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DOI:10.1016/j.jde.2021.02.005

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Abstract

In this paper, we study some potential-theoretic aspects of the eikonal and infinity Laplace operator on a Finsler manifold M. Our main result shows that the forward completeness of M can be detected in terms of Liouville properties and maximum principles at infinity for subsolutions of suitable inequalities, including $\Delta_\infty^N u \ge g(u)$. Also, an ∞-capacity criterion and a viscosity version of Ekeland principle are proved to be equivalent to the forward completeness of M. Part of the proof hinges on a new boundary-to-interior Lipschitz estimate for solutions of $\Delta_\infty^N u = g(u)$ on relatively compact sets, that implies a uniform Lipschitz estimate for certain entire, bounded solutions without requiring the completeness of M.

Contents

1 Introduction

The main purpose of the present work is to study the relationship between the metric properties of a Finsler manifold and the potential-theoretic properties of the ∞-Laplace operator

$$\Delta_\infty u := \text{Hess}\, u(\nabla u, \nabla u)$$

and its normalized version

$$\Delta_\infty^N u := \text{Hess}\, u\left(\frac{\nabla u}{|\nabla u|}, \frac{\nabla u}{|\nabla u|}\right).$$

Our investigation arises in connection to the fully nonlinear potential theory developed in [37, 38] in a Riemannian setting (cf. also [50, 49]), whose main goal is to recast, in a unified framework, various maximum principles at infinity available in the literature: the celebrated Ekeland [24, 25] and Omori-Yau ones [45, 57, 19], as well as those coming from stochastic geometry (the weak maximum principles of Pigola-Rigoli-Setti [48], related to the parabolicity and the stochastic and martingale completeness of a Riemannian manifold). The appearance of first order conditions in the statements of Ekeland and Omori-Yau principles calls for a theory that includes eikonal equations, and opens the way to encompass the ∞-Laplace operator, tightly related to the eikonal one.

The infinity Laplacian has received great attention after the pioneering work of G. Arronson [7, 8] in the 1960s, and showed intriguing connections with pure and applied mathematical issues, as for example, Tug-of-war games [12, 46, 51], mass transportation problems [27] and others. The study of the infinity Laplacian is strictly related with an L^∞ minimization problem: given a bounded domain $\Omega \subset \mathbb{R}^m$ and a Lipschitz function $\zeta : \partial\Omega \to \mathbb{R}$, to find an extension u of ζ in Ω such that the Lipschitz constant $\text{Lip}(u, A) \le \text{Lip}(h, A)$ for any $A \Subset \Omega$ and h which agrees with u on ∂A. Such a function is called an absolutely minimizing Lipschitz extension, shortly AMLE [21, 18]. Jensen in [31] showed that u is AMLE if and only if u solves $\Delta_\infty u = 0$ in the viscosity sense, and by [22, 31] AMLE functions are also characterized by the comparison principle with cone functions

$$C_x(y) = a + b|x - y| \quad a, b \in \mathbb{R} \text{ and } y \in \mathbb{R}^m,$$

which are fundamental solutions of the homogeneous infinity Laplacian. All these properties are foundational in the theory of infinity harmonic functions on $\mathbb{R}^m$. Since then, various works have been devoted to the analysis of Δ_∞ on more general spaces, and an account can be found in [21, 9]. Especially, on domains of $\mathbb{R}^m$ equipped with a Finsler norm, the AMLE problem and the associated ∞-Laplace operator have been studied in [55, 29, 41, 42].

In [37], building on the above characterizations of ∞-harmonic functions, the geodesic completeness of a Riemannian manifold was characterized in terms of various potential-theoretic properties of Δ_∞, among them a suitable version of Ekeland principle for subsolutions of Eikonal type equations and the validity of the following Liouville theorem:

$$\textit{entire viscosity solutions of } \Delta_\infty u \ge 0 \textit{ with } \sup_M u < \infty \textit{ are constant.} \tag{1}$$

In the present work we improve results from [37] on different aspects. First of all, we consider more general inhomogeneous inequalities of the type

$$\Delta_\infty^N u \ge g(u) \tag{2}$$

for continuous non-negative g. This class includes reaction-diffusion equations with strong absorption as those investigated in [4]. Such general setting brings extra difficulties: for example, to overcome the lack of AMLE properties tailored to solutions of (2) we shall prove a boundary-to-interior Lipschitz estimates for solutions of (2) (with the equality sign) which only depends on the L^1 norm of g. The result might be of independent interest (cf. Section 1.1). Among other things, we study whether a Liouville property for bounded solutions of (2) for *some* non-negative g still detects the completeness of M or rather a weaker property. In this respect, the ∞-Laplacian behaves differently from other operators, notably from the p-Laplacian for $1 < p < \infty$, despite their relation showed by the formal limit

$$\Delta_\infty^N u = \lim_{p\to\infty} \frac{|\nabla u|^{2-p}}{p} \Delta_p u.$$

Indeed, in a geodesically complete manifold the validity of the Liouville theorem as in (1) for the differential inequality $\Delta_p u \geq g(u)$ is achieved from sharp geometric criteria that vary accordingly to the vanishing or positiveness of the function g. When $g \equiv 0$, a sharp sufficient condition is given by

$$\int^\infty \left(\frac{s\mathrm{d}s}{|B_s|} \right)^{\frac{1}{p-1}} = \infty,$$

where $|B_r|$ is the volume of a geodesic ball centered at a fixed origin (see [48]). On the other hand, for $g > 0$ a sharp threshold is given by

$$\liminf_{r\to\infty} \frac{\log |B_r|}{r^2} < \infty,$$

see Theorem 2.24 and Proposition 7.4 in [14]. In the semilinear case $p = 2$, this difference is clarified in terms of stochastic geometry (cf. [1, 28, 48] for a detailed account): briefly, if $g \equiv 0$, the Liouville theorem (1) for $\Delta u \geq 0$ turns out to be equivalent to the *parabolicity* of M, which means that almost surely any Brownian path visits every compact set infinitely often, while the case $g > 0$ on $\mathbb{R}^+$, $g(0) = 0$, ties to the *stochastic completeness* of M, that is, to the property that Brownian paths on M have infinity lifetime almost surely, see [47, 48, 3].

A source of motivation for preseting our results in the framework of general Finsler manifolds comes from causality theory in General Relativity. In fact, as showed in [17], there is a correspondence between stationary Lorentzian manifolds and Finsler manifolds of Randers type: to any $(m + 1)$-dimensional Lorentzian manifold $\bar{M} = \mathbb{R} \times M$ that is stationary, in the sense that its metric can be written as

$$-\mathrm{d}t^2 + \pi^*\omega \otimes \mathrm{d}t + \mathrm{d}t \otimes \pi^*\omega + \pi^* g_0$$

for some Riemannian manifold (M, g_0) and some 1-form ω on M (with $t : \bar{M} \to \mathbb{R}$ and $\pi^* : \bar{M} \to M$ the projections onto the first and second factors), the correspondence associates a Finsler structure of Randers type on M by setting

$$F(p) := \sqrt{g_0(p,p) + \omega(p) \otimes \omega(p)} + \omega(p)$$

Remarkably, the causal geometry of $\bar{M}$ can be grasped by studying the metric geometry of (M, F), and in this respect it is therefore useful to find criteria to guarantee the forward completeness of (M, F). For instance, in [17, Thm. 4.4] the authors proved that the forward and backward completeness of (M, F) is equivalent to the fact that M is a Cauchy hypersurface (see also [17, Rmk. 4.5]), while in [17, Thm. 4.3] they showed the equivalence between the global hyperbolicity of $\bar{M}$ and the precompactness of symmetrized balls in (M, F).

1.1 Main results

Let (M, F) be a Finsler manifold (the basics of Finsler Geometry are recalled in Section 2). We assume the Finsler norm $F : TM \to [0, \infty)$ be positively homogeneous of degree 1, and F^2 be strictly convex when restricted on each fiber of $TM \to M$. For smooth u, the Chern connection associated to F allows to define the Hessian of a function and, consequently, a Finsler ∞-Laplacian. Also, the norm F induces a pseudo-distance d on M that is, d satisfies all of the requirements of a distance function but, possibly, its symmetry. The lack of symmetry introduces further issues, among them the need to distinguish which properties relate to the *forward* completeness of M rather than to its *backward* one. The forward completeness for (M, F) is defined by asking that forward Cauchy sequences converge, i.e. if $\{x_i\}$ satisfies the following Cauchy condition:

$$\forall\, \varepsilon > 0,\ \exists N = N(\varepsilon) \in \mathbb{N}\ :\ N \le i < j \Longrightarrow \mathrm{d}(x_i, x_j) < \varepsilon,$$

then $\{x_i\}$ converges. Following [18], we define the Lipschitz constant of u on a set A to be

$$\mathrm{Lip}(u, A) \doteq \inf \Big\{ L \in [0, \infty]\ :\ u(y) - u(x) \le L\mathrm{d}(x, y) \quad \forall\, x, y \in A \Big\}. \tag{3}$$

Let $\varrho^+(x) = \mathrm{d}(o, x)$ denotes the distance from a fixed origin $o \in M$. We are ready to state our main result. Note that solutions are meant to be in the viscosity sense, see [23].

Theorem 1.1. *Let (M, F) be a connected Finsler manifold. Then, the following properties are equivalent:*

1) (M, F) is forward complete.

2) Having denoted with ϱ^+ the forward distance from a fixed origin,

$$\begin{cases} \Delta^N_\infty u \ge 0 \quad \text{on } M, \\ u_+(x) = o\big(\varrho^+(x)\big) \quad \text{as } \varrho^+(x) \to +\infty \end{cases} \qquad \Longrightarrow \qquad u \text{ is constant.} \tag{4}$$

3) For some/every $g \in C(\mathbb{R})$ with $g(0) = 0$ and $g \ge 0$ on $\mathbb{R}^+$, the following holds:

$$\begin{cases} \Delta^N_\infty u \ge g(u) \quad \text{on } M, \\ 0 < \sup_M u < +\infty \end{cases} \qquad \Longrightarrow \qquad u \text{ is constant.}$$

4) For some/every $g \in C(\mathbb{R})$ with $g(0) = 0$ and $g \ge 0$ on $\mathbb{R}^+$, the following holds: for every open subset $\Omega \subset M$,

$$\begin{cases} \Delta^N_\infty u \ge g(u) \quad \text{on } \Omega, \\ 0 < \sup_\Omega u < +\infty \end{cases} \qquad \Longrightarrow \qquad \sup_\Omega u = \sup_{\partial\Omega} u. \tag{5}$$

5) For some/every $\theta \in (0, 1)$ and $\lambda > 0$, it holds

$$\begin{cases} \Delta^N_\infty u \ge \lambda u_+^\theta \quad \text{on } M, \\[2ex] \displaystyle\limsup_{\varrho^+(x)\to+\infty} \frac{u_+(x)}{\varrho^+(x)^{\frac{2}{1-\theta}}} < \sqrt[1-\theta]{\lambda \frac{(1-\theta)^2}{2(1+\theta)}} \end{cases} \qquad \Longrightarrow \qquad u \text{ is a (nonpositive) constant.} \tag{6}$$

6) *For some/every $K \subset M$ compact, it holds*

$$\inf_{u \in \mathscr{L}(K,M)} \mathrm{Lip}(u, M) = 0,$$

where

$$\mathscr{L}(K, M) = \Big\{ u \in \mathrm{Lip}_c(M),\ u \le -1 \ \ on \ \ K \Big\}. \tag{7}$$

7) *For some/every $K \subset M$ compact, the ∞-capacity of K vanishes:*

$$\mathrm{cap}_\infty(K) := \inf_{u \in \mathscr{L}(K,M)} \|F(\nabla u)\|_{L^\infty(M)} = 0,$$

where $\mathscr{L}(K, M)$ is defined in (7)*.*

8) *For some/every $0 < G \in C(\mathbb{R})$, the following holds: for every open subset $\Omega \subset M$, and for every viscosity subsolution of*

$$\begin{cases} G(u) - F(\nabla u) = 0 \quad on \ \Omega, \\ \sup_\Omega u < \infty \end{cases} \qquad \Longrightarrow \qquad \sup_\Omega u = \sup_{\partial\Omega} u. \tag{8}$$

9) *(**Ekeland principle**). For every $u \in \mathrm{USC}(M)$ with $\sup_M u < \infty$, for every $\varepsilon > 0$ and $x_0 \in M$ such that $u(x_0) > \sup_M u - \varepsilon$, and for every $\delta > 0$, there exists $\bar{x} \in M$ such that*

$$u(\bar{x}) \ge u(x_0), \quad \mathrm{d}(x_0, \bar{x}) \le \delta, \quad and \quad u(y) \le u(\bar{x}) + \tfrac{\varepsilon}{\delta}\mathrm{d}(\bar{x}, y) \quad \forall\, y \in M.$$

Some remarks on the equivalences in Theorem 1.1 are in order:

The some/every alternative. *Property* 3)*, as well as* 4)*, holds for* every *g as in the statement provided that it holds for* some *such g. In particular, in view of our assumption on g, the* every *alternative is equivalent to require* 3) *for the smallest choice $g \equiv 0$. Therefore, unlikely the case of Δ_p with $p < \infty$, for the ∞-Laplacian the Liouville theorems for $\Delta_\infty^N u \ge g(u)$ under the assumptions $g \equiv 0$ or $g(0) = 0$, $g > 0$ on $\mathbb{R}^+$ are equivalent.*

Backward completeness. *The notion of backward completeness for (M, F), demanding that backward Cauchy sequences converge, corresponds to the forward completeness of the dual Finsler structure*

$$\widetilde{F}(p) := F(-p), \qquad p \in TM,$$

hence it can be described via the eikonal and normalized ∞-Laplacian $\widetilde{\Delta}_\infty^N$ associated to $\widetilde{F}$. In view of the identity

$$\widetilde{\Delta}_\infty^N u = -\Delta_\infty^N(-u),$$

the backward completeness of (M, F) can be detected by minimum principles for solutions of $\Delta_\infty^N u \le g(u)$. We leave the statement to the interested reader.

On implication 1) ⇒ 2). *In Euclidean space, this implication was shown in [33, 22] in a different way, namely as a consequence of the Harnack inequality for ∞-subharmonic equations (see [33, 34, 32], and also [26]).*

On conditions 8), 9) - a viscosity Ekeland principle. *Implication* 1) $\Rightarrow$ 9) *is the celebrated Ekeland principle [25, 24], originally stated for metric spaces, while* 9) $\Rightarrow$ 1) *has been pointed out by J.D. Weston [54] and F. Sullivan [53]. Extension to the Finsler setting is straightforward, since Weston-Sullivan arguments as well as the proof of* 9) *provided in [25, p.444] do not use the symmetry of* d *at any step. We included* 9) *for the sake of completeness, and to emphasize that* 8) *can be interpreted as a viscosity version of Ekeland principle.*

On condition 5). *Reaction-diffusion equations with strong absorption as in* 5) *were investigated in [4], where the authors proved regularity for the unnormalized case* $\Delta_\infty u = \lambda u_+^\gamma$ *in* $\mathbb{R}^m$, $0 \le \gamma < 3$, *and related Liouville theorems for entire solutions satisfying*

$$u(x) = O(|x|^{\frac{4}{3-\gamma}}) \quad as \quad |x| \to \infty. \tag{9}$$

In the limit $\gamma \to 0$, *this relates to the* ∞*-obstacle problem. The constant bounding the limsup in* (6) *is sharp, as readily seen on flat Euclidean space by noting that*

$$u(x) = \sqrt[1-\theta]{\lambda \frac{(1-\theta)^2}{2(1+\theta)}}|x|^{\frac{2}{1-\theta}}$$

solves $\Delta_\infty^N u = \lambda u^\theta$.

