

Distributed Control for Non-Cooperative Systems Under Conjugation Conditions

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Abstract

In this paper, the distributed control for non-cooperative elliptic systems under conjugation conditions is established. First, the existence and uniqueness of the state for these systems with Dirichlet and conjugation conditions is proved, then the set of equations and inequalities that characterizes the distributed control of these systems is found. The non-cooperative Neumann systems with conjugation conditions is also discussed.

Keywords

Non cooperative elliptic systems - Conjugation conditions - Dirichlet and Neumann conditions - Existence and uniqueness of solutions - Distributed control

1. Introduction

The necessary and sufficient conditions of optimality for systems governed by partial differential equations have been studied by Lions [11]. The control problems described by either infinite order operators or operators with an infinite number of variables were established by Gali et. al. [3, 4, 5]. These results have been extended to cooperative systems [1, 2, 6, 13, 16] or non-cooperative systems [10, 17].

Serag et. al. discussed the optimal control for systems involving schrodinger operators [12, 14]. The existence results have been proved for some non linear systems in [8, 9, 14, 15]. Some applications for control problems have been introduced for example in [7, 10].

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New optimal control problems of distributed systems described by an elliptic, parabolic and hyperbolic operators with conjugation conditions and by a quadratic cost functional have been studied by Sergienko and Deineka [18-20].

In the present work, using the theory of Lions [11], Sergienko and Deineka [18-20], the distributed control for $n \times n$ non cooperative Dirichlet elliptic systems is discussed . First the existence and uniqueness of the state for these systems is proved, then the set of equations and inequalities that characterizes the distributed control of these systems is found. The optimal control of distributed type for non-cooperative Neumann problems under conjugation conditions is also studied.

2 Distributed control for non-cooperative Dirichlet elliptic systems under conjugation conditions

In this section, we study the distributed control for the following $n \times n$ non cooperative Dirichlet elliptic systems:

$$\begin{cases} -\Delta h_i + \sum_{j=1}^n a_{ij} h_j = f_i & \text{in } \Omega, \\ h_i = 0 & \text{on } \Gamma, \quad i = 1, 2, \dots, n, \end{cases} \quad (1)$$

under conjugation conditions:

$$\begin{cases} R_1 \left\{ \frac{\partial h_i}{\partial v_A} \right\}^- + R_2 \left\{ \frac{\partial h_i}{\partial v_A} \right\}^+ = [h_i] + \delta & \text{on } \gamma, \\ \left[\frac{\partial h_i}{\partial v_A} \right] = \left[\sum_{i,j=1}^n \frac{\partial h_i}{\partial x_j} \cos(v, x_i) \right] = w_i & \text{on } \gamma, \quad i = 1, 2, \dots, n, \end{cases} \quad (2)$$

where Ω_1 and Ω_2 , with boundary $\partial\Omega_1$ and $\partial\Omega_2$ respectively, are bounded, continuous and strictly Lipchitz domains from R^n such that :

$$\Omega = \Omega_1 \cup \Omega_2, \quad \Omega_1 \cap \Omega_2 = \phi,$$

$$\Gamma = (\partial\Omega_1 \cup \partial\Omega_2) / \gamma, \text{ is boundary of } \Omega, \gamma = \partial\Omega_1 \cap \partial\Omega_2 \neq \phi, \gamma = \gamma^+ \cup \gamma^-$$

$$\partial\Omega_1 \cap \gamma = \gamma^+, \partial\Omega_2 \cap \gamma = \gamma^-, \quad f_i \in L^2(\Omega), (i = 1, 2, \dots, n),$$

$$R_1, R_2, w, \delta \in C(\gamma), R_1, R_2 \geq 0, R_1 + R_2 \geq R_0 > 0, R_0 = \text{constant}, \quad (3)$$

\vec{n} is an ort of an outer normal to γ , $[\varphi] = \varphi^+ - \varphi^-$,

$$\begin{aligned} \varphi^+ &= \{\varphi\}^+ = \varphi(x) & \text{for } x \in \gamma^+ \\ \varphi^- &= \{\varphi\}^- = \varphi(x) & \text{for } x \in \gamma^-. \end{aligned}$$

System (1) is called cooperative system if $a_{ij} > 0 \quad \forall i \neq j$, otherwise is called non-cooperative system.

