $1^{m+i} q 1^{m+i}$ is a subpalindrome of rws, and we calculate:

$$\begin{aligned}|1^{m+i}q1^{m+i}| &= 2i + |1^m q 1^m| \geqslant 2i + |w|_1 + j - i + 3 = |w|_1 + i + j + 3\\ &= |rws|_1 + 3 > |rws|_1 + 2 \geqslant \left\lceil \frac{|rws|}{3} \right\rceil\end{aligned}$$

(the last inequality follows from Proposition 3.1). Contradiction, since the word rws is minimal-palindromic. This proves the lemma. ■

Lemma 3.10. *Let $w \in \{0,1,2\}^*$, and let (r,s) be an MP-extension of w. If w is economic, then $|rws| > 6|w|_1$.*

Proof. The proof is a straightforward computation that relies on Proposition 3.1 and the previous lemma:

$$\begin{aligned}|rws| &= |rws|_0 + |rws|_1 + |rws|_2 \geqslant 3|rws|_1 - 2 = 3|w|_1 + 3|rs|_1 - 2\\ &\geqslant 3|w|_1 + 3(|w|_1 + 1) - 2 > 6|w|_1.\end{aligned}$$

■

Lemma 3.11. *Let w_0 be an economic word and let the sequence $(w_i)_{i\geqslant 0}$ be defined recursively by $w_{i+1} = w_i 1^{t_i} w_i$, where $(t_i)_{i\geqslant 0}$ is a given sequence of positive integers. If for each nonnegative integer i we have $t_i < |w_i|_0$, then all the words w_i are economic.*

Proof. We proceed by induction on i. The base is clear (there is nothing to prove for $i = 0$). We now assume that w_i is economic and prove that then w_{i+1} is also economic. We should prove that w_{i+1} is k-economic for each k, $k = 0, 1, \ldots, |w_{i+1}|_1$.

Assume first $0 \leqslant k \leqslant |w_i|_1$. By the inductive assumption, w_i is k-economic. Let (q,m) be a k-witness of w_i. Recall that $m \leqslant k$ and $2m + |q| \geqslant |w_i|_1 + k + 3$. Let

$$p = 1^m q 1^{t_i+m} q 1^m.$$

Since $1^m q \in \mathrm{Subw}(w_i)$ and $1^m \in \mathrm{Subw}(1^k)$, we have $p \in \mathrm{Subw}(w_i 1^{t_i} w_i 1^k) = \mathrm{Subw}(w_{i+1}1^k)$. Furthermore,

$$\begin{aligned}|p| &= 3m + 2|q| + t_i = 2(2m + |q|) - m + t_i \geqslant 2(|w_i|_1 + k + 3) - k + t_i\\ &= |w_{i+1}|_1 + k + 6 > |w_{i+1}|_1 + k + 3.\end{aligned}$$

This gives that w_{i+1} is k-economic.

Assume now $k = |w_i|_1+1$. Then the word w_i is $(k-1)$-economic. Let (q, m) be a $(k-1)$-witness of w_i (now $m \leqslant k-1$). Let (again) $p = 1^m q 1^{t_i+m} q 1^m$. Then $p \in \mathrm{Subw}(w_{i+1}1^k)$ and

$$\begin{aligned}|p| &= 3m + 2|q| + t_i = 2(2m + |q|) - m + t_i \\ &\geqslant 2(|w_i|_1 + k - 1 + 3) - (k-1) + t_i = |w_{i+1}|_1 + k + 5 > |w_{i+1}|_1 + k + 3;\end{aligned}$$

therefore, w_{i+1} is k-economic.

Let now $|w_i|_1 + 1 < k \leqslant |w_i|_1 + t_i$. Then we write

$$p = 1^k w_i 1^k.$$

Clearly, $p \in \mathrm{Subw}(w_{i+1}1^k)$, and since $t_i + 1 \leqslant |w_i|_0$ and $|w_i|_1 + 2 \leqslant k$, we have

$$|p| = 2k + |w_i|_1 + |w_i|_0 \geqslant k + 2|w_i|_1 + 2 + t_i + 1 = |w_{i+1}|_1 + k + 3,$$

which means that w_{i+1} is k-economic.

Finally, assume $|w_i|_1 + t_i < k \leqslant |w_{i+1}|_1$. Let $j = |w_i|_1 + t_i$ and $l = k - j$. Since $k - j \leqslant |w_i|_1$, we conclude that w_i is l-economic. Let (q, m) be an l-witness of w_i. Write

$$p = 1^{j+m} q 1^{j+m}.$$

Since $1^j \in \mathrm{Subw}(w_i 1^{t_i})$, $1^m q \in \mathrm{Subw}(w_i)$ and $j + m \leqslant k$, we have $p \in \mathrm{Subw}(w_{i+1}1^k)$. Furthermore,

$$|p| = 2j + |1^m q 1^m| \geqslant 2j + |w_i|_1 + l + 3 = |w_{i+1}|_1 + k + 3.$$

Therefore, w_{i+1} is economic also in this case, which completes the proof. ■

For a sequence $(t_i)_{i \geqslant 0}$, let $w(t_0, t_1, \ldots, t_{j-1})$ denote the word w_j from the statement of Lemma 3.11, with the initial term $w_0 = 0000$ (we observe that w_0 is economic as a ternary word; indeed, since $|w_0|_1 = 0$, we only have to check whether w_0 is 0-economic, and it clearly is since w_0 itself is a palindrome of length 4). Note that, if the sequence $(t_i)_{i \geqslant 0}$ satisfies $2^i \leqslant t_i < 2^{i+2}$ for each i, then we easily see $t_j < 2^{j+2} = |w(t_0, t_1, \ldots, t_{j-1})|_0$, and thus, by Lemma 3.11, the word $w(t_0, t_1, \ldots, t_{j-1})$ is economic (for each j and for each sequence $(t_i)_{i \geqslant 0}$ satisfying the required property). As we have seen in Lemma 1.17, for every large enough integer k there exists a word, say v_k, that can be obtained by the described construction, such that $|v_k| = k$; further, we have

$$\lim_{k\to\infty} \frac{|v_k|_1}{|v_k|} = 1. \tag{3.4}$$

We now have enough prerequisites to prove the main theorem of this section.

Theorem 3.12. *Let $R_3(n)$ denote the maximal MP-ratio over all the words $w \in \{0,1,2\}^*$, $|w| = n$. We have*

$$\lim_{n\to\infty} R_3(n) = 6.$$

Proof. Given a positive real number η, choose an integer k_0 such that, for each $k \geqslant k_0$, we have

$$\frac{|v_k|_1}{|v_k|} > 1 - \frac{\eta}{6}$$

(such k_0 exists because of (3.4)). Let a pair (r, s) be an MP-extension of v_k, $k \geqslant k_0$. By Lemma 3.10, due to the fact that the word v_k is economic, we have

$$\frac{|rv_ks|}{|v_k|} > \frac{6|v_k|_1}{|v_k|} > 6 - \eta;$$

therefore, the MP-ratio of v_k is greater than $6 - \eta$. This completes the proof. ■

3.3 A postponed technical theorem

Theorem 3.13. *Let $u \in \{1,2\}^*$, let $t, v \in 2^*$, and let p and q be subpalindromes of tu and uv, respectively. If*

$$|p| + |q| > 2|u|,$$

then

$$|u|_1 \leqslant \frac{|tv| - 1}{|tv|}|tuv|_2. \tag{3.5}$$

Before we begin the proof, we shall show that it is enough to prove the theorem in the special case when the subpalindrome p, respectively q, starts (and ends) with t, respectively v. Assume that this case of the theorem is proved. Let now t, u, v, p and q be as in the statement of the theorem, but not satisfying the conditions of the described special case. Let t_0, respectively v_0, be the longest prefix (and suffix) of p, respectively q, that is a subword of t, respectively v; note that $|t_0v_0| < |tv|$. Then p and q are subpalindromes of t_0u and uv_0, respectively, $|p| + |q| > 2|u|$, and t_0, u, v_0, p and q satisfy the condition of the described special case. Since the theorem is assumed to hold in this case, we have

$$|u|_1 \leqslant \frac{|t_0v_0| - 1}{|t_0v_0|}|t_0uv_0|_2 < \frac{|tv| - 1}{|tv|}|tuv|_2 \tag{3.6}$$

(where the second inequality follows from $\frac{|t_0v_0|-1}{|t_0v_0|} = 1 - \frac{1}{|t_0v_0|} < 1 - \frac{1}{|tv|} = \frac{|tv|-1}{|tv|}$ and $|t_0uv_0|_2 < |tuv|_2$); therefore, the theorem holds for t, u, v, p and q.

From now onward we assume that p, respectively q, contains all the letters from t, respectively v.

In the following two subsections we shall give two (very) different proofs of Theorem 3.13. The second proof is (much) shorter than the first one, and many would probably agree that it is also more elegant. However, we feel that the second proof is a neat little "trick" that works almost by a coincidence, while the first proof presents a deep structural analysis and gives some insight into *why* the theorem is true (we actually feel that the first proof is more intuitive than the second one, despite some quite heavy expressions at some places). In case that a result similar to Theorem 3.13 turns out to be needed to deal with (for example) the MP-ratio for alphabets of larger arities, we think that it would not be surprising if the (suitably modified) first proof would then still work, but the second one would not. Therefore, it is our belief that, despite the evident disparity in their lengths, both proofs have their own merits, and thus we decided to present them both.

3.3.1 First proof

We first define sequences $P_1, P_2, \ldots, P_{|p|}$ and $Q_1, Q_2, \ldots, Q_{|q|}$ such that $1 \leqslant P_1 < P_2 < \cdots < P_{|p|} \leqslant |tu|$ and $|t| + 1 \leqslant Q_1 < Q_2 < \cdots < Q_{|q|} \leqslant |tuv|$,

$$p = (tuv)[P_1](tuv)[P_2] \ldots (tuv)[P_{|p|}]$$

and

$$q = (tuv)[Q_1](tuv)[Q_2] \ldots (tuv)[Q_{|q|}].$$

We write $P = \{P_1, P_2, \ldots, P_{|p|}\}$ and $Q = \{Q_1, Q_2, \ldots, Q_{|q|}\}$.

We define $\sigma_P : P \to P$ by $\sigma_P : P_s \mapsto P_{|P|-s+1}$ and $\sigma_Q : Q \to Q$ by $\sigma_Q : Q_s \mapsto Q_{|Q|-s+1}$. Note that σ_P and σ_Q are bijections, and their squares are identical mappings.

For $1 \leqslant n \leqslant |t|$, let $\sigma_0(n) = n$ and

$$\sigma_{i+1}(n) = \begin{cases} \sigma_P(\sigma_i(n)), & \text{for } 2 \mid i \text{ and } \sigma_i(n) \in P; \\ \sigma_Q(\sigma_i(n)), & \text{for } 2 \nmid i \text{ and } \sigma_i(n) \in Q; \\ \text{undefined}, & \text{otherwise.} \end{cases}$$

In a similar manner, for $|tu| + 1 \leqslant n \leqslant |tuv|$, let $\sigma_0(n) = n$ and

$$\sigma_{i+1}(n) = \begin{cases} \sigma_Q(\sigma_i(n)), & \text{for } 2 \mid i \text{ and } \sigma_i(n) \in Q; \\ \sigma_P(\sigma_i(n)), & \text{for } 2 \nmid i \text{ and } \sigma_i(n) \in P; \\ \text{undefined}, & \text{otherwise.} \end{cases}$$

Note. Before we proceed further, we would like to mention that, although we do believe (as we have already said) that the proof that is about to follow is quite natural in its essence, an unfortunate occurrence is that among its "repulsive aspects" is not only its length, but also some quite heavy expressions at some places. Because of both of these aspects, we find it possible that the reader gets overwhelmed and misses its substance. For that reason, in the Appendix at the end of this section we sketch the main ideas of the proof, showing them on some examples. The Appendix is not, by any means, necessary for understanding the proof, and it can be skipped altogether, but we believe that it makes the task of understanding the proof much less demanding. Therefore, we recommend the reader at this point to read the Appendix first, and then return here and continue reading the formal account of the proof ("enriched" with all the technical details).

We now show a few properties of the defined notions.

Proposition 3.14. *a) For every $n, m \in Q$ (respectively, $n, m \in P$), if $n < m$, then $\sigma_Q(n) > \sigma_Q(m)$ (respectively, $\sigma_P(n) > \sigma_P(m)$).*

b) We have $\sigma_0(n) > \sigma_2(n) > \sigma_4(n) > \cdots$ and $\sigma_1(n) < \sigma_3(n) < \sigma_5(n) < \cdots$ for $n \geqslant |tu| + 1$, and $\sigma_0(n) < \sigma_2(n) < \sigma_4(n) < \cdots$ and $\sigma_1(n) > \sigma_3(n) > \sigma_5(n) > \cdots$ for $n \leqslant |t|$. (The inequalities are extended as long as the terms are defined.)

c) If one of the following holds:

1) n and m simultaneously belong to the interval $[1,|t|]_{\mathbb{N}}$ or the interval $[|tu|+1,|tuv|]_{\mathbb{N}}$, and i and j are of the same parity; or

2) n and m are in different intervals and i and j are of opposite parities,

then $\sigma_i(n)=\sigma_j(m)$ implies $n=m$ and $i=j$. (In particular, $\sigma_i(n)=\sigma_j(m)$ is impossible in the second case.)

d) For each n such that $n \leqslant |t|$ or $n \geqslant |tu|+1$, there exists $z \in \mathbb{N}$ such that $\sigma_z(n)$ is the last defined term in the sequence $\sigma_0(n), \sigma_1(n), \sigma_2(n)\dots$

Proof. a) Let $n, m \in Q$ and $n<m$. Write $n=Q_s$ and $m=Q_r$. Since $s<r$, we have $|Q|-s+1>|Q|-r+1$, and thus

$$\sigma_Q(n)=\sigma_Q(Q_s)=Q_{|Q|-s+1}>Q_{|Q|-r+1}=\sigma_Q(Q_r)=\sigma_Q(m),$$

which was to be proved. The proof of the claim for $n, m \in P$ and σ_P is analogous.

b) We consider only the case $n \geqslant |tu|+1$ (the other one is analogous). Since $\sigma_2(n)=\sigma_P(\sigma_Q(n)) \in P$ (assuming, of course, that $\sigma_2(n)$ is defined), we have

$$\sigma_2(n) \leqslant |tu| < |tu|+1 \leqslant n = \sigma_0(n).$$

Iteratively applying a), we get

$$\sigma_3(n)=\sigma_Q(\sigma_2(n))>\sigma_Q(\sigma_0(n))=\sigma_1(n),$$

then

$$\sigma_4(n)=\sigma_P(\sigma_3(n))<\sigma_P(\sigma_1(n))=\sigma_2(n)$$

etc., which was to be proved.

c) Let $\sigma_i(n)=\sigma_j(m)$. Assume first that 1) holds. Without loss of generality, let $m, n \geqslant |tu|+1$ (the case $m, n \leqslant |t|$ is analogous), and let $i \geqslant j$. If $i>j=0$, then $2 \mid i$, and we have $\sigma_P(\sigma_{i-1}(n))=\sigma_i(n)=\sigma_j(m)=m$, which is impossible since the left-hand side is in P, and thus no greater than $|tu|$, while $m \geqslant |tu|+1$. Therefore, if $j=0$, then $i=0$, and we then immediately have $n=m$, which was to be proved. Assume now $i \geqslant j>0$. If i and j are even, then $\sigma_P(\sigma_{i-1}(n))=\sigma_i(n)=\sigma_j(m)=\sigma_P(\sigma_{j-1}(m))$, which implies $\sigma_{i-1}(n)=\sigma_{j-1}(m)$; if i and j are odd, we get the same conclusion in a similar manner. Iterating this, we obtain $\sigma_{i-j}(n)=\sigma_0(m)$. By the previous case, we get $n=m$ and $i-j=0$, that is, $i=j$, which was to be proved.

Assume now that 2) holds and let $i > j$. In the same way as in the previous paragraph we conclude that $\sigma_i(n) = \sigma_j(m)$ implies $\sigma_{i-j}(n) = \sigma_0(m) = m$. However, if $|n| \geqslant |tu| + 1$ (and then $m \leqslant |t|$), then, since $2 \nmid i - j$, we have $\sigma_{i-j}(n) = \sigma_Q(\sigma_{i-j-1}(n)) \geqslant |t| + 1$, a contradiction; if $n \leqslant |t|$, we again get a contradiction in a similar manner.

Therefore, we have proved that, under any of the assumptions 1) or 2), $\sigma_i(n) = \sigma_j(m)$ implies $n = m$ and $i = j$.

d) This is a direct consequence of b). ∎

The following lemma will be useful.

Lemma 3.15. *a) For each $n \in Q$ such that $\sigma_P(\sigma_Q(n))$ is defined (that is, $\sigma_Q(n) \in P$), we have*

$$\begin{aligned} n - \sigma_P(\sigma_Q(n)) \leqslant 2(|t| + |v|) - 1 \\ - |[n, |tuv|]_{\mathbb{N}} \setminus Q| - |[\sigma_Q(n), |tuv|]_{\mathbb{N}} \setminus Q| \\ - |[1, \sigma_Q(n)]_{\mathbb{N}} \setminus P| - |[1, \sigma_P(\sigma_Q(n))]_{\mathbb{N}} \setminus P|. \end{aligned}$$

Also, for each $n \in P$ such that $\sigma_Q(\sigma_P(n))$ is defined (that is, $\sigma_P(n) \in Q$), we have

$$\begin{aligned} \sigma_Q(\sigma_P(n)) - n \leqslant 2(|t| + |v|) - 1 \\ - |[1, n]_{\mathbb{N}} \setminus P| - |[1, \sigma_P(n)]_{\mathbb{N}} \setminus P| \\ - |[\sigma_P(n), |tuv|]_{\mathbb{N}} \setminus Q| - |[\sigma_Q(\sigma_P(n)), |tuv|]_{\mathbb{N}} \setminus Q|. \end{aligned}$$

b) For each $n \in Q$ such that $\sigma_P(\sigma_Q(n))$ is undefined (that is, $\sigma_Q(n) \notin P$), we have

$$n \leqslant 2(|t| + |v|) + |P| - \sigma_Q(n) - |[n, |tuv|]_{\mathbb{N}} \setminus Q| - |[\sigma_Q(n), |tuv|]_{\mathbb{N}} \setminus Q|.$$

Also, for each $n \in P$ such that $\sigma_Q(\sigma_P(n))$ is undefined (that is, $\sigma_P(n) \notin Q$), we have

$$\begin{aligned} |tuv| + 1 - n \leqslant 2(|t| + |v|) + |Q| - (|tuv| + 1 - \sigma_P(n)) \\ - |[1, n]_{\mathbb{N}} \setminus P| - |[1, \sigma_P(n)]_{\mathbb{N}} \setminus P|. \end{aligned}$$

Proof. a) We shall prove only the first statement (the second one is analogous).