On conditions 7), 8). *The equivalence between* 1) *and* 7), 8) *was observed in [50, Thms. 2.28 and 2.29] in a Riemannian setting: it is inspired by the characterization of parabolic Riemannian manifolds by means of the vanishing of the* 2*-capacity* $\mathrm{cap}_2(K)$ *of some/every compact set* K *(cf. [28]), and to equivalent ones for the p-Laplacian,* $p \in (1, \infty)$ *in terms of the p-capacity*

$$\mathrm{cap}_p(K) := \left\{ \int_M |\nabla u|^p \ : \ u \in \mathrm{Lip}_c(M),\ u \ge 1 \ \ on \ \ K \right\}.$$

Observe that, to detect the forward *completeness, we had to switch signs and define our class* $\mathscr{L}(K, M)$ *by requiring* $u \le -1$ *on* K.

Normalized vs unnormalized ∞-Laplacian. *The equivalence between items* 1), …, 5) *could be rephrased for the unnormalized* ∞*-Laplacian with minor changes, replacing* $\Delta_\infty^N u \ge g(u)$ *with the inequality*

$$\Delta_\infty u \ge g(u)|\nabla u|^2,$$

and 5) *with the following statement:*

5') for some/every $\theta \in (0, 3)$ and $\lambda > 0$, it holds

$$\begin{cases} \Delta_\infty u \ge \lambda u_+^\theta \qquad \text{on } M, \\[1em] \limsup\limits_{\varrho^+(x)\to+\infty} \dfrac{u_+(x)}{\varrho^+(x)^{\frac{4}{3-\theta}}} < \sqrt[3-\theta]{\lambda \dfrac{(3-\theta)^4}{64(1+\theta)}} \end{cases} \qquad \Longrightarrow \qquad u \text{ is a (nonpositive) constant.}$$

The fact that the forward completeness of (M, F) implies any of 2), …, 4) is not difficult to prove, and might be well-known among specialists, although we found no precise reference; on the other hand, 1) $\Rightarrow$ 5) is more subtle, due to the possibility that the limsup in (6) be positive, and inspired by [4]. We briefly comment on implications 8) $\Rightarrow$ 1) and 3) $\Rightarrow$ 1), that are the technical core of the present work.

The proof of 8) $\Rightarrow$ 1) exploits results in [40, 37], namely it uses the Ahlfors-Khas'minskii duality (AK-duality, for short). Roughly speaking, for a large class of fully nonlinear inequalities

$$\mathscr{F}(x, u, \mathrm{d}u, \mathrm{Hess}\, u) \geq 0, \tag{10}$$

the AK-duality establishes the equivalence between a maximum principle at infinity for solutions of (10), in the form given by (5) (called there the Ahlfors property), and the existence of solutions of the *dual* inequality

$$\widetilde{\mathscr{F}}(x, u, \mathrm{d}u, \mathrm{Hess}\, u) \geq 0, \qquad \text{with } \widetilde{\mathscr{F}}(x, r, p, A) = -\mathscr{F}(x, -r, -p, -A),$$

that decay to $-\infty$ as slow as we wish[1] (named Khas'minskii potentials). The eikonal equation

$$G(u) - F(\nabla u) = 0$$

falls into the class of PDEs for which the AK-duality holds, thus we can construct a Khas'minskii potential w that is a subsolution of the dual equation $\widetilde{F}(\widetilde{\nabla} w) - \widetilde{G}(w) = 0$, with $\widetilde{F}$ the dual Finsler structure, $\widetilde{\nabla}$ the gradient induced by $\widetilde{F}$ and $\widetilde{G}(t) := G(-t)$. The existence of w easily implies the forward completeness of M. The construction of w proceeds, as in [37, 38], by stacking solutions of obstacle problems, and has independent interest.

A key point in our arguments is related to the proof of implication 3) $\Rightarrow$ 1). The sought conclusion is obtained by constructing a sequence $\{u_j\}$ of functions which solve equation $\Delta^N_\infty u = g(u)$ on an increasing family of relatively compact sets Ω_j. The main issue is then to guarantee that such sequence locally converges to a limit solution u_∞ on $M \backslash K$, where K is a fixed small compact set. This requires to prove a uniform global Lipschitz bound for u_j on possibly incomplete manifolds, obtained in the following result.

Theorem 1.2. *Let $\Omega \Subset (M, F)$, and let $u \in C(\overline{\Omega})$ satisfying*

$$\Delta^N_\infty u = g(u) \quad \text{on } \Omega, \tag{11}$$

for a continuous, non-decreasing and non-negative function g defined on $u(\overline{\Omega})$. If u is Lipschitz on $\partial\Omega$, then u is Lipschitz on $\overline{\Omega}$. Furthermore,

$$\mathrm{Lip}(u, \Omega) \leq \sqrt{\mathrm{Lip}(u, \partial\Omega)^2 + 2 \int_{\inf_\Omega u}^{\sup_\Omega u} g(s)\mathrm{d}s}. \tag{12}$$

Inequality (12) is boundary-to-interior, and its relevance is motivated by the fact that standard Lipschitz estimates for infinity subharmonic functions as those in [22, Lemma 2.5] are not suited to our purposes. Indeed, such estimates are local on relatively compact balls B_R, and blow up as $R \to 0$. Hence, they cannot be turned into a global Lipschitz bound on M unless all forward balls of some fixed radius are relatively compact in M, which is the case if and only if M is forward complete. In [37], for $g \equiv 0$, the authors reach the goal by exploiting the absolutely minimizing property of the ∞-harmonic functions u_j, a characterization that currently

[1] We say that u decays to $-\infty$ if upper level sets of u have compact closure in M.

seems unavailable[2] for solutions of (11). We overcome the problem by showing a Lipschitz bound directly via comparison with radial solutions g (hereafter called g-cones), extending an elegant argument in [9, Prop. 2.1]. Note that in the particular case $g \equiv 0$, this suitably reduces to the AMLE condition $\mathrm{Lip}(u, \Omega) = \mathrm{Lip}(u, \partial\Omega)$.

The paper is organized as follows: in Section 2 we collect some preliminary material and main properties of Finsler manifolds. In Sections 3 and 4, we define viscosity solutions of ∞-Laplace equations, state their main comparison results with forward and backward g-cones, and prove Theorem 1.2. Eventually, in Section 5 we prove Theorem 1.1. Appendices I and II contain some ancillary results adapted to the Finsler setting.

Acknowledgements. The authors would like to express their gratitude to Andrea Mennucci, for interesting discussions, and to Miguel Angel Javaloyes, for suggesting reference [17] and for pointing out to us the relation with the causality theory in Lorentzian manifolds. The first and third authors are partially supported by CNPq-Brazil, grant 2019/0014 Paraiba State Research Foundation (FAPESQ) and PROMISSÕES-UFPI (010/2018), respectively. They also would like to thank the worm hospitality of the Abdus Salam International Centre for Theoretical Physics (ICTP), and of the Mathematisches Forschungsinstitut Oberwolfach (MFO), where part of this work was conducted.

2 Basics on Finsler manifolds

Let M be an m-dimensional smooth manifold. As usual we denote by $TM \doteq \cup_{x \in M} T_x M$ the tangent bundle of M, where $T_x M$ means the tangent space at $x \in M$. Each element of TM has the form (x, p), where $x \in M$ and $p = p^i \frac{\partial}{\partial x^i} \in T_x M$. A Finsler structure on M (cf. [10]) is a function $F : TM \to [0, \infty)$ satisfying the following properties:

i) *Regularity:* F is smooth on $TM \backslash 0$, with 0 the zero section.

ii) *Positive homogeneity:* $F(x, \lambda p) = \lambda F(x, p)$ for all $\lambda > 0$.

iii) *Strong convexity:* The fundamental tensor

$$g_{ij}(x, p) := \frac{1}{2} \frac{\partial^2 F^2(x, p)}{\partial p^i \partial p^j}$$

is positive definite at every $(x, p) \in TM \backslash 0$.

Note that the expression $g_{ij}(x, p) p^i p^j$ is invariant by a change of coordinates. We call a Finsler manifold the pair (M, F), where M is a smooth manifold and F is a Finsler structure on

[2] In this respect, note that (11) is not included in the class of PDEs considered in [13], where the authors compute the Euler-Lagrange equations of absolute minimizers for

$$\mathcal{J}(u, \Omega) = \operatorname*{ess\,sup}_{x \in \Omega} f(x, u(x), \mathrm{d}u(x))$$

In our case (say, even in a Riemannian setting), the PDE $\Delta_\infty u = g(u)|\nabla u|^2$ for the unnormalized ∞-Laplacian would be, formally, the Euler-Lagrange equation for the choice

$$f(x, s, p) = |p|^2 - 2 \int_0^s g(t) \mathrm{d}t,$$

a function that does not satisfy all of the assumptions in Theorem 3.5 of [13].

M. Riemannian manifolds (M, g) are a particular subclass of Finsler manifolds, obtained by choosing

$$F(x,p) := \sqrt{g_{ij}(x)p^i p^j}.$$

The induced Finsler structure $F^* : T^*M \to [0,\infty)$ on the cotangent bundle is defined by

$$F(x,\xi) \doteq \sup_{p \in T_x M \setminus 0} \frac{\xi(p)}{F(x,p)} = \sup_{F(x,p)=1} \xi(p),$$

and gives rise to a family of Minkowski norms $F^* = \{F_x^*\}_{x \in M}$ with corresponding fundamental tensor

$$g^{*kl}(\xi) = \frac{1}{2}\frac{\partial^2 F^{*2}(\xi)}{\partial \xi_k \partial \xi_l}.$$

Hereafter, we write $F(p), F^*(\xi)$ for notational convenience, suppressing the dependence on x. We will use the Chern connection of (M, F), defined on the vector bundle $\pi^* TM$, where $\pi : TM\setminus 0 \to M$ is the natural projection. Its connection forms are torsion free, that is,

$$dx^j \wedge \omega_j^i = 0,$$

which means that dp^k are absent in the definition of ω_j^i, namely,

$$\omega_j^i = \Gamma_{jk}^i dx^k, \quad \text{and} \quad \Gamma_{jk}^i = \Gamma_{kj}^i.$$

Let $\Omega \subset M$ be open and consider a coordinate system $(x^i, \frac{\partial}{\partial x^i})$ on $T\Omega$. Given a non-vanishing vector field $v = v_i \frac{\partial}{\partial x^i}$ on Ω, we introduce a Riemannian metric g_v and a linear connection ∇^v on $T\Omega$ by setting, for $p = p^i \frac{\partial}{\partial x^i}$ and $q = q^i \frac{\partial}{\partial x^i}$ in $T_x\Omega$,

$$g_v(p,q) \doteq p^i q^j g_{ij}(x,v), \quad \text{and} \quad \nabla^v_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} \doteq \Gamma_{ij}^k(x,v)\frac{\partial}{\partial x^k}.$$

We define the Legendre transformation $\ell : TM \to T^*M$ by

$$\ell(p) = \begin{cases} g_p(p,\cdot), & p \neq 0, \\ 0, & p = 0. \end{cases}$$

Remarkably, $\ell : TM\setminus 0 \to TM^*\setminus 0$ is a smooth diffeomorphism and

$$F^*(\ell(p)) = F(p), \quad \text{for all } p \in TM.$$

Consequently, $g^{*ij}(\ell(p))$ coincides with the inverse of $g_{ij}(p)$ (see [10], [52]), and the map $\ell^{-1} : T^*M \to TM$ does exist. Given a smooth function $f : M \to \mathbb{R}$, we therefore define the gradient of f as

$$\nabla f = \ell^{-1}(\mathrm{d}f).$$

In particular, note that

$$\mathrm{d}f(p) \le F^*(\mathrm{d}f)F(p) = F(\nabla f)F(p) \quad \forall f \in C^1(M),\ p \in TM,$$
$$\mathrm{d}f(p) = g_{\nabla f}(\nabla f, p) \ \text{ on } \mathcal{R}_f = \left\{x : \mathrm{d}_x f \neq 0\right\}, \text{for all } p \in TM.$$

Following [56], given a smooth function f we define its Hessian Hess f on $\mathcal{R}_f$ by

$$\operatorname{Hess} f(V,W) \doteq VW(f) - \nabla_V^{\nabla f} W(f), \quad \text{for all } V, W \in T\mathcal{R}_f.$$

It is easy to see that Hess f is symmetric and can be rewritten as

$$\mathrm{Hess}\, f(V,W) = g_{\nabla f}\left(\nabla_V^{\nabla f}\nabla f, W\right).$$

An alternative construction is proposed in [52], where the Hessian of f is defined as the map

$$D^2 f : TM \to \mathbb{R}, \qquad D^2 f(p) \doteq \frac{d^2}{ds^2}(f\circ\gamma)_{|_{s=0}},$$

with $\gamma : (-\varepsilon, \varepsilon) \to M$ the geodesic satisfying $\gamma'(0) = p$. In [56], the authors point out that

$$D^2 f(V) \equiv \mathrm{Hess}\, f(V,V), \quad \text{for all } V \in T\mathcal{R}_f.$$

2.1 Forward and backward completeness

For $x_0, x_1 \in M$, denote by $\Gamma(x_0, x_1)$ the collection of all piecewise smooth curves $\gamma : [a,b] \to (M,F)$ with $\gamma(a) = x_0$ and $\gamma(b) = x_1$. The distance $\mathrm{d} : M \times M \to [0,\infty)$ is defined by

$$\mathrm{d}(x_0,x_1) \doteq \inf_{\Gamma(x_0,x_1)} L(\gamma), \qquad \text{with } L(\gamma) := \int_a^b F(\gamma'(t))\mathrm{d}t$$

the length of γ. Despite d is not a metric, the space (M,d) satisfies the first two axioms of a metric space:

1. $\mathrm{d}(x_0,x_1) \ge 0$, with equality holding iff $x_0 = x_1$.
2. $\mathrm{d}(x_0,x_2) \le \mathrm{d}(x_0,x_1) + \mathrm{d}(x_1,x_2)$.

The symmetry $\mathrm{d}(x_0,x_1) = \mathrm{d}(x_1,x_0)$ is satisfied whenever the Finsler structure F is absolutely homogeneous, that is $F(\lambda p) = \lambda F(p)$ for every $\lambda \in \mathbb{R}$. In this case, (M,d) is a genuine metric space.

For $\bar{x} \in M$ fixed, and $r > 0$, we define on $T_{\bar{x}}M$ the tangent balls and spheres of radius r

$$B_{\bar{x}}(r) := \left\{p \in T_{\bar{x}}M : F(\bar{x},p) < r\right\}, \quad S_{\bar{x}}(r) := \left\{p \in T_{\bar{x}}M : F(\bar{x},p) = r\right\},$$

and the corresponding *forward metric balls and spheres*

$$\mathcal{B}^+_{\bar{x}}(r) := \left\{x \in M : \mathrm{d}(\bar{x},x) < r\right\}, \quad \mathcal{S}^+_{\bar{x}}(r) := \left\{x \in M : \mathrm{d}(\bar{x},x) = r\right\}.$$

The associated backward balls and spheres

$$\mathcal{B}^-_{\bar{x}}(r) := \left\{x \in M : \mathrm{d}(x,\bar{x}) < r\right\}, \quad \mathcal{S}^-_{\bar{x}}(r) := \left\{x \in M : \mathrm{d}(x,\bar{x}) = r\right\}$$

coincide with the forward balls of the dual Finsler structure $\widetilde{F}$. As proved in Section 6.2 C of [10], the topology of the underlying manifold and that generated by the forward balls coincide. Hence we can state that a sequence $x_i \to x$ in M if, given any open set $O \ni x$, there is a positive integer N (depending on O) such that $x_i \in O$ whenever $i \ge N$. According to [10, Lemma 6.2.1], for a fixed point $x_0 \in M$ there exist an open neighbourhood U and a constant $\alpha > 1$, depending on x_0 and U, such that

$$\frac{1}{\alpha}\mathrm{d}(x_2,x_1) \le \mathrm{d}(x_1,x_2) \le \alpha\mathrm{d}(x_2,x_1) \qquad \forall\, x_1,x_2 \in U. \tag{13}$$

Therefore, the statements

$$x_i \to x, \qquad \mathrm{d}(x,x_i) \to 0, \qquad \mathrm{d}(x_i,x) \to 0$$

are equivalent. However, this is not the case in general for Cauchy sequences.

