




Design and Investigation of a Touch Gesture for Dividing in a Virtual Manipulative Model for Equation-solving

Thomas Janßen¹  · Estela Vallejo-Vargas¹  · Angelika Bikner-Ahsbahs¹  · David A. Reid^{1,2} 

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Abstract

Physical models for equation solving typically lack feedback regarding their appropriate use. Such feedback is possible in virtual environments and could be implemented in hybrid models. Based on an epistemological analysis, this article presents a touch gesture as a way for users to signal they want to divide both sides of an equation and a design for feedback on the use of this so-called ‘division gesture’. The design is investigated by contrasting a case study, in which students used an app with the division gesture, with a preparatory study where students had to perform corresponding actions on physical manipulatives. This investigation revealed insight into feedback functions, steps of understanding dividing with this touch gesture and, furthermore, showed problems that students have with the boundary case where the dividend is 0. The study informs possible improvements of the design of the division gesture and of the overall learning environment. The results are reflected on, in order to illuminate known problems of learning how to solve linear equations, and theorized to contribute to the wider discussion around the design of digital and physical manipulatives, in particular the design of modes of interaction enabled by new technologies.

Keywords Division · Touchscreen · Touch gesture · Feedback · Solving linear equations · Digital manipulatives

For many students, solving equations is a challenge, because of the changing role of the equal sign from an operation sign to a relational one (Kieran 1981, 2014; Alibali et al. 2007). Several (physical and virtual) manipulative models have been introduced,

✉ Thomas Janßen
janssent@uni-bremen.de

¹ University of Bremen, Bibliothekstraße 5, 28359 Bremen, Germany

² University of Agder, Universitetsveien 25, 4630 Kristiansand, Norway

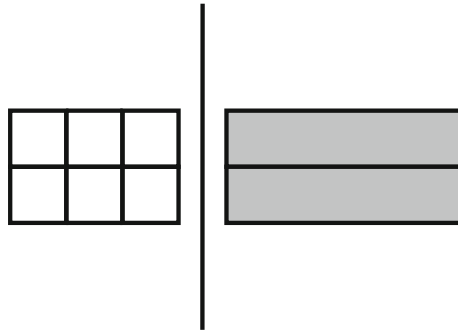


Fig. 1 Redrawing of a figure used by Wah and Picciotto (1994) to show how, at the end of a solution process, tiles can be arranged “to make it easy to see the solution of the equation” (p. 212): the tiles represent the equation $6 = 2x$

including balances and algebra tiles, to help students to overcome this challenge. A balance model associates the physical act of placing or removing objects on each side of the balance with the mathematical operations of adding to or subtracting from each side of an equation. A physical balance provides feedback if differing numbers of objects are placed or removed, but it is limited to representing natural numbers. In the algebra tiles model, it is possible to represent negative integers and variables, but no physical feedback is offered. In both models, division is difficult to represent and so division has mostly been limited to the symbolic realm (e.g. Affolter et al. 2003). In some cases (e.g. Wah and Picciotto 1994), division is envisaged as the final step, thus inhibiting students’ flexibility in choosing appropriate solving approaches (see Fig. 1, cf. Star and Rittle-Johnson 2008).

This article contributes to filling this gap by describing a more flexible design approach for division with a system developed in the *Multimodal Algebra Lernen* (Multimodal Algebra Learning) (MAL) project (Reinschlüssel et al. 2018), which explores ways to bring digital feedback to algebra tiles, both with tangible tiles and touchscreen representations. As well as offering balance-style feedback when adding and subtracting, touchscreens enable an expansion of the existing implementation of division as an equivalence transformation. A touch gesture separating equal groups was designed, referred to as the ‘division gesture’. In the process of addressing specific research questions with regard to the division gesture, this article pursues the more general aim of informing the wider discussion around the design of digital and physical manipulatives and, in particular, the design of modes of interaction enabled by new technologies.

After a short literature review making the case for the *well-prepared* use of *properly designed* manipulatives, we first present an epistemological analysis of division by a natural number as an equivalence transformation, relating it to division by natural numbers in arithmetic, and then present the design of the division gesture. Subsequently, the following questions are addressed by contrasting a case study where students used an app with the division gesture against the background of a preparatory study where students had to perform corresponding actions on physical manipulatives:

- In what ways was the division gesture adopted by the students when solving equations?