The following equalities will be used repeatedly: for any $x \in [1, |P|]_{\mathbb{N}}$ we have

$$P_x = x + |[1, P_x]_{\mathbb{N}} \setminus P|,$$

and for any $x \in [1, |Q|]_{\mathbb{N}}$ we have

$$\begin{aligned}
Q_x &= |t| + x + |[|t|+1, Q_x]_{\mathbb{N}} \setminus Q| \\
&= |t| + x + (|[|t|+1, |tuv|]_{\mathbb{N}} \setminus Q| - |[Q_x, |tuv|]_{\mathbb{N}} \setminus Q|) \\
&= |t| + x + ((|uv| - |Q|) - |[Q_x, |tuv|]_{\mathbb{N}} \setminus Q|) \\
&= |tuv| + x - |Q| - |[Q_x, |tuv|]_{\mathbb{N}} \setminus Q|.
\end{aligned}$$

Let us now proceed to the proof. Since $n \in Q$ and $\sigma_Q(n) \in P$, we may write $n = Q_s$ and $\sigma_Q(n) = Q_{|Q|-s+1} = P_r$ (and also $\sigma_P(\sigma_Q(n)) = P_{|P|-r+1}$). We then have:

$$\begin{aligned}
n - \sigma_P(\sigma_Q(n)) &= Q_s - P_{|P|-r+1} \\
&= |tuv| + s - |Q| - |[n, |tuv|]_{\mathbb{N}} \setminus Q| \\
&\quad - (|P| - r + 1 + |[1, \sigma_P(\sigma_Q(n))]_{\mathbb{N}} \setminus P|) \\
&= |tuv| + s - |Q| - |[n, |tuv|]_{\mathbb{N}} \setminus Q| - |P| - 1 \\
&\quad - |[1, \sigma_P(\sigma_Q(n))]_{\mathbb{N}} \setminus P| + (P_r - |[1, P_r]_{\mathbb{N}} \setminus P|) \\
&= |tuv| + s - |Q| - |[n, |tuv|]_{\mathbb{N}} \setminus Q| - |P| - 1 \\
&\quad - |[1, \sigma_P(\sigma_Q(n))]_{\mathbb{N}} \setminus P| + Q_{|Q|-s+1} - |[1, \sigma_Q(n)]_{\mathbb{N}} \setminus P| \\
&= |tuv| + s - |Q| - |[n, |tuv|]_{\mathbb{N}} \setminus Q| - |P| - 1 \\
&\quad - |[1, \sigma_P(\sigma_Q(n))]_{\mathbb{N}} \setminus P| - |[1, \sigma_Q(n)]_{\mathbb{N}} \setminus P| \\
&\quad + (|tuv| + |Q| - s + 1 - |Q| - |[\sigma_Q(n), |tuv|]_{\mathbb{N}} \setminus Q|) \\
&= 2(|t| + |v|) + 2|u| - |P| - |Q| - |[n, |tuv|]_{\mathbb{N}} \setminus Q| \\
&\quad - |[1, \sigma_P(\sigma_Q(n))]_{\mathbb{N}} \setminus P| - |[1, \sigma_Q(n)]_{\mathbb{N}} \setminus P| \\
&\quad - |[\sigma_Q(n), |tuv|]_{\mathbb{N}} \setminus Q| \\
&\leqslant 2(|t| + |v|) - 1 - |[n, |tuv|]_{\mathbb{N}} \setminus Q| - |[1, \sigma_P(\sigma_Q(n))]_{\mathbb{N}} \setminus P| \\
&\quad - |[\sigma_Q(n), |tuv|]_{\mathbb{N}} \setminus Q| - |[1, \sigma_Q(n)]_{\mathbb{N}} \setminus P|,
\end{aligned} \tag{3.7}$$

which was to be proved.

b) We shall prove only the first statement (the second one is analogous). In addition to the two equalities from the part a), we shall also use the following one: for any $x \in [1, |Q|]_{\mathbb{N}}$ we have

$$\begin{aligned}
\sigma_Q(Q_x) = Q_{|Q|-x+1} &= |tuv| + (|Q| - x + 1) - |Q| - |[\sigma_Q(Q_x), |tuv|]_{\mathbb{N}} \setminus Q| \\
&= |tuv| - x + 1 - |[\sigma_Q(Q_x), |tuv|]_{\mathbb{N}} \setminus Q|.
\end{aligned}$$

Let us now proceed to the proof. Since $n \in Q$, we may write $n = Q_s$. We then have:

$$\begin{aligned}
n = Q_s &= |tuv| + s - |Q| - |[n, |tuv|]_{\mathbb{N}} \setminus Q| \\
&= |tuv| + (|tuv| - \sigma_Q(n) + 1 - |[\sigma_Q(n), |tuv|]_{\mathbb{N}} \setminus Q|) \\
&\quad - |Q| - |[n, |tuv|]_{\mathbb{N}} \setminus Q| \\
&= 2(|t| + |v|) + 2|u| + 1 \\
&\quad - |Q| - \sigma_Q(n) - |[n, |tuv|]_{\mathbb{N}} \setminus Q| - |[\sigma_Q(n), |tuv|]_{\mathbb{N}} \setminus Q| \\
&\leqslant 2(|t| + |v|) + |P| - \sigma_Q(n) - |[n, |tuv|]_{\mathbb{N}} \setminus Q| - |[\sigma_Q(n), |tuv|]_{\mathbb{N}} \setminus Q|,
\end{aligned}$$

which was to be proved. ■

Given a number n, $n \leqslant |t|$ or $n \geqslant |tu| + 1$, let $\mathrm{end}(n)$ denote the number z whose existence was shown in Proposition 3.14d). We say that n, $n \leqslant |t|$ or $n \geqslant |tu| + 1$, *dies* if and only if $|t| + 1 \leqslant \sigma_{\mathrm{end}(n)}(n) \leqslant |tu|$. Let n be such that n dies and that $\sigma_{\mathrm{end}(n)}(n) \notin P$. Note that it is impossible for any m, $m \neq n$, to have $\sigma_{\mathrm{end}(m)}(m) = \sigma_{\mathrm{end}(n)}(n)$ (and thus $\sigma_{\mathrm{end}(m)}(m) \notin P$). Indeed, in that case n and m would satisfy one of the conditions 1) or 2) from Proposition 3.14c), which would imply $m = n$, a contradiction. Therefore,

$$\begin{aligned}
|\{n : n \text{ dies and } \sigma_{\mathrm{end}(n)}(n) \notin P\}| &\leqslant |[|t| + 1, |tu|]_{\mathbb{N}} \setminus P| \\
&= |[1, |tu|]_{\mathbb{N}} \setminus P| = |tu| - |p|.
\end{aligned}$$

Analogously, we prove

$$\begin{aligned}
|\{n : n \text{ dies and } \sigma_{\mathrm{end}(n)}(n) \notin Q\}| &\leqslant |[|t| + 1, |tu|]_{\mathbb{N}} \setminus Q| \\
&= |[|t| + 1, |tuv|]_{\mathbb{N}} \setminus Q| = |uv| - |q|.
\end{aligned}$$

Therefore, there are at most $|tu| - |p| + |uv| - |q| < |t| + |v|$ numbers n that die. This implies that there exists n, $n \leqslant |t|$ or $n \geqslant |tu| + 1$, that does not die. Let n_0 be any such number. Without loss of generality, we may assume $n_0 \geqslant |tu| + 1$. By the choice of n_0, we have either $\sigma_{\mathrm{end}(n_0)}(n_0) \leqslant |t|$ or $\sigma_{\mathrm{end}(n_0)}(n_0) \geqslant |tu| + 1$.

Lemma 3.16. *Let i, $i \geqslant 0$, be such that $\sigma_{2i+2}(n_0)$ is defined.*

a) For each m, $m \geqslant |tu| + 1$, we have one of the following:

- *there exists j such that $\sigma_{2i+2}(n_0) < \sigma_j(m) \leqslant \sigma_{2i}(n_0)$;*
- $2 \mid \mathrm{end}(m)$ *and* $\sigma_{\mathrm{end}(m)}(m) > \max\{\sigma_{2i}(n_0), \sigma_{2i+1}(n_0)\}$*;*

- $2 \nmid \mathrm{end}(m)$ *and* $\sigma_{\mathrm{end}(m)}(m) < \min\{\sigma_{2i+1}(n_0), \sigma_{2i+2}(n_0)\}$.

Also, for each m, $m \leqslant |t|$, we have one of the following:

- *there exists j such that $\sigma_{2i+2}(n_0) < \sigma_j(m) \leqslant \sigma_{2i}(n_0)$;*
- $2 \nmid \mathrm{end}(m)$ *and* $\sigma_{\mathrm{end}(m)}(m) > \max\{\sigma_{2i}(n_0), \sigma_{2i+1}(n_0)\}$*;*
- $2 \mid \mathrm{end}(m)$ *and* $\sigma_{\mathrm{end}(m)}(m) < \min\{\sigma_{2i+1}(n_0), \sigma_{2i+2}(n_0)\}$.

b) *To each m, $m \leqslant |t|$ or $m \geqslant |tu| + 1$, for which there exists j described in a) we can assign one such j in such a way that all the corresponding values $\sigma_j(m)$ are different.*

Proof. a) We shall prove only the first assertion (the second one is analogous). Choose the least even j such that $\sigma_j(m) \leqslant \sigma_{2i}(n_0)$, or the least odd j such that $\sigma_j(m) > \sigma_{2i+2}(n_0)$, assuming that there exists j that satisfies either of these two conditions. We claim that in that case we have

$$\sigma_{2i+2}(n_0) < \sigma_j(m) \leqslant \sigma_{2i}(n_0).$$

Assume first that j is even. If $j = 0$, then

$$\sigma_{2i+2}(n_0) = \sigma_P(\sigma_{2i+1}(n_0)) \leqslant |tu| < m = \sigma_j(m),$$

which was to be proved. If $j > 0$, then $\sigma_{2i}(n_0) < \sigma_{j-2}(m)$ (by the minimality of j), and Proposition 3.14a) (applied twice) now gives

$$\sigma_{2i+2}(n_0) < \sigma_j(m),$$

which was to be proved. Assume now that j is odd. If $j = 1$, then, having in mind that $\sigma_{2i+2}(n_0)$ is defined, that is, $\sigma_{2i+1}(n_0) \in P$, we obtain $m > |tu| \geqslant \sigma_{2i+1}(n_0)$, and now Proposition 3.14a) gives

$$\sigma_1(m) = \sigma_Q(m) < \sigma_Q(\sigma_{2i+1}(n_0)) = \sigma_Q(\sigma_Q(\sigma_{2i}(n_0))) = \sigma_{2i}(n_0),$$

which was to be proved. If $j > 1$, then $\sigma_{j-2}(m) \leqslant \sigma_{2i+2}(n_0)$ (by the minimality of j), and Proposition 3.14a) (applied twice) now gives

$$\begin{aligned}\sigma_j(m) &= \sigma_Q(\sigma_P(\sigma_{j-2}(m))) \leqslant \sigma_Q(\sigma_P(\sigma_{2i+2}(n_0))) \\ &= \sigma_Q(\sigma_P(\sigma_P(\sigma_Q(\sigma_{2i}(n_0))))) = \sigma_{2i}(n_0),\end{aligned}$$

which was to be proved.

Assume now that j from the previous paragraph does not exist. Let $2 \mid \operatorname{end}(m)$. Then clearly

$$\sigma_{\operatorname{end}(m)}(m) > \sigma_{2i}(n_0),$$

since otherwise there would exist the even j from the previous paragraph. We now prove

$$\sigma_{\operatorname{end}(m)}(m) > \sigma_{2i+1}(n_0).$$

Suppose $\sigma_{2i+1}(n_0) \geqslant \sigma_{\operatorname{end}(m)}(m)$. In fact, the inequality must be strict, since the right-hand side does not belong to Q (by the definition of $\operatorname{end}(m)$), while the left-hand side does. Then, by Proposition 3.14a), we have

$$\begin{aligned}\sigma_{2i+2}(n_0) &= \sigma_P(\sigma_{2i+1}(n_0)) < \sigma_P(\sigma_{\operatorname{end}(m)}(m)) \\ &= \sigma_P(\sigma_P(\sigma_{\operatorname{end}(m)-1}(m))) = \sigma_{\operatorname{end}(m)-1}(m),\end{aligned}$$

and thus there would exist the odd j from the previous paragraph, a contradiction. The case $2 \nmid \operatorname{end}(m)$ is similar. Indeed, the inequality

$$\sigma_{\operatorname{end}(m)}(m) \geqslant \sigma_{2i+2}(n_0)$$

is impossible, since it would have to be strict (the right-hand side is in P, the left-hand side is not), and thus the odd j from the previous paragraph would exist; the inequality

$$\sigma_{\operatorname{end}(m)}(m) \geqslant \sigma_{2i+1}(n_0)$$

is also impossible, since applying σ_Q to the both sides gives, by Proposition 3.14a), $\sigma_{\operatorname{end}(m)-1}(m) \leqslant \sigma_{2i}(n_0)$, and thus the even j from the previous paragraph would exist. This completes the proof.

b) Define the following relation on the set $[1, |t|]_{\mathbb{N}} \cup [|tu|+1, |tuv|]_{\mathbb{N}}$:

$$m \sim m' \text{ if and only if there exists } l \text{ such that } m' = \sigma_l(m).$$

Let us show that "$\sim$" is an equivalence relation. Indeed, it is clearly reflexive and symmetric (if $m' = \sigma_l(m)$, then it is easily checked that $m = \sigma_l(m')$), and if $m' = \sigma_l(m)$ and $m'' = \sigma_{l'}(m')$, then $m'' = \sigma_{l'}(\sigma_l(m))$, while it is not hard to see that either $\sigma_{l'}(\sigma_l(m)) = \sigma_{l'+l}(m)$ or $\sigma_{l'}(\sigma_l(m)) = \sigma_{|l'-l|}(m)$; this proves the assertion.

We claim that each equivalence class is of size either 1 or 2. This is implied by the following observation: if $m' \neq m$, $m' = \sigma_l(m)$ and m' and m simultaneously belong to the interval $[1, |t|]_{\mathbb{N}}$ or the interval $[|tu|+1, |tuv|]_{\mathbb{N}}$, then l is odd, while if m' and m are in different intervals, then l is even.

We are now ready to prove b). In the rest of the proof, m (and also m') will always denote a value from $[1, |t|]_{\mathbb{N}} \cup [|tu|+1, |tuv|]_{\mathbb{N}}$ such that there exists j for which $\sigma_{2i+2}(n_0) < \sigma_j(m) \leqslant \sigma_{2i}(n_0)$.

Note that, if $\sigma_j(m) = \sigma_{j'}(m')$, then $m \sim m'$ (since either $m' = \sigma_{j+j'}(m)$ or $m' = \sigma_{|j-j'|}(m)$). Therefore, if m is alone in its class, then its assigned j (whichever we choose if there is a choice) will not collide with the other assignments. Let now $m \sim m'$, $m \neq m'$. We prove that there exist j and j' such that

$$\sigma_{2i+2}(n_0) < \sigma_j(m), \sigma_{j'}(m') \leqslant \sigma_{2i}(n_0)$$

and

$$\sigma_j(m) \neq \sigma_{j'}(m').$$

Note that from $m \sim m'$ we get that neither m nor m' dies. It is enough to prove that for any m that does not die there exist both an even j and an odd j that satisfy the requirement. Let us first show why this is enough. Since neither m nor m' dies, if, say, both $m, m' \geqslant |tu|+1$ (the other cases are similar), then we can choose j and j' to be of the same parity, and Proposition 3.14c) implies $\sigma_j(m) \neq \sigma_{j'}(m')$, which was to be proved.

Therefore, assume that m does not die, and let, without loss of generality, $m \geqslant |tu|+1$. If $2 \mid \mathrm{end}(m)$, then, because of $\sigma_{\mathrm{end}(m)}(m) \in P$, we have $\sigma_{\mathrm{end}(m)}(m) \leqslant |t|$. Since $\sigma_{2i}(n_0) \in Q$ (because $\sigma_{2i+1}(n_0)$ is defined) and $\sigma_{2i+1}(n_0) \in Q$ (because it equals $\sigma_Q(\sigma_{2i}(n_0))$), we have that $\sigma_{\mathrm{end}(m)}(m)$ is less than both of these values. But this implies, as seen during the proof of the part a), that there exist both an even j and an odd j that satisfy the requirement, which was to be proved. The case $2 \nmid \mathrm{end}(m)$ is analogous. This completes the proof. ∎

Finally, we shall need the following lemma.

Lemma 3.17. *Let n be such that $n \geqslant |tu|+1$ and $\sigma_{\mathrm{end}(n)}(n) \geqslant |tu|+1$. Then:*

$$\begin{aligned} &2|tuv| + 1 - n - \sigma_{\mathrm{end}(n)}(n) \\ &\qquad \geqslant |\{m : m \geqslant |tu|+1,\ \mathrm{end}(m) \geqslant \mathrm{end}(n) \text{ and } \sigma_{\mathrm{end}(m)}(m) \geqslant |tu|+1\}| \\ &\qquad\quad + |\{m : m \geqslant \sigma_{\mathrm{end}(n)}(n) \text{ and } m \text{ dies}\}|. \end{aligned}$$

Proof. Let us first prove the following: if $|tu|+1 \leqslant m < m'$ and both $\sigma_{\mathrm{end}(m)}(m), \sigma_{\mathrm{end}(m')}(m') \geqslant |tu|+1$, then $\mathrm{end}(m) \leqslant \mathrm{end}(m')$. Suppose the contrary: $\mathrm{end}(m) > \mathrm{end}(m')$. We get $2 \nmid \mathrm{end}(m')$ (because of $\sigma_{\mathrm{end}(m')}(m') \notin$

P); therefore, by Proposition 3.14a), from $m < m'$ we obtain $\sigma_{\mathrm{end}(m')}(m) > \sigma_{\mathrm{end}(m')}(m')$ (the left-hand side is defined since $\mathrm{end}(m) > \mathrm{end}(m')$). However, this implies $\sigma_{\mathrm{end}(m')}(m) > |tu|+1$ and thus $\sigma_{\mathrm{end}(m')}(m) \notin P$, which contradicts $\mathrm{end}(m) > \mathrm{end}(m')$. This proves the assertion.

Now, let n be as in the lemma's statement. In the calculations below we shall need only one more observation: the function $m \mapsto \sigma_{\mathrm{end}(m)}(m)$ bijectively maps the set

$$\{m : m > n,\ \mathrm{end}(m) = \mathrm{end}(n) \text{ and } \sigma_{\mathrm{end}(m)}(m) \geqslant |tu| + 1\}$$

to the set

$$\{m : m < \sigma_{\mathrm{end}(n)}(n),\ \mathrm{end}(m) = \mathrm{end}(n) \text{ and } \sigma_{\mathrm{end}(m)}(m) \geqslant |tu| + 1\}$$

(indeed, this follows by Proposition 3.14a), having in mind that $2 \nmid \mathrm{end}(n)$, and we see that the considered function is its own inverse). For the sake of brevity, we say that m, $m \geqslant |tu|+1$, is *pleasant* if and only if $\mathrm{end}(m) > \mathrm{end}(n)$ and $\sigma_{\mathrm{end}(m)}(m) \geqslant |tu|+1$, and is *delightful* if and only if $\mathrm{end}(m) = \mathrm{end}(n)$ and $\sigma_{\mathrm{end}(m)}(m) \geqslant |tu| + 1$. Note that, by the assertion from the first paragraph, there are no pleasant numbers less than n, nor less than $\sigma_{\mathrm{end}(n)}(n)$ (since $\mathrm{end}(\sigma_{\mathrm{end}(n)}(n)) = \mathrm{end}(n)$). We finally have:

$$\begin{aligned}
&2|tuv| + 1 - n - \sigma_{\mathrm{end}(n)}(n)\\
&\quad = |[n+1, |tuv|]_{\mathbb{N}}| + |[\sigma_{\mathrm{end}(n)}(n), |tuv|]_{\mathbb{N}}|\\
&\quad \geqslant |\{m : m > n,\ m \text{ is pleasant or delightful}\}|\\
&\qquad + |\{m : m \geqslant \sigma_{\mathrm{end}(n)}(n),\ m \text{ is pleasant or delightful, or } m \text{ dies}\}|\\
&\quad = |\{m : m > n,\ m \text{ is delightful}\}| + 2|\{m : m \text{ is pleasant}\}|\\
&\qquad + |\{m : m \geqslant \sigma_{\mathrm{end}(n)}(n),\ m \text{ is delightful or } m \text{ dies}\}|\\
&\quad = |\{m : m < \sigma_{\mathrm{end}(n)}(n),\ m \text{ is delightful}\}| + 2|\{m : m \text{ is pleasant}\}|\\
&\qquad + |\{m : m \geqslant \sigma_{\mathrm{end}(n)}(n),\ m \text{ is delightful or } m \text{ dies}\}|\\
&\quad = |\{m : m \text{ is delightful}\}| + 2|\{m : m \text{ is pleasant}\}|\\
&\qquad + |\{m : m \geqslant \sigma_{\mathrm{end}(n)}(n) \text{ and } m \text{ dies}\}|\\
&\quad \geqslant |\{m : m \text{ is pleasant or delightful}\}|\\
&\qquad + |\{m : m \geqslant \sigma_{\mathrm{end}(n)}(n) \text{ and } m \text{ dies}\}|,
\end{aligned} \tag{3.8}$$

which was to be proved. ■

Finally, we are ready to prove Theorem 3.13.