Definition 2.1. *A sequence* $\{x_i\}$ *in* M *is called a* forward (resp., backward) Cauchy sequence *if, for all* $\varepsilon > 0$*, there exists a positive integer* j_ε *(depending on* ε*) such that*

$$j_\varepsilon \le i < j \Longrightarrow \mathrm{d}(x_i, x_j) < \varepsilon \qquad [\mathit{resp.}, \mathrm{d}(x_j, x_i) < \varepsilon].$$

Definition 2.2. *A Finsler manifold* (M, F) *is said to be* forward complete *if every forward Cauchy sequence converges in* M*. It is said to be* backward complete *if every backward Cauchy sequence converges.*

A geodesic γ from $\bar{x}$ to x is a curve that is stationary for L. It can (and will henceforth) be reparametrized via an affine map to have constant velocity $F(\gamma') \equiv 1$. The exponential map $\exp_{\bar{x}}$ associates to $v \in T_{\bar{x}}M$ the value $\gamma_v(1)$ of the unique forward geodesic γ_v issuing from $\bar{x}$ with constant velocity $F(v)$. The following result summarizes the minimizing properties of short geodesics that we need.

Theorem 2.3. *Let* (M, F) *be a Finsler manifold. Then, for a given compact set* K*, there exists* $\varepsilon > 0$ *such that*

1) [10, pp. 126-127] The map

$$\exp \ : \ \left\{ v \in TK \ : \ F(v) < \varepsilon \right\} \to M, \qquad \exp(x, v) = \exp_x(v)$$

is a C^1*-diffeomorphism onto its image, and* C^∞ *outside of the zero section.*

Fix a point $\bar{x}$ *and suppose that, for some* $r, \varepsilon > 0$*,* $\exp_{\bar{x}}$ *is a* C^1*-diffeomorphism from the tangent ball* $B_{\bar{x}}(r + \varepsilon)$ *onto its image (we call these balls* regular*). Then:*

2) [10, Thm. 6.3.1] Each radial geodesic $\exp_{\bar{x}}(tv)$*,* $0 \le t \le r$*,* $F(\bar{x}, v) = 1$ *is the unique curve that minimizes distance among all piecewise* C^∞ *curves in* M *with the same endpoits.*

The corresponding behaviour of the distance function from (or towards) a fixed origin $\bar{x} \in M$ on small balls has been described in [52, Lemma 3.2.4], and in [56, Eq. (4.1)]. Summarizing, we have

Proposition 2.4. *[52, 56] Let* (M, F) *be a Finsler manifold, let* $r > 0$ *be such that* $\mathcal{B}^+_{\bar{x}}(r)$ *and* $\mathcal{B}^-_{\bar{x}}(r)$ *are regular geodesic balls. Then, the functions*

$$\varrho^+(y) = \mathrm{d}(\bar{x}, y), \qquad \varrho^-(y) = -\mathrm{d}(y, \bar{x})$$

are smooth on, respectively, $\mathcal{B}^+_{\bar{x}}(r)\backslash\{\bar{x}\}$ *and* $\mathcal{B}^-_{\bar{x}}(r)\backslash\{\bar{x}\}$*, and there they satisfy*

$$F(\nabla\varrho^\pm) = 1, \qquad \mathrm{Hess}\, \varrho^\pm(\nabla\varrho^\pm, \nabla\varrho^\pm) = 0.$$

Indeed, the identity $F(\nabla\varrho^\pm) = 1$ is proved in [52, Lemma 3.2.4], while for the Hessian identity we observe the following: if $\gamma \ : \ [0, \mathrm{d}(y, \bar{x})] \to \mathcal{B}^-_{\bar{x}}(r)$ is a geodesic from y to $\bar{x}$ with initial velocity $\nabla\varrho^-(y)$, then $\varrho^-(\gamma(t)) = -\mathrm{d}(\gamma(t), \bar{x}) = t - \mathrm{d}(y, \bar{x})$ and thus

$$\mathrm{Hess}\, \varrho^-(\nabla\varrho^-, \nabla\varrho^-) = \frac{\mathrm{d}^2}{\mathrm{d}t^2}\varrho^-(\gamma(t)) = 0.$$

Regarding the behaviour of long minimizing geodesics, we have the following Hopf-Rinow type theorem due to Cohn-Vossen [20] (cf. also [43, 44] for more general statements, also considering Finsler metrics constructed from Hamilton-Jacobi equations).

Theorem 2.5 ([20], see Theorem 6.6.1 in [10])**.** *Let (M, F) be a connected Finsler manifold. The following properties are equivalent:*

1. *(M, F) is forward complete.*

2. *(M, F) is forward geodesically complete, that is, every geodesic $\gamma(t)$, $a \leq t \leq b$, parametrized to have constant speed, can be extended to a geodesic defined on $a \leq t < \infty$.*

3. *For some/every $x \in M$, $\exp_x$ is defined on all of $T_x M$.*

4. *Every closed and forward bounded subset $K \subset M$ (in the sense that K is contained into some forward ball) is compact.*

Furthermore, if any of the above holds, then every pair of points in M can be joined by a minimizing geodesic.

3 Viscosity solutions

Hereafter, given a test function ϕ regular enough, with $\phi \prec_x u$ (resp., $\phi \succ_x u$) we mean that ϕ is defined in a neighbourhood of x, $\phi \leq u$ (resp. $\phi \geq u$) and $\phi(x) = u(x)$. We start by recalling the definition of subsolutions for the eikonal equations.

Definition 3.1. *Given $\Omega \subset M$ open and $G \in C(\Omega \times \mathbb{R})$, we say that*

1. *$u \in \mathrm{USC}(\Omega)$ is a viscosity subsolution of*

$$F(\nabla u) - G(x, u) = 0 \quad \textit{on } \Omega$$

if, for every $x \in \Omega$ and test function $\phi \succ_x u$ of class C^1 it holds $F(\nabla\phi) - G(x, \phi) \leq 0$ at x.

2. *$u \in \mathrm{USC}(\Omega)$ is a viscosity subsolution of*

$$G(x, u) - F(\nabla u) = 0 \quad \textit{on } \Omega$$

if, for every $x \in \Omega$ and test function $\phi \succ_x u$ of class C^1 it holds $G(x, \phi) - F(\nabla\phi) \leq 0$ at x.

Next, for $\phi \in C^2(\Omega)$ we define

$$\Delta_\infty^{N,+}\phi(x) = \begin{cases} \mathrm{Hess}\,\phi\left(\frac{\nabla\phi}{F(\nabla\phi)}, \frac{\nabla\phi}{F(\nabla\phi)}\right), & \text{if } \mathrm{d}_x\phi \neq 0, \\ \max\left\{D^2\phi(p,p) \,:\, F(p) = 1\right\}, & \text{if } \mathrm{d}_x\phi = 0. \end{cases}$$

and

$$\Delta_\infty^{N,-}\phi(x) = \begin{cases} \mathrm{Hess}\,\phi\left(\frac{\nabla\phi}{F(\nabla\phi)}, \frac{\nabla\phi}{F(\nabla\phi)}\right), & \text{if } \mathrm{d}_x\phi \neq 0, \\ \min\left\{D^2\phi(p,p) \,:\, F(p) = 1\right\}, & \text{if } \mathrm{d}_x\phi = 0. \end{cases}$$

Definition 3.2. *Let $\Omega \subset M$ be open, and let $f : \mathbb{R} \times T^*\Omega \to \mathbb{R}$ be a continuous function (the dependence of f on $x \in \Omega$ is implicit when writing $T^*\Omega$).*

1. *A function $u \in \mathrm{USC}(\Omega)$ is said to solve $\Delta_\infty^N u \geq f(u, \mathrm{d}u)$*

- *in the* viscosity sense *if, for every $x \in \Omega$ and every test function $\phi \succ_x u$ of class C^2,*
$$\Delta_\infty^{N,+}\phi \geq f(\phi(x), \mathrm{d}\phi(x));$$
- *in the* barrier sense *if, for every $x \in \Omega$, there exists $u_\varepsilon \in C^2$ with $u_\varepsilon \prec_x u$ and*
$$\Delta_\infty^{N,+}u_\varepsilon \geq f(u_\varepsilon(x), \mathrm{d}u_\varepsilon(x)) - \varepsilon.$$

In these cases, we also say that u is a subsolution (in the viscosity/barrier sense).

2. *A function $u \in \mathrm{LSC}(\Omega)$ is said to solve $\Delta_\infty^N u \leq f(u, \mathrm{d}u)$*
   - *in the* viscosity sense *if, for every $x \in \Omega$ and every test function $\phi \prec_x u$ of class C^2,*
   $$\Delta_\infty^{N,-}\phi \leq f(\phi(x), \mathrm{d}\phi(x));$$
   - *in the* barrier sense *if, for every $x \in \Omega$, there exists $u_\varepsilon \in C^2$ with $u_\varepsilon \succ_x u$ and*
   $$\Delta_\infty^{N,-}u_\varepsilon \leq f(u_\varepsilon(x), \mathrm{d}u_\varepsilon(x)) + \varepsilon.$$

   In these cases, we also say that u is a supersolution (in the viscosity/barrier sense).

3. *A function $u \in C(\Omega)$ is said to solve*
$$\Delta_\infty^N u = f(u, \mathrm{d}u) \quad \text{on } \Omega \tag{14}$$
(in the viscosity/barrier sense) if it is both a subsolution and a supersolution.

Remark 3.3. If u is a subsolution (resp. a supersolution) in the barrier sense, and f is continuous, then u is also a subsolution (supersolution) in the viscosity sense. However, the converse is not necessarily true.

In the following proposition we state useful properties satisfied by ∞-Laplacian subsolutions, that in our needed generality (the operator is discontinuous) can be found in [30, Thm. 2.6] and [41, Prop. 3.7].

Proposition 3.4. *Let $\Omega \subset M$ be a bounded subset and $f \in C(\mathbb{R} \times T^*\Omega)$.*

i) If $u, v \in \mathrm{USC}(\Omega)$ are subsolutions of (14)*, then* $\max\{u, v\}$ *is also a subsolution of* (14)*.*

ii) (Stability) *If $\{u_k\} \subset \mathrm{USC}(\Omega)$ is a sequence of* viscosity *subsolutions of* (14)*, and $u_k \to u$ converges locally uniformly in Ω, then u is also a viscosity subsolution of* (14)*.*

3.1 Calabi's trick

We begin with a chain rule for the ∞-Laplacian. Let $\eta \in C^2(\mathbb{R})$ and $\phi \in C^2(\Omega)$, where $\Omega \subset M$ is an open set. Since the function $w = \eta \circ \phi$ solves

$$\Delta_\infty^{N,\pm} w = \eta''(\phi)F^2(\nabla\phi) + \eta'(\phi)\Delta_\infty^{N,\pm}\phi \qquad \text{on } \Omega^* = \Big\{ x \in \Omega \, : \, \eta'(\phi(x)) > 0 \Big\}, \tag{15}$$

a direct check shows the following

Proposition 3.5. *Let $u \in \mathrm{USC}(\Omega)$ (resp., $\mathrm{LSC}(\Omega)$) be a subsolution (resp., a supersolution) of* (14)*, and let $\eta \in C^2(\mathbb{R})$. On the set $\Omega^* = \{x \in \Omega \, : \, \eta'(u) > 0\}$, the function $w = \eta \circ u$ is a viscosity subsolution (resp., supersolution) of*

$$\Delta_\infty^N w = \eta''(u)F^2(\nabla u) + \eta'(u)f(u, \mathrm{d}u).$$

The following Lemma is a form of the classical Calabi's trick [15] adapted to the Finsler setting. By slightly modifying the original argument, we are able to avoid the assumption that the underlying manifold be forward complete, a fact that will be important in what follows.

Lemma 3.6 (Calabi's trick)**.** *Let (M,F) be a Finsler manifold, fix $\bar{x} \in M$ and define*

$$\varrho^+(y) = \mathrm{d}(\bar{x}, y), \qquad \varrho^-(y) = -\mathrm{d}(y, \bar{x}) \qquad \forall\, y \in M.$$

Let $x \in M \backslash \{\bar{x}\}$. Then, for every $\varepsilon > 0$ small enough there exist functions $\varrho_\varepsilon^+, \varrho_\varepsilon^-$ satisfying the following properties:

$$\begin{cases} \varrho_\varepsilon^+, \varrho_\varepsilon^- \quad \text{are smooth in a neighbourhood } U_\varepsilon \text{ of } x, \\ \varrho_\varepsilon^+ \succ_x \varrho^+, \quad \varrho_\varepsilon^- \prec_x \varrho^- \\ F(\nabla \varrho_\varepsilon^\pm) = 1, \quad \operatorname{Hess} \varrho_\varepsilon^\pm \left(\nabla \varrho_\varepsilon^\pm, \nabla \varrho_\varepsilon^\pm\right) = 0 \ \text{ on } \ U_\varepsilon. \end{cases} \tag{16}$$

In particular, for every $\eta \in C^2(\mathbb{R})$, the functions $w_\varepsilon^\pm = \eta(\varrho_\varepsilon^\pm)$ satisfy

$$F(\nabla w_\varepsilon^\pm) = \eta'(\varrho_\varepsilon^\pm), \quad \Delta_\infty^{N,\pm} w_\varepsilon^\pm = \eta''(\varrho_\varepsilon^\pm) \qquad \text{on} \quad U^* \doteq \{x \in U_\varepsilon \, : \, \eta'(\varrho_\varepsilon^\pm) > 0\}. \tag{17}$$

Proof. We first prove the statement for ϱ^+. Fix a small $\varepsilon > 0$ in such a way that

(i) the backward geodesic ball $\mathcal{B}_x^-(2\varepsilon)$ is relatively compact.

(ii) for every $y \in \mathcal{B}_x^-(2\varepsilon)$, $\exp_y : B_y^+(2\varepsilon) \subset T_y M \to \mathcal{B}_y^+(2\varepsilon)$ is a diffeomorphism.

Choose $x_\varepsilon \in \mathcal{S}_x^-(\varepsilon)$ to be the minimum point of ϱ^+ restricted to $\mathcal{S}_x^-(\varepsilon)$, and define

$$\varrho_\varepsilon^+(y) \doteq \mathrm{d}(\bar{x}, x_\varepsilon) + \mathrm{d}(x_\varepsilon, y) \qquad \forall\, y \in M.$$

By the triangle inequality, $\varrho_\varepsilon^+ \geq \varrho^+$ on M. We claim that equality holds at $y = x$. Indeed, assume by contradiction that $\varrho_\varepsilon^+(x) = \varrho_+(x) + c_\varepsilon$ for some $c_\varepsilon > 0$. Let $\{\gamma_j\}$ be a sequence of unit speed curves from $\bar{x}$ to x with $L(\gamma_j) \leq \varrho^+(x) + j^{-1}$ and, for every j, define

$$t_j = \inf \Big\{ t \in [0, L(\gamma_j)] \ : \ \gamma_j\big((t_j, L(\gamma_j)]\big) \subset \mathcal{B}_x^-(\varepsilon) \Big\}.$$

Note that $x_j = \gamma(t_j) \in \mathcal{S}_x^-(\varepsilon)$. Then,

$$\begin{aligned} \mathrm{d}(\bar{x}, x) + \tfrac{1}{j} \quad &\geq \quad L(\gamma_j) = L\Big((\gamma_j)_{[0,t_j]}\Big) + L\Big((\gamma_j)_{[t_j, L(\gamma_j)]}\Big) \\ &\geq \quad \mathrm{d}(\bar{x}, x_\varepsilon) + \mathrm{d}(x_j, x) = \mathrm{d}(\bar{x}, x_\varepsilon) + \mathrm{d}(x_\varepsilon, x) > \mathrm{d}(\bar{x}, x) + c_\varepsilon, \end{aligned}$$

a contradiction if j is chosen to be large enough.