- How did the epistemological potential of the division gesture unfold in the boundary case where the dividend was 0?

As the exploration of feedback is an overall issue of the project, a particular focus will be on feedback in the investigation of the two research questions. In the conclusions, insights from the study (a) are reflected on in order to illuminate known problems of learning to solve linear equations and (b) are theorized to bring forward the greater field of the design and investigation of digital and physical tools for the teaching and learning of mathematics.

Literature Review

Given the well-known problems of students when dealing with the symbolic language of algebra (Küchemann 1981; MacGregor and Stacey 1997), providing manipulatives as a representation of equations may help visualize the relational nature of the equal sign in physical space and offer ways of seeing and grasping equivalence transformations as operations on physical or virtual objects. Leaving the realm of letters and numbers could furthermore encourage a deviation from the left-to-right search that Kieran (1984, p. 90) observed in novices (see also Alibali et al. 2007).

An approach relying on graspable material is furthermore theoretically supported by theories of grounded/embodied cognition (Lakoff and Núñez 2000; Manches and O'Malley 2012; Hayes and Kraemer 2017; Tran et al. 2017), as well as by activity-theoretical (Radford 2014; Ladel and Kortenkamp 2016) and semiotic (Mariotti 2002) views on learning. Sometimes, psychological constructs, such as cognitive load, are cited as supporting the use of manipulatives (Manches and O'Malley 2012; Antle and Wise 2013).

However, meta-analyses and research review articles report inconsistent results regarding the measured learning benefits of manipulatives in mathematics education (Sowell 1989; Carbonneau et al. 2013; Hayes and Kraemer 2017; Tran et al. 2017). Carbonneau et al. (2013) conclude that “simply incorporating manipulatives into mathematics instruction may not be enough to increase student achievement” (p. 396). (See also McNeil and Jarvin 2007, pp. 312–3 and Tran et al. 2017, p. 16.) This is well in line with the more general discussion on didactical models in mathematics education (Fischbein 1987; English and Halford 1995; Meira 1998; Gravemeijer 2011). It is therefore necessary to design and test the material and the tasks carefully, with special regard to the purpose and the target group.

A particular focus should lie on the interaction with the (virtual or physical) material, which is realized by touching and moving it and, in the case at hand, performing touch gestures on a touchscreen. Spontaneous gestures in mathematics teaching and learning have been studied intensively in the past years (Krause 2016; Chen and Herbst 2013; Goldin-Meadow 2010; Edwards 2009; Arzarello et al. 2009). Touch gestures, however, are different as they are defined in the design of digital environments. Drawing on Peirce's material semiotics, Sinclair and de Freitas (2014) theorize touch gestures as indexical signs and highlight “the feedback mechanism of digital technologies, which can talk, push and show back” (p. 371). The trace left on the screen by touch gestures “is important in drawing attention to the material engagement of the

children’s gestures [which can] result in material reconfigurations that can *give rise* to new movements of the hand” (pp. 371–2; *emphasis in original*) – and, we would like to add, encourages the re-interpretation of and reflection on the movements and the concept at hand.

The investigation of the interaction between manipulatives and students has been a strength of the referenced activity-theoretical and semiotic approaches, sometimes enriched by the theory of instrumentation (Artigue 2002). Taking an alternative approach, the present design study is set in a conceptual rather than a theoretical framework (Eisenhart 1991). We adopt Sinclair and de Freitas’ conceptualization of touch gestures and, in the design of the touch gesture, rely on principles adopted from the fields of interface design, didactical modeling and embodied cognition that will be outlined in the following paragraphs.

Design of Manipulatives and their Fit with Mathematical Concepts

In a previous article (Janßen et al. 2019), we brought together theories regarding interface design (Goguen 1999) and models in mathematics education (English and Halford 1995). Building on the work of Dedre Gentner on analogical reasoning, Goguen and English and Halford investigated the quality of an interface or of a model, respectively, by looking at the mappings defining them. We found particularly helpful Goguen’s approach of seeing interfaces and what they represent as sign systems. This allows us to investigate rigorously the representation of a given mathematical sign system (such as linear equations with the equivalence transformation of division within it) by a certain manipulative environment (such as algebra tiles with an accompanying set of allowed actions, gestures as conceptualized by Sinclair and de Freitas 2014). English and Halford (1995) contribute