First proof of Theorem 3.13. First of all, we make a (trivial) observation that, whenever $m \in [1, |t|]_{\mathbb{N}} \cup [|tu|+1, |tuv|]_{\mathbb{N}}$ and $\sigma_i(m)$ is defined, then holds $(tuv)[\sigma_i(m)] = 2$.

Assume first $2 \mid \mathrm{end}(n_0)$ (where n_0 is chosen as described earlier). Then $\sigma_{\mathrm{end}(n_0)}(n_0) \in P$, and therefore $\sigma_{\mathrm{end}(n_0)}(n_0) \leqslant |t|$. By Proposition 3.14b), we may write

$$\begin{aligned} |tuv|_2 &= \big|(tuv)[n_0+1, |tuv|]\big|_2 \\ &\quad + \sum_{i=0}^{\frac{\mathrm{end}(n_0)}{2}-1} \big|(tuv)[\sigma_{2i+2}(n_0)+1, \sigma_{2i}(n_0)]\big|_2 + \big|(tuv)[1, \sigma_{\mathrm{end}(n_0)}(n_0)]\big|_2 \\ &= (|tuv| - n_0) + \sum_{i=0}^{\frac{\mathrm{end}(n_0)}{2}-1} \big|(tuv)[\sigma_{2i+2}(n_0)+1, \sigma_{2i}(n_0)]\big|_2 + \sigma_{\mathrm{end}(n_0)}(n_0). \end{aligned} \tag{3.9}$$

Write $k = |tv|$. Let us first prove that, for each i such that $\sigma_{2i+2}(n_0)$ is defined, we have

$$|(tuv)[\sigma_{2i+2}(n_0)+1, \sigma_{2i}(n_0)]|_2 \geqslant \frac{k}{k-1} |(tuv)[\sigma_{2i+2}(n_0)+1, \sigma_{2i}(n_0)]|_1. \tag{3.10}$$

It is enough to prove

$$|(tuv)[\sigma_{2i+2}(n_0)+1, \sigma_{2i}(n_0)]|_2 \geqslant \frac{\sigma_{2i}(n_0) - \sigma_{2i+2}(n_0) + 1}{2};$$

indeed, we note that then we would have $|(tuv)[\sigma_{2i+2}(n_0)+1, \sigma_{2i}(n_0)]|_1 \leqslant \frac{\sigma_{2i}(n_0) - \sigma_{2i+2}(n_0) - 1}{2}$, and thus

$$\begin{aligned} \frac{|(tuv)[\sigma_{2i+2}(n_0)+1, \sigma_{2i}(n_0)]|_2}{|(tuv)[\sigma_{2i+2}(n_0)+1, \sigma_{2i}(n_0)]|_1} &\geqslant \frac{\sigma_{2i}(n_0) - \sigma_{2i+2}(n_0) + 1}{\sigma_{2i}(n_0) - \sigma_{2i+2}(n_0) - 1} \\ &= 1 + \frac{2}{\sigma_{2i}(n_0) - \sigma_{2i+2}(n_0) - 1} \\ &\geqslant 1 + \frac{2}{(2k-1)-1} = 1 + \frac{1}{k-1} = \frac{k}{k-1} \end{aligned} \tag{3.11}$$

(the last inequality follows by Lemma 3.15a) for $n = \sigma_{2i}(n_0)$), which is what we want to prove. Therefore, let us prove the claimed inequality.

We shall use Lemma 3.16 here. If $\sigma_{\mathrm{end}(m)}(m) > \max\{\sigma_{2i}(n_0), \sigma_{2i+1}(n_0)\}$ for either $m \geqslant |tu| + 1$ and $2 \mid \mathrm{end}(m)$, or $m \leqslant |t|$ and $2 \nmid \mathrm{end}(m)$, then we have $\sigma_{\mathrm{end}(m)}(m) \notin Q$; therefore, there are at most

$$\min\{|[\sigma_{2i}(n_0), |tuv|]_{\mathbb{N}} \setminus Q|, |[\sigma_{2i+1}(n_0), |tuv|]_{\mathbb{N}} \setminus Q|\}$$

such values m (we recall that, for any two such different values m and m', we have $\sigma_{\mathrm{end}(m)}(m) \neq \sigma_{\mathrm{end}(m')}(m')$, which was necessary for the last conclusion). In a similar manner, we see that there are at most

$$\min\{|[1, \sigma_{2i+1}(n_0)]_{\mathbb{N}} \setminus P|, |[1, \sigma_{2i+2}(n_0)]_{\mathbb{N}} \setminus P|\}$$

values m such that $\sigma_{\mathrm{end}(m)}(m) < \min\{\sigma_{2i+1}(n_0), \sigma_{2i+2}(n_0)\}$ and either $m \geqslant |tu| + 1$ and $2 \nmid \mathrm{end}(m)$, or $m \leqslant |t|$ and $2 \mid \mathrm{end}(m)$. Altogether, in the set $[1, |t|]_{\mathbb{N}} \cup [|tu| + 1, |tuv|]_{\mathbb{N}}$ there are at least

$$\begin{aligned} k &- \min\{|[\sigma_{2i}(n_0), |tuv|]_{\mathbb{N}} \setminus Q|, |[\sigma_{2i+1}(n_0), |tuv|]_{\mathbb{N}} \setminus Q|\} \\ &- \min\{|[1, \sigma_{2i+1}(n_0)]_{\mathbb{N}} \setminus P|, |[1, \sigma_{2i+2}(n_0)]_{\mathbb{N}} \setminus P|\} \end{aligned}$$

values m for which there exists a corresponding j from Lemma 3.16. But then Lemma 3.16b) immediately implies that this bound is also a lower bound for $|(tuv)[\sigma_{2i+2}(n_0) + 1, \sigma_{2i}(n_0)]|_2$. And now, by Lemma 3.15a) for $n = \sigma_{2i}(n_0)$, we obtain

$$\begin{aligned} &\frac{\sigma_{2i}(n_0) - \sigma_{2i+2}(n_0) + 1}{2} \\ &\qquad \leqslant k - \frac{|[\sigma_{2i}(n_0), |tuv|]_{\mathbb{N}} \setminus Q| + |[\sigma_{2i+1}(n_0), |tuv|]_{\mathbb{N}} \setminus Q|}{2} \\ &\qquad\qquad - \frac{|[1, \sigma_{2i+1}(n_0)]_{\mathbb{N}} \setminus P| + |[1, \sigma_{2i+2}(n_0)]_{\mathbb{N}} \setminus P|}{2} \\ &\qquad \leqslant k - \min\{|[\sigma_{2i}(n_0), |tuv|]_{\mathbb{N}} \setminus Q|, |[\sigma_{2i+1}(n_0), |tuv|]_{\mathbb{N}} \setminus Q|\} \\ &\qquad\qquad - \min\{|[1, \sigma_{2i+1}(n_0)]_{\mathbb{N}} \setminus P|, |[1, \sigma_{2i+2}(n_0)]_{\mathbb{N}} \setminus P|\} \\ &\qquad \leqslant |(tuv)[\sigma_{2i+2}(n_0) + 1, \sigma_{2i}(n_0)]|_2, \end{aligned}$$

which proves the claim.

Using (3.10), from (3.9) we get

$$
\begin{aligned}
|tuv|_2 &> \frac{k}{k-1} \sum_{i=0}^{\frac{\mathrm{end}(n_0)}{2}-1} |(tuv)[\sigma_{2i+2}(n_0)+1, \sigma_{2i}(n_0)]|_1 \\
&= \frac{k}{k-1}|(tuv)[\sigma_{\mathrm{end}(n_0)}(n_0), n_0]|_1 = \frac{k}{k-1}|u|_1
\end{aligned} \tag{3.12}
$$

(the inequality is strict since the rightmost summand at (3.9) is nonzero; the last equality follows from $\sigma_{\mathrm{end}(n_0)}(n_0) \leqslant |t|$ and $n_0 \geqslant |tu|+1$), which completes the case $2 \mid \mathrm{end}(n_0)$.

We can now assume that not only $2 \nmid \mathrm{end}(n_0)$, but also $2 \nmid \mathrm{end}(n)$ for any n that does not die (otherwise, if there were such n, we could take it for n_0, and the above proof would work). We still assume $n_0 \geqslant |tu|+1$, without loss of generality. Finally, a further assumption we can make is that $\mathrm{end}(n_0)$ is no greater than $\mathrm{end}(n)$ for any n that does not die and that $n \geqslant |tu|+1$ (since otherwise we could again rechoose n_0). With all these assumptions, we proceed to the rest of the proof.

Using (3.10), we get

$$
\begin{aligned}
|tuv|_2 &= \big|(tuv)[n_0+1, |tuv|]\big|_2 + \sum_{i=0}^{\frac{\mathrm{end}(n_0)-3}{2}} \big|(tuv)[\sigma_{2i+2}(n_0)+1, \sigma_{2i}(n_0)]\big|_2 \\
&\qquad + \big|(tuv)[1, \sigma_{\mathrm{end}(n_0)-1}(n_0)]\big|_2 \\
&\geqslant (|tuv| - n_0) + \frac{k}{k-1} \sum_{i=0}^{\frac{\mathrm{end}(n_0)-3}{2}} |(tuv)[\sigma_{2i+2}(n_0)+1, \sigma_{2i}(n_0)]|_1 \\
&\qquad + |(tuv)[1, \sigma_{\mathrm{end}(n_0)-1}(n_0)]|_2 \\
&= |tuv| - n_0 + |(tuv)[1, \sigma_{\mathrm{end}(n_0)-1}(n_0)]|_2 \\
&\qquad + \frac{k}{k-1}|(tuv)[\sigma_{\mathrm{end}(n_0)-1}(n_0)+1, n_0]|_1.
\end{aligned}
$$

Therefore, it is enough to prove

$$
|tuv| - n_0 + |(tuv)[1, \sigma_{\mathrm{end}(n_0)-1}(n_0)]|_2 \geqslant \frac{k}{k-1}|(tuv)[1, \sigma_{\mathrm{end}(n_0)-1}(n_0)]|_1
$$

(in that case the calculations above would give $|tuv|_2 \geqslant \frac{k}{k-1}|u|_1$, which is what we need).

The following sets will be needed through the proof, and for the sake of brevity, we name them as follows:

$$A = \{m : |tu|+1 \leqslant m < \sigma_{\mathrm{end}(n_0)}(n_0) \text{ and } |t|+1 \leqslant \sigma_{\mathrm{end}(m)}(m) \leqslant \sigma_{\mathrm{end}(n_0)-1}(n_0)\};$$

$$B = \{m : |tu|+1 \leqslant m < \sigma_{\mathrm{end}(n_0)}(n_0) \text{ and } \sigma_{\mathrm{end}(n_0)-1}(n_0) < \sigma_{\mathrm{end}(m)}(m) \leqslant |tu|\};$$

$$C = \{m : m \geqslant \sigma_{\mathrm{end}(n_0)}(n_0) \text{ and } m \text{ dies}\};$$

$$D = \{m : m \geqslant |tu|+1,\ \mathrm{end}(m) \geqslant \mathrm{end}(n_0) \text{ and } \sigma_{\mathrm{end}(m)}(m) \geqslant |tu|+1\}$$
$$= \{m : m \geqslant |tu|+1 \text{ and } m \text{ does not die}\};$$

$$E = [\sigma_{\mathrm{end}(n_0)-1}(n_0)+1, |tu|]_{\mathbb{N}} \setminus P;$$

$$F = [\sigma_{\mathrm{end}(n_0)-1}(n_0)+1, |tu|]_{\mathbb{N}} \setminus Q = [\sigma_{\mathrm{end}(n_0)-1}(n_0), |tuv|]_{\mathbb{N}} \setminus Q.$$

The equality between the two forms of D follows by the assumption introduced above, and the one between the two forms of F is clear. We shall also use the equality

$$|A| + |B| + |C| + |D| = |v|$$

(which is easily seen), as well as the inequality

$$|E| + |F| \geqslant |B|$$

(which follows by the observation that the function $m \mapsto \sigma_{\mathrm{end}(m)}(m)$ injectively maps the set B to the set $E \cup F$).

Note that, for each m where $\sigma_{\mathrm{end}(n_0)}(n_0) \leqslant m \leqslant |tuv|$, we have $|t| + 1 \leqslant \sigma_Q(m) \leqslant \sigma_{\mathrm{end}(n_0)-1}(n_0)$; that makes for $|tuv| - \sigma_{\mathrm{end}(n_0)}(n_0) + 1$ letters 2 in the word $(tuv)[|t| + 1, \sigma_{\mathrm{end}(n_0)-1}(n_0)]$. Further, for each $m \in A$, the value $\sigma_{\mathrm{end}(m)}(m)$ marks the position of another letter 2 in the word $(tuv)[|t| + 1, \sigma_{\mathrm{end}(n_0)-1}(n_0)]$ (and all these positions are pairwise different, and also different from the positions from the previous sentence, since we recall that all the positions "generated" by a number that dies are unique). Therefore:

$$|(tuv)[1, \sigma_{\mathrm{end}(n_0)-1}(n_0)]|_2 \geqslant |t| + (|tuv| - \sigma_{\mathrm{end}(n_0)}(n_0) + 1) + |A|.$$

From this inequality we get

$$\begin{aligned} |tuv| - n_0 + &|(tuv)[1, \sigma_{\mathrm{end}(n_0)-1}(n_0)]|_2 \\ &\geqslant |tuv| - n_0 + |t| + (|tuv| - \sigma_{\mathrm{end}(n_0)}(n_0) + 1) + |A| \\ &\geqslant |D| + |C| + |t| + |A| = |t| + |v| - |B| = k - |B| \end{aligned} \tag{3.13}$$

(the second inequality is due to Lemma 3.17), and

$$\begin{aligned}
&|(tuv)[1,\sigma_{\mathrm{end}(n_0)-1}(n_0)]|_1 \\
&\qquad = \sigma_{\mathrm{end}(n_0)-1}(n_0) - |(tuv)[1,\sigma_{\mathrm{end}(n_0)-1}(n_0)]|_2 \\
&\qquad \leqslant 2(|t|+|v|) + |P| - \sigma_{\mathrm{end}(n_0)}(n_0) - |[\sigma_{\mathrm{end}(n_0)-1}(n_0), |tuv|]_{\mathbb{N}} \setminus Q| \\
&\qquad\quad - (|t| + (|tuv| - \sigma_{\mathrm{end}(n_0)}(n_0) + 1) + |A|) \\
&\qquad = |t| + |v| + (|P| + |v| - |tuv|) - |F| - 1 - |A| \\
&\qquad = k - |[1,|tu|]_{\mathbb{N}} \setminus P| - |F| - 1 - |A| \\
&\qquad \leqslant k - |E| - |F| - 1 - |A| \leqslant k - 1 - |A| - |B|
\end{aligned} \tag{3.14}$$

(the first inequality has been obtained with the help of Lemma 3.15b) for $n = \sigma_{\mathrm{end}(n_0)-1}(n_0)$; note that then $|[\sigma_Q(n), |tuv|]_{\mathbb{N}} \backslash Q| = |[\sigma_{\mathrm{end}(n_0)}(n_0), |tuv|]_{\mathbb{N}} \backslash Q| = 0$, since $\sigma_{\mathrm{end}(n_0)}(n_0) \geqslant |tu| + 1$). Finally,

$$\begin{aligned}
\frac{|tuv| - n_0 + |(tuv)[1,\sigma_{\mathrm{end}(n_0)-1}(n_0)]|_2}{|(tuv)[1,\sigma_{\mathrm{end}(n_0)-1}(n_0)]|_1} &\geqslant \frac{k-|B|}{k-1-|A|-|B|} \geqslant \frac{k-|B|}{k-1-|B|} \\
&= 1 + \frac{1}{k-1-|B|} \\
&\geqslant 1 + \frac{1}{k-1} = \frac{k}{k-1},
\end{aligned} \tag{3.15}$$

which completes the proof. ∎

It turns out that the case when the equality in (3.5) is reached has an interesting characterization, which can be obtained from the above proof in a (more or less) straightforward manner.

Proposition 3.18. *Under the conditions of Theorem 3.13, the equality in (3.5) is reached if and only if, for a positive integer k and a nonnegative integer l, we have $u = (1^{k-1}2^k)^l 1^{k-1}$, $t = \varepsilon$ and $v = 2^k$ (or vice versa).*

Proof. Assume that u, t, v, p and q are such that the equality is reached. We may also assume that p and q are *longest* subpalindromes of tu and uv, respectively.

We first note that p, respectively q, contains all the letters from t, respectively v (since otherwise the equality cannot be reached because of the strict inequality in (3.6)). Let n_0 be as in the proof. Since in the case $2 \mid \mathrm{end}(n_0)$

we have a strict inequality in (3.12), we conclude $2 \nmid \mathrm{end}(n_0)$, and we then also recall all the assumptions from the paragraph following (3.12). Now, in order for the second and the third inequalities in (3.15) to be equalities, we conclude $A = \varnothing$ and $B = \varnothing$, and then, in order for the last inequality in (3.14) to be equality, we conclude $|E| + |F| = |B|$, that is, $E = F = \varnothing$. The penultimate inequality in (3.14) (when converted to equality) now gives $|[1, |tu|]_{\mathbb{N}} \setminus P| = |E| = 0$, that is,

$$p = tu. \tag{3.16}$$

Since Lemma 3.17 for n_0 was used in (3.13), we need to have equality in that lemma. Looking at (3.8), we see that there must not be any pleasant number (in order to reach the equality at the end), as well as that there must not be any number greater than n_0 that dies (in order to reach the equality at the beginning). In particular, $|tuv|$ does not die (and this implies $2 \nmid \mathrm{end}(|tuv|)$, that is, $\sigma_{\mathrm{end}(|tuv|)}(|tuv|) \geqslant |tu| + 1$, because of the recalled assumptions about n_0), $\mathrm{end}(|tuv|) \leqslant \mathrm{end}(n_0)$ (because there are no pleasant numbers), and from this we conclude that the only possibility is $\mathrm{end}(|tuv|) = \mathrm{end}(n_0)$ (again because of the assumptions about n_0). Therefore, $|tuv|$ satisfies all the same assumptions as n_0 does, and thus for the rest of the proof we may assume $n_0 = |tuv|$ (we rechoose n_0 if necessary).

Assume first $\mathrm{end}(|tuv|) = 1$, that is, $\sigma_Q(|tuv|) \geqslant |tu| + 1$. Then it is easy to see that $q = v$, and furthermore, u contains only 1s (since otherwise $2^{|uv|_2}$ would be a subpalindrome of uv longer than q). Now from (3.16) we get $t = \varepsilon$, and thus the equality in (3.5) reduces to $|u| = \frac{|v|-1}{|v|}|uv|_2 = \frac{|v|-1}{|v|}|v| = |v| - 1$; in other words, $v = 2^k$ and $u = 1^{k-1}$, which was to be proved.