Having shown that ϱ_ε^+ touches ϱ^+ from above at x, by (ii) we deduce that ϱ_ε^+ is smooth on $\mathcal{U}_\varepsilon \doteq \mathcal{B}_{x_\varepsilon}^+(2\varepsilon) \backslash \{x_\varepsilon\}$, that is a neighbourhood of x. Moreover, by Proposition 2.4

$$F(\nabla \varrho_\varepsilon^+) = 1, \qquad \operatorname{Hess} \varrho_\varepsilon^+ \left(\nabla \varrho_\varepsilon^+, \nabla \varrho_\varepsilon^+\right) = 0 \qquad \text{on } \mathcal{U}_\varepsilon,$$

as required. The argument is analogous for the signed distance ϱ^-: we choose ε small enough to match

(i) the forward geodesic ball $\mathcal{B}_x^+(2\varepsilon)$ is relatively compact.

(ii) for every $y \in \mathcal{B}_x^+(2\varepsilon)$, $\exp_y : B_y^-(2\varepsilon) \subset T_y M \to \mathcal{B}_y^-(2\varepsilon)$ is a diffeomorphism.

Choose then $x_\varepsilon \in \mathcal{S}_x^+(\varepsilon)$ minimizing $-\varrho^- = \mathrm{d}(\cdot,\bar{x})$ on $\mathcal{S}_x^+(\varepsilon)$ and define ϱ_ε^- according to the identity

$$-\varrho_\varepsilon^-(y) := \mathrm{d}(y,x_\varepsilon) + \mathrm{d}(x_\varepsilon,\bar{x}) \ge -\varrho^-(y) \qquad \forall\, y \in M.$$

With the same argument as above, we can show that $\varrho_\varepsilon^- \prec_x \varrho^-$, and the third condition in (16) follows from Proposition (2.4) as well. To conclude, on U^* it holds $F(\nabla w_\varepsilon^\pm) = \eta'(\varrho_\varepsilon^\pm)F(\nabla\varrho_\varepsilon^\pm) = \eta'(\varrho_\varepsilon^\pm)$, while from equation (15),

$$\Delta_\infty^{N,\pm} w_\varepsilon^\pm \;=\; \eta''(\varrho_\varepsilon^\pm)F^2(\nabla\varrho_\varepsilon^\pm) + \eta'(\varrho_\varepsilon^\pm)\Delta_\infty^{N,\pm}\varrho_\varepsilon^\pm = \eta''(\varrho_\varepsilon^\pm).$$

□

Corollary 3.7. *Let (M,F) be a Finsler manifold, and $\eta \in C^2(\mathbb{R})$. Fix $\bar{x} \in M$ and consider the signed distance functions*

$$\varrho^+(\cdot) = \mathrm{d}(\bar{x},\cdot), \qquad \varrho^-(\cdot) = -\mathrm{d}(\cdot,\bar{x}).$$

Then, $v := \eta(\varrho^+)$ is a viscosity supersolution of $F(\nabla v) - \eta'(\varrho^+) = 0$ on $\{\eta'(\varrho^+) > 0\}\backslash\{\bar{x}\}$ (that is, $F(\nabla\phi) - \eta'(\varrho^+) \ge 0$ holds at x whenever $\phi \prec_x v$), and there it satisfies

$$\Delta_\infty^N v \le \eta''(\varrho^+)$$

in the barrier sense. Similarly, the function $u := \eta(\varrho^-)$ is a viscosity subsolution of $F(\nabla u) - \eta'(\varrho^+) = 0$, and it satisfies

$$\Delta_\infty^N u \ge \eta''(\varrho^-)$$

in the barrier sense on $\{\eta'(\varrho^-) > 0\}\backslash\{\bar{x}\}$.

Proof. We will just prove it for ϱ^+. Let ϱ_ε^+ be defined as in Lemma 3.6 and smooth in a neighbourhood U_ε. Up to reducing ε, we can further assume that $\eta'(t) > 0$ for every $t \in [\varrho^+(y), \varrho_\varepsilon^+(y)]$ and $y \in U_\varepsilon$. Therefore, $v_\varepsilon \doteq \eta(\varrho_\varepsilon^+) \succ_x v$ and

$$F(\nabla v_\varepsilon) = \eta'(\varrho_\varepsilon^+) = \eta'(\varrho^+), \qquad \Delta_\infty^{N,-} v_\varepsilon = \eta''(\varrho_\varepsilon^+) = \eta''(\varrho^+) \qquad \text{at } x.$$

If $\phi \prec_x v$, then $\nabla\phi(x) = \nabla v_\varepsilon(x)$ and thus $F(\nabla\phi) - \eta'(\varrho^+) = 0$ at x.

□

4 Comparison with g-cones and Lipschitz regularity

In this section, we will consider bounded sub-and supersolutions of the equation

$$\Delta_\infty^N u = g(u) \qquad \text{on } \Omega \Subset M,$$

where g is a function whose restriction to $[u_*, u^*]$ is non-decreasing and continuous, and $u_* = \inf_\Omega u$, $u^* = \sup_\Omega u$.

For given $b \ge 0$, consider a solution η_b of

$$\begin{cases} \eta_b''(t) = g(\eta_b(t)) & \text{on a maximal interval } [0,T), \\ \eta_b(0) = u_*, \quad \eta_b'(0) = b. \end{cases} \tag{18}$$

Multiplying the equation by $2\eta'$ and integrating we deduce

$$[\eta_b'(t)]^2 - b^2 = G\big(\eta_b(t)\big), \qquad \text{where } G(s) = 2\int_{u_*}^{s} g(\sigma)\mathrm{d}\sigma. \tag{19}$$

If

$$b > \sqrt{\max\{-G_*, 0\}}, \tag{20}$$

where $G_* \doteq \inf_{[u_*,u^*]} G$, then $\eta_b' > 0$ and a second integration shows that η_b is implicitly defined by the identity

$$t = \int_{u_*}^{\eta_b(t)} \frac{\mathrm{d}s}{\sqrt{b^2 + G(s)}} \qquad \text{on } [0,T). \tag{21}$$

In particular, note that the family $\{\eta_b\}$ is increasing in b, whenever it is valued on $[u_*, u^*]$.

Given $a \in [u_*, u^*]$ we define

$$R_b(a) \doteq \inf\Big\{ t \in [0,T) \ : \ \eta_b(t) \geq a \Big\}.$$

This constant encompasses the non translational invariance character of the inhomogeneous equation, and it helps us to deduce "how far" the g-cones can be defined. In view of (19), for any values $u_* \leq a_1 < a_2 \leq u^*$ we have

$$\|\eta_b'\|_{L^\infty(R_b(a_1),R_b(a_2))} \leq \sqrt{b^2 + 2\int_{a_1}^{a_2} g_+} \leq \sqrt{b^2 + 2\int_{u_*}^{u^*} g_+}. \tag{22}$$

Remark 4.1. We recall that when the function g is constant, let us say $g \equiv c$ for some $c \in \mathbb{R}$, the solutions of (18) are the quadratic functions $\eta_b(t) = u_* + bt + \frac{c}{2}t^2$ considered in [46, 36, 6, 41].

Remark 4.2. If $g \geq 0$ on $[u_*, u^*]$, we will also consider the limit case of (21) for $b = 0$. Under the validity of the Keller-Osserman condition

$$\int_{u_*^+} \frac{\mathrm{d}s}{\sqrt{G(s)}} < \infty, \tag{KO}$$

uniqueness for (18) does not hold, and we select η_0 as being the one defined by the limit identity

$$t = \int_{u_*}^{\eta_0(t)} \frac{\mathrm{d}s}{\sqrt{G(s)}} \qquad \text{on } [0,T).$$

If (KO) fails, necessarily $g(u_*) = 0$ and the only solution of (18) with $b = 0$ is the function $\eta_0 \equiv u_*$. In this case, we set $R_0(a) \doteq +\infty$ for every $a \in (u_*, u^*]$.

For $z \in M$ fixed, we define the *forward and backward g-cones* centered at z as being, respectively,

$$\begin{aligned} C^+_{z,b}(w) &= \eta_b\big(\mathrm{d}(z,w) + R_b(u(z))\big) \qquad \text{on } \mathcal{B}^+_z\big(R_b(u^*) - R_b(u(z))\big), \\ C^-_{z,b}(w) &= \eta_b\big(R_b(u(z)) - \mathrm{d}(w,z)\big) \qquad \text{on } \mathcal{B}^-_z\big(R_b(u(z))\big). \end{aligned}$$

Example 4.3. For instance, if $g = 0$,

$$C^+_{z,b}(w) = u(z) + b\mathrm{d}(z,w), \qquad C^-_{z,b}(w) = u(z) - b\mathrm{d}(w,z)$$

are the standard forward and backward cones. If $g \equiv c \neq 0$, then

$$\begin{aligned} C^+_{z,b}(w) &= u(z) + \big(b + cR_b(u(z))\big)\mathrm{d}(z,w) + \tfrac{c}{2}\mathrm{d}(z,w)^2 \qquad \text{on } \mathcal{B}^+_z\big(R_b(u^*) - R_b(u(z))\big), \\ C^-_{z,b}(w) &= u(z) - \big(b + cR_b(u(z))\big)\mathrm{d}(w,z) + \tfrac{c}{2}\mathrm{d}(w,z)^2 \qquad \text{on } \mathcal{B}^-_z\big(R_b(u(z))\big). \end{aligned}$$

Since $\eta_b' > 0$ on $(0, R_b(u^*))$, because of Corollary 3.7, $C^+_{z,b}$ and $C^-_{z,b}$ satisfy, respectively,

$$\begin{cases} \Delta_\infty C^+_{z,b} \le g(C^+_{z,b}) & \text{on } \mathcal{B}^+_z\big(R_b(u^*) - R_b(u(z))\big)\backslash\{z\}, \\ C^+_{z,b}(z) = u(z), \\ C^+_{z,b} = u^* & \text{on } \mathcal{S}^+_z\big(R_b(u^*) - R_b(u(z))\big), \end{cases}$$

and

$$\begin{cases} \Delta_\infty C^-_{z,b} \ge g(C^-_{z,b}) & \text{on } \mathcal{B}^-_z\big(R_b(u(z))\big)\backslash\{z\}, \\ C^-_{z,b}(z) = u(z), \\ C^-_{z,b} = u_* & \text{on } \mathcal{S}^-_z\big(R_b(u(z))\big). \end{cases}$$

Extend $C^+_{z,b}$ and $C^-_{z,b}$ outside of the respective domains by setting them equal to, respectively, u^* and u_*, and call the resulting extensions $\bar{C}^+_{z,b}$ and $\bar{C}^-_{z,b}$. Note that the extensions are Lipschitz continuous on the entire M, and in view of (22) they satisfy

$$\mathrm{Lip}(\bar{C}^+_{z,b}, M) \le \sqrt{b^2 + 2\int_{u_*}^{u^*} g_+(s)\mathrm{d}s}, \qquad \mathrm{Lip}(\bar{C}^-_{z,b}, M) \le \sqrt{b^2 + 2\int_{u_*}^{u^*} g_+(s)\mathrm{d}s}. \tag{23}$$

Our next result extends the celebrated comparison with cones theorem (cf. [22, 18, 36, 41] and references therein) for g-cones.

Theorem 4.4. *Let $\Omega \subset M$ be a bounded open set.*

i) Suppose that $u \in \mathrm{USC}(\overline{\Omega}) \cap L^\infty(\Omega)$ satisfies

$$\Delta^N_\infty u \ge g(u) \quad in\ \ \Omega, \tag{24}$$

and assume

$g \in C(u(\overline{\Omega}))$ be non-decreasing, and b satisfy (20).

Then, for any relatively compact, open set $K \subset \Omega$, and any forward g-cone $\bar{C}^+_{z,b}$ centered at $z \in \Omega\backslash K$, we have

$$u \le \bar{C}^+_{z,b} \quad on\ \partial K \quad \Longrightarrow \quad u \le \bar{C}^+_{z,b} \quad on\ \overline{K}.$$

ii) Suppose that $v \in \mathrm{LSC}(\overline{\Omega}) \cap L^\infty(\Omega)$ satisfies

$$\Delta^N_\infty v \le g(v) \quad in\ \ \Omega, \tag{25}$$

and assume

$g \in C(v(\overline{\Omega}))$ be non-decreasing, and b satisfy (20).

Then, for any relatively compact, open set $K \subset \Omega$ and any backward g-cone $\bar{C}^-_{z,b}$ centered at $z \in \Omega\backslash K$, we have

$$v \ge \bar{C}^-_{z,b} \quad on\ \partial K \quad \Longrightarrow \quad v \ge \bar{C}^-_{z,b} \quad on\ \overline{K}.$$

Proof. The argument follows the standard comparison strategy. For *i*), first observe that the statement is obvious if u is constant. We argue by contradiction and assume that $\gamma := \max_{\overline{K}}(u - \bar{C}^+_{z,b}) > 0$. For $\varepsilon > 0$ small enough we define

$$\phi_\varepsilon(t) = \eta_b(t + R_b(u(z))) - \frac{\varepsilon}{2}t^2,$$

and set $\varrho^+(x) = \mathrm{d}(z,x)$. Up to reducing ε, we can assume that

$$
\begin{aligned}
&\gamma_\varepsilon \doteq \max_{\overline{K}}(u - \phi_\varepsilon(\varrho^+)) > \max\left\{\tfrac{\gamma}{2}, \max_{\partial K}(u - \phi_\varepsilon(\varrho^+))\right\}, \\
&\phi_\varepsilon' > 0 \quad \text{on } [0, R_b(u^*)],
\end{aligned}
\tag{26}
$$

where the second line follows from the strict inequality in (20). Let $x_0 \in \mathrm{Int}(K)$ realize γ_ε, and note that $\phi_\varepsilon(\varrho^+) < u^*$ in a sufficiently small neighbourhood of x_0. Choose $\varrho_\varepsilon^+ \succ_{x_0} \varrho^+$ as in Lemma 3.6, and reduce ε to satisfy $\varepsilon(\rho_\varepsilon^+)^2 < \gamma$. By construction, $\gamma_\varepsilon + \phi_\varepsilon(\varrho_\varepsilon^+) \succ_{x_0} u$ and therefore, at the point x_0,

$$
g\big(\frac{\gamma}{2} + \phi_\varepsilon(\varrho_\varepsilon^+)\big) \le g\big(\gamma_\varepsilon + \phi_\varepsilon(\varrho_\varepsilon^+)\big) \le \Delta_\infty^{N,-}\big(\gamma_\varepsilon + \phi_\varepsilon(\varrho_\varepsilon^+)\big)
$$

On the other hand, by Lemma 3.6

$$
\Delta_\infty^{N,-}\big(\gamma_\varepsilon + \phi_\varepsilon(\varrho_\varepsilon^+)\big) = \phi_\varepsilon''(\varrho_\varepsilon^+) = g\big(\phi_\varepsilon(\varrho_\varepsilon^+) + \frac{\varepsilon}{2}(\varrho_\varepsilon^+)^2\big) - \varepsilon < g\big(\phi_\varepsilon(\varrho_\varepsilon^+) + \frac{\gamma}{2}\big),
$$

yielding to a contradiction. Case *ii*) follows similarly. □

When g is constant, with the same argument we deduce the following comparison with quadratic cones, well-known in the Riemannian setting (cf. [35, 41]), and a related local Lipschitz regularity result. For $z \in \Omega$ we set

$$
\mathrm{d}^+(z) \doteq \sup\big\{r > 0 \,:\, \mathcal{B}_z^+(r) \Subset \Omega\big\}, \qquad \mathrm{d}^-(z) \doteq \sup\big\{r > 0 \,:\, \mathcal{B}_z^-(r) \Subset \Omega\big\},
$$

and

$$
\delta_\Omega^+(z) \doteq \max\big\{\mathrm{d}(z,w) \,:\, w \in \overline{\Omega}\big\}, \qquad \delta_\Omega^-(z) \doteq \max\big\{\mathrm{d}(w,z) \,:\, z \in \overline{\Omega}\big\}.
$$

Corollary 4.5. *Let $\Omega \subset M$ be a bounded open set, and let $c \in \mathbb{R}$.*

i) Suppose $u \in \mathrm{USC}(\Omega) \cap L^\infty(\Omega)$ solves

$$
\Delta_\infty^N u \ge c \quad \text{in } \Omega.
$$

Then, for any relatively compact, open set $K \subset \Omega$, and any forward quadratic cone $C_{z,b}^+$ centered at $z \in \Omega\backslash K$, and $b + cR_b(u(z)) \ge c_-\delta_K^+(z)$, we have

$$
\max_{\overline{K}}\left(u - C_{z,b}^+\right) = \max_{\partial K}\left(u - C_{z,b}^+\right).
$$

Moreover, for every $r \in (0, \mathrm{d}^+(z))$ and every $w \in \mathcal{B}_z^+(r)$ it holds

$$
\frac{u(w) - u(z)}{\mathrm{d}(z,w)} \le \max\left\{c_- r, \frac{c_-}{2}r + \sup_{\xi \in \mathcal{S}_z^+(r)} \frac{u(\xi) - u(z)}{r}\right\} + \frac{c_-}{2}\mathrm{d}(z,w). \tag{27}
$$

ii) Suppose $v \in \mathrm{LSC}(\Omega)$ satisfies

$$
\Delta_\infty^N v \le c \quad \text{in } \Omega.
$$

For any relatively compact, open set $K \subset \Omega$ and any backward quadratic cone $C_{z,b}^-$ centered at $z \in \Omega\backslash K$, and $b + cR_b(u(z)) \ge c_+\delta_K^-(z)$, we have

$$
\min_{\overline{K}}\left(v - C_{z,b}^-\right) = \min_{\partial K}\left(v - C_{z,b}^-\right). \tag{28}
$$

Moreover, for every $r \in (0, \mathrm{d}^-(z))$ and every $w \in \mathcal{B}_z^-(r)$ it holds

$$\frac{v(z) - v(w)}{\mathrm{d}(w, z)} \leq \max\left\{ c_+ r, \frac{c_+}{2} r + \sup_{\xi \in \mathcal{S}_z^-(r)} \frac{v(z) - v(\xi)}{r} \right\} + \frac{c_+}{2} \mathrm{d}(w, z).$$

In particular, u and v are locally Lipschitz.