In our work, we assume

$$a_{ij} = \begin{cases} 1 & \text{if } i \geq j, \\ -1 & \text{if } i < j, \end{cases} \quad i, j = 1, 2, \dots, n.$$

(i.e. non-cooperative systems).

We first prove the existence of the state for system (1) under conjugation conditions (2). Then, we discuss the existence of distributed control for this system; and we find the set of equations and inequalities that characterizes this distributed control.

Existence and uniqueness of the state

Since

$$H_0^1(\Omega) \subseteq L^2(\Omega) \subseteq H^{-1}(\Omega),$$

then by Cartesian product, we have chain of the form

$$(H_0^1(\Omega))^n \subseteq (L^2(\Omega))^n \subseteq (H^{-1}(\Omega))^n.$$

On $(H_0^1(\Omega))^n$, we introduce the bilinear form:

$$\begin{aligned} a(h, \psi) &= \sum_{i=1}^n \int_{\Omega} \nabla h_i \nabla \psi_i dx + \sum_{i,j=1}^n \int_{\Omega} a_{ij} h_i \psi_j dx \\ &+ \sum_{i=1}^n \int_{\gamma} \frac{[h_i][\psi_i]}{R_1 + R_2} d\gamma. \end{aligned} \quad (4)$$

It is easy to check that

$$|a(h, \psi)| \leq k_1 \|h\| \|\psi\|. \quad (5)$$

The bilinear form (4) is coercive on $(H_0^1(\Omega))^n$; that is, there exists a positive constant C such that

$$a(h, h) \geq C \|h\|_{(H_0^1(\Omega))^n}^2 \quad \forall h = \{h_1, h_2, \dots, h_n\} \in (H_0^1(\Omega))^n \quad (6)$$

Proof.

$$\begin{aligned} a(h, h) &= \sum_{i=1}^n \int_{\Omega} |\nabla h_i|^2 dx + \sum_{i=1}^n \int_{\Omega} |h_i|^2 dx + \sum_{i=1}^n \int_{\gamma} \frac{[h_i]^2}{R_1 + R_2} d\gamma \\ &= \sum_{i=1}^n \int_{\Omega} (|\nabla h_i|^2 + |h_i|^2) dx + \sum_{i=1}^n \int_{\gamma} \frac{[h_i]^2}{R_1 + R_2} d\gamma \end{aligned}$$

(3), implies

$$a(h, h) \geq C \sum_{i=1}^n \int_{\Omega} (|\nabla h_i|^2 + |h_i|^2) dx,$$

therefore

$$\begin{aligned} a(h, h) &\geq C \sum_{i=1}^n \|h_i\|_{H_0^1(\Omega)}^2 \\ &= C \|h\|_{(H_0^1(\Omega))^n}^2, \end{aligned}$$

□

which proves the coerciveness condition of the bilinear form (4) on $(H_0^1(\Omega))^n$.
Now, let

$$L(\psi) = \sum_{i=1}^n \int_{\Omega} f_i(x) \psi_i(x) dx + \sum_{i=1}^n \int_{\gamma} \frac{(R_2 w - \delta) \psi_i}{R_1 + R_2} d\gamma - \sum_{i=1}^n \int_{\gamma} w \psi_i^+ d\gamma \quad (7)$$

be a linear form on $(H_0^1(\Omega))^n$, this linear form is continuous, since :

$$|L(\psi)| \leq K \|\psi\|_{(H_0^1(\Omega))^n} \quad \forall \psi \in (H_0^1(\Omega))^n, K \text{ is a constant.} \quad (8)$$

Then using Lax Milgram lemma, there exists a unique solution $h \in (H_0^1(\Omega))^n$ such that:

$$a(h, \psi) = L(\psi) \quad \forall \psi = (\psi_i)_{i=1}^n \in (H_0^1(\Omega))^n. \quad (9)$$

Then, we have proved the following theorem

For a given $f = \{f_i\}_{i=1}^n \in (L^2(\Omega))^n$ there exists a unique solution $h = \{h_i\}_{i=1}^n \in (H_0^1(\Omega))^n$ for non-cooperative Dirichlet system (1) with conjugation conditions (2)