Assume now $\mathrm{end}(|tuv|) > 1$. Let $k = |tv|$. Then the second inequality in (3.11) for $i = 0$ (when converted to equality) gives $|tuv| - \sigma_2(|tuv|) = 2k - 1$, and the first inequality gives

$$\big|(tuv)[\sigma_2(|tuv|) + 1, |tuv|]\big|_2 = \frac{|tuv| - \sigma_2(|tuv|) + 1}{2} = k \tag{3.17}$$

and

$$\big|(tuv)[\sigma_2(|tuv|) + 1, |tuv|]\big|_1 = \frac{|tuv| - \sigma_2(|tuv|) - 1}{2} = k - 1. \tag{3.18}$$

Also note that the second inequality in (3.11) is an application of Lemma 3.15a), and in order for the equality to hold in that lemma, from the last lines of (3.7) we see that $|p| + |q| = 2|u| + 1$ must hold; (3.16) reduces this to $|t| + |q| = |u| + 1$, which implies

$$|uv| - |q| = |t| + |v| - 1 = k - 1. \tag{3.19}$$

Further, since (3.16) implies that tu ends with $|t|$ letters 2, that is, tuv ends with k letters 2, by (3.17) and (3.18) we conclude that there is an array of $k-1$ letters 1 immediately preceding those 2s. In other words, $1^{k-1}2^{|t|} \in \mathrm{Suff}(tu)$, and now because of (3.16) we have $2^{|t|}1^{k-1} \in \mathrm{Pref}(tu)$, that is, $1^{k-1} \in \mathrm{Pref}(u)$. Those 1s clearly do not participate in the palindrome q, and now (3.19) implies that everything else has to, that is,

$$q = (uv)[k, |uv|]. \tag{3.20}$$

Starting from $tuv = 2^{|t|}1^{k-1} \dots 1^{k-1}2^k$, by (3.20) we get

$$tuv = 2^{|t|}1^{k-1}2^k1^{k-1} \dots 1^{k-1}2^k,$$

then by (3.16)

$$tuv = 2^{|t|}1^{k-1}2^k1^{k-1} \dots 1^{k-1}2^k1^{k-1}2^k,$$

then we again use (3.20) etc. To conclude,

$$tuv = 2^{|t|}(1^{k-1}2^k)^l1^{k-1}2^k \quad \text{for a nonnegative integer } l.$$

Now we evaluate $|u|_1 = (l+1)(k-1)$ and $|tuv|_2 = |t| + (l+1)k$. The equality case in (3.5) is now reduced to $(l+1)(k-1) = \frac{k-1}{k}(|t| + (l+1)k)$, that is, $k(l+1) = |t| + (l+1)k$, which gives $|t| = 0$, that is, $t = \varepsilon$. We then have $v = 2^k$ and $u = (1^{k-1}2^k)^l1^{k-1}$, and it is straightforward to check that these words indeed satisfy the required equality. ■

3.3.2 Second proof

Second proof of Theorem 3.13. We prove the theorem by induction on $|u|$. If $|u| = 0$, then (3.5) trivially holds, since the left-hand side is 0 while the right-hand side is always nonnegative. Now we assume that the assertion holds for each word shorter than u, and prove that it holds for u. Let v' denote the shortest prefix of uv such that $|v'|_2 = |v|$, and let t' denote the shortest suffix of tu such that $|t'|_2 = |t|$.

Assume first that $|v'| + |t'| < |u|$. Let $u = v'u't'$, and let p' and q' be longest subpalindromes of $u't$ and vu', respectively (note that we now put

t to the right and v to the left of u', not vice versa, as it was before!). We claim that

$$|q'| \geqslant |p| - 2|t| - 2(|v'| - |v|).$$

We can write $p = 2^{|t|}p_1p_22^{|t|}$, where $p_1 \in \mathrm{Subw}(v')$ and $p_2 \in \mathrm{Subw}(u')$. We have that p_1p_2 is a palindrome of length $|p|-2|t|$; erasing all the letters 1 from p_1 (and there are at most $|v'|_1$, which is $|v'|-|v|$, of them), and additionally erasing (if necessary) all the "mirror images" (with respect to the midpoint of p) of all these 1s, we obtain a subpalindrome of vu' of length at least $|p| - 2|t| - 2(|v'| - |v|)$, which proves the claim. Analogously, we also obtain

$$|p'| \geqslant |q| - 2|v| - 2(|t'| - |t|).$$

We aim to use the inductive assumption on the words v, u' and t. Clearly, $|u'| = |u| - |v'| - |t'| \leqslant |u| - |v| - |t| < |u|$. Let us now show that the condition of the theorem is satisfied:

$$\begin{aligned} |q'| + |p'| &\geqslant |p| - 2|t| - 2(|v'| - |v|) + |q| - 2|v| - 2(|t'| - |t|) \\ &= |p| + |q| - 2|v'| - 2|t'| > 2|u| - 2|v'| - 2|t'| = 2|u'|. \end{aligned}$$

Therefore, by the inductive assumption, we get

$$|u'|_1 \leqslant \frac{|vt| - 1}{|vt|}|vu't|_2. \tag{3.21}$$

Note that, since 1s inside the word v' do not participate in the palindrome q (since the first $|v|$ letters of q are 2), we have $|v'|_1 \leqslant |uv| - |q|$. Analogously, $|t'|_1 \leqslant |tu| - |p|$. Therefore,

$$|v'|_1 + |t'|_1 \leqslant |uv| - |q| + |tu| - |p| < |t| + |v|. \tag{3.22}$$

Now, (3.21) and (3.22) yield:

$$\begin{aligned} |u|_1 = |v'|_1 + |u'|_1 + |t'|_1 \leqslant |tv| - 1 + |u|_1 \leqslant \frac{|tv| - 1}{|tv|}(|tv| + |vu't|_2) \\ = \frac{|tv| - 1}{|tv|}(|t'|_2 + |v'|_2 + |vu't|_2) = \frac{|tv| - 1}{|tv|}|tuv|_2, \end{aligned}$$

which was to be proved.

Finally, we need to take care of the case $|v'| + |t'| \geqslant |u|$. Then we have:

$$|u|_1 \leqslant |v'|_1 + |t'|_1 \leqslant |t| + |v| - 1 = \frac{|tv| - 1}{|tv|}|tv| \leqslant \frac{|tv| - 1}{|tv|}|tuv|_2,$$

(where the second inequality was already seen at (3.22)), which was to be proved. ∎

Appendix: Examples

The mappings σ_i are illustrated in Figure 3.1, where $m_1 = 23$, $m_2 = 22$, $m_3 = 21$ and $m_4 = 1$. The positions in P are marked by the underlines, and the positions in Q by the overlines. Note that the sequence $\sigma_0(m_i), \sigma_1(m_i), \sigma_2(m_i) \dots$ can end either in the same part (that is, t or v) where it started from (such is the case for m_4, which returns exactly to itself, and also for m_1 and m_2, which end at each other), or inside u (such is the case for m_3), or at the opposite end (t or v) with respect to the one in started from (no such examples in Figure 3.1, but this is the case, for example, for the 1st and the 18th position in Figure 3.3).

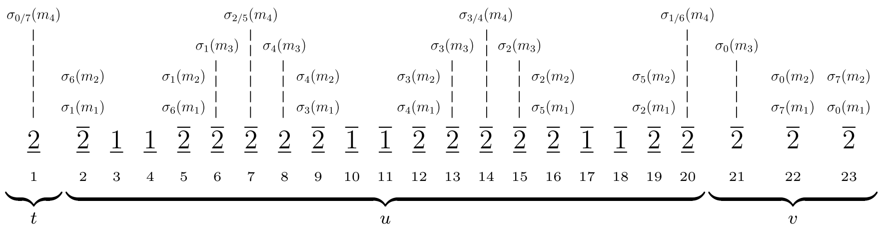

Figure 3.1: The mappings σ_i.

Let us now show the idea of the proof on the example in Figure 3.2 (note that, in that example, $t = \varepsilon$). The two vertical dashed lines are exactly in the center of the palindrome p (the one on the left) and the palindrome q (the one on the right). All the dashed arrows present the mapping σ_P (which is the reflection with respect to the center of p), and all the solid arrows present the mapping σ_Q (which is the reflection with respect to the center of q). Let $k = |t| + |v|$; therefore, our aim is to prove that the ratio of the number of 2s in tuv and the number of 1s in tuv is at least $\frac{k}{k-1}$.

We choose a position n_0 in t or v for which the sequence of mappings does not end in u (it will be shown that there always exist such one); in our example, $n_0 = 19$.

In Proposition 3.14 we show some basic properties of σ's; one of the most important ones is the fact that the sequence $\sigma_0(n_0), \sigma_2(n_0), \sigma_4(n_0) \dots$ is strictly decreasing (if n_0 is in v, which is the case in our example; otherwise, it would be strictly increasing, which is analogous). Owing to this property, we can partition the word tuv into a number of intervals, plus some additional letters to the left of the leftmost interval and to the right of the rightmost

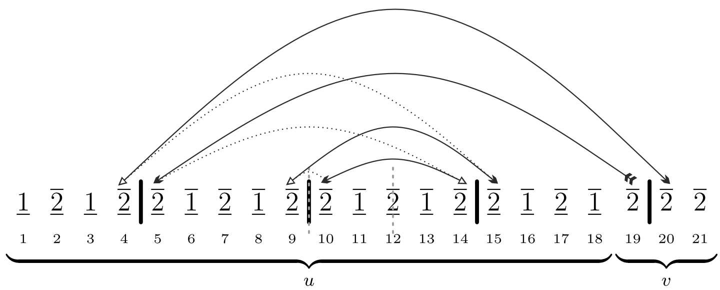

Figure 3.2: An example for the proof of Theorem 3.13.

interval; in Figure 3.2, each such interval is the part between two vertical thick lines. The proof is then over once we show the following two assertions.

- *Within each interval, the ratio of the number of* 2*s and the number of* 1*s is at least* $\frac{k}{k-1}$.

  We prove this by first showing an upper bound on the length of each interval. The bound is shown in Lemma 3.15a), and it equals $2k-1$ minus some "noise." In our example, there is no "noise," that is, all the intervals are of length 5 (this is the case in which the inequality is tightest; if the intervals were shorter, the considered ratio would be even greater than needed). Therefore, it is enough to show that there are at least k letters 2 in each interval, and we show this by noting that all k letters 2 from t and v, while "jumping around," leave k "footprints" in each interval, which accounts for k letters 2 in each of them. This is precisely what Lemma 3.16b) is about. (Note: if there is a letter 2 that ends the jumping sequence inside u, then there might exist an interval with less than k letters 2. But then the calculations show that the "noise" mentioned above shortens the interval enough so that the number of 2s inside it will still be sufficient with respect to its length.)

- *The ratio of the number of* 2*s and the number of* 1*s not belonging to any of the intervals is at least* $\frac{k}{k-1}$.

  This is probably the most technical part of the whole proof. Lemma 3.15b) gives an upper bound on the length of the part to the left of the leftmost interval. Lemma 3.17 provides another technical inequality that will be needed in this case (in particular, it will be of use, later in the proof, to bound from below the number of 2s not belonging to any of the intervals). Direct (albeit messy) calculations then show what was needed.

In the example that we have just shown, the position n_0 belongs to v, and its sequence of mappings ends again in v. It turns out that this is the harder case; if n_0 can be chosen that is in v but ends in t (or vice versa), the proof is easier (an example will be shown in a moment). For that reason, the case presented above includes the assumption that no n_0 as in the previous sentence can be chosen, and not only that, but also, among all the possible choices for n_0, it is required to choose the one whose mapping sequence is of the minimal possible length. This assumption is indeed necessary: the reader may check that, if we choose $n_0 = 21$ in the example from Figure 3.2 (its mapping sequence is of length 9, while note that for $n_0 = 19$ its mapping sequence was of length 7), the proof sketched above will not work (in particular, the statement from the lower bullet point will be false).

Consider now the example from Figure 3.3 and $n_0 = 18$. We draw the intervals as before. The part of the proof that shows the first bullet point from the previous example is the same, but in this case this is actually everything we need, since there are no 1s at all that do not belong to any interval (this is so not only in the example from Figure 3.3, but it is easy to see that this property always holds in this case).

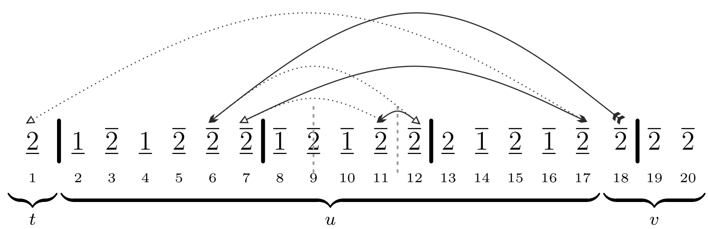

Figure 3.3: Another example for the proof of Theorem 3.13.

As a bonus feature, note that in this example the intervals are of length less than $2k - 1$ (that is, less than 7, since $k = 4$), and there is an interval with less than k letters 2 (which we mentioned earlier as a possibility if there exists a position in t or v which ends the jumping sequence inside u, which is indeed the case here, for the 19^{th} position), but of course, in each interval the ratio between the number of 2s and the number of 1s is at least $\frac{k}{k-1}$ (actually, even greater than that).

Finally, in order to avoid any confusion of the reader, we mention that, in the formal write-up of the proof in Subsection 3.3.1, the "easier" case is treated before the "harder" one (that is, in reversed order with respect to the one in which they are presented here).

3.4 Another postponed technical theorem

Theorem 3.19. *Let $w \in \{1,2\}^*$ be such that $2|w|_2 \geqslant |w|_1$. Let p and q be two nonempty subpalindromes of w. Let w_p, v, w_q and t be such that $w = w_p v = t w_q$, p is a subword of w_p, and q is a subword of w_q. Then*

$$|p| + 2|v|_2 + |q| + 2|t|_2 \leqslant 4|w|_2 + |w|_1.$$

Proof. We distinguish two cases:

- Case 1°: $|w_p| \leqslant |t|$;
- Case 2°: $|t| < |w_p|$.

Case 1°. In this case we have $|p|_1 + |q|_1 \leqslant |w|_1$. Furthermore, we have:

$$|p|_2 + 2|v|_2 \leqslant |w|_2 + |v|_2 \leqslant 2|w|_2.$$

In an analogous way we obtain $|q|_2 + 2|t|_2 \leqslant 2|w|_2$. Now, we get the required inequality directly:

$$\begin{aligned} |p| + 2|v|_2 + |q| + 2|t|_2 &\leqslant |p|_1 + |q|_1 + |p|_2 + 2|v|_2 + |q|_2 + 2|t|_2 \\ &\leqslant |w|_1 + 4|w|_2. \end{aligned}$$

Case 2°. In this case we may write $w = tuv$, where u is a nonempty word. Now suppose that the required inequality does not hold, that is,

$$|p| + 2|v|_2 + |q| + 2|t|_2 > |w|_1 + 4|w|_2.$$

This reduces to

$$|p| + |q| > |w|_1 + 2|w|_2 + 2|u|_2 \geqslant 2|w|_1 + 2|u|_2. \tag{3.23}$$

Let $\widehat{t}$ and $\widehat{v}$ be the words obtained from the words t and v, respectively, by erasing all the letters 1 (or, equivalently, $\widehat{t} = 2^{|t|_2}$ and $\widehat{v} = 2^{|v|_2}$), and let

$\widehat{p}$ and $\widehat{q}$ be the longest subpalindromes of $\widehat{t}u$ and $u\widehat{v}$, respectively. We then have:

$$\begin{aligned}|\widehat{p}| &\geqslant |p| - 2|t|_1;\\ |\widehat{q}| &\geqslant |q| - 2|v|_1.\end{aligned}$$

Therefore,

$$|\widehat{p}| + |\widehat{q}| \geqslant |p| - 2|t|_1 + |q| - 2|v|_1 > (2|w|_1 + 2|u|_2) - 2|t|_1 - 2|v|_1 = 2|u|,$$

which means that the conditions of Theorem 3.13 are satisfied (for u, $\widehat{t}$, $\widehat{v}$, $\widehat{p}$ and $\widehat{q}$); by that theorem we obtain

$$|u|_1 \leqslant \frac{|\widehat{t}\widehat{v}| - 1}{|\widehat{t}\widehat{v}|}|\widehat{t}u\widehat{v}|_2 < |\widehat{t}u\widehat{v}|_2 \leqslant |tuv|_2 = |w|_2.$$

On the other hand, since $|p| + |q| \leqslant |w| + |u|$, by the first inequality in (3.23) we have $|w| + |u| > |w|_1 + 2|w|_2 + 2|u|_2$, that is, $|u|_1 > |w|_2 + |u|_2 \geqslant |w|_2$, a contradiction. The proof is completed. ■

4 Conclusion

In this thesis we collected several results related to some reversal-invariant complexity measures of words. The two such measures we were investigating are the palindromic defect and the MP-ratio of a given word.

One of our main results is the introduction of the class of *generalized highly potential words.* We proved that their defect is always finite, and in many cases positive, that their set of factors is closed under reversal, and that each of them is either periodic, or recurrent but not uniformly recurrent. The significance of this class of words lies in the fact that this conjunction of properties is rarely met in words, which makes our words a very good supply of examples and counterexamples for various problems of words.

The other main result of this thesis is the extension of the definition of the *MP-ratio* to ternary words. The notion of the MP-ratio was defined originally only for binary words ten years ago, the possibility of generalization for larger alphabets was left as an open question, and there was no progress at all until now. We showed that the MP-ratio is well-defined in the ternary case, that it is bounded from above by 6, and that this is the best possible upper bound. Finding a sharp upper bound on the MP-ratio in general case is still open, but we believe that our work represents a significant step in this direction.

Note. After the main text of this thesis was completed, we have proved that the MP-ratio for n-ary alphabet is well-defined for any n. An article is in preparation.

Prošireni izvod

U oblasti kombinatorike na rečima definisane su razne funkcije koje predstavljaju mere složenosti neke date reči. U ovoj tezi posmatramo neke od njih koje su invarijantne u odnosu na operaciju preokretanja reči. Videćemo (kao što i jeste očekivano) da su te mere složenosti tesno povezane s pojmom palindroma. Izdvajamo dva pravca istraživanja i dajemo odgovor na više pitanja u vezi s njima.

Jedan pravac istraživanja se bazira na rezultatima Droubaya, Justina i Pirilla [30], koji su primetili da reč dužine n sadrži najviše $n + 1$ različitih palindromskih faktora. Razlika između ovog gornjeg ograničenja i ukupnog broja palindromskih faktora se zove *palindromski defekt* (ili samo *defekt*) date reči [19] (koji je, po definiciji, uvek nenegativan). Iako je definicija suštinski zasnovana na konačnosti date reči, ispostavlja se da se definicija defekta može prirodno uopštiti i na beskonačne reči (defekt beskonačne reči se definiše kao supremum defekata njenih konačnih faktora). Reči defekta 0 se nazivaju *pune* ili *bogate*, i o njima se u literaturi može naći priličan broj rezultata [47, 21, 50, 59, 36, 60, 53, 51].