Proof. To prove (27) and (28) we just compare u and v with the cones

$$C_{z,b}^+(w) = u(z) + (b + R_b(u(z)))\mathrm{d}(z, w) + \frac{c}{2} \mathrm{d}(z, w)^2,$$

and

$$C_{z,b}^-(w) = u(z) - (b + R_b(u(z)))\mathrm{d}(w, z) + \frac{c}{2} \mathrm{d}(w, z)^2,$$

either on K or, respectively, on the balls $\mathcal{B}_z^+(r)$ and $\mathcal{B}_z^-(r)$. The restrictions $b + cR_b(u(z)) \geq c_- \delta_K^+(z)$ and $b + cR_b(u(z)) \geq c_+ \delta_K^-(z)$ enable us to apply Corollary 3.7 on the entire K. □

Remark 4.6. Corollary 4.5 shall be compared with Theorems 4.1 and 4.7 in [41]. We remark that our quadratic cones are parametrized in a different way.

This comparison with cones theory allows us to assert the validity of the following strong finite maximum principle which will be crucial in the proof of our main results.

Corollary 4.7. *Let $\Omega \subset M$ be a connected open subset. If $u \in \mathrm{USC}(\Omega)$ is a subsolution of $\Delta_\infty^N u = 0$ in Ω, then u cannot attain an interior maximum point, unless u is constant. If $v \in \mathrm{LSC}(\Omega)$ is a supersolution of $\Delta_\infty^N v = 0$ in Ω, then v cannot attain a interior minimum point, unless v is constant.*

Proof. We only describe the proof for subsolutions, since the other case follows along similar lines. Let $y \in \Omega$ be a maximum point, fix a forward ball $\mathcal{B}_y^+(r) \subset \Omega$ and $\alpha > 1$ as in (13) for $U = \mathcal{B}_y^+(r)$. Let $z \in \mathcal{B}_y^+(\alpha^{-1}r/2)$, and note that the triangle inequality and (13) imply $y \in \mathcal{B}_z^+(r/2) \subset \mathcal{B}_y^+(r)$. Applying Corollary 4.8 on $\mathcal{B}_z^+(r)\backslash\{z\}$ to u and the forward linear cone

$$C_z^+(w) = u(z) + \frac{2(u(y) - u(z))}{r} \mathrm{d}(z, w),$$

we conclude that

$$0 \leq \Big(u(y) - u(z)\Big) \left(\frac{r}{2} - \mathrm{d}(z, y)\right) \leq 0,$$

hence u is constant on $\mathcal{B}_y^+(\alpha^{-1}r/2)$, and the conclusion follows by an open-closed argument. □

Another important consequence of Corollary 4.5 is the following comparison theorem for the homogeneous case. Its proof, for Euclidean space with its flat Riemannian metric, was first given by Jensen [31] with a delicate procedure (see also [9, 11]). A subsequent short and elegant argument has been provided by Armstrong and Smart [5], and in Appendix I below we describe the necessary changes to adapt their proof to the Finsler setting.

Theorem 4.8. *Let $\Omega \Subset M$ and assume that $u \in \mathrm{USC}(\overline{\Omega})$, $v \in \mathrm{LSC}(\overline{\Omega})$ satisfy*

$$\Delta_\infty^N u \geq 0, \quad \textit{and} \quad \Delta_\infty^N v \leq 0 \quad \textit{in the viscosity sense on } \Omega.$$

Then,

$$\max_{\overline{\Omega}}(u - v) = \max_{\partial\Omega}(u - v).$$

Comparison with standard linear cones is fundamental in the theory of the ∞-Laplace equation, and provides the bridge to show the equivalence between ∞-harmonicity and the absolutely minimizing Lipschitz property (see [9, 18, 21], and references therein).

Definition 4.9. *Let Ω be a proper subset of M. We say that $u \in \mathrm{Lip}(\Omega)$ is an absolutely minimizing Lipschitz function on Ω if, for all open subset $A \subset \Omega$,*

$$\mathrm{Lip}(u, A) = \mathrm{Lip}(u, \partial A).$$

As recalled in the introduction, a characterization of $\Delta_\infty^N u = g(u)$ in terms of certain absolutely minimizing properties seems still unavailable. In order to achieve a uniform, global Lipschitz regularity without using the completeness of M, we introduce the following

Definition 4.10. *Given $\Omega \subset M$, $u \in C(\overline{\Omega})$ and a compact subset $A \subset \overline{\Omega}$, we define the sliding slope*

$$b_A \doteq \inf \left\{ b > \sqrt{\max\{-G_*, 0\}} \ : \ \forall\, z \in A,\ \bar{C}^-_{z,b} \le u \le \bar{C}^+_{z,b} \ on \ A \right\}.$$

If the set is empty, we define $b_A \doteq +\infty$.

It is easy to see that $b_A < +\infty$ if and only if $u_{|A}$ is Lipschitz.

Example 4.11. *If $g = 0$, since $C^+_{z,b}(w) = u(z) + b\mathrm{d}(z, w)$ and $C^-_{z,b}(w) = u(z) - b\mathrm{d}(w, z)$ we have $b_A = \mathrm{Lip}(u, A)$.*

Remark 4.12. If $g(u(\overline{\Omega})) \ge 0$, the convexity of η solving (18) implies that the set

$$\left\{ b > 0 \ : \ \forall\, z \in A,\ \bar{C}^-_{z,b} \le u \le \bar{C}^+_{z,b} \text{ on } A \right\}$$

is the half-line (b_A, ∞).

Lemma 4.13. *If $g(u(\overline{\Omega})) \ge 0$ then*

$$b_A \le \mathrm{Lip}(u, A).$$

Proof. Let $b \doteq \mathrm{Lip}(u, A)$, so upward linear cones $L^+_{z,b} = u(z) + b\mathrm{d}(z, \cdot)$ and downward linear cones $L^-_{z,b} = u(z) - b\mathrm{d}(\cdot, z)$ can be slid along $z \in A$ remaining, respectively, above and below the graph of u on A. Since η is convex up until it reaches value u^*, a forward g-cone $\bar{C}^+_{z,b}$ lies above $L^+_{z,b}$ up until the latter reaches the value u^*, hence $\bar{C}^+_{z,b} \ge u$ on A. Again by the convexity of η, a downward g-cone $\bar{C}^-_{z,b}$ with vertex at $z \in A$ and slope b lies below the linear cone $L^-_{z,b}$ until the latter reaches value u_*, hence $\bar{C}^-_{z,b} \le u$ on A. By its very definition, $b_A \le b$. $\square$

We will state now our main result of this section, Theorem 1.2, in the following strengthened form:

Theorem 4.14. *Let $\Omega \Subset M$, and let $u \in C(\overline{\Omega})$ satisfy*

$$\Delta_\infty^N u = g(u) \qquad on \ \Omega,$$

where g is continuous, non-decreasing and non-negative on $u(\overline{\Omega})$. If u is Lipschitz on $\partial\Omega$, then $u \in \mathrm{Lip}(\overline{\Omega})$ and

$$\mathrm{Lip}(u, \Omega) \le \sqrt{b_{\partial\Omega}^2 + 2\int_{u_*}^{u^*} g(s)\mathrm{d}s}.$$

In particular,

$$\mathrm{Lip}(u, \Omega) \le \sqrt{\mathrm{Lip}(u, \partial\Omega)^2 + 2\int_{u_*}^{u^*} g(s)\mathrm{d}s}.$$

Proof. Pick $b > b_{\partial\Omega}$ and set for convenience

$$L_b = \sqrt{b^2 + 2\int_{u_*}^{u^*} g(s)\mathrm{d}s}.$$

For $x, y \in \overline{\Omega}$, it is sufficient to show that

$$u(x) \leq u(y) + L_b \mathrm{d}(y, x),$$

since the thesis follows by letting $b \downarrow b_{\partial\Omega}$. By Remark 4.12,

$$\forall\, z \in \partial\Omega, \quad \bar{C}_{z,b} \leq u \leq \bar{C}^+_{z,b} \quad \text{on } \partial\Omega,$$

thus comparison with g-cones implies $\bar{C}^-_{z,b} \leq u \leq \bar{C}^+_{z,b}$ on $\overline{\Omega}$, that is,

$$\bar{C}^-_{z,b}(w) \leq u(w) \leq \bar{C}^+_{z,b}(w) \qquad \text{for every } w \in \overline{\Omega},\ z \in \partial\Omega.$$

If $y \in \partial\Omega$, then setting $z = y$, $w = x$ and using (22) we get

$$u(x) \leq \bar{C}^+_{y,b}(x) \leq \bar{C}^+_{y,b}(y) + L_b \mathrm{d}(y, x) = u(y) + L_b \mathrm{d}(y, x).$$

On the other hand, if $x \in \partial\Omega$ and $y \in \overline{\Omega}$, setting $z = x$ and $w = y$ we deduce

$$u(y) \geq \bar{C}^-_{x,b}(y) \geq \bar{C}^-_{x,b}(x) - L_b \mathrm{d}(y, x) = u(x) - L_b \mathrm{d}(y, x).$$

It remains to investigate the case $x, y \in \Omega$. Choose

$$b' = \inf\left\{ h \geq 0 \ : \ u \geq \bar{C}^-_{x,h} \ \text{ on } \partial\Omega \right\}.$$

Since $\Delta^N_\infty u \geq 0$ on Ω, $u \in \mathrm{Lip}_{\mathrm{loc}}(\Omega)$. In particular, the set defining b' is non-empty, thus $b' < \infty$ and, by a compactness argument together with Remark 4.2, b' is attained. The compactness of $\partial\Omega$, and the fact that $\bar{C}^-_{x,k} \geq \bar{C}^-_{x,h}$ if $k \leq h$, guarantee the existence of $z_0 \in \partial\Omega$ such that $\bar{C}^-_{x,b'}(z_0) = u(z_0)$ and $C^-_{x,b'}(z) \leq u(z)$ for every $z \in \partial\Omega$. Therefore, by comparison

$$\bar{C}^-_{x,b'} \leq u \qquad \text{on } \overline{\Omega}.$$

We examine the cone $\bar{C}^+_{z_0,b}$. Since it lies above the graph of u, hence above $C^-_{z,b'}$, its initial slope at z_0 must be, at least, the slope of the solution $\eta_{u_*,b'}$ of the ODE corresponding to $C^-_{x,b'}$ at the point $R_{b'}(u(z_0))$. The latter is not smaller than the slope b' (because $\eta_{u_*,b'}$ is convex), therefore we infer the inequality

$$b \geq b'.$$

By comparison, $u \geq \bar{C}^-_{x,b'}$ on Ω, implying

$$\begin{aligned} u(y) &\geq u(x) - \mathrm{Lip}(\bar{C}^-_{x,b'}, M)\mathrm{d}(y, x) \\ &\geq u(x) - L_{b'}\mathrm{d}(y, x) \geq u(x) - L_b \mathrm{d}(y, x). \end{aligned}$$

This concludes the proof. □

5 Proof of Theorem 1.1

When the "some/every" alternative occurs in 3), 4), 6), 7), 8), we will always assume the weaker and prove the stronger. For instance, when considering implication 2) $\Rightarrow$ 4), we will show the validity of 4) for *every* choice of g as in the statement. On the other hand, in implication 4) $\Rightarrow$ 1), for instance, we will only assume the validity of 4) for *some* choice of g. In what follows, we set $u^* = \sup_M u$ and $u_* = \inf_M u$.

1) $\Rightarrow$ **2)**.
Suppose, by contradiction, that there exists a solution u of $\Delta_\infty^N u \ge 0$ on M with sublinear growth $u(x) = o(\varrho^+(x))$ as $\varrho^+(x) \to \infty$. Fix a compact set K. In view of the strong maximum principle, $u_K := \max_K u < u^*$. Because of Corollary 3.7, for every $\varepsilon > 0$ the function $w_\varepsilon := u_K + \varepsilon\varrho_+$ satisfies $\Delta_\infty^N w_\varepsilon \le 0$. Furthermore, our growth requirement on u implies that $u < w_\varepsilon$ outside of a relatively compact, open set U. The comparison theorem in Appendix I on $U \backslash K$ yields to

$$u \le w_\varepsilon = u_K + \varepsilon w_\varepsilon \qquad \text{on } U\backslash K, \text{ hence on } M\backslash K,$$

and letting $\varepsilon \to 0$ we infer $u \le u_K$ on M, contradiction.

2) $\Rightarrow$ **3)** is obvious, for every choice of such g.

2) $\Rightarrow$ **4)**.
By contradiction, assume that there exist $g \in C(\mathbb{R})$, and u satisfying

$$\begin{cases} \Delta_\infty^N u \ge g(u) \ge 0 & \text{on } \Omega, \\ \sup_\Omega u < +\infty \end{cases} \qquad \text{with} \qquad \sup_\Omega u > \sup_{\partial\Omega} u.$$

Note that $u \in \mathrm{Lip}_{\mathrm{loc}}(\Omega)$ because of Corollary 4.5, so choosing $\gamma \in (\sup_{\partial\Omega} u, \sup_\Omega u)$ the function

$$v := \begin{cases} \max\{\gamma, u\} & \text{on } \Omega, \\ \gamma & \text{on } M\backslash\Omega \end{cases}$$

is bounded, non-constant and coincides with γ in a neighbourhood of $\partial\Omega$, thus $\Delta_\infty^N v \ge 0$ on M by Proposition 3.4. This contradicts 2).