Formulation of the control problem

The space $U = (L^2(\Omega))^n$ is the space of controls. For a control $u = \{u_1, u_2, \dots, u_n\} \in (L^2(\Omega))^n$, the state $h(u) = \{h_1(u), h_2(u), \dots, h_n(u)\}$ of the system is given by the solution of

$$\begin{cases} -\Delta h_i(u) + \sum_{j=1}^n a_{ij} h_j(u) = f_i(u) + u_i & \text{in } \Omega, \\ h_i(u) = 0 & \text{on } \Gamma, \quad i = 1, 2, \dots, n, \end{cases} \quad (10)$$

under conjugation conditions:

$$\begin{cases} R_1 \left\{ \frac{\partial h_i(u)}{\partial v_A} \right\}^- + R_2 \left\{ \frac{\partial h_i(u)}{\partial v_A} \right\}^+ = [h_i(u)] + \delta & \text{on } \gamma, \\ \left[\frac{\partial h_i(u)}{\partial v_A} \right] = \left[\sum_{j=1}^n \frac{\partial h_j(u)}{\partial x_j} \cos(v, x_j) \right] = w_i & \text{on } \gamma, \quad i = 1, 2, \dots, n. \end{cases} \quad (11)$$

Specify the observation equation by

$$\begin{aligned} z(u) &= \{z_1(u), z_2(u), \dots, z_n(u)\} = Ch(u) = C\{h_1(u), h_2(u), \dots, h_n(u)\} \\ &= \{h_1(u), h_2(u), \dots, h_n(u)\}. \end{aligned}$$

For a given $z_d = \{z_{1d}, z_{2d}, \dots, z_{nd}\} \in (L^2(\Omega))^n$, the cost functional is given by

$$J(v) = \sum_{i=1}^n \|h_i(v) - z_{id}\|_{L^2(\Omega)}^2 + (Nv, v)_{(L^2(\Omega))^n}, \quad (12)$$

where N is a hermitian positive definite operator such that :

$$(Nv, v)_{(L^2(\Omega))^n} \geq M \|v\|_{(L^2(\Omega))^n}^2, \quad M > 0, \quad \forall v \in U. \quad (13)$$

The control problem then is to find :

$$\begin{cases} u = \{u_1, u_2, \dots, u_n\} \in U_{ad} \\ J(u) = \inf J(v) \quad \forall v \in U_{ad}, \end{cases} \quad \text{such that:} \quad (14)$$

where U_{ad} is a closed convex subset of $(L^2(\Omega))^n$.

The cost functional (12) can be written as

$$J(v) = \pi(v, v) - 2H(v) + \sum_{i=1}^n \|z_{id} - h_i(0)\|_{L^2(\Omega)}^2,$$

where

$$\pi(u, v) = \sum_{i=1}^n (h_i(u) - h_i(0), h_i(v) - h_i(0))_{L^2(\Omega)} + (Nu, v)_{(L^2(\Omega))^n}, \quad (15)$$

is a continuous bilinear form and from (13), it is coercive, that is:

$$\pi(v, v) \geq N\|v\|_{(L^2(\Omega))^n}^2 \text{ and}$$

$$H(v) = \sum_{i=1}^n (z_{id} - h_i(0), h_i(v) - h_i(0))_{L^2(\Omega)}, \quad (16)$$

is a continuous linear form on $(L^2(\Omega))^n$. Then, using the theory of Lions [11], there exists a unique optimal control of problem (14); Moreover it is characterized by Let us suppose that (6) holds and the cost functional is given by (12), then the distributed control u is characterized by

$$\left\{ \begin{array}{ll} -\Delta p_i(u) + \sum_{j=1}^n a_{ij} p_j(u) = h_i(u) - z_{id} & \text{in } \Omega, \\ p_i(u) = 0 & \text{on } \Gamma, \\ \left[\frac{\partial p_i(u)}{\partial v_{A^*}} \right] = 0 & \text{on } \gamma, \\ \left\{ \frac{\partial p_i(u)}{\partial v_{A^*}} \right\}^{\pm} = \frac{1}{R_1 + R_2} [p_i(u)] & \text{on } \gamma, \\ \sum_{i=1}^n (p_i(u), v_i - u_i) + (Nu, v - u)_{(L^2(\Omega))^n} \geq 0 & , i = 1, 2, \dots, n, \end{array} \right. \quad (17)$$

together with (10), where $p(u) = \{p_1(u), p_2(u), \dots, p_n(u)\}$ is the adjoint state.