Nasuprot tome, beskonačne reči konačnog pozitivnog defekta su znatno manje istražene. Jedan od razloga leži u tome što se eksplicitne konstrukcije takvih reči, uz možda neke dodatne uslove (pre svega aperiodičnost, budući da su periodične reči uglavnom pravolinijske za ispitivanje), ispostavljaju kao relativno deficitarne u literaturi. Dugo nije bio poznat nijedan primer beskonačne reči koja je aperiodična, koja ima konačan pozitivan defekt, i čiji je skup faktora zatvoren za preokretanje, a za reči s ovim sklopom osobina se smatralo da mogu bar do neke mere rasvetliti hipoteze koje su bile postavljene

[17, 20]. Veliki napredak se pojavio u radu [15], gde je definisana konstrukcija beskonačne familije beskonačnih reči, tzv. visokopotencijalnih reči, koje su sve aperiodične, imaju skup faktora zatvoren za preokretanje, i imaju konačan pozitivan defekt (štaviše, prikazana konstrukcija zapravo predstavlja metod kojim se takva reč dobija od bilo koje konačne reči koja nije palindrom). Kako se može videti u tom radu, te reči deluju kao vrlo korisna zaliha primera i kontrapimera za razne probleme o rečima (po čemu su i dobile naziv). Pritom, treba reći da je hronološki prvi primer aperiodične beskonačne reči koja ima konačan pozitivan defekt i čiji je skup faktora zatvoren za preokretanje ipak viđen nešto ranije: videti [9, Example 3.4], gde je konstruisana jedna takva reč, koja je pritom uniformno rekurentna. U radu [14] je konstruisana jedna takva reč koja nije uniformno rekurentna i pomoću koje je demonstriran suštinski propust u jednom dokazu iz [10]; ova reč, iako očigledno ima dosta zajedničkog s familijom visokopotencijalnih reči, ipak zapravo ne pripada toj familiji.

U ovoj tezi konstruišemo novu familiju beskonačnih reči čiji je defekt konačan, i u mnogo slučaja pozitivan (daćemo tačnu karakterizaciju kada im je defekt 0). Ta konstruisana familija, štaviše, kao specijalne slučajeve odjednom obuhvata i visokopotencijalne reči (stoga je radni naziv za nju uopštene visokopotencijalne reči) i pomenutu specijalno konstrusanu reč iz [14]. Dalje, u [34, Proposition 2.10] autori su pokazali egzistenciju bogatih beskonačnih reči koje su rekurentne ali ne i uniformno rekurentne, navodeći tri primera; ispostavlja se da ova tri primera takođe pripadaju klasi uopštenih visokopotencijalnih reči. Verujemo da sve ovo svedoči o visokom nivou generalnosti naše konstrukcije. Sve reči naše familije su zatvorene za preokretanje i jesu ili periodične (što je manje zanimljiv slučaj i eksplicitno okarakterisan), ili rekurentne ali ne i uniformno rekurentne. Činjenica da nisu uniformno rekurentne (osim ako su periodične) posebno je značajna zbog: prvo, u literaturi se tu i tamo još i mogu naći neki rezultati i primeri u vezi sa uniformno rekurentnim rečima (videti, na primer, [34, Proposition 4.8], ili [9], ili kontraprimer za tzv. hipotezu nultog defekta iz [23], koji je definisan kao fiksna tačka određenog primitivnog morfizma, a poznato je [6, Theorem 10.9.5] da su fiksne tačke primitivnih morfizama su uvek uniformno rekurentne), dok se o aperiodičnim rečima koje nisu uniformno rekurentne ne zna praktično ništa; drugo, u [49, Theorem 2] je pokazano da svaka uniformno rekurentna reč konačnog defekta zapravno mora biti morfična slika neke reči defekta nula (dok je rezultat koji autori dobijaju bez pretpostavke uniformne rekurentnosti slabiji, i u poslednjoj sekciji analiziraju značaj te pretpostavke, i ostavljaju kao otvoreno pitanje da li jači rezultat važi bez nje), što sve sugeriše da je sa uniformno rekurentnim rečima donekle lakše raditi, dok one koje nisu uniformne rekurentne predstavljaju slabije istraženu teritoriju koja zaslužuje da se podrobnije izuči.

Holub i Saari [39] su uveli još jedan način za merenje koliko je neka reč „bogata" palindromima, tzv. *MP-razmeru* date reči. MP-razmera je racionalan broj veći od ili jednak sa 1 takav da, što je MP-razmera veća, reč je „bogatija" palindromima (autori rada [39] za takve reči kažu da su „jako palindromične"); za reč MP-razmere 1 kažemo da je *minimalno-palindromična*. Ispostavilo se da su neke osobine MP-razmere znatno dublje nego što bi se moglo pomisliti na osnovu prvog utiska jer, kao što je pokazano u [13], MP-razmera se ponekad ponaša na vrlo nepredvidiv način. Pojam MP-razmere se bazira na palindromičnim *podrečima* (a ne faktorima) date reči, što je znatno manje izučavano u literaturi. Međutim, i one imaju neke zanimljive osobine. Kao što je pokazano u [39], svaka binarna reč, do na preokretanje, može se rekonstruisati na osnovu skupa njenih palindromičnih podreči. Takođe u [39], definisan je pojam *abelovske neomeđenosti*, i pokazano je da svaka binarna minimalno-palindromična reč jeste abelovski neomeđena (što je jaka verzija neomeđenosti); abelovska (ne)omeđenost reči privlači sve veću pažnju u poslednje vreme [28, 35, 26, 7, 16]. Međutim, najveći nedostatak koncepta MP-razmere je činjenica da je MP-razmera definisana samo za binarni alfabet. Iako postoji prirodno proširenje definicije i na veći alfabet, nije jasno da li će u tom slučaju MP-razmera uopšte biti dobro definisana. Zbog toga su autori rada [39] ostavili problem dobre definisanosti MP-razmere za veće alfabete kao otvoreno pitanje. U ovoj tezi rešavamo ovaj problem za ternarni alfabet. Pokazaćemo da MP-razmera jeste dobro definisana u ternarnom slučaju, da je ograničena s gornje strane kontantom 6, i da je ta granica najbolja moguća.

Teza je organizovana na sledeći način.

Glava 1 je uvodna. U sekciji 1.1 navodimo osnovne pojmove i teoreme o rečima generalno. U sekcijama 1.2 i 1.3 predstavljamo relevantne rezultate vezane za defekt i za MP-razmeru, respektivno. Svi rezultati iz ove glave su poznati u literaturi i dati sa odgovarajućim referencama.

Glave 2 i 3 predstavljaju u potpunosti originalan doprinos, i najvećim delom su sadržane u radovima [4], odnosno [2] i [3].

Glava 2 je posvećena uopštenim visokopotencijalnim rečima. Njihova definicija i neki tehnički preliminarni rezultati su dati u sekcijama 2.1 i 2.2. U sekciji 2.3 dajemo potreban i dovoljan uslov za periodičnost uopštenih visokopotencijalnih reči, pokazujemo da je skup njihovih faktora zatvoren za preokretanje (što implicira da su one rekurentne), i još pokazujemo da one koje nisu periodične nisu ni uniformno rekurentne. U sekciji 2.4 pokazujemo da je njihov defekt uvek konačan i dajemo potreban i dovoljan uslov kada je pozitivan. Zasebno, u sekciji 2.5 na kraju se bavimo periodičnim uopštenim visokopotencijalnim rečima (što je manje interesantan slučaj).

Glava 3 se bavi MP-razmerom. U sekciji 3.1 pokazujemo da za bilo koju ternarnu reč w uvek postoji MP-proširenje (r, s); štaviše, pošto za našu konstrukciju važi $|rws| = 6|w|$, dobijamo da je MP-razmera ograničena s gornje strane sa 6. Tokom dokaza potrebna su nam dva tehnička rezultata, koji su dati izdvojeno u sekcijama 3.3 i 3.4 (gde je sekcija 3.3 nezavisna od prethodnih sekcija, a sekcija 3.4 se oslanja jedino na sekciju 3.3, pa verujemo da čitalac neće doći u zabunu); osim toga, ova dva rezultata su u suštini rezultati vezane za *binarne* reči (i postoji mala šansa da se mogu upotrebiti i negde drugde), pa je i zbog toga prirodnije dati ih odvojeno od dokaza u sekciji 3.1. U sekciji 3.2 pokazujemo da MP-razmera može biti proizvoljno blizu konstanti 6, što zapravo daje da je 6 najbolja moguća gornja granica MP-razmere u ternarnom slučaju.

1 Uvod

1.1 O rečima

U ovoj sekciji uvodimo osnovne definicije i osobine koje se pominju u tezi. Sve što je navedeno je poznato i može se pronaći recimo u [6].

Neka je Σ konačan neprazan skup simbola, koji zovemo *alfabet*, a njegove elemente zovemo *slovima*. Konačne (respektivno beskonačne) nizove slova nazivamo *reči* (respektivno *beskonačne reči*) nad alfabetom Σ. (Ponekad ćemo zloupotrebiti terminologiju i reći samo „reč“ kada je jasno iz konteksta da je u pitanju beskonačna reč, ili dodatno naglašavati „konačna reč“, kada smatramo da je tako zgodnije.) Označimo sa Σ^* skup konačnih reči, a sa Σ^∞ skup konačnih ili beskonačnih reči. U slučaju $|\Sigma| = 2$ govorimo o binarnim rečima, u slučaju $|\Sigma| = 3$ o ternarnim rečima, i generalno, u slučaju $|\Sigma| = n$ govorimo o n-arnim rečima. Za $w = a_1a_2...a_n$, gde $a_1, a_2, \dots, a_n \in \Sigma$, kažemo da je n *dužina* reči w, i pišemo $|w| = n$. Jedinstvenu reč dužine 0 zovemo *prazna reč*, i označavamo sa ε.

Konkatenacije (ili *proizvod*) reči u i v, $u = a_1a_2 \dots a_n$ i $v = b_1b_2 \dots b_m$, jeste reč $a_1a_2 \dots a_nb_1b_2 \dots b_m$, i označavamo je sa uv. Proizvod uv za $u \in \Sigma^*$

i $v \in \Sigma^\infty \backslash \Sigma^*$ je slično definisan. Za reč w i za prirodan broj k, sa w^k označavamo reč $\underbrace{ww\dots w}_{k}$, a sa w^∞ beskonačnu reč $wwww\dots$; takođe, definišemo $w^0 = \varepsilon$ za bilo koju reč w. Reč $w \in \Sigma^*$ je *primitivna* ako i samo ako nije oblika z^k za neko $z \in \Sigma^* \backslash \{\varepsilon\}$ i ceo broj k, $k \geqslant 2$.

Za $A \subseteq \Sigma$, sa A^* označavamo skup

$$\{a_1 a_2 \dots a_k : k \geqslant 0 \text{ i } a_i \in A \text{ za svako } i\},$$

a $A^+ = A^* \backslash \{\varepsilon\}$. Ako skup A sadrži samo jedan element, recimo $A = \{a\}$, pišemo a^* i a^+ umesto $\{a\}^*$ i $\{a\}^+$. Ako su A i B skupovi reči, pišemo $AB = \{uv : u \in A, v \in B\}$. Kako je konkatenacija reči asocijativna operacija, i proizvod više od dva skupa reči takođe je dobro definisan.

Reč $u \in \Sigma^*$ se zove *faktor* (respektivno *prefiks*, *sufiks*) reči $w \in \Sigma^\infty$ ako i samo ako postoje reči $x \in \Sigma^*$ i $y \in \Sigma^\infty$ takve da $w = xuy$ (respektivno $w = uy$, $w = xu$). Reč $u \in \Sigma^*$ je *podreč* reči $w \in \Sigma^*$ ako i samo ako postoje reči $x_1, x_2, \dots, x_n, x_{n+1} \in \Sigma^*$ i $y_1, y_2, \dots, y_n \in \Sigma^*$ takve da $u = y_1 y_2 \dots y_n$ i $w = x_1 y_1 x_2 y_2 \dots x_n y_n x_{n+1}$ (ili, ekvivalentno, u je podreč reči w ako je u njen podniz). Ako je w proizvoljna reč, skup svih faktora (respektivno prefiksa, sufiksa, podreči) označavamo sa $\mathrm{Fact}(w)$ (respektivno $\mathrm{Pref}(w)$, $\mathrm{Suff}(w)$, $\mathrm{Subw}(w)$).

Koristićemo oznaku $w[i]$ za i-to slovo u reči w, a za par (i, j) celih brojeva, gde $1 \leqslant i \leqslant j \leqslant |w|$, sa $w[i, j]$ ćemo označavati faktor $w[i]w[i+1]\dots w[j]$ (jasno, $w[i, i] = w[i]$). U slučaju $i > j$, kao i u slučajevima $i > |w|$ ili $j < 1$, definišemo $w[i, j] = \varepsilon$. Po konvenciji, ova operacija ima prednost nad konkatenacijom; drugim rečima, $uv[i]$ (i slično $uv[i, j]$) označava $u(v[i])$, a ne $(uv)[i]$.

Za prirodne brojeve i i j, $i \leqslant j$, sa $[i, j]_{\mathbb{N}}$ označavamo skup $\{i, i+1, i+2, \dots, j\}$. (Sa $\mathbb{N}$ označavamo pozitivne cele brojeve, dok sa $\mathbb{N}_0$ označavamo nenegativne cele brojeve.)

Za reči u i v sa $|u|_v$ označavamo broj različitih pojavljivanja reči v unutar u, to jest,

$$|u|_v = |\{i : 1 \leqslant i \leqslant |u| - |v| + 1, u[i, i + |v| - 1] = v\}|.$$

Kažemo da je slovo c *dominantno* u reči w ako i samo ako $|w|_c = \max\{|w|_a : a \in \Sigma\}$. (Primetimo da dominantno slovo nije nužno jedninstveno.)

Preokretanje reči $w = a_1 a_2 \dots a_n$, gde $a_1, a_2, \dots, a_n \in \Sigma$, definisano je sa $\widetilde{w} = a_n a_{n-1} \dots a_1$. Kažemo da je skup faktora w *zatvoren za preokretanje* ako i samo ako za proizvoljno $v \in \mathrm{Fact}(w)$ važi $\widetilde{v} \in \mathrm{Fact}(w)$. Za reč w kažemo da je *palindrom* ako i samo ako $w = \widetilde{w}$. (Praznu reč takođe smatramo palindromom.) Palindromsku podreč date reči zovemo *potpalindrom*. Koristimo oznaku $\mathrm{Pal}(w) = \{u \in \mathrm{Fact}(w) : u = \widetilde{u}\}$.

Preslikavanje $\varphi : \Sigma^* \to \Sigma^*$ nazivamo *morfizam* ako i samo ako $\varphi(uv) = \varphi(u)\varphi(v)$ za sve $u, v \in \Sigma^*$.

Pre naredne teoreme [45, Proposition 1.3.2] uvodimo sledeći pojam: reč w' je *konjugat* reči w ako i samo ako postoje reči x i y takve da $w = xy$ i $w' = yx$.

Teorema 1.1. *Neka $x, y \in \Sigma^* \backslash \{\varepsilon\}$. Tada $xy = yx$ ako i samo ako postoje $t \in \Sigma^*$ i prirodni brojevi p, q takvi da $x = t^p$, $y = t^q$. Drugim rečima, ako je reč jednaka svom konjugatu (različitom od same date reči), tada ona mora biti potpun stepen sa eksponentom bar dva.*

Beskonačna reč w je *periodična* ako i samo ako je oblika $w = u^\infty$ za neko $u \in \Sigma^*$, *eventualno periodična* ako i samo ako je oblika vu^∞ za neko $u, v \in \Sigma^*$ i *aperiodična* ako i samo ako nije eventualno periodična. Prirodan broj p se naziva *period* reči w ako i samo ako $w[i] = w[i+p]$ za svako $i \geqslant 1$. (Period ne mora biti jedinstven.) Beskonačna reč w je *rekurentna* ako i samo ako se svaki faktor reči w pojavljuje beskonačno mnogo puta u reči w, a *uniformno rekurentna* ako i samo ako za svaki konačan faktor u reči w postoji ceo broj n takav da $u \in \mathrm{Fact}(v)$ za svako $v \in \mathrm{Fact}(w)$ takvo da $|v| = n$.

Sledeće tri teoreme su poznate i mogu se naći u [6], Exercise 10.50a), Example 10.9.1 i Exercise 10.37, respektivno.

Teorema 1.2. *Za beskonačnu reč w, ako je* Fact(w) *zatvoreno za preokretanje, tada je w rekurentna.*

Teorema 1.3. *Svaka periodična reč je uniformno rekurentna.*

Teorema 1.4. *Svaka rekurentna, eventualno periodična reč je periodična.*

1.2 Palindromski defekt

Sledeću nejednakost su primetili Droubay, Justin i Pirillo [30, Proposition 2].

Teorema 1.5. *Neka je w konačna reč. Tada:*

$$|\mathrm{Pal}(w)| \leqslant |w| + 1.$$

Ova nejednakost je inspirisala Brleka et al. [19] da definišu *palindromski defekt* (ili samo *defekt*) reči w, u oznaci $D(w)$:

$$D(w) = |w| + 1 - |\mathrm{Pal}(w)|.$$

Oni su uočili da defekt faktora date reči nikada nije veći od defekta polazne reči; drugim rečima:

Teorema 1.6. *Neka je w konačna reč i $v \in \mathrm{Fact}(w)$. Tada*

$$D(v) \leqslant D(w).$$

Ovo daje motivaciju za uopštenje definicije defekta i na beskonačne reči: za $w \in \Sigma^\infty \backslash \Sigma^*$ definišemo

$$D(w) = \sup_{v \in \mathrm{Fact}(w)} D(v).$$

(Naravno, ova nejednakost takođe važi i za konačne reči.) Primetimo da je defekt bilo koje konačne ili beskonačne reči uvek nenegativan ili beskonačan.

1.2.1 Defekt nekih periodičnih reči

U ovoj podsekciji posmatramo specijalan slučaj periodičnih reči u kom je defekt uvek konačan i lako izračunljiv. Teoreme 1.7 (videti [19, Lemma 5] i [45, Proposition 1.3.4]) i 1.8 (videti [19, Theorem 6]) predstavljaju tehničke rezultate koji vode ka glavnoj teoremi ove podsekcije, teoremi 1.9 (iz [19, Corollary 8]).

Teorema 1.7. *Neka $w = xy = yz$. Ako je w palindrom, tada postoje palindromi u i v takvi da $x = uv$, $z = vu$ i $y = (uv)^{i-1}u$ za neki prirodan broj i. Dalje, tada je xyz takođe palindrom.*

Teorema 1.8. *Neka je p primitivna reč koja je proizvod dva palindroma u i v, $|u| \geqslant |v|$. Tada:*

$$D(p^\infty) = D\left(p^\infty\left[1, |uv| + \left\lfloor \frac{|u| - |v|}{3} \right\rfloor\right]\right).$$

Teorema 1.9. *Ako je p primitivna reč koja je proizvod dva palindroma (od kojih jedan može biti i prazna reč), tada postoji neki konjugat q od p takav da*

$$D(p^\infty) = D(q).$$

Komentar. Zapravo, zbog definicije defekta beskonačne reči, u prethodnoj teoremi za q možemo (i moramo) birati onaj konjugat za koji je $D(q)$ maksimalan (ili bilo koji takav, ako ih ima više).

1.2.2 Visokopotencijalne reči

Familija beskonačnih reči, tzv. *visokopotencijalnih reči*, uvedena je u [15]. Neka je data reč w koja nije palindrom i slovo c koje se ne pojavljuje u w, i definišimo:

$$w_0 = w;$$

$$w_i = w_{i-1}c^i\widetilde{w_{i-1}},\ i \in \mathbb{N};$$

$$\mathrm{hpw}(w) = \lim_{i\to\infty} w_i.$$

Beskonačna reč $\mathrm{hpw}(w)$ se zove *visokopotencijalna reč generisana sa* w. (Limes je dobro definisan jer je svako w_i prefiks od w_{i+1}.)

Glavne osobine visokopotencijalnih reči su date u sledeće dve teoreme.

Teorema 1.10. *Neka je* $\mathrm{hpw}(w)$ *visokopotencijalna reč. Tada:*

- $\mathrm{hpw}(w)$ *je aperiodična;*
- $\mathrm{Fact}(\mathrm{hpw}(w))$ *je zatvoreno za preokretanje;*
- $\mathrm{hpw}(w)$ *je rekurentna;*
- $\mathrm{hpw}(w)$ *nije uniformno rekurentna.*

Teorema 1.11. *Neka je* $\mathrm{hpw}(w)$ *visokopotencijalna reč. Tada:*

$$D(\mathrm{hpw}(w)) = D(w) + 1.$$

1.2.3 Još nekoliko aperiodičnih reči konačnog defekta

Sledeća reč je definisana u [14].