3) $\Rightarrow$ **1)** and **4)** $\Rightarrow$ **1)**.
We prove both of the implications with the same strategy, and split the proof only at the last step. Assume that either 3) or 4) holds for some choice of g. First, we redefine g on an interval, say $[0, 1]$ as follows: $g(t) \equiv g(1)$ for $t \ge 1$ and $g(t) = 0$ for $t \le 0$. In this way, the validity of 3) and 4) restricts to functions u valued in $[0, 1]$. Next, set

$$\bar{g}(t) = \sup_{s \le t} g(s).$$

Then, $\bar{g} \in C(\mathbb{R})$, $\bar{g} \ge g$, $\bar{g}(0) = 0$ and $\bar{g}$ is non-decreasing. Therefore, the validity of 3) or 4) for g (and $u \in [0, 1]$) implies its validity for $\bar{g}$, under the same restriction on u. Hence, up to replacing g with $\bar{g}$, we can assume that g be non-decreasing. Fix a point $x \in M$ and a small, forward regular ball $\mathcal{B}$ centered at x. Consider a smooth exhaustion $\{\Omega_j\} \uparrow M$ with $\mathcal{B} \Subset \Omega_j$ for each j. Set $A_j \doteq \Omega_j \backslash \overline{\mathcal{B}}$, and let u_j be a solution of

$$\begin{cases} \Delta_\infty u_j = g(u_j) & \text{on } A_j, \\ u_j = f_j & \text{on } \partial A_j, \end{cases} \tag{29}$$

where $f_j = 0$ on $\partial\mathcal{B}$ and $f_j = 1$ on $\partial\Omega_j$ (its existence follows from Perron method, using 0 as a subsolution and 1 as a supersolution, and is proved in Appendix II; note that $0 \le u_j \le 1$). Theorem 4.14 guarantees that

$$\mathrm{Lip}(u_j, A_j) \le \sqrt{b^2_{\partial A_j} + 2\int_0^1 g(s)\mathrm{d}s},$$

With $b_{\partial A_j}$ the sliding slope of ∂A_j. We claim that $\{b_{\partial A_j}\}$ is decreasing, hence uniformly bounded, as $j \to \infty$. Indeed, since $\partial\mathcal{B}$ separates M and $\Omega_j \Subset \Omega_{j+1}$, every curve from $x \in \partial\mathcal{B}$ to a point $y \in \partial\Omega_{j+1}$ must cross $\partial\Omega_j$. Therefore,

$$\mathrm{d}(\partial\mathcal{B}, \partial\Omega_{j+1}) \ge \mathrm{d}(\partial\mathcal{B}, \partial\Omega_j),$$

and thus any forward g-cone $\bar{C}^+_{x,b}$ that lies above 1 on $\partial\Omega_j$ (i.e., it satisfies $R_b(1) \le \mathrm{d}(\partial\mathcal{B}, \partial\Omega_j)$) also lies above 1 on $\partial\Omega_{j+1}$. Similarly, to every backward g-cone $\bar{C}^-_{y,b}$ that can be slid along $y \in \partial\Omega_j$ remaining below 0 on $\partial\mathcal{B}$, the cones $\bar{C}^-_{z,b}$ centered at $z \in \partial\Omega_{j+1}$ and with the same b remain below 0 on $\partial\mathcal{B}$. This suffices to conclude $b^2_{\partial A_{j+1}} \le b^2_{\partial A_j}$. Therefore, $\{u_j\}$ is equi-Lipschitz, say with constant L. Extend u_j with values 0 on $\mathcal{B}$ and 1 outside of Ω_j. Up to subsequences, $\{u_j\}$ converges locally uniformly to a Lipschitz limit $u_\infty \ge 0$. By Proposition 3.4, u_∞ satisfies $\Delta_\infty u_\infty = g(u_\infty)$ and $u_\infty = 0$ on $\partial\mathcal{B}$. We now exploit our assumptions. If 4) holds, applying the principle to u_∞ on $\Omega = M\setminus\overline{\mathcal{B}}$ we deduce $u_\infty \equiv 0$. On the other hand, if 3) holds, first extend u_∞ with $u_\infty \doteq 0$ on $\mathcal{B}$, and note that the resulting extension solves $\Delta_\infty u_\infty \ge g(u_\infty)$ on M. Apply then 3) to conclude that u_∞ is constant, hence $u_\infty \equiv 0$. To show the forward completeness of M, pick a unit speed geodesic $\gamma : [0, T) \to M$ issuing from the center o of $\mathcal{B}$, and assume by contradiction that $T < +\infty$. Consider the functions $w_j = u_j \circ \gamma$, and note that $w_j = 1$ after some $T_j < T$. From

$$\frac{w_j(t) - w_j(s)}{t - s} \le \frac{u_j(\gamma(t)) - u_j(\gamma(s))}{\mathrm{d}(\gamma(s), \gamma(t))} \le \mathrm{Lip}(u_j, M) \le L \qquad \forall\, 0 < s < t < T,$$

letting $t \to T^-$ we deduce

$$1 - w_j(s) \le L(T - s).$$

However, $w_j \to 0$ locally uniformly, a contradiction if s is chosen to be close enough to T.

5) $\Rightarrow$ **2)** is obvious, with the choice $g(u) = \lambda u^\theta_+$.

1) $\Rightarrow$ **5)**.

The argument follows the ideas in [4]. Let ϱ^+ be the forward distance from $o \in M$. For each $r > 0$ we define the function v_r on $\mathcal{B}^+_o(r) \subset M$ by $v_r(x) = \eta(\varrho^+(x))$, with

$$\eta(t) = \tau(\lambda, \theta)\left[t - r + \left(\frac{\sup_{\partial\mathcal{B}^+_o(r)} u}{\tau(\lambda, \theta)}\right)^{\frac{1-\theta}{2}}\right]_+^{\frac{2}{1-\theta}},$$

and

$$\tau(\lambda, \theta) = \sqrt[1-\theta]{\frac{\lambda(1-\theta)^2}{2(1+\theta)}}.$$

Note that $\eta \in C^2(\mathbb{R})$ since $\theta \in (0, 1)$. Using Corollary 3.7, v_r satisfies

$$\begin{cases} \Delta^N_\infty v_r \le \lambda (v_r)^\theta_+ & \text{on } \mathcal{B}^+_o(r) \qquad \text{in the barrier sense,} \\ v_r = \sup_{\partial\mathcal{B}^+_o(r)} u, & \text{on } \partial\mathcal{B}^+_o(r). \end{cases}$$

Since $u \le v_r$ on $\partial\mathcal{B}_o^+(r)$, and u is a subsolution of the above problem (in viscosity sense), we claim that $u \le v_r$ on $\mathcal{B}_o^+(r)$. In fact, if $u - v_r$ has a positive maximum c at $x \in \mathcal{B}_o^+(r)$, let $\varrho_\varepsilon^+ \succ_x \varrho^+$ be an upper barrier for ϱ^+ guaranteed by Calabi's trick. Then, $\phi := c + \eta(\varrho_\varepsilon^+) \succ_x u$ and thus

$$\lambda\phi_+^\theta \le \Delta_\infty^{N,+}\phi = \eta''(\varrho_\varepsilon^+) = \lambda\eta(\varrho_\varepsilon^+)_+^\theta < \lambda\phi_+^\theta \qquad \text{at } x,$$

contradiction. Next, by the growth assumption on u, we can find $0 < \delta < 1$ such that

$$\sup_{\partial\mathcal{B}_o^+(r)} u \le \delta\tau(\lambda,\theta)r^{\frac{2}{1-\theta}}.$$

Summarizing, we can write

$$u(x) \le \tau(\lambda,\theta)\left[\varrho^+(x) - \left(1 - \delta^{\frac{1-\theta}{2}}\right)r\right]_+^{\frac{2}{1-\theta}}.$$

Letting $r \to +\infty$ we deduce that $u \le 0$ on M. To conclude, we apply 1) $\Rightarrow$ 3) to obtain that u is constant.

1) $\Rightarrow$ **6**) and **1**) $\Rightarrow$ **7**).
Let $K \Subset M$ be compact, fix $o \in M$, $\varrho^+(x) = \mathrm{d}(o,x)$ and choose R large enough that $K \subset \mathcal{B}_o^+(R)$. For $r > R$, the functions

$$u_r(x) = \min\left\{-1 + \frac{R}{r}(\varrho_+ - R), 0\right\} \in \mathscr{L}(K,M)$$

satisfy

$$\mathrm{Lip}(u_r, M) = \frac{R}{r}, \qquad F(\nabla u_r) \le \frac{R}{r} \quad \text{a.e. on } M,$$

so letting $r \to \infty$ we deduce both 6) and 7).

7) $\Rightarrow$ **6**) for some compact K.
The implication follows from the inequality

$$\mathrm{Lip}(u, M) \le \|F(\nabla u)\|_\infty \qquad \forall\, x \in \mathrm{Lip}(M).$$

Indeed, for every unit speed curve $\gamma : [0,\ell] \to M$ joining x to y, and for every $u \in C^1(M)$, integrating the inequality $\mathrm{d}u(\gamma') \le F^*(\mathrm{d}u)F(\gamma') = F(\nabla u) \le \|F(\nabla u)\|_\infty$ on $[0,\ell]$ we infer

$$u(y) = u(x) + \int_0^\ell \mathrm{d}u(\gamma'(t))\mathrm{d}t \le u(x) + \|F(\nabla u)\|_\infty\ell.$$

Choosing ℓ such that $\ell = \mathrm{d}(x,y) + j^{-1}$, and letting $j \to \infty$, we deduce $u(y) \le u(x) + \|F(\nabla u)\|_\infty \mathrm{d}(x,y)$. The case $u \in \mathrm{Lip}(M)$ follows by approximation.

6) $\Rightarrow$ **1**).
Fix a compact set $K \subset M$ and a sequence of functions $\bar{u}_j \in \mathrm{Lip}_c(M)$ with $\mathrm{Lip}(\bar{u}_j, M) \to 0$ and $\bar{u}_j \le -1$ on K. Up to replacing $\bar{u}_j$ with $\max\{\bar{u}_j, 1\}$, we can assume that $-1 \le \bar{u}_j \le 0$ on M and $\bar{u}_j = -1$ on K. By Ascoli-Arzelá theorem, up to subsequences, $\bar{u}_j \to \bar{u}_\infty$ locally uniformly, for some $\bar{u}_\infty \in \mathrm{Lip}(M)$, and from $\mathrm{Lip}(\bar{u}_\infty, M) \le \liminf_j \mathrm{Lip}(\bar{u}_j, M) = 0$ we deduce that $\bar{u}_\infty = -1$ on M. Now, the proof concludes exactly as the one for 3) $\Rightarrow$ 1), up to defining $u_j = \bar{u}_j + 1$.

1) $\Rightarrow$ **8**).
By contradiction, if u is a subsolution of

$$G(u) - F(\nabla u) = 0 \quad \text{on } \Omega,$$

and $\sup_{\partial\Omega} u < \sup_\Omega u < \infty$, the function

$$v(x) = \int_0^{u(x)} \frac{\mathrm{d}s}{G(s)}$$

would be a subsolution of

$$\begin{cases} 1 - F(\nabla v) = 0 \quad \text{on } \Omega, \\ v_0 \doteq \sup_{\partial\Omega} v < \sup_\Omega v < \infty. \end{cases}$$

Let ϱ^+ be the forward distance from a fixed origin, and set $w_\varepsilon \doteq v_0 + \varepsilon\varrho^+$ for $\varepsilon \in (0,1)$. We claim that $v \le w_\varepsilon$ on Ω. Once this is shown, letting $\varepsilon \to 0$ we would have $v \le v_0$, which is absurd. Assume therefore that $U \doteq \{v > w_\varepsilon\}$ be non-empty. Since M is forward complete, $w_\varepsilon(x) \to +\infty$ as x diverges, thus U is relatively compact and does not meet $\partial\Omega$. Pick a point $x \in U$ where $u - w_\varepsilon$ attains a (positive) maximum value c, and let $\varrho^+_\varepsilon >_x \varrho^+$ be a barrier at x. Then, $\phi \doteq v_0 + c + \varepsilon\varrho^+_\varepsilon$ would touch v from above at x, that would imply $0 \ge 1 - F(\nabla\phi) = 1 - \varepsilon F(\nabla\varrho^+_\varepsilon) = 1 - \varepsilon$, contradiction.

8) $\Rightarrow$ **1**).
Let $0 < G \in C(\mathbb{R})$ such that 8) holds. We define

$$\hat{G}(t) = \min_{[0,t]} G(s).$$

Then, $\hat{G}$ is non-increasing and positive on $\mathbb{R}^+$, and from $\hat{G} \le G$ on $\mathbb{R}^+$ we deduce that 8) still holds, with $\hat{G}$ replacing G, provided that u be non-negative on Ω. Summarizing, we can assume that G is non-increasing on $\mathbb{R}^+$, up to restricting the validity of 5) to nonnegative u. Fix a small, regular forward ball $\mathcal{B} = \mathcal{B}^+_{x_0}(3\varepsilon)$, denote with $\widetilde{\nabla}$ the gradient induced by the dual Finsler structure $\widetilde{F}$, and define

$$\widetilde{G}(t) = G(-t).$$

We aim to prove the existence of a function satisfying

$$\begin{cases} w \in C(M\backslash\mathcal{B}), \quad w \le 0, \\ w(x) \to -\infty \;\text{ as } x \text{ diverges }, \\ w \text{ is a viscosity subsolution of } \;\widetilde{F}(\widetilde{\nabla} w) - \widetilde{G}(w) = 0 \;\text{ on }\; M\backslash\overline{\mathcal{B}}. \end{cases} \tag{30}$$

Here, the writing $w(x) \to -\infty$ as x diverges means that w has compact upper level sets in $M\backslash\mathcal{B}$. Once this is shown, we conclude that M must be forward complete as follows: set

$$h(x) \doteq \int_0^{w(x)} \frac{\mathrm{d}s}{\widetilde{G}(s)},$$

then $h \le 0$ and, since G is non-increasing, $h(x) \to -\infty$ as x diverges. Furthermore, h is a viscosity subsolution of $\widetilde{F}(\widetilde{\nabla}h) - 1 = 0$ on $M\backslash\mathcal{B}$. By Proposition 4.3 in [16], h is Lipschitz continuous in the pseudo-distance $\tilde{\mathrm{d}}$ induced by $\widetilde{F}$:

$$h(y) \le h(x) + L\tilde{\mathrm{d}}(x,y) = h(x) + L\mathrm{d}(y,x) \qquad \forall\, x, y \in M\backslash\mathcal{B}.$$

for some constant $L > 0$. Take a maximal, forward geodesic $\gamma : [0,T) \to M$ issuing from x_0, and suppose by contradiction that $T < +\infty$. Define $v(t) \doteq h(\gamma(t))$ on $[3\varepsilon, T)$. By assumption, $v(t) \to -\infty$ as $t \to T^-$. On the other hand,

$$v(t) \ge v(3\varepsilon) - L\mathrm{d}(\gamma(3\varepsilon), \gamma(t)) \ge v(3\varepsilon) + L(3\varepsilon - t),$$

contradiction.