Proof. The optimal control $u = \{u_i\}_{i=1}^n \in (L^2(\Omega))^n$ is characterized by [11] :

$$\pi(u, v - u) \geq H(v - u) \quad \forall v = \{v_1, v_2, \dots, v_n\} \in U_{ad}.$$

From (15), and (16):

$$\begin{aligned} \pi(u, v - u) - H(v - u) &= \sum_{i=1}^n (h_i(u) - z_{id}, h_i(v) - h_i(u))_{L^2(\Omega)} \\ &+ \sum_{i=1}^n (Nu_i, v_i - u_i)_{L^2(\Omega)} \geq 0. \end{aligned} \quad (18)$$

Since the model A of the system is given by

$$Ah(u) = A(h_1(u), h_2(u), \dots, h_n(u)) = \sum_{i=1}^n (-\Delta h_i(u) + \sum_{j=1}^n a_{ij} h_j(u)),$$

and since

$$(p(u), Ah(u)) = (A^* p(u), h(u)),$$

then

$$\begin{aligned} (p(u), Ah(u))_{(L^2(\Omega))^n} &= \sum_{i=1}^n (p_i(u), -\Delta h_i(u) + \sum_{j=1}^n a_{ij} h_j(u)) \\ &= \sum_{i=1}^n (-\Delta p_i(u) + \sum_{j=1}^n a_{ji} p_j(u), h_i(u))_{L^2(\Omega)}, \end{aligned}$$

hence (18) is equivalent to

$$\begin{aligned} &\sum_{i=1}^n (-\Delta p_i(u) - \sum_{j=1}^n a_{ji} p_j(u), h_i(v-u) - h_i(0))_{L^2(\Omega)} \\ &+ \sum_{i=1}^n (Nu_i, v_i - u_i)_{L^2(\Omega)} \geq 0. \end{aligned}$$

Therefore

$$\begin{aligned} &\sum_{i=1}^n (A^* p_i(u), h_i(v-u) - h_i(0))_{L^2(\Omega)} \\ &+ \sum_{i=1}^n (Nu_i, v_i - u_i)_{L^2(\Omega)} \geq 0. \end{aligned}$$

Applying Green's formula, we obtain

$$\begin{aligned} &\sum_{i=1}^n (p_i(u), Ah_i(v-u) - Ah_i(0))_{L^2(\Omega)} \\ &+ \sum_{i=1}^n (Nu_i, v_i - u_i)_{L^2(\Omega)} \geq 0. \end{aligned}$$

Using equation (10), we get

$$\sum_{i=1}^n (p_i(u), v_i - u_i) + \sum_{i=1}^n (Nu_i, v_i - u_i)_{L^2(\Omega)} \geq 0,$$

i.e

$$\sum_{i=1}^n \int_{\Omega} (p_i(u) + Nu_i)(v_i - u_i) dx \geq 0.$$

□

3 Distributed control for non-cooperative Neumann elliptic systems with conjugation conditions

In this section, we consider the following non-cooperative Neumann elliptic system

$$\begin{cases} -\Delta h_i + \sum_{j=1}^n a_{ij} h_j = f_i & \text{in } \Omega, \\ \frac{\partial h_i}{\partial \nu_A} = g_i & \text{on } \Gamma, \end{cases} \quad (19)$$

with conjugation conditions (2), where $g = \{g_1, g_2, \dots, g_n\} \in (L^2(\Gamma))^n$ is given function. We introduce again the bilinear form (4) which is coercive on $(H^1(\Omega))^n$, since

$$((H_0^1(\Omega))^n) \subseteq ((H^1(\Omega))^n).$$

Then based on (6), (8) and Lax-Milgram lemma, there exists a unique solution h for system (19) such that :

$$a(h, \psi) = L_N(\psi), \quad \forall \psi \in (H^1(\Omega))^n,$$

where

$$\begin{aligned} L_N(\psi) &= \sum_{i=1}^n \int_{\Omega} f_i(x) \psi_i(x) dx + \sum_{i=1}^n \int_{\Gamma} g_i(x) \psi_i(x) d\Gamma \\ &+ \sum_{i=1}^n \int_{\gamma} \frac{(R_2 w - \delta) \psi_i}{R_1 + R_2} d\gamma - \sum_{i=1}^n \int_{\gamma} w \psi_i^+ d\gamma, \end{aligned}$$

is a continuous linear form defined on $(H^1(\Omega))^n$.