Teorema 1.12. *Neka je f morfizam definisan sa $f(1) = 1213$, $f(2) = \varepsilon$ i $f(3) = 23$. Neka $f^\infty(1) = \lim_{i\to\infty} f^i(1)$. Beskonačna reč $f^\infty(1)$ ima sledeće osobine:*

- $f^\infty(1)$ *je aperiodična;*
- $\mathrm{Fact}(f^\infty(1))$ *je zatvoreno za preokretanje;*
- $f^\infty(1)$ *je rekurentna ali nije uniformno rekurenta;*
- $D(f^\infty(1))$ *je konačan i pozitivan.*

Ovo je bio i prvi konstruisan primer u literaturi reči koja ima skup faktora zatvoren za preokretanje, nije uniformno rekurentna, a ima konačan i pozitivan defekt. Prethodno su u [34] viđeni i sledeći primeri reči koje nisu uniformno rekurentne a imaju defekt 0: 1) $\varphi_1^\infty(a)$ gde $\varphi_1 : a \mapsto aba$, $b \mapsto bb$ (primer uzet iz [25], gde je bio razmatran u druge svrhe); 2) Cantorova reč (takođe poznata i kao reč Sierpińskog), to jest, $\varphi_2^\infty(a)$ gde $\varphi_2 : a \mapsto aba$, $b \mapsto bbb$ (poznata reč; videti, na primer, [52], što i autori citiraju); 3) $\varphi_3^\infty(a)$ gde $\varphi_3 : a \mapsto abab$, $b \mapsto b$ (primer od autorâ); dokaz da one imaju navedena svojstva takodje će biti specijalan slučaj rezultata iz ove teze.

1.3 MP-razmera

Jasno, svaka binarna reč sadrži potpalindrom dužine bar $\lceil \frac{|w|}{2} \rceil$ (recimo potpalindrom koji je sačinjen od dominantnog slova reči w). Kažemo da je binarna reč w *minimalno-palindromična* ako ne sadrži potpalindrom duži od $\lceil \frac{|w|}{2} \rceil$. Za $w \in \{0,1\}^*$ uređeni par (r,s), gde $r, s \in \{0,1\}^*$, takav da je rws minimalno-palindromična zove se *MP-proširenje* reči w, a ako je dužina $|r|+|s|$ najmanja moguća, tada se (r,s) zove *najkraće MP-proširenje*, ili *SMP-proširenje* reči w. Racionalan broj $\frac{|rws|}{|w|}$, gde (r,s) predstavlja SMP-proširenje reči w, zove se *MP-razmera* reči w. Kako je pokazano u [39], za svaku binarnu reč postoji MP-proširenje (i time i SMP-proširenje, pa je MP-razmera dobro definisana); dalje, MP-razmera proizvoljne binarne reči je ograničena s gornje strane sa 4, i ova granica je najbolja moguća.

Teorema 1.13. *MP-razmera bilo koje binarne reči je najviše* 4.

Može se pokazati da je 4 najbolja moguća gornja granica za MP-razmeru u binarnom slučaju. Prvo treba da uvedemo pojam k-ekonomične i ekonomične reči i takođe nam treba i nekoliko pomoćnih lema.

Kažemo da je reč $w \in \{0,1\}^*$ k-*ekonomična* (u odnosu na slovo 1) ako i samo ako je w palindrom i ako reč $w1^k$ sadrži potpalindrom dužine bar $|w|_1 + k + 2$. Svaki takav potpalindrom može se napisati u obliku $1^m q 1^m$ za $0 \leqslant m \leqslant k$ i $1^m q \in \mathrm{Subw}(w)$; uređeni par (q,m) se zove k-*svedok* od w. Kažemo da je w *ekonomična* ako i samo ako je k-ekonomična za sve k, $k = 0, 1, \ldots, |w|_1$.

Lema 1.14. *Neka $w \in \{0,1\}^*$ i neka je (r,s) MP-proširenje od w. Ako je w ekonomična, tada $|rs|_1 > |w|_1$.*

Lema 1.15. *Neka $w \in \{0,1\}^*$ i neka je (r,s) MP-proširenje od w. Ako je w ekonomična, tada $|rws| > 4|w|_1$.*

Lema 1.16. *Neka je w_0 ekonomična reč. Definišemo niz $(w_i)_{i\geqslant 0}$ rekurzivno na sledeći način:*

$$w_{i+1} = w_i 1^{t_i} w_i,$$

gde je $(t_i)_{i\geqslant 0}$ neki unapred zadat niz prirodnih brojeva. Ako za svaki pozitivan broj i važi $t_i < |w_i|_0$, tada su sve reči w_i ekonomične.

U sledećoj lemi pokazujemo da postoje ekonomične reči proizvoljno velike dužine.

Lema 1.17. *Za niz $(t_i)_{i\geqslant 0}$ označavamo sa $w(t_0, t_1, \ldots, t_{j-1})$ reč w_j definisanu rekurzivnom formulom datoj u lemi 1.16 za početni term $w_0 = 0000$. Za svako k, $k \geqslant 448$, postoji ekonomična reč v_k dužine k, takva da $v_k = w(t_0, t_1, \ldots, t_{n-1})$ za neko n, $n \geqslant 6$, i neke brojeve $t_0, t_1, \ldots, t_{n-1}$ koje zadovoljavaju $2^i \leqslant t_i < 2^{i+2}$ za sve i, $i = 0, 1, \ldots, n-1$.*

Ove leme su dovoljne da se pokaže da je navedena gornja granica optimalna [39, Theorem 5].

Teorema 1.18. *Označimo sa $R_2(n)$ maksimalnu MP-razmeru nad svim rečima $w \in \{0,1\}^*$, $|w| = n$. Tada:*

$$\lim_{n\to\infty} R_2(n) = 4.$$

2 Uopštene visokopotencijalne reči

2.1 Konstrukcija

Definicija 2.1. Neka $w, u, v \in \Sigma^*$, gde $wuv \neq \varepsilon$ i u i v su palindromi, i neka je $A = (a_i)_{i\in\mathbb{N}}$ strogo rastući niz prirodnih brojeva. Definišemo rekurzivno:

$$w_0 = w;$$

$$w_i = w_{i-1}(uv)^{a_i} u \widetilde{w_{i-1}}, \ \ i \in \mathbb{N};$$

i označimo:

$$\mathrm{ghpw}(w, u, v, A) = \lim_{i\to\infty} w_i.$$

(Limes je dobro definisan jer je svako w_i prefiks od w_{i+1}.) Beskonačna reč $\mathrm{ghpw}(w, u, v, A)$ se zove *uopštena visokopotencijalna reč generisana sa w, u, v i A.*

Napomena 2.2. Primetimo da uopštene visokopotencijalne reči zaista predstavljaju uopštenje visokopotencijalnih reči: ako je w reč koja nije palindrom, c slovo koje se ne pojavljuje u w, a I niz $(i)_{i\in\mathbb{N}}$, tada, jasno, važi

$$\mathrm{hpw}(w) = \mathrm{ghpw}(w, \varepsilon, c, I).$$

Dalje, za reč iz teoreme 1.12 u podsekciji 1.2.3 važi

$$f^{\infty}(1) = \mathrm{ghpw}(1213121, 3, 2, I).$$

Takođe, lako se vidi da se preostale tri reči iz iste podsekcije mogu zapisati kao:

$$\begin{aligned}
\varphi_1^{\infty}(a) &= \mathrm{ghpw}(a, \varepsilon, b, (2^{i-1})_{i\in\mathbb{N}}),\\
\varphi_2^{\infty}(a) &= \mathrm{ghpw}(a, \varepsilon, b, (3^{i-1})_{i\in\mathbb{N}}),\\
\varphi_3^{\infty}(a) &= \mathrm{ghpw}(a, \varepsilon, b, I).
\end{aligned}$$

2.2 Standardni oblik

Razne uređene četvorke (w, u, v, A) mogu da generišu istu uopštenu visokopotencijalnu reč. U sledećoj lemi je pokazano da se za svaku uopštenu visokopotencijalnu reč može odabrati uređena četvorka sa nekim određenim osobinama koje će biti od koristi.

Lema 2.3. *Neka je* $\mathrm{ghpw}(w, u, v, A)$ *uopštena visokopotencijalna reč. Tada postoje reči* w^S, u^S, v^S *i niz* A^S *takvi da je* w^S *palindrom,* $u^S v^S$ *je primitivna reč i*

$$\mathrm{ghpw}(w, u, v, A) = \mathrm{ghpw}(w^S, u^S, v^S, A^S).$$

Ideja dokaza. Lako se vidi da

$$\mathrm{ghpw}(w, u, v, A) = \mathrm{ghpw}(w_1, u, v, B),$$

za $w_1 = w(uv)^{a_1}u\widetilde{w}$ (w_1 je palindrom) i $B = (b_i)_{i\in\mathbb{N}}$, $b_i = a_{i+1}$.

Takođe, pokazuje se da

$$\mathrm{ghpw}(w, u, v, A) = \mathrm{ghpw}(w, u', v', C),$$

gde je $u'v'$ primitivna reč takva da $uv = (u'v')^n$, $u = (u'v')^k u'$, a $C = (c_i)_{i\in\mathbb{N}}$, $c_i = na_i + k$. ■

Ako je u uređenoj četvorci (w, u, v, A) w palidrom a uv primitivna reč, kažemo da je $\mathrm{ghpw}(w, u, v, A)$ u *standardnom obliku*. Po prethodnoj lemi, svaka uopštena visokopotencijalna reč se može zapisati u standardnom obliku.

Napomena 2.4. Pretpostavka da je uv primitivna će biti korišćena često, uglavnom u obliku sledeće posledice: na osnovu teoreme 1.1 sledi da se uv pojavljuje kao faktor u $uvuvuv\ldots$ samo na „očiglednim mestima“ (drugim rečima, $|uvuv|_{uv} = 2$; još preciznije, ovu terminologiju ćemo koristiti za pojavljivanja uv unutar $uvuvuv\ldots$ koja počinju na poziciji i za $i \equiv 1 \pmod{|uv|}$); dalje, slično tvrđenje važi i za svaki konjugat od uv (svaki konjugat primitivne reči je takođe primitivan, što lako sledi na osnovu teoreme 1.1).

Sledeća (tehnička) lema je takođe posledica primitivnosti reči uv, i biće korisna u nastavku.

Lema 2.5. *Neka su u i v palindromi, $uv \neq \varepsilon$, takvi da je reč uv primitivna. Neka je x palindrom takav da $|x| \geqslant 2|uv| - 1$ i $x\big[1, \lfloor \frac{|x|}{2} \rfloor + |uv|\big] = (vu)^\infty\big[1, \lfloor \frac{|x|}{2} \rfloor + |uv|\big]$. Tada $x = (vu)^m v$ za neki prirodan broj m.*

2.3 Osnovne osobine

Prvo prezentujemo potreban i dovoljan uslov za periodičnost uopštene visokopotencijalne reči.

Teorema 2.6. *Neka je reč $\mathrm{ghpw}(w, u, v, A)$ data u standardnom obliku. Tada je $\mathrm{ghpw}(w, u, v, A)$ periodična ako i samo ako ili $w = (vu)^m v$ za neki nenegativan ceo broj m, ili je tačno jedna od reči w, u i v neprazna.*

Sledeća propozicija će biti značajna.

Propozicija 2.7. $\mathrm{Fact}(\mathrm{ghpw}(w, u, v, A))$ *je zatvoreno za preokretanje.*

Sada, teorema 1.2 odmah daje sledeću posledicu.

Posledica 2.8. *Svaka uopštena visokopotencijalna reč je rekurentna.*

Što se uniformne rekurentnosti tiče, imamo sledeće:

Propozicija 2.9. *Uopštena visokopotencijalna reč je uniformno rekurentna ako i samo ako je periodična.*

Konačno, važi sledeće:

Propozicija 2.10. *Ako uopštena visokopotencijalna reč nije periodična, onda je aperiodična.*

2.4 Defekt uopštenih visokopotencijalnih reči

U ovoj sekciji pokazujemo da je defekt uopštene visokopotencijalne reči uvek konačan. Pre glavne teoreme, potrebne su dve tehničke leme.

Lema 2.11. *Neka je neperiodična reč* $\mathrm{ghpw}(w,u,v,A)$ *data u standardnom obliku, gde* $vu \notin \mathrm{Pref}(wuv)$. *Pretpostavimo da postoji ceo broj* i *takav da* $i \geqslant 3$ *i*

$$|w_i|_{(uv)^{a_i}u} = 1.$$

Tada

$$|w_{i+1}|_{w_i} = 2$$

i

$$|w_{i+1}|_{(uv)^{a_{i+1}-1}u} = 2 + 2|w_i|_{(uv)^{a_{i+1}-1}u}.$$

Lema 2.12. *Neka je neperiodična reč* $\mathrm{ghpw}(w,u,v,A)$ *data u standardnom obliku, gde* $vu \notin \mathrm{Pref}(wuv)$. *Tada postoji prirodan broj* i *takav da:*

1) $|w_i|_{(uv)^{a_i}u} = 1$ *(gde je to jedno pojavljivanje* $(uv)^{a_i}u$ *na sredini od* w_i*);*

2) $|w_{i+1}|_{w_i} = 2$ *(gde su ta dva pojavljivanja* w_i *na početku i na kraju od* w_{i+1}*);*

3) $|w_{i+1}|_{(uv)^{a_{i+1}-1}u} = 2+2|w_i|_{(uv)^{a_{i+1}-1}u}$ *(što iznosi ili* 2 *ili* 4*, u zavisnosti od toga da li je* $a_{i+1}-1$ *veće od ili jednako sa* a_i*, respektivno).*

Štaviše, ako je i *bilo koji broj koji zadovoljava 1), 2) i 3), tada i svako* k*,* $k \geqslant i$*, zadovoljava te osobine.*

Sledeća teorema je glavna teorema ove sekcije, i u suštini i cele glave:

Teorema 2.13. *Neka je* $\mathrm{ghpw}(w,u,v,A)$ *uopštena visokopotencijalna reč. Tada*

$$D(\mathrm{ghpw}(w,u,v,A)) < \infty$$

Ideja dokaza. Pretpostavimo da je reč $\mathrm{ghpw}(w,u,v,A)$ data u standardnom obliku, da nije periodična i da $vu \notin \mathrm{Pref}(wuv)$. Neka je i broj čije postojanje sledi iz leme 2.12. Pokazujemo $D(\mathrm{ghpw}(w,u,v,A)) = D(w_{i+1})$, a za to je zapravo dovoljno ustanoviti $D(w_{i+2}) = D(w_{i+1})$. Da bismo pokazali ovu jednakost, treba naći $|w_{i+2}| - |w_{i+1}| = |w_{i+1}| + a_{i+2}|uv| + |u|$ palindroma u w_{i+2} koji se ne pojavljuju u w_{i+1}. Razlikujemo sledeće klase palindroma.

- Palindroma koji se mogu dobiti „širenjem“ (nalevo i nadesno) uokvirenog faktora $(uv)^{a_{i+2}}u$ u
$$w_{i+1}\boxed{(uv)^{a_{i+2}}u}w_{i+1}$$
ima ukupno $|w_{i+1}|$.

- Palindroma koji se mogu dobiti „širenjem“ w_i u
$$w_{i+1}(uv)^{a_{i+2}}u\boxed{w_i}(uv)^{a_{i+1}}uw_i$$
ima ukupno $a_{i+1}|uv| + |u| + t$, za $t = \max\{|p| : p \in \mathrm{Pref}(wuv) \cap \mathrm{Pref}(vu)\}$.

- Palindroma koji se mogu dobiti „širenjem“ $(uv)^{a_{i+2}-1}u$ u
$$w_{i+1}uv\boxed{(uv)^{a_{i+2}-1}u}w_{i+1}$$
ima t u slučaju $a_{i+2} > a_{i+1} + 1$, a inače ovi palindromi nisu novi.

- Konačno, nabrajamo nove palindrome koji su faktori od $(uv)^{a_{i+2}}u$. Može se pokazati da njih ima $(a_{i+2} - a_{i+1})|uv| - 2t$ u slučaju $a_{i+2} > a_{i+1} + 1$, a $|uv| - t$ u slučaju $a_{i+2} = a_{i+1} + 1$.

Sabiranjem ovih brojeva dobija se traženo.

Treba još razmotriti slučaj $vu \in \mathrm{Pref}(wuv)$. Označimo:
$$\begin{aligned}
s &= \min\{j : (wuv)[j] \neq (vu)^\infty[j]\} - |uv|;\\
e &= (|u| + 2s) \bmod |uv|;\\
l &= (-s) \bmod |uv|;\\
w' &= w[s+1, |w| - s];\\
u' &= (uvuv)[l+1, l+e];\\
v' &= (uvuv)[l+e+1, l+|uv|];\\
A' &= \left(a_i + \frac{|u| + 2s - e}{|uv|}\right)_{i=1}^\infty = (a_i')_{i=1}^\infty.
\end{aligned}$$

Tada važi $\mathrm{ghpw}(w,u,v,A) = w[1,s]\,\mathrm{ghpw}(w',u',v',A')$, gde je za reč $\mathrm{ghpw}(w',u',v',A')$ ispunjeno da je w' palindrom, $u'v'$ je primitivno i $v'u' \notin \mathrm{Pref}(w'u'v')$, pa na osnovu prvog dela dokaza sledi da je $D(\mathrm{ghpw}(w',u',v',A'))$ konačan. Kako $\mathrm{Fact}(\mathrm{ghpw}(w,u,v,A)) = \mathrm{Fact}(\mathrm{ghpw}(w',u',v',A'))$, sledi
$$D(\mathrm{ghpw}(w,u,v,A)) = D(\mathrm{ghpw}(w',u',v',A')) < \infty.$$

■

Kako je rečeno u predgovoru, o beskonačnim rečima defekta 0 mnogo više se zna nego o onima koje imaju konačan pozitivan defekt. Zbog toga ima smisla okarakterisati kada je defekt uopštenih visokopotencijalnih reči (ne)nula. Takva karakterizacija lako sledi iz dokaza teoreme 2.13. Dajemo je u sledećoj posledici (gde pretpostavljamo da je reč data u standardnom obliku, a ako to nije slučaj, uvek možmo reizabrati parametre kao u dokazu leme 2.3 i time prevesti reč u standardni oblik).

Posledica 2.14. *Neka reč* $\mathrm{ghpw}(w, u, v, A)$ *nije periodična i neka je data u standardnom obliku. Tada:*

1° *Ako* $vu \notin \mathrm{Pref}(wuv)$*, biramo najmanji ceo broj* i *koji zadovoljava 1), 2) i 3) iz formulacije leme 2.12, i tada važi* $D(\mathrm{ghpw}(w, u, v, A)) = D(w_{i+1})$*.*

2° *Ako* $vu \in \mathrm{Pref}(wuv)$*, biramo* w'*,* u'*,* v' *i* A' *kao malopre, i tada važi* $D(\mathrm{ghpw}(w, u, v, A)) = D(\mathrm{ghpw}(w', u', v', A'))$*, što se izračunava kao pod* 1° *gore.*

Specijalno, ovako možemo odrediti da li je $D(\mathrm{ghpw}(w, u, v, A))$ *pozitivan ili* 0*, što daje karakterizaciju uopštenih visokopotencijalnih reči (ne)nula defekta.*

Takođe primetimo da lako možemo konstruisati uopštene visokopotencijalne reči pozitivnog defekta. Najjednostavnije je prosto uzeti da neka od reči w, u ili v ima pozitivan defekt, i tada će i $\mathrm{ghpw}(w, u, v, A)$ imati pozitivan defekt. Ovo je dovoljan ali ne i potreban uslov: na primer, mogu sve reči w, u i v biti bogate, ali ako pritom wu ili uv ima pozitivan defekt, tada $\mathrm{ghpw}(w, u, v, A)$ opet ima pozitivan defekt.