The idea to prove the existence of w is inspired by [37, 40]. Let $\Omega_j \uparrow M$ be an increasing exhaustion of M by means of relatively compact open sets with smooth boundary, satisfying $\overline{\mathcal{B}} \Subset \Omega_1$. We will construct a sequence of functions $\{w_j\}$ such that

$$\begin{cases} w_j \in C(M\backslash\mathcal{B}), \qquad w_j \le 0 \ \text{ on } \ M\backslash\mathcal{B}, \ w_j > -1/2 \quad \text{on } \partial\mathcal{B} \\ w_{j+1} \le w_j \qquad \text{on } \ M\backslash\overline{\mathcal{B}}, \\ \|w_{j+1} - w_j\|_{L^\infty(\Omega_j\backslash\mathcal{B})} < 2^{-j}, \\ w_j \equiv -j \qquad \text{outside of some compact set } C_j, \\ w_j \text{ is a viscosity subsolution of } \ \widetilde{F}(\widetilde{\nabla} w_j) - \widetilde{G}(w_j) = 0 \ \text{ on } \ M\backslash\overline{\mathcal{B}}. \end{cases} \tag{31}$$

Once this is done, $\{w_j\}$ locally uniformly converges to some $w \in C(M\backslash\mathcal{B})$, and from $w \le w_j = -j$ outside of C_j we deduce that $w(x) \to -\infty$ as x diverges. By stability of viscosity solutions, w satisfies all of the properties in (30). Fix a sequence $\{\lambda_j\} \subset C(M)$ such that

$$\begin{aligned} &0 \ge \lambda_j \ge -1, \quad \lambda_j = 0 \ \text{ on } \mathcal{B}, \quad \lambda_j \equiv -1 \ \text{ on } \ M\backslash\Omega_j, \\ &\lambda_{j+1} \ge \lambda_j \ \text{ on } \ M, \quad \text{and} \quad \lambda_j \uparrow 0 \quad \text{locally uniformly on } \ M. \end{aligned}$$

We proceed inductively. Set $w_0 \equiv 0$ and define the forward balls $\mathcal{B}_1 = \mathcal{B}^+_{x_0}(\varepsilon)$ and $\mathcal{B}_2 = \mathcal{B}^+_{x_0}(2\varepsilon)$, so that $\mathcal{B}_1 \Subset \mathcal{B}_2 \Subset \mathcal{B}$. Fix a smooth cutoff $\psi \in C^\infty_c(\mathcal{B})$ satisfying $\psi \equiv 1$ on $\mathcal{B}_2$, and denote with $\varrho^+(x) = \mathrm{d}(x_0, x)$ the forward distance to x_0 in M. For each j, define the Lipschitz function

$$s_j(x) = j \cdot \max\left\{\frac{\varepsilon - \varrho^+}{\varepsilon}, -1\right\}.$$

Since $-\varrho^+(x)$ coincides with the signed backward distance to x_0 in $\widetilde{F}$, applying Corollary 3.7 to $(M, \widetilde{F})$ we deduce that s_j is a viscosity subsolution of

$$\widetilde{F}(\widetilde{\nabla} s_j) - \widetilde{G}(s_j) - \frac{j}{\varepsilon}\psi(x) = 0 \qquad \text{on } \ M\backslash\overline{\mathcal{B}_1}.$$

We will construct $\{w_j\}$ in such a way that $w_j \ge s_j$ on M, in particular, $w_j = 0$ on $\partial\mathcal{B}_1$. This is trivial for w_0. Having fixed $w = w_j$, we define the obstacles $g_i = w + \lambda_i$ for $i > j$. For each i, we consider the following Perron class:

$$\mathscr{F}[g_i] = \left\{ v \in C(\overline{\Omega_i\backslash\mathcal{B}_1}) \ : \ \begin{array}{c} v \le g_i, \ \text{ and } v \text{ is a viscosity subsolution of} \\ \widetilde{F}(\widetilde{\nabla} v) - \widetilde{G}(v) - \frac{j+1}{\varepsilon}\psi(x) = 0 \ \text{ on } \ \Omega_i\backslash\overline{\mathcal{B}_1} \end{array} \right\},$$

and the envelope

$$u_i(x) \doteq \sup\Big\{ v(x) \ : \ v \in \mathscr{F}[g_i]\Big\},$$

namely the solution of the obstacle problem on $\Omega_i\backslash\mathcal{B}_1$ with obstacle g_i. Perron class is non-empty, since it contains the constant $-j-1$. Furthermore, since $\lambda_i = 0$ on $\mathcal{B}$, we have $g_i \ge s_j + \lambda_i \ge s_{j+1}$, and from $\psi \equiv 0$ outside of $\mathcal{B}$ we deduce $s_{j+1} \in \mathscr{F}[g_i]$. This and $0 \ge u_i \ge s_{j+1}$ guarantee that $u_i = 0$ on $\partial\mathcal{B}_1$. For $v \in \mathscr{F}[g_i]$, the function

$$h_v = \int_0^{v(x)} \frac{\mathrm{d}s}{\widetilde{G}(s)}$$

is a subsolution of

$$\widetilde{F}(\widetilde{\nabla} h_v) - 1 - \frac{j+1}{\varepsilon} \cdot \frac{1}{\inf_{[-j-1,0]} \widetilde{G}} = 0$$

on $M \backslash \mathcal{B}_1$. Proposition 4.3 in [16] guarantees that h_v is Lipschitz with constant L_j only depending on j. Thus, functions $v \in \mathcal{F}[g_i]$ with $v \geq -j-1$ are equiLipschitz, in particular $u_i \in \mathrm{Lip}(\Omega_i \backslash \mathcal{B}_1)$. By stability, u_i is still a viscosity subsolution of

$$\widetilde{F}(\widetilde{\nabla} u_i) - \widetilde{G}(u_i) - \frac{j+1}{\varepsilon} \psi(x) = 0 \qquad \text{on } \Omega_i \backslash \mathcal{B}_1,$$

and in fact it is also a viscosity supersolution of the same equation on the open set $\{u_i < g_i\}$. For i large enough to satisfy $C_j \Subset \Omega_i$,

$$-j-1 \leq u_i \leq g_i = -j-1 \qquad \text{on } \Omega_i \backslash C_j.$$

Thus, $u_i = -j-1$ in a neighbourhood of $\partial\Omega_i$. Extending u_i with $-j-1$ outside of Ω_i produces a subsolution (still named u_i) of

$$\widetilde{F}(\widetilde{\nabla} u_i) - \widetilde{G}(u_i) = 0 \quad \text{on } M \backslash \mathcal{B}.$$

Clearly, by construction $u_i \in \mathscr{F}[g_{i'}]$ for every $i' > i$. Therefore, the sequence $\{u_i\}$ is monotone increasing and equiLipschitz, and hence converges to a limit function $u \in \mathrm{Lip}(M \backslash \mathcal{B}_1)$ that vanishes on $\partial \mathcal{B}_1$.

Claim: $u \equiv w$.

We first prove that $u \geq -j$ on $M \backslash \mathcal{B}_1$. We proceed by contradiction, assuming that the open set $U = \{u < -j - \delta\}$ be non-empty for some $\delta > 0$. Note that U might intersect $\mathcal{B}$, where the term ψ does not vanish, but $\overline{U} \subset M \backslash \overline{B_1}$ since $u = 0$ on ∂B_1. Choose i_0 large enough that

$$U_{i_0} = \{u < g_{i_0} - \delta\} \neq \emptyset.$$

This is possible since $g_i \uparrow w$ locally uniformly. By monotonicity, $u_i < g_i - \delta$ on U_{i_0} for every $i \geq i_0$, meaning that the solution of the obstacle problem u_i detaches from the obstacle g_i on U_{i_0}. Therefore, u_i is also a supersolution of

$$\widetilde{F}(\widetilde{\nabla} u_i) - \widetilde{G}(u_i) - \frac{j+1}{\varepsilon} \psi(x) = 0 \qquad \text{on } U_{i_0}$$

and, by stability, so is u on U_{i_0}. From $U = \bigcup_{i_0} U_{i_0}$, we deduce that u is a supersolution of

$$\widetilde{F}(\widetilde{\nabla} u) - \widetilde{G}(u) - \frac{j+1}{\varepsilon} \psi(x) = 0 \qquad \text{on } U,$$

and, as a consequence, a supersolution of $\widetilde{F}(\widetilde{\nabla} u) - \widetilde{G}(u) = 0$ on U. At this stage, we use property 8) to $v := -u$, that is a subsolution of

$$G(\nabla v) - F(\nabla v) = 0 \qquad \text{on } U,$$

to deduce that $\sup_U v = \sup_{\partial U} v$, contradicting the very definition of U and proving the claim. Next, fix i_0 with $C_j \Subset \Omega_{i_0}$, and $\delta > 0$ small. From $u \geq -j$ and $w = -j$ on $M \backslash C_j$, we deduce that $u_i \uparrow -j$ uniformly on $\partial \Omega_{i_0}$. Choose $i >> i_0$ such that

$$u_i > -j - \frac{\delta}{2} \qquad \text{on } \partial\Omega_{i_0}.$$

It follows that the function

$$v_i = \begin{cases} \max\{w - \delta, u_i\} & \text{on } \Omega_{i_0}, \\ u_i & \text{on } \Omega_i \backslash \Omega_{i_0} \end{cases}$$

belongs to $\mathscr{F}[g_i]$, and therefore $u_i \geq v_i$ on Ω_i by the maximality of u_i. In particular, $u_i \geq w - \delta$ holds on $\Omega_{i_0} \backslash \mathcal{B}$ for i large enough. By the arbitrariness of i_0 and δ, this proves that $u_i \uparrow w$ locally uniformly on $M \backslash \mathcal{B}$, hence $u \equiv w$.

To conclude, from $u_i \uparrow u \equiv w$ locally uniformly we can choose i large enough such that, setting $w_{j+1} \doteq u_i$, w_{j+1} satisfies all of the requirements in (31).

1) $\Rightarrow$ **9**).
As stated in the introduction, the proof of Ekeland principle given in [25, p.444], see also [2, p.85], does not use the symmetry of d, and can therefore be repeated verbatim.

9) $\Rightarrow$ **1**).
The argument is due to [54, 53], and we reproduce it here for the sake of completeness. Let $\{x_j\}$ be a forward Cauchy sequence, and define the function

$$f \ : \ M \to [-\infty, 0], \qquad f(x) = -\limsup_j \mathrm{d}(x, x_j).$$

The goal is to prove the existence of $\bar{x} \in M$ such that $f(\bar{x}) = 0$. Fix $\varepsilon > 0$ and j_ε guaranteed by the Cauchy condition. From

$$\mathrm{d}(x_{j_\varepsilon}, x_j) < \varepsilon \qquad \forall\, j > j_\varepsilon$$

we deduce $f(x_{j_\varepsilon}) \geq -\varepsilon$, hence $\sup_M f = 0$. Furthermore, the triangle inequality implies $f(y) \leq f(x) + \mathrm{d}(x, y)$, hence f is locally Lipschitz and finite everywhere. Fix $\delta \in (0, 1)$ and, by 9), let $\bar{x}$ satisfy

$$f(\bar{x}) \geq -\delta, \qquad f(y) \leq f(\bar{x}) + \delta \mathrm{d}(\bar{x}, y).$$

Choosing $y = x_j$ for $j > j_\varepsilon$ we deduce

$$-\varepsilon \leq f(\bar{x}) + \delta \mathrm{d}(\bar{x}, x_j).$$

Thus, letting $j \to \infty$ along a sequence realizing $f(\bar{x})$, and then letting $\varepsilon \to 0$, we get

$$0 \leq f(\bar{x}) - \delta f(\bar{x}) = (1 - \delta) f(\bar{x}) \leq 0,$$

and we conclude $f(\bar{x}) = 0$.

6 Appendix I: A homogeneous comparison

Theorem 6.1. *Let $\Omega \Subset M$ and assume that $u \in \mathrm{USC}(\overline{\Omega})$, $v \in \mathrm{LSC}(\overline{\Omega})$ are bounded on Ω and satisfy*

$$\Delta_\infty^N u \geq 0, \quad \text{and} \quad \Delta_\infty^N v \leq 0 \quad \text{in the viscosity sense on } \Omega.$$

Then,

$$\max_{\overline{\Omega}} (u - v) = \max_{\partial\Omega} (u - v).$$

Proof: sketch. Since the Finsler structure is non-symmetric, we need to adapt some notation from [5]. First of all, by a compactness argument, we fix $\alpha > 1$ satisfying (13) on the whole of Ω. For any $\varepsilon > 0$ and $\Omega \Subset M$ let us denote

$$\Omega^+_\varepsilon = \{x \in \Omega : \overline{\mathcal{B}^+_x(\varepsilon)} \subset \Omega\}, \quad \text{and} \quad \Omega^-_\varepsilon = \{x \in \Omega : \overline{\mathcal{B}^-_x(\varepsilon)} \subset \Omega\}.$$

We set $\Omega_\varepsilon \doteq \Omega^-_\varepsilon \cap \Omega^+_\varepsilon$. Up to reducing ε, we will assume that $B^+_x(2\varepsilon)$ and $B^-_x(2\varepsilon)$ are relatively compact for all $x \in \Omega$.

For $x \in \Omega^+_\varepsilon$ and $y \in \Omega^-_\varepsilon$, define

$$u^\varepsilon(x) \doteq \max_{\overline{\mathcal{B}^+_x(\varepsilon)}} u \qquad \text{and} \qquad v_\varepsilon(y) \doteq \min_{\overline{\mathcal{B}^-_y(\varepsilon)}} v.$$

As in [5], applying Corollary 4.5 we can prove that u^ε and v_ε are solutions of the following finite difference inequalities

$$S^-_\varepsilon u^\varepsilon(x) - S^+_\varepsilon u^\varepsilon(x) \ \le\ 0 \ \le\ S^-_\varepsilon v_\varepsilon(x) - S^+_\varepsilon v_\varepsilon(x) \tag{32}$$

for every $x \in \Omega^+_{2\alpha\varepsilon}$, where S^ε and S_ε are defined as follows

$$S^+_\varepsilon u(x) \doteq \max_{y\in\overline{\mathcal{B}^+_x(\varepsilon)}} \frac{u(y) - u(x)}{\varepsilon}, \quad \text{and} \quad S^-_\varepsilon u(x) \doteq \max_{y\in\overline{\mathcal{B}^-_x(\varepsilon)}} \frac{u(x) - u(y)}{\varepsilon}.$$

Now, arguing as in [5, Lem 4] we can conclude that

$$\sup_{\Omega^+_{\alpha\varepsilon}} \left(u^\varepsilon - v_\varepsilon\right) = \sup_{\Omega^+_{\alpha\varepsilon}\setminus\Omega^+_{2\alpha\varepsilon}} \left(u^\varepsilon - v_\varepsilon\right).$$

The conclusion then follows by passing to the limit $\varepsilon \to 0$. □

7 Appendix II: The Dirichlet problem

Let $\Omega \subset M$ be relatively compact, and let $g : \mathbb{R} \times T^*\overline{\Omega} \to \mathbb{R}$ with the following properties:

$$\begin{aligned} &(i) && g \in C(\mathbb{R} \times T^*\overline{\Omega}), \\ &(ii) && \sup_{(t,v)\in I\times T^*\overline{\Omega}} |g| < \infty \qquad \text{for every compact } I \subset \mathbb{R}. \end{aligned} \tag{33}$$

Theorem 7.1. *Let g satisfying* (33)*, and let $u_1, u_2 \in C(\overline{\Omega})$ solving*

$$\begin{cases} \Delta^N_\infty u_1 \ge g(u_1, \mathrm{d}u_1) & on\ \Omega, \\ \Delta^N_\infty u_2 \le g(u_2, \mathrm{d}u_2) & on\ \Omega, \\ u_1 \le u_2 & on\ \overline{\Omega}. \end{cases}$$

Then, for every $\zeta \in C(\partial\Omega)$ with $u_1 \le \zeta \le u_2$, there exists $u \in C(\overline{\Omega})$ such that

$$\begin{cases} \Delta^N_\infty u = g(u, \mathrm{d}u) & on\ \Omega, \\ u_1 \le u \le u_2 & on\ \overline{\Omega}, \\ u = \zeta & on\ \partial\Omega. \end{cases}$$

Remark 7.2. Note that the above existence result does not need any comparison theorem.