Let us multiply both sides of first equation of (19) by $\psi \in (H^1(\Omega))^n$ and integrate over Ω , we obtain

$$\sum_{i=1}^n \int_{\Omega} (-\Delta h_i + \sum_{j=1}^n a_{ij} h_j) \psi_i(x) dx = \sum_{i=1}^n \int_{\Omega} f_i \psi_i dx.$$

Applying Green's formula,

$$\begin{aligned} &\sum_{i=1}^n \int_{\Omega} (-\Delta h_i + \sum_{j=1}^n a_{ij} h_j) \psi_i(x) dx + \sum_{i=1}^n \int_{\Gamma} \left(\frac{\partial h_i}{\partial \nu_A} \right) \psi_i(x) d\Gamma + \\ &\sum_{i=1}^n \int_{\gamma} \left(\frac{\partial h_i}{\partial \nu_A} \right) \psi_i(x) d\gamma + a(h, \psi) = \sum_{i=1}^n \int_{\Omega} f_i \psi_i dx. \end{aligned}$$

Then, from

$$a(h, \psi) = L_N(\psi),$$

we deduce the Neumann conditions

$$\frac{\partial h_i}{\partial \nu_A} = g_i, \quad \text{on } \Gamma.$$

So we can formulate the corresponding control problem:

The space $U = (L^2(\Omega))^n$ is the space of controls. For a control $u = \{u_1, u_2, \dots, u_n\} \in (L^2(\Omega))^n$, the state $h(u) = \{h_1(u), h_2(u), \dots, h_n(u)\}$ of the system is given by the solution of

$$\begin{cases} -\Delta h_i(u) + \sum_{j=1}^n a_{ij} h_j(u) = f_i(u) + u_i & \text{in } \Omega, \\ \frac{\partial h_i(u)}{\partial \nu_A} = g_i & \text{on } \Gamma, \end{cases} \quad (20)$$

under conjugation conditions (11). For a given $z_d = \{z_{1d}, z_{2d}, \dots, z_{nd}\} \in (L^2(\Omega))^n$, the cost functional is again given by (12), then there exists a unique optimal control $u \in U_{ad}$ such that:

$$\begin{cases} u = \{u_1, u_2, \dots, u_n\} \in U_{ad} & \text{such that:} \\ J(u) = \inf J(v) & \forall v \in U_{ad}. \end{cases}$$

Moreover it is characterized by the following equations and inequalities

$$\left\{ \begin{array}{ll} -\Delta p_i(u) + \sum_{j=1}^n a_{ij} p_j(u) = h_i(u) - z_{id} & \text{in } \Omega, \\ \frac{\partial p_i(u)}{\partial v_{A^*}} = 0 & \text{on } \Gamma, \\ \left[\frac{\partial p_i(u)}{\partial v_{A^*}} \right] = 0 & \text{on } \gamma, \\ \left\{ \frac{\partial p_i(u)}{\partial v_{A^*}} \right\}^{\pm} = \frac{1}{R_1 + R_2} [p_i(u)] & \text{on } \gamma \\ \sum_{i=1}^n (p_i(u), v_i - u_i) + (Nu, v - u)_{(L^2(\Omega))^n} \geq 0 & , i = 1, 2, \dots, n, \end{array} \right.$$

together with (20), where $p(u) = \{p_1(u), p_2(u), \dots, p_n(u)\}$ is the adjoint state.