Zapravo, ako su bilo koje dve od reči w, u i v takve da ne mogu biti faktori iste bogate reči, tada $\mathrm{ghpw}(w, u, v, A)$ ima pozitivan defekt. Otvoren problem iz rada [51] je bio sledeće: da li je odlučiv problem odrediti mogu li dve bogate reči biti faktori iste bogate reči; taj problem je rešen (potvrdno) vrlo skoro [54], no algoritam nije baš praktičan. Jedan elegantan dovoljan uslov za to da dve bogate reči budu faktori iste bogate reči sledi iz [22, Theorem 6]: nikoja dva faktora bogate reči ne mogu imati isti najduži palindromski prefiks i istovremeno najduži palindromski sufiks (prema tome, ako želimo da $\mathrm{ghpw}(w, u, v, A)$ ima pozitivan defekt, dovoljno je da ovaj uslov bude narušen za neke dve od reči w, u, v). U [59, Open problem 6.2] je postavljeno pitanje da li je ovaj uslov i potreban (to jest, da li dve bogate reči koje imaju različit najduži palindromski prefiks ili sufiks moraju biti faktori iste bogate reči); ako bi se ispostavilo da je ovo tačno, to bi znatno pojednostavilo pomenuti algoritam iz [54], ali koliko je poznato autoru ove teze, taj problem je i dalje otvoren.

2.5 Periodični slučaj

Teorema 2.15. *Periodične uopštene visokopotencijalne reči imaju konačan defekt.*

3 MP-razmera u ternarnom slučaju

Posmatrajmo n-arni alfabet $\Sigma = \{0, 1, \ldots, n-1\}$. Jasno, svaka reč $w \in \Sigma^*$ sadrži potpalindrom dužine bar $\lceil \frac{|w|}{n} \rceil$. Dakle, prirodno je reći da je reč $w \in \Sigma^*$ *minimalno palindromična* ako ne sadrži potpalindrom duži od $\lceil \frac{|w|}{n} \rceil$. Za reč $w \in \Sigma^*$ uređeni par (r, s), gde $r, s \in \Sigma^*$, takav da je rws minimalno palindromična, zove se *MP-proširenje* reči w. SMP-proširenje, kao i MP-razmera, definišu se slično kao u binarnom slučaju. Međutim, kako je rečeno ranije, u slučaju višearnih alfabeta nije jasno da li MP-proširenje uvek postoji, pa time i da li je MP-razmera dobro definisana. U ovoj glavi pokazujemo da MP-razmera jeste dobro definisana u slučaju ternarnog alfabeta.

Sledeća propozicija će biti korisna u nastavku

Propozicija 3.1. *Neka $w \in \{0, 1, 2\}^*$, neka je (r, s) SMP-proširenje od w i neka $|rs| \geqslant 2$. Tada $|rws| = 3k - 2$ za neki prirodan broj k, a vrednosti $|rws|_0, |rws|_1, |rws|_2$ su jednake (u nekoj permutaciji) $k-1, k-1, k$ ili $k-2, k, k$.*

3.1 Gornja granica za MP-razmeru

Naš je cilj u ovoj sekciji da pokažemo da je MP-razmera bilo koje ternarne reči w najviše 6. Fiksirajmo alfabet $\Sigma = \{0, 1, 2\}$.

Trebaju nam sledeće funkcije. Za $w \in \Sigma^*$ i $a, b \in \Sigma$, označimo

$$\gamma(w, a, b) = \min \left\{ 2 \big| w[i, |w|] \big|_a - \big| w[i, |w|] \big|_b : i = 1, 2, \ldots, |w| + 1 \right\},$$

$$g(w, a, b) = \max \left\{ 2 \big| w[i, |w|] \big|_a - \big| w[i, |w|] \big|_b : i = 1, 2, \ldots, |w| + 1 \right\}.$$

Dalje, neka $j(a,w)$ označava poziciju poslednjeg pojavljivanja slova a unutar w (to jest, $w[j(a,w)] = a$ i $w[k] \neq a$ za sve k, $k > j(a,w)$), i $j(a,w) = 0$ ako se a ne pojavljuje u w. Definišimo

$$g'(w,a,b) = \max\left(\left\{2\big|w[i,|w|]\big|_a - \big|w[i,|w|]\big|_b : i = 1,2,\ldots,j(a,w)\right\} \cup \{0\}\right).$$

Prvo navodimo dve jednostavne osobine ovih funkcija.

Lema 3.2. *Neka je w konačna reč i neka su a i b dva različita slova. Tada:*

a) $g'(w,a,b) \leqslant g(w,a,b)$;

b) $\gamma(w,a,b) + g(\widetilde{w},a,b) = g(w,a,b) + \gamma(\widetilde{w},a,b) = 2|w|_a - |w|_b$.

Sledeća osobina funkcije g je manje očigledna, ali biće korisna.

Lema 3.3. *Neka $w \in \Sigma^*$, neka je b dominantno slovo u w i neka je a slovo različito od b. Tada:*

$$g(w,a,b) + g(\widetilde{w},a,b) \leqslant 3|w|_a.$$

Sada smo spremni da konstruišemo MP-proširenje date reči w. Do kraja ove sekcije, bez umanjenja opštosti, pretpostavimo $|w|_0 \leqslant |w|_1 \leqslant |w|_2$. Zapravo, daćemo dva proširenja reči w, $f(w)$ i $f'(w)$, i pokazati da je bar jedno od njih MP-proširenje. Ta dva proširenja su

$$\begin{aligned} f(w) &= 0^{2|w|-|w|_0} 2^{2|w|-|w|_2-g'(w,0,2)} \ \ w \ \ 2^{g'(w,0,2)} 1^{2|w|-|w|_1};\\ f'(w) &= 1^{2|w|-|w|_1} 2^{g'(\widetilde{w},0,2)} \ \ w \ \ 2^{2|w|-|w|_2-g'(\widetilde{w},0,2)} 0^{2|w|-|w|_0}. \end{aligned}$$

Primetimo, $f'(w) = \widetilde{f(\widetilde{w})}$. Sa r i s, respektivno r' i s', označavamo prefiks i sufiks koji smo dodali na w u $f(w)$, respektivno u $f'(w)$.

Drugim rečima, slova 1 i 0 su stavljena na krajeve, dok je slovo 2 raspoređeno oko w na asimetričan način. Preciznije, važi

$$(2|w| - |w|_2 - g'(w,0,2)) - g'(w,0,2) \geqslant |w|_2$$

(i slično za drugo proširenje sa $\widetilde{w}$ umesto w).

Komentar. Date konstrukcije nisu jedine moguće. Još jedna mogućnost je koristiti funkciju g umesto g' (ili zapravo bilo koju međuvrednost), i dokaz bi bio u potpunosti isti. Verzija sa g' zapravo predstavlja granični slučaj u smislu da su dvojke raspoređene najasimetričnije moguće; drugim rečima, ako bismo samo jednu dvojku pomerili sa manje „gomile“ na veću, više ne bismo imali MP-proširenje.

Kako smo rekli, bar jedan od uređenih parova (r, s) i (r', s') predstavlja MP-proširenje reči w, to jest, bar jedna od reči $f(w)$ i $f'(w)$ ne sadrži potpalindrom duži od $2|w|$ (primetimo, $|f(w)| = |f'(w)| = 6|w|$). Dokaz se sastoji od nekoliko tvrdnji.

Lema 3.4. *Dužina proizvoljnog potpalindroma oblika* $0p0$ *unutar obe reči* $f(w)$ *i* $f'(w)$ *je najviše* $2|w|$.

Lema 3.5. *Dužina proizvoljnog potpalindroma oblika* $1p1$ *unutar obe reči* $f(w)$ *i* $f'(w)$ *je najviše* $2|w|$.

Lema 3.6. *Neka su* p *i* q *dva neprazna potpalindroma reči* w*. Neka su* w_p, v, w_q *i* t *reči takve da* $w = w_p v = t w_q$, p *je podreč od* w_p *i* q *je podreč od* w_q*. Tada imamo:*

$$|p| + 2|v|_2 + |q| + 2|t|_2 \leqslant 4|w|_2 + |w|_1 + |w|_0.$$

Ideja dokaza. Na reč w', $|w'| = |w|$, koja je definisana na sledeći način:

$$w'[i] = \begin{cases} 1, & \text{ako } w[i] = 0 \text{ ili } w[i] = 1; \\ 2, & \text{ako } w[i] = 2. \end{cases}$$

primenimo teoremu 3.19 (koja je formulisana i dokazana kasnije u sekciji 3.4). ■

Lema 3.7. *Dužina proizvoljnog potpalindroma oblika* $2p2$ *unutar bar jedne reči* $f(w)$ *i* $f'(w)$ *je najviše* $2|w|$.

Ideja dokaza. Pretpostavimo suprotno: najduži potpalindromi u obe reči su duži od $2|w|$. Tada su oni oblika $2^{l+|s|_2} p_w 2^{l+|s|_2}$ i $2^{l'+|r'|_2} q_w 2^{l'+|r'|_2}$, gde su p_w i q_w palindromi i važi $p_w 2^l \in \mathrm{Subw}(w)$, odnosno $2^{l'} q_w \in \mathrm{Subw}(w)$. Može se pokazati sledeće:

$$|p_w| + 2l + |q_w| + 2l' > 4|w| - 2g'(w, 0, 2) - 2g'(\widetilde{w}, 0, 2).$$

Sada je leva strana nejednakosti, na osnovu leme 3.6, manja od ili jednaka sa $4|w|_2 + |w|_1 + |w|_0$, dok je desna strana jednakosti na osnovu lema 3.2a) i 3.3 veća od ili jednaka sa $4|w|_2 + |w|_1 + |w|_0$, što je kontradikcija. ■

Sad smo spremni za glavnu teoremu ove sekcije.

Teorema 3.8. *MP-razmera bilo koje ternarne reči je najviše* 6.

Komentar. Ne tvrdimo da je pomenuto proširenje SMP-proširenje. Zapravo, imajući u vidu propoziciju 3.1, možemo zaključiti da sigurno nije; u stvari, brisanjem proizvoljna dva slova iz r i s dobili bismo još kraće proširenje, što pokazuje da je MP-razmera proizvoljne ternarne reči strogo manja od 6. No, kao što se vidi u sledećoj sekciji, ova činjenica suštinski nije relevantna. Dokaz je ispisan na prikazani način jer je bilo nešto manje tehnički zahtevno ako se u reči rws svako slovo pojavljuje isti broj puta. U svakom slučaju, brisanje dva slova iz našeg proširenja i dalje ne mora da vodi do SMP-proširenja. Konstrukcija SMP-proširenja je mnogo zahtevnija i deluje da je van domašaja čak i u binarnom slučaju [13].

3.2 Optimalnost gornje granice

Sada ćemo pokazati da je konstanta 6 iz prethodne sekcije optimalna.

U sekciji 1.3 smo uveli pojam ekonomičnosti i k-ekonomičnosti za binarne reči. Modifikovaćemo malo te definicije kako bismo ih prilagodili za ternarni slučaj. Kažemo da je reč $w \in \{0,1,2\}^*$ *k-ekonomična* (u odnosu na slovo 1) ako je w palindrom i reč $w1^k$ sadrži potpalindrom dužine bar $|w|_1 + k + 3$. Ovi potpalindromi se mogu zapisati u obliku $1^m q 1^m$ gde $0 \leqslant m \leqslant k$ i $1^m q \in \mathrm{Subw}(w)$; uređeni par (q, m) se zove *k-svedok* od w. Kažemo da je w *ekonomična* ako i samo ako je k-ekonomična za sve k, $k = 0, 1, \ldots, |w|_1$.

Sledeće tri leme su (manje-više) direktne adaptacije lema 1.14, 1.15 i 1.16.

Lema 3.9. *Neka $w \in \{0,1,2\}^*$ i neka je (r,s) MP-proširenje od w. Ako je w ekonomična, tada $|rs|_1 > |w|_1$.*

Lema 3.10. *Neka $w \in \{0,1,2\}^*$ i neka je (r,s) MP-proširenje od w. Ako je w ekonomična, tada $|rws| > 6|w|_1$.*

Lema 3.11. *Neka je w_0 ekonomična reč. Definišemo niz $(w_i)_{i \geqslant 0}$ rekurzivno na sledeći način:*

$$w_{i+1} = w_i 1^{t_i} w_i,$$

gde je $(t_i)_{i \geqslant 0}$ neki unapred zadat niz prirodnih brojeva. Ako za svaki pozitivan broj i važi $t_i < |w_i|_0$, tada su sve reči w_i ekonomične.

Za niz $(t_i)_{i \geqslant 0}$ označavamo sa $w(t_0, t_1, \ldots, t_{j-1})$ reč w_j definisanu rekurzivnom formulom datoj u lemi 3.11 za početni term $w_0 = 0000$. (Primetimo da je reč 0000 ekonomična i kao ternarna reč.) Dalje, primetimo da, ako niz $(t_i)_{i \geqslant 0}$ zadovoljava $2^i \leqslant t_i < 2^{i+2}$ za sve i, tada $t_j < 2^{j+2} =$

$|w(t_0, t_1, \ldots, t_{j-1})|_0$, i na osnovu leme 3.11 reč $w(t_0, t_1, \ldots, t_{j-1})$ je ekonomična (za sve j i svaki niz $(t_i)_{i\geqslant 0}$ koji zadovoljava pomenuti uslov). Tada se može pokazati da za dovoljno veliko k postoji reč v_k koja se može dobiti opisanom konstrukcijom, takva da $|v_k| = k$, pored toga važi

$$\lim_{k\to\infty} \frac{|v_k|_1}{|v_k|} = 1.$$

Ovo je dovoljno da se pokaže glavna teorema ove sekcije.

Teorema 3.12. *Označimo sa $R_3(n)$ maksimalnu MP-razmeru nad svim rečima $w \in \{0, 1, 2\}^*$, $|w| = n$. Tada:*

$$\lim_{n\to\infty} R_3(n) = 6.$$

Ideja dokaza. Za pozitivan realan broj η, biramo k_0 takvo da za sve $k \geqslant k_0$ važi

$$\frac{|v_k|_1}{|v_k|} > 1 - \frac{\eta}{6}.$$

∎

3.3 Odložena tehnička teorema

Teorema 3.13. *Neka $u \in \{1, 2\}^*$, neka su $t, v \in 2^*$ i neka su p i q potpalindromi od tu i uv, respektivno. Ako*

$$|p| + |q| > 2|u|,$$

tada

$$|u|_1 \leqslant \frac{|tv| - 1}{|tv|} |tuv|_2.$$

Pre dokaza napomenimo, može se pokazati da, bez umanjenja opštosti, možemo raditi pod pretpostavkom da potpalindromi p i q sadrže sva slova reči t i v, resprektivno.

U sledeće dve podsekcije dajemo dva (vrlo) različita dokaza teoreme 3.13. Drugi dokaz je (znatno) kraći od prvog, i mnogi bi se složili da je i elegantniji. No, smatramo da drugi dokaz radi skoro pa slučajno, dok prvi dokaz daje duboku strukturnu analizu, i preko njega možemo videti zašto je teorema zaista tačna (zapravo, mislimo da je prvi dokaz intuitivniji nego drugi, iako se na nekim mestima javljaju prilično glomazne formule). Rekli bismo da ne bi bilo veliko iznenađenje ako bi se za neko tvrđenje slično teoremi 3.13 (vezano, recimo, za MP-razmeru za neki veći alfabet) ispostavilo da adekvatna modifikacija prvog dokaza i dalje radi, dok za drugi dokaz to ne bi bio slučaj. Dakle, verujemo da oba dokaza, bez obzira na očiglednu razliku između njihovih dužina, imaju svoje prednosti, i zato smo odlučili da ih prezentujemo oba.

3.3.1 Prvi dokaz

Definišemo najpre nizove $P_1, P_2, \dots, P_{|p|}$ i $Q_1, Q_2, \dots, Q_{|q|}$ takve da $1 \leqslant P_1 < P_2 < \dots < P_{|p|} \leqslant |tu|$ i $|t|+1 \leqslant Q_1 < Q_2 < \dots < Q_{|q|} \leqslant |tuv|$, i

$$p = (tuv)[P_1](tuv)[P_2]\dots(tuv)[P_{|p|}]$$

i

$$q = (tuv)[Q_1](tuv)[Q_2]\dots(tuv)[Q_{|q|}].$$

Pišemo $P = \{P_1, P_2, \dots, P_{|p|}\}$ i $Q = \{Q_1, Q_2, \dots, Q_{|q|}\}$.

Definišemo $\sigma_P : P \to P$ tako da $\sigma_P : P_s \mapsto P_{|P|-s+1}$ i $\sigma_Q : Q \to Q$ tako da $\sigma_Q : Q_s \mapsto Q_{|Q|-s+1}$. Primetimo da su σ_P i σ_Q bijekcije, i da je njihov kvadrat identičko preslikavanje.

Za $1 \leqslant n \leqslant |t|$ označimo $\sigma_0(n) = n$ i

$$\sigma_{i+1}(n) = \begin{cases} \sigma_P(\sigma_i(n)), & \text{za } 2 \mid i \text{ i } \sigma_i(n) \in P; \\ \sigma_Q(\sigma_i(n)), & \text{za } 2 \nmid i \text{ i } \sigma_i(n) \in Q; \\ \text{nedefinisano}, & \text{inače.} \end{cases}$$

Slično, za $|tu|+1 \leqslant n \leqslant |tuv|$ označimo $\sigma_0(n) = n$ i

$$\sigma_{i+1}(n) = \begin{cases} \sigma_Q(\sigma_i(n)), & \text{za } 2 \mid i \text{ i } \sigma_i(n) \in Q; \\ \sigma_P(\sigma_i(n)), & \text{za } 2 \nmid i \text{ i } \sigma_i(n) \in P; \\ \text{nedefinisano}, & \text{inače.} \end{cases}$$

Važe sledeće osobine.

Propozicija 3.14. *a) Za svako $n, m \in Q$ (respektivno, $n, m \in P$), ako $n < m$, tada $\sigma_Q(n) > \sigma_Q(m)$ (respektivno, $\sigma_P(n) > \sigma_P(m)$).*

b) Važi $\sigma_0(n) > \sigma_2(n) > \sigma_4(n) > \cdots$ i $\sigma_1(n) < \sigma_3(n) < \sigma_5(n) < \cdots$ za $n \geqslant |tu|+1$, i $\sigma_0(n) < \sigma_2(n) < \sigma_4(n) < \cdots$ i $\sigma_1(n) > \sigma_3(n) > \sigma_5(n) > \cdots$ za $n \leqslant |t|$. (Nejednakosti nastavljamo sve dok su termovi definisani.)

c) Ako važi jedan od sledeća dva uslova:

1) n i m su istovremeno u intervalu $[1,|t|]_{\mathbb{N}}$ ili $[|tu|+1,|tuv|]_{\mathbb{N}}$, a i i j su iste parnosti; ili

2) n i m su u različitim intervalima, a i i j su suprotne parnosti,

tada $\sigma_i(n)=\sigma_j(m)$ implicira $n=m$ i $i=j$. (Zapravo, $\sigma_i(n)=\sigma_j(m)$ je nemoguće u drugom slučaju.)

d) Za svako n takvo da $n \leqslant |t|$ ili $n \geqslant |tu|+1$ postoji $z \in \mathbb{N}$ takvo da je $\sigma_z(n)$ poslednji definisan term u nizu $\sigma_0(n), \sigma_1(n), \sigma_2(n) \dots$

Sledeća lema će biti korisna.