Proof. We will employ the Perron method. Fix $I = [\min_{\overline{\Omega}} u_1, \max_{\overline{\Omega}} u_2]$ and choose $c \in \mathbb{R}^+$ such that

$$c > \max_{T\overline{\Omega}\times I} |g|. \tag{34}$$

Consider the Perron class

$$\mathscr{P} = \Big\{ v \in C(\overline{\Omega}) \ : \ u_1 \le v \le u_2, \ \Delta_\infty^N v \ge g(v, \mathrm{d}v), \ v \le \zeta \ \text{ on } \partial\Omega \Big\},$$

and the Perron envelope $u = \sup\{v : v \in \mathscr{P}\}$ on Ω. By (34), $-c \le \Delta_\infty^N v \le c$ for every $v \in \mathscr{P}$. Because of Corollary 4.5, $\mathscr{P}$ is uniformly locally Lipschitz continuous, hence $u \in \mathrm{Lip}_{\mathrm{loc}}(\Omega)$. Given $x \in \partial\Omega$ and $\delta > 0$, let $\varepsilon > 0$ small enough that the signed distance $\varrho^-(y) = -\mathrm{d}(y,x)$ is smooth on $\mathcal{B}_x^-(\varepsilon)\backslash\{x\}$ and that

$$\begin{aligned} u_2 &> \zeta(x) - \delta \quad \text{on } \overline{B_x^-(\varepsilon)\cap\Omega}, & \zeta &> \zeta(x) - \delta \quad \text{on } \overline{B_x^-(\varepsilon)\cap\partial\Omega}, \\ u_1 &< \zeta(x) + \delta \quad \text{on } \overline{B_x^+(\varepsilon)\cap\Omega}, & \zeta &< \zeta(x) + \delta \quad \text{on } \overline{B_x^+(\varepsilon)\cap\partial\Omega}. \end{aligned}$$

Set $\zeta_\delta^-(x) \doteq \zeta(x) - \delta$, and let $b >> 1$ large enough in such a way that the backward quadratic cone

$$C_{b,x}^-(y) \doteq \zeta_\delta(x) - (b + R_b(\zeta_\delta^-(x)))\mathrm{d}(y,x) + \frac{c}{2}\mathrm{d}(y,x)^2,$$

defined on $\mathcal{B}_x^-(\varepsilon)$, satisfies $C_{b,x}^- < u_1$ on $\mathcal{S}_x^-(\varepsilon)\cap\overline{\Omega}$. By Corollary 4.5 we then have $C_{b,x}^- \le u_2$ on $\overline{\mathcal{B}_x^-(\varepsilon)\cap\Omega}$, and

$$\Delta_\infty^N C_{b,x}^- \ge c \ge g(C_{b,x}^-, \mathrm{d}C_{b,x}^-) \qquad \text{on } \mathcal{B}_x^-(\varepsilon)\cap\{C_{b,x}^- > u_1\}.$$

It follows that

$$w := \begin{cases} \max\{C_{b,x}^-, u_1\} & \text{on } \overline{\mathcal{B}_x^-(\varepsilon)\cap\Omega}, \\ u_1 & \text{otherwise} \end{cases}$$

lies in $\mathscr{P}$ and therefore

$$\liminf_{y\to x} u(y) \ge \liminf_{y\to x} w(y) \ge \liminf_{y\to x} C_{b,x}^-(y) = \zeta(x) - \delta. \tag{35}$$

Similarly, setting $\zeta_\delta^+(x) = \zeta(x) + \delta$, we consider the forward quadratic cone

$$C_{b,x}^+(x) \doteq \zeta_\delta^+(x) + (b + R_b(\zeta_\delta^+(x)))\mathrm{d}(x,y) - \frac{c}{2}\mathrm{d}(x,y)^2$$

that for large enough b solves

$$\begin{cases} \Delta_\infty^N C_{b,x}^+ \le -c & \text{on } \mathcal{B}_x^+(\varepsilon), \\ \quad C_{b,x}^+ > u_2 & \text{on } \mathcal{S}_x^+(\varepsilon)\cap\overline{\Omega}. \end{cases}$$

We claim that $v < C_{b,x}^+$ on $\mathcal{B}_x^+(\varepsilon)$ for every $v \in \mathscr{P}$. Indeed, this holds by construction on $\mathcal{S}_x^+(\varepsilon)\cap\overline{\Omega}$, while on $\partial\Omega\cap\overline{\mathcal{B}_x^+(\varepsilon)}$ we have

$$v \le \zeta < \zeta_\delta^+(x) \le C_{b,x}^+,$$

thus $v < C^+_{b,x}$ on $\partial(\mathcal{B}^+_x(\varepsilon)\cap\Omega)$. If $v - C^+_{b,x}$ attains a non-negative maximum m_0 at $x_0 \in \mathcal{B}^+_x(\varepsilon)\cap\Omega$, then $C^+_{b,x} + m_0$ is a smooth function that touches v from above and satisfies $\Delta^N_\infty C^+_{b,x}(x_0) \le -c < g(u(x_0), \mathrm{d}u(x_0))$, contradiction. Thus, $v \le C^+_{b,x}$ on $\mathcal{B}^+_x(\varepsilon)\cap\Omega$ and, taking supremum, $u \le C^+_{b,x}$ there. Hence,

$$\limsup_{y\to x} u(y) \le \limsup_{y\to x} C^+_{b,x}(y) = \zeta(x) + \delta,$$

thus coupling with (35) and letting $\delta \to 0$ we infer $u \in C(\overline{\Omega})$ with $u = \zeta$ on $\partial\Omega$. By the stability of subsolutions with respect to uniform convergence (Proposition 3.4), $\Delta^N_\infty u \ge g(u, \mathrm{d}u)$ on Ω. We are left to prove that u is also a supersolution. Suppose, by contradiction, that there exist $x_0 \in \Omega$ and $\phi \prec_{x_0} u$ defined in a small, relatively compact neighbourhood $U \Subset \Omega$ of x_0 such that $\Delta^N_\infty \phi(x_0) > g(\phi, \mathrm{d}\phi)(x_0)$. If $u(x_0) = u_2(x_0)$, then $\phi \prec_{x_0} u_2$, contradicting the fact that u_2 is a supersolution. Therefore, $u(x_0) < u_2(x_0)$. Up to subtracting to ϕ a function $\psi \succ_{x_0} 0$ that is positive on $U\backslash\{x_0\}$ and vanishes at x_0 at second order, we can assume that $\phi < u$ on $U\backslash\{x_0\}$. By continuity of ζ and since ϕ is smooth, up to shrinking U and choosing ε small we can satisfy any of the following properties:

$$\begin{cases} \phi + \varepsilon < u & \text{on } \partial U, \\ \phi + \varepsilon \le u_2 & \text{on } U, \\ \Delta^N_\infty \phi > g(\phi + \varepsilon, \mathrm{d}\phi) & \text{on } U. \end{cases}$$

It follows that

$$\hat{u} := \begin{cases} \max\{u, \phi + \varepsilon\} & \text{on } U, \\ u & \text{on } \overline{\Omega}\backslash U \end{cases}$$

lies in $\mathscr{P}$, and since $\hat{u}(x_0) > u(x_0)$ this contradicts the definition o u. □

References

[1] L.V. Ahlfors and L. Sario, *Riemann surfaces*. Princeton mathematical series 26, Princeton Univ. Press, 1960.

[2] L.J. Alías, P. Mastrolia, and M. Rigoli, *Maximum principles and geometric applications.* Springer Monographs in Mathematics. Springer, Cham, 2016. xvii+570 pp.

[3] L.J. Alías, J. Miranda, and M. Rigoli, *A new open form of the weak maximum principle and geometric applications.* Comm. Anal. Geom. 24 (2016), no. 1, 1-43.

[4] D.J. Araújo, R.. Leitão, and E.V. Teixeira, *Infinity Laplacian equation with strong absorptions.* J. Funct. Anal. 270 (2016), no. 6, 2249-2267.

[5] S.N. Armstrong and C. Smart, *An easy proof of Jensen's theorem on the uniqueness of infinity harmonic functions.* Calc. Var. Partial Differential Equations 37 (2010), no. 3-4, 381-384.

[6] S.N. Armstrong and C. Smart, *A finite difference approach to the infinity Laplace equation and tug-of-war games.* Trans. Amer. Math. Soc. 364 (2012), no. 2, 595-636.

[7] G. Aronsson, *Extension of functions satisfying Lipschitz conditions.* Ark. Mat. 6 (1967), 551-561.

[8] G. Aronsson, *On the partial differential equation $u_{x^2}u_{xx} + 2u_x u_y u_{xy} + u_{y^2}u_{yy} = 0$.* Ark. Mat. 7 (1968) 395-425.

[9] G. Aronsson, M.G. Crandall and P. Juutinen, *A tour of the theory of absolutely minimizing functions.* Bull. Amer. Math. Soc. 41, n.4, (2004), 439-505.

[10] D. Bao, S.-S. Chern and Z. Shen, *An introduction to Riemann-Finsler geometry.* Springer-Verlag New York 2000. p. 431. (Graduate texts in mathematics; 200).

[11] G. Barles and J. Busca, *Existence and comparison results for fully nonlinear degenerate elliptic equations without zeroth-order term.* Comm. Partial Differential Equations 26 (2001), no. 11-12, 2323-2337.

[12] E.N. Barron, L.C. Evans and R. Jensen, *The infinity laplacian, Aronsson's equation and their generalizations*. Trans. Amer. Math. Soc. 360 (2008), 77-101.

[13] E.N. Barron, R.R. Jensen, and C.Y. Wang, *The Euler Equation and Absolute Minimizers of L^∞ Functionals.* Arch. Rational Mech. Anal. 157 (2001) 255-283.

[14] B. Bianchini, L. Mari, P. Pucci and M. Rigoli, *Geometric Analysis of Quasilinear Inequalities on Noncompact Manifolds.* Frontiers in Mathematics, Birkhäuser, 2021.

[15] E. Calabi, *An extension of E. Hopf's maximum principle with an application to Riemannian geometry.* Duke Math. J. 25 (1958), no. 1, 45-56.

[16] F. Camilli and A. Siconolfi, *Maximal subsolutions for a class of degenerate Hamilton-Jacobi problems.* Indiana University Math. J. 48 (1999), 1111-1131.

[17] E. Caponio, M.A. Javaloyes, M. Sánchez, *On the interplay between Lorentzian causality and Finsler metrics of Randers type.* Rev. Mat. Iberoam. 27 (2011), no. 3, 919-952.

[18] T. Champion and L. de Pascale, *Principles of comparison with distance functions for absolutely minimizers.* Journal of Convex Analysis, 14 (2007), n.3, 515-541.

[19] S.Y. Cheng and S.T. Yau, *Differential equations on Riemannian manifolds and their geometric applications.* Comm. Pure Appl. Math. 28 (1975), no. 3, 333-354.

[20] S. Cohn-Vossen. *Existenz kürzester Wege.* Compositio math., Groningen, 3 (1936), 441-452.

[21] M.G. Crandall, *A Visit with the ∞-Laplace Equation.* Calculus of Variations and Nonlinear Partial Differential Equations. Lecture Notes in Mathematics, vol. 1927, pp. 75-122. Springer, Berlin (2008).

[22] M.G. Crandall, L.C. Evans and R.F. Gariepy, *Optimal Lipschitz extensions and the infinity Laplacian.* Calc. Var. P.D.E. 13 (2001), no. 2, 123-139.

[23] M.G. Crandall, H. Ishii and P.L. Lions, *User's guide to viscosity solutions of second-order partial differential equations.* Bull. Am. Math. Soc. 27 (1992), 1-67.

[24] I. Ekeland, *On the variational principle.* J. Math. Anal. Appl. 47 (1974), 324-353.

[25] I. Ekeland, *Nonconvex minimization problems.* Bull. Amer. Math. Soc. (N.S.) 1 (1979), no. 3, 443-474.

[26] L.C. Evans, *Estimates for smooth absolutely minimizing Lipschitz extensions.* Electron. J. Differential Equations 1993, No. 03, approx. 9 pp. (electronic only).

[27] J. García-Azorero, J.J. Manfredi, I. Peral and J.D. Rossi, *The Neumann problem for the ∞-Laplacian and the Monge-Kantorovich mass transfer problem*. Nonlinear Anal. 66 (2007), 349-366.

[28] A. Grigor'yan, *Analytic and geometric background of recurrence and non-explosion of the Brownian motion on Riemannian manifolds.* Bull. Amer. Math. Soc. 36 (1999), 135-249.

[29] C.-Y. Guo, C.-L. Chang and D. Yang, *L^∞-variational problems associated to measurable Finsler structures.* Nonlinear Anal. 132 (2016), 126-140.

[30] F.R. Harvey and H.B. Lawson Jr., *Dirichlet Duality and the Nonlinear Dirichlet Problem on Riemannian Manifolds.* J. Differential Geom. 88 (2011), 395-482.

[31] R. Jensen, *Uniqueness of Lipschitz extensions: minimizing the sup norm of the gradient.* Arch. Rational Mech. Anal. 123 (1993), no. 1, 51-74.

[32] P. Juutinen, *Minimization problems for Lipschitz functions via viscosity solutions*. Ann. Acad. Sci. Fenn. Math. Diss. No. 115 (1998), 53 pp.

[33] P. Lindqvist and J.J. Manfredi, *The Harnack inequality for ∞-harmonic functions.* Electron. J. Differential Equations 1995, no. 04, approx. 5 pp. (electronic).

[34] P. Lindqvist and J.J. Manfredi, *Note on ∞-superharmonic functions.* Rev. Mat. Univ. Complut. Madrid 10 (1997), no. 2, 471-480.

[35] G. Lu and P. Wang, *Inhomogeneous infinity Laplace equation.* Advances in Mathematics 217 (2008) 1838-1868.

[36] G. Lu and P. Wang, *A PDE perspective of the normalized infinity Laplacian.* Comm. Partial Differ. Equ. 33 (2008), no. 10-12, 1788-1817.

[37] L. Mari and L.F. Pessoa, *Duality between Ahlfors-Liouville and Khas'minskii properties for non-linear equations.* Comm. Anal. Geom. 28 (2020), no. 2, 395-497.

[38] L. Mari and L.F. Pessoa, *Maximum principles at infinity and the Ahlfors-Khas'minskii duality: an overview.* Contemporary research in elliptic PDEs and related topics, 419-455, Springer INdAM Ser., 33, Springer, Cham, 2019;

[39] L. Mari, M. Rigoli and A.G. Setti, *On the $1/H$-flow by p-Laplace approximation: new estimates via fake distances under Ricci lower bounds*. Available at arXiv:1905.00216.

[40] L. Mari and D. Valtorta, *On the equivalence of stochastic completeness, Liouville and Khas'minskii condition in linear and nonlinear setting*. Trans. Amer. Math. Soc. 365 (2013), no. 9, 4699-4727.

[41] B. Mebrate and A. Mohammed, *Infinity-Laplacian type equations and their associated Dirichlet problems.* Complex Variables and Elliptic Equations (2018), 31p.

[42] B. Mebrate and A. Mohammed, *Comparison principles for infinity-Laplace equations in Finsler metrics*. Nonlinear Anal. 190 (2020), 111605.

[43] A.G.C. Mennucci, *Regularity and Variationality of Solutions to Hamilton-Jacobi Equations. Part II: Variationality, Existence, Uniqueness.* Appl. Math. Opt. 63 (2011), 191-216.

[44] A.G.C. Mennucci, *On asymmetric distances.* Anal. Geom. Metr. Spaces 1 (2013), 200-231.

[45] H. Omori, *Isometric immersions of Riemannian manifolds.* J. Math. Soc. Japan 19 (1967), 205-214.

[46] Y. Peres, O. Schramm, S. Sheffield and D. Wilson, *Tug-of-war and the infinity Laplacian.* J. Amer. Math. Soc. 22 (2009), 167-210.

[47] S. Pigola, M. Rigoli and A.G. Setti, *A remark on the maximum principle and stochastic completeness.* Proc. Amer. Math. Soc. 131 (2003), no. 4, 1283-1288.

[48] S. Pigola, M. Rigoli and A.G. Setti, *Maximum principles on Riemannian manifolds and applications.* Mem. Amer. Math. Soc. 174 (2005), no. 822.

[49] S. Pigola, M. Rigoli and A.G. Setti, *Maximum principles at infinity on Riemannian manifolds: an overview.* Workshop on Differential Geometry (Portuguese), Mat. Contemp. 31 (2006), 81-128.

[50] S. Pigola and A.G. Setti, *Global divergence theorems in nonlinear PDEs and geometry.* Ensaios Matemáticos [Mathematical Surveys], 26. Sociedade Brasileira de Matemática, Rio de Janeiro, 2014. ii+77 pp.

[51] J.D. Rossi, *The Infinity Laplacian and Tug-of-War Games (Games that PDE People like to play).* www.capde.cl/infty-Lapla-Pisa.pdf

[52] Z. Shen, *Lectures on Finsler Geometry.* World Scientific Publishing, Singapore 2001. p. 307.

[53] F. Sullivan, *A characterization of complete metric spaces.* Proc. Amer. Math. Soc. 83 (1981), no. 2, 345-346.

[54] J.D. Weston, *A characterization of metric completeness.* Proc. Amer. Math. Soc. 64 (1977), no. 1, 186-188.

[55] Y. Wu, *Absolute minimizers in Finsler metrics.* Ph.D. dissertation, UC Berkeley, 1995.

[56] B.Y. Wu and Y.L. Xin, *Comparison theorems in Finsler Geometry and their applications.* Math. Ann. 337 (2007), 177–196.

[57] S.T. Yau, *Harmonic functions on complete Riemannian manifolds.* Comm. Pure Appl. Math. 28 (1975), 201-228.