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References

- [1] El-Saify, H. A., Serag, H. M., and Shehata, M. A., Time optimal control problem for cooperative hyperbolic systems involving the Laplace operator, *J. of Dynamics and Control Systems*, Vol. 15, No. 3, (2009), pp. 405-423.
- [2] Fleckinger J., and Serag, H. M., Semilinear cooperative elliptic systems on R^n , *Rend. Mat. Appl.*, Vol. 15, No.1, (1995), pp. 89-108.
- [3] Gali, I.M., Optimal control of system governed by elliptic operators of infinite order, *Ordinary and Partial Diff. Eqns.*, Proc. Dundee Scotland, Springer-Verlag Ser. Lecture Notes in Maths., Vol. 964, (1982), pp. 263-272.
- [4] Gali, I. M. and El-Saify, H. A., Optimal control of a system governed by hyperbolic operator with an infinite number of variables, *J. of Mathematical Analysis and Applications*, Vol. 85, No. 1, (1982), pp. 24-30.
- [5] Gali, I. M. and EL-Saify H. A., Distributed control of a system governed by Dirichlet and Neumann problems for a self adjoint elliptic operator with an infinite number of variables, *J. of Optimization Theory and Applications*, Vol. 39, No. 2, (1983), pp. 293-298.
- [6] Gali, I. M. and Serag, H. M., Optimal control of cooperative elliptic systems defined on R^n , *J. of the Egyptian Mathematical Society*, Vol. 3, (1995), pp.33-39.
- [7] Hassan, H. M. and Serag, H. M., Boundary control of quasi-static problem with viscous boundary conditions, *Indian J. pure and Applied Math.*, Vol. 31, No. 7, (2000), pp. 767-772.
- [8] Khafagy, S. and Serag, H. M., Stability results of positive weak solution for singular p-Laplacian nonlinear system, *J. Appl. Math. Inf.* 36, 173-179 (2018).
- [9] Khafagy, S. and Serag, H. M., On the existence of positive weak solution for nonlinear system with singular weights, *Journal of Contemporary Mathematical Analysis (Armenian Academy of Sciences)*, Vol. 55, No. 4, (2020), pp. 259-267.

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- [10] Kotarski, W. and El- Saify, H. A., Optimality of the boundary control problem for $n \times n$ parabolic lag system, *J. Math. Anal. Appl.* 319, (2006), pp. 61-73.
 - [11] Lions, J. L., *Optimal control of a system governed by partial differential equations*, Springer-Verlag, New York 170, (1971).
 - [12] Serag, H. M. , *Optimal control of systems involving Schrodinger operators*, *Int. J. of Control and Intelligent Systems* , Canada , Vol. 32, No. 3, (2004), pp. 154-159.
 - [13] Serag, H. M. , *Distributed control for cooperative systems involving parabolic operators with an infinite number of variables*, *IMA J. of Mathematical Control and Information*, Vol. 24, No. 2, (2007), pp. 149-161.
 - [14] Serag, H. M. and Qamlo, A. H. , *On elliptic systems involving Schrodinger operator*, *The Mediterranean J. of Measurement and control*, Vol. 1, No. 2, (2005), pp. 91-96.
 - [15] Serag, H.M. and Khafagy, S., *On non - homogeneous $n \times n$ elliptic systems involving p- Laplacian with different weights*, *J. of Advanced Research in Differential Equations*, Vol. 33, (2009), pp.1-13.
 - [16] Serag, H. M., EL-Zahaby, S. A. and Abd Elrhman, L. M., *Distributed control for cooperative parabolic systems with conjugation conditions* *Journal of progressive research in mathematics* ,Vol. 4, No. 3, (2015), pp. 348-365.
 - [17] Serag, H. M., and Qamlo, A. H., *Boundary control of non-cooperative elliptic system*, *Advances in Modeling Analysis*, Vol. 38, No. 3, (2001), pp. 31-42.
 - [18] Sergienko I. V. and Deineka V. S. , *The Dirichlet and Neumann problems for elliptical equations with conjugation conditions and high-precision algorithms of their discretization*, *Cybernetics and Systems Analysis*, Vol. 37, No. 3, (2001), pp. 323-347.
 - [19] Sergienko I. V. and Deineka V. S. , *Optimal control of an elliptic system with conjugation conditions and Neumann boundary conditions*, *Cybernetics and Systems Analysis*, Vol. 40, No. 6, (2004), pp. 865-882.
 - [20] Sergienko, I. V. and Deineka, V. S., *Optimal control of distributed systems with conjugation conditions*, Springer, (2005).