Lema 3.15. *a) Za svako $n \in Q$ takvo da je $\sigma_P(\sigma_Q(n))$ definisano (to jest, $\sigma_Q(n) \in P$), imamo*

$$\begin{aligned} n - \sigma_P(\sigma_Q(n)) \leqslant 2(|t|+|v|) - 1 \\ - |[n,|tuv|]_{\mathbb{N}} \setminus Q| - |[\sigma_Q(n),|tuv|]_{\mathbb{N}} \setminus Q| \\ - |[1,\sigma_Q(n)]_{\mathbb{N}} \setminus P| - |[1,\sigma_P(\sigma_Q(n))]_{\mathbb{N}} \setminus P|. \end{aligned}$$

Takođe, za svako $n \in P$ takvo da je $\sigma_Q(\sigma_P(n))$ definisano (to jest, $\sigma_P(n) \in Q$), imamo

$$\begin{aligned} \sigma_Q(\sigma_P(n)) - n \leqslant 2(|t|+|v|) - 1 \\ - |[1,n]_{\mathbb{N}} \setminus P| - |[1,\sigma_P(n)]_{\mathbb{N}} \setminus P| \\ - |[\sigma_P(n),|tuv|]_{\mathbb{N}} \setminus Q| - |[\sigma_Q(\sigma_P(n)),|tuv|]_{\mathbb{N}} \setminus Q|. \end{aligned}$$

b) Za svako $n \in Q$ takvo da $\sigma_P(\sigma_Q(n))$ nije definisano (to jest, $\sigma_Q(n) \notin P$), imamo

$$n \leqslant 2(|t|+|v|) + |P| - \sigma_Q(n) - |[n,|tuv|]_{\mathbb{N}} \setminus Q| - |[\sigma_Q(n),|tuv|]_{\mathbb{N}} \setminus Q|.$$

Takođe, za svako $n \in P$ takvo da $\sigma_Q(\sigma_P(n))$ nije definisano (to jest, $\sigma_P(n) \notin Q$), imamo

$$\begin{aligned} |tuv| + 1 - n \leqslant 2(|t|+|v|) + |Q| - (|tuv| + 1 - \sigma_P(n)) \\ - |[1,n]_{\mathbb{N}} \setminus P| - |[1,\sigma_P(n)]_{\mathbb{N}} \setminus P|. \end{aligned}$$

Za prirodan broj n, $n \leqslant |t|$ ili $n \geqslant |tu|+1$, neka $\mathrm{end}(n)$ označava broj z čije je postojanje pokazano u propoziciji 3.14d). Kažemo da n *umire* ako $|t|+1 \leqslant \sigma_{\mathrm{end}(n)}(n) \leqslant |tu|$. Može se pokazati da strogo manje od $|t|+|v|$ brojeva n umire. To znači da postoji n, $n \leqslant |t|$ ili $n \geqslant |tu|+1$, koje ne umire. Neka je n_0 proizvoljan takav broj. Bez umanjenja opštosti možemo pretpostaviti $n_0 \geqslant |tu|+1$. Na osnovu odabira n_0 važi ili $\sigma_{\mathrm{end}(n_0)}(n_0) \leqslant |t|$ ili $\sigma_{\mathrm{end}(n_0)}(n_0) \geqslant |tu|+1$.

Lema 3.16. *Neka je i, $i \geqslant 0$, takvo da je $\sigma_{2i+2}(n_0)$ definisano.*

a) Za svako m, $m \geqslant |tu|+1$, važi jedna od sledećih stvari:

- *postoji j takvo da $\sigma_{2i+2}(n_0) < \sigma_j(m) \leqslant \sigma_{2i}(n_0)$;*
- *$2 \mid \mathrm{end}(m)$ i $\sigma_{\mathrm{end}(m)}(m) > \max\{\sigma_{2i}(n_0), \sigma_{2i+1}(n_0)\}$;*
- *$2 \nmid \mathrm{end}(m)$ i $\sigma_{\mathrm{end}(m)}(m) < \min\{\sigma_{2i+1}(n_0), \sigma_{2i+2}(n_0)\}$.*

Takođe, za svako m, $m \leqslant |t|$, važi jedna od sledećih stvari:

- *postoji j takvo da $\sigma_{2i+2}(n_0) < \sigma_j(m) \leqslant \sigma_{2i}(n_0)$;*
- *$2 \nmid \mathrm{end}(m)$ i $\sigma_{\mathrm{end}(m)}(m) > \max\{\sigma_{2i}(n_0), \sigma_{2i+1}(n_0)\}$;*
- *$2 \mid \mathrm{end}(m)$ i $\sigma_{\mathrm{end}(m)}(m) < \min\{\sigma_{2i+1}(n_0), \sigma_{2i+2}(n_0)\}$.*

b) Svakom m, $m \leqslant |t|$ ili $m \geqslant |tu|+1$, za koje postoji j opisano pod a) možemo pridružiti jedno takvo j tako da sve odgovarajuće vrednosti $\sigma_j(m)$ budu različite.

Konačno, potrebna je sledeća lema.

Lema 3.17. *Neka je n takvo da $n \geqslant |tu|+1$ i $\sigma_{\mathrm{end}(n)}(n) \geqslant |tu|+1$. Tada:*

$$\begin{aligned}
&2|tuv|+1-n-\sigma_{\mathrm{end}(n)}(n)\\
&\qquad \geqslant |\{m : m \geqslant |tu|+1,\ \mathrm{end}(m) \geqslant \mathrm{end}(n) \text{ i } \sigma_{\mathrm{end}(m)}(m) \geqslant |tu|+1\}|\\
&\qquad\quad + |\{m : m \geqslant \sigma_{\mathrm{end}(n)}(n) \text{ i } m \text{ umre}\}|.
\end{aligned}$$

Sada smo spremni da dokažemo teoremu 3.13.

Ideja prvog dokaza teoreme 3.13. Prvo primetimo, ako $m \in [1, |t|]_{\mathbb{N}} \cup [|tu|+1, |tuv|]_{\mathbb{N}}$ i $\sigma_i(m)$ je definisano, tada $(tuv)[\sigma_i(m)] = 2$.

Pretpostavimo prvo da $2 \mid \mathrm{end}(n_0)$. Tada na osnovu propozicije 3.14b) možemo pisati

$$
\begin{aligned}
|tuv|_2 &= \big|(tuv)[n_0+1,|tuv|]\big|_2 \\
&\quad + \sum_{i=0}^{\frac{\mathrm{end}(n_0)}{2}-1} \big|(tuv)[\sigma_{2i+2}(n_0)+1,\sigma_{2i}(n_0)]\big|_2 + \big|(tuv)[1,\sigma_{\mathrm{end}(n_0)}(n_0)]\big|_2 \\
&= (|tuv|-n_0) + \sum_{i=0}^{\frac{\mathrm{end}(n_0)}{2}-1} \big|(tuv)[\sigma_{2i+2}(n_0)+1,\sigma_{2i}(n_0)]\big|_2 + \sigma_{\mathrm{end}(n_0)}(n_0).
\end{aligned}
$$

Označimo $k = |tv|$. Koristeći leme 3.15 i 3.16 može se pokazati da za svako i takvo da je $\sigma_{2i+2}(n_0)$ definisano važi

$$
|(tuv)[\sigma_{2i+2}(n_0)+1,\sigma_{2i}(n_0)]|_2 \geqslant \frac{k}{k-1}|(tuv)[\sigma_{2i+2}(n_0)+1,\sigma_{2i}(n_0)]|_1.
$$

Konačno, imamo

$$
\begin{aligned}
|tuv|_2 &> \frac{k}{k-1}\sum_{i=0}^{\frac{\mathrm{end}(n_0)}{2}-1} |(tuv)[\sigma_{2i+2}(n_0)+1,\sigma_{2i}(n_0)]|_1 \\
&= \frac{k}{k-1}|(tuv)[\sigma_{\mathrm{end}(n_0)}(n_0),n_0]|_1 = \frac{k}{k-1}|u|_1.
\end{aligned}
$$

Sada možemo pretpostaviti ne samo da $2 \nmid \mathrm{end}(n_0)$, nego i da $2 \nmid \mathrm{end}(n)$ za bilo koje n koje ne umire (inače bismo mogli reizabrati n_0). Dalje, možemo pretpostaviti da $\mathrm{end}(n_0)$ nije veće od $\mathrm{end}(n)$ za bilo koje n koje ne umire i za koje važi $n \geqslant |tu|+1$ (inače bismo mogli ponovo reizabrati n_0).

Slično kao u prethodnom slučaju, važi

$$
\begin{aligned}
|tuv|_2 &= \big|(tuv)[n_0+1, |tuv|]\big|_2 + \sum_{i=0}^{\frac{\mathrm{end}(n_0)-3}{2}} \big|(tuv)[\sigma_{2i+2}(n_0)+1, \sigma_{2i}(n_0)]\big|_2 \\
&\qquad + \big|(tuv)[1, \sigma_{\mathrm{end}(n_0)-1}(n_0)]\big|_2 \\
&\geqslant (|tuv| - n_0) + \frac{k}{k-1} \sum_{i=0}^{\frac{\mathrm{end}(n_0)-3}{2}} |(tuv)[\sigma_{2i+2}(n_0)+1, \sigma_{2i}(n_0)]|_1 \\
&\qquad + |(tuv)[1, \sigma_{\mathrm{end}(n_0)-1}(n_0)]|_2 \\
&= |tuv| - n_0 + |(tuv)[1, \sigma_{\mathrm{end}(n_0)-1}(n_0)]|_2 \\
&\qquad + \frac{k}{k-1} |(tuv)[\sigma_{\mathrm{end}(n_0)-1}(n_0)+1, n_0]|_1.
\end{aligned}
$$

Dakle, za kompletiranje dokaza treba pokazati

$$
|tuv| - n_0 + |(tuv)[1, \sigma_{\mathrm{end}(n_0)-1}(n_0)]|_2 \geqslant \frac{k}{k-1} |(tuv)[1, \sigma_{\mathrm{end}(n_0)-1}(n_0)]|_1,
$$

što se može dobiti korišćenjem (između ostalog) leme 3.17. ■

Na osnovu prethodnog dokaza mogu se tačno okarakterisati reči za koje se u teoremi 3.13 dostiže jednakost.

Propozicija 3.18. *Pod uslovima teoreme 3.13, u navedenoj nejednakosti dostiže se jednakost ako i samo ako za neki prirodan broj k i nenegativan ceo broj l imamo $u = (1^{k-1}2^k)^l 1^{k-1}$, $t = \varepsilon$ i $v = 2^k$ (ili obratno).*

3.3.2 Drugi dokaz

Ideja drugog dokaza teoreme 3.13. Koristimo indukciju po dužini $|u|$. Slučaj $|u| = 0$ je trivijalan, dakle, pretpostavimo da je tvrđenje tačno za sve reči u' gde $|u'| < |u|$, i dokazujemo da tada mora važiti i za u.

Označimo sa v' najkraći prefiks od uv za koji važi $|v'|_2 = |v|$, a sa t' najkraći sufiks od tu za koji važi $|t'|_2 = |t|$.

Posmatramo prvo slučaj $|v'| + |t'| < |u|$. Zapišimo $u = v'u't'$, i neka su p' i q' najduži potpalindromi od $u't$ i vu', respektivno. (Primetimo, t je sa desne a v sa leve strane u', a ne obratno kao ranije!) Može se pokazati da tada važi

$$|q'| \geqslant |p| - 2|t| - 2(|v'| - |v|),$$

kao i

$$|p'| \geqslant |q| - 2|v| - 2(|t'| - |t|).$$

Koristimo indukcijsku hipotezu za v, u' i t. Lako se pokazuje $|u'| < |u|$, a na osnovu malopre dobijenih jednakosti dobija se i

$$|q'| + |p'| \geqslant 2|u'|.$$

Sada na osnovu indukcijske hipoteze sledi

$$|u'|_1 \leqslant \frac{|vt| - 1}{|vt|} |vu't|_2.$$

Direktno se pokazuje:

$$|v'|_1 + |t'|_1 \leqslant |uv| - |q| + |tu| - |p| < |t| + |v|.$$

Konačno, sve zajedno imamo

$$\begin{aligned} |u|_1 = |v'|_1 + |u'|_1 + |t'|_1 \leqslant |tv| - 1 + |u|_1 \leqslant \frac{|tv| - 1}{|tv|}(|tv| + |vu't|_2) \\ = \frac{|tv| - 1}{|tv|}(|t'|_2 + |v'|_2 + |vu't|_2) = \frac{|tv| - 1}{|tv|}|tuv|_2, \end{aligned}$$

što je i trebalo pokazati.

Slučaj $|v'| + |t'| \geqslant |u|$ se rešava (manje-više) direktno. ∎

3.4 Još jedna odložena tehnička teorema

Teorema 3.19. *Neka $w \in \{1, 2\}^*$, pri čemu važi $2|w|_2 \geqslant |w|_1$. Neka su p i q dva neprazna potpalindroma od w. Neka su w_p, v, w_q i t reči takve da $w = w_p v = t w_q$, p je podreč od w_p, i q je podreč w_q. Tada*

$$|p| + 2|v|_2 + |q| + 2|t|_2 \leqslant 4|w|_2 + |w|_1.$$

Ideja dokaza. Posmatramo dva slučaja u zavisnosti od toga da li $|w_p| \leqslant |t|$ ili $|t| < |w_p|$. U prvom slučaju nakon nekoliko tehničkih koraka zaključak sledi direktno, dok u drugom slučaju (takođe nakon nekoliko tehničkih koraka) koristimo teoremu 3.13. ∎

4 Zaključak

U ovoj tezi smo objedinili nekoliko rezultata vezanih za neke reverznoinvarijantne mere složenosti reči. Dve takve mere koje smo posmatrali su palindromski defekt i MP-razmera date reči.

Jedan od naših glavnih rezultata jeste uvođenje klase *uopštenih visokopotencijalnih reči*. Pokazali smo da je njihov defekt uvek konačan, i u mnogo slučajeva pozitivan, da je njihov skup faktora zatvoren u odnosu na operaciju preokretanja reči, i da je svaka od njih ili periodična, ili rekurentna ali ne i uniformno rekurentna. Značaj ove klase reči ogleda se u činjenici da se ova kombinacija osobina veoma retko sreće kod reči, što ih čini vrlo korisnom zalihom primera ili kontraprimera za razne probleme na rečima.

Drugi glavni rezultat koji je izložen u ovoj tezi je proširenje definicije MP-razmere na ternarne reči. Pojam MP-razmere je originalno, pre deset godina, bio definisan samo za binarne reči, mogućnost proširenja definicije na veće alfabete je bila ostavljena kao otvoreno pitanje, i od tada nije bilo nikakvog pomaka sve do sada. Pokazali smo da je MP-razmera dobro definisana u ternarnom slučaju, da je ograničena s gornje strane sa 6 i da je ovo najbolja gornja granica. Nalaženje optimalne gornje granice za MP-razmeru u opštem slučaju i dalje je otvoren problem, ali verujemo da naš rezultat predstavlja značajan korak u tom smeru.

Komentar. Nakon kompletiranja glavnog teksta ove teze, dokazali smo da je MP-razmera za n-arni alfabet dobro definisana za svako n. Rad je u fazi pripreme.

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Kristina Ago Balog je rođena 24. februara 1991. godine u Subotici. Srednju školu „Gimnazija sa domom učenika za talentovane učenike Boljai“ u Senti je završila 2010. godine. Po završetku srednje škole upisala se na osnovne studije na Prirodno-matematičkom fakultetu u Novom Sadu, smer matematika. Osnovne studije je završila 2013. godine sa prosečnom ocenom 10,00 i upisala se na master studije teorijske matematike. Master studije je završila 2015. godine sa prosečnom ocenom 9,95. Iste godine se upisala na doktorske studije matematike. Položila je sve ispite sa doktorskih studija sa prosečnom ocenom 10,00. Od 2015. godine je angažovana u izvođenju nastave na Departmanu za matematiku i informatiku Prirodno-matematičkog fakulteta Univerziteta u Novom Sadu. Od 2016. godine drži dodatnu nastavu u gimnaziji u Senti. Član je naučnog projekta Algebarske, logičke i kombinatorne metode sa primenama u teorijskom računarstvu Ministarstve prosvete, nauke i tehnološkog razvoja Republike Srbije. Ima tri naučna rada objavljena ili pod recenzijom u međunarodnim časopisima.

Novi Sad, maj 2020 Kristina Ago Balog

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Izvod: Izučavamo dve mere složenosti reči koje su invarijantne u odnosu na operaciju preokretanja reči: palindromski defekt i MP-razmeru date reči.

Palindromski defekt reči w definiše se kao $|w|+1-|\mathrm{Pal}(w)|$, gde $|\mathrm{Pal}(w)|$ predstavlja broj palindromskih faktora reči w. Mi izučavamo beskonačne reči, na koje se ova definicija može prirodno proširiti. Postoje mnogobrojni rezultati u vezi sa tzv. *bogatim rečima* (reči čije je defekt 0), dok se o rečima sa konačnim pozitivnim defektom relativno malo zna; tokom jednog perioda (donedavno) nije bilo poznato ni da li uopšte postoje takve reči koje su, dodatno, aperiodične i imaju skup faktora zatvoren za preokretanje. Među prvim primerima koji su se pojavili u literaturi su bile tzv. *visokopotencijalne reči*. U disertaciji ćemo predstaviti znatno opštiju konstrukciju, kojom se dobija značajno šira klasa reči, nazvanih *uopštene visokopotencijalne reči*, i analiziraćemo njihov značaj u okvirima kombinatorike na rečima.

MP-razmera date n-arne reči w definiše se kao količnik $\frac{|rws|}{|w|}$, gde su r i s takve da je reč rws minimalno-palindromična, i dužina $|r|+|s|$ je najmanja moguća; ovde, za n-arnu reč kažemo da je *minimalno-palindromična* ako ne sadrži palindromsku podreč dužine veće od $\left\lceil\frac{|w|}{n}\right\rceil$. U binarnom slučaju dokazano je da je MP-razmera dobro definisana i da je ograničena odozgo konstantom 4, što je i najbolja moguća granica. Dobra definisanost MP-razmere za veće alfabete je ostavljena kao otvoren problem. U ovoj tezi rešavamo taj problem u ternarnom slučaju: pokazaćemo da MP-razmera jeste dobro definisana u ternarnom slučaju, da je ograničena odozgo sa 6, i da se ta granica

ne može poboljšati.
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Abstract: We focus on two complexity measures of words that are invariant under the operation of reversal of a word: the palindromic defect and the MP-ratio.

The *palindromic defect* of a given word w is defined by $|w| + 1 - |\mathrm{Pal}(w)|$, where $|\mathrm{Pal}(w)|$ denotes the number of palindromic factors of w. We study infinite words, to which this definition can be naturally extended. There are many results in the literature about the so-called *rich words* (words of defect 0), while words of finite positive defect have been studied significantly less; for some time (until recently) it was not known whether there even exist such words that additionally are aperiodic and have their set of factors closed under reversal. Among the first examples that appeared were the so-called *highly potential words.* In this thesis we present a much more general construction, which gives a wider class of words, named *generalized highly potential words,* and analyze their significance within the frames of combinatorics on words.

The *MP-ratio* of a given n-ary word w is defined as the quotient $\frac{|rws|}{|w|}$, where r and s are words such that the word rws is minimal-palindromic and that the length $|r| + |s|$ is minimal possible; here, an n-ary word is called *minimal-palindromic* if it does not contain palindromic subwords of length greater than $\lceil \frac{|w|}{n} \rceil$. In the binary case, it was proved that the MP-ratio is well-defined and that it is bounded from above by 4, which is the best possible upper bound. The question of well-definedness of the MP-ratio for larger

alphabets was left open. In this thesis we solve that question in the ternary case: we show that the MP-ratio is indeed well-defined in the ternary case, that it is bounded from above by the constant 6 and that this is the best possible upper bound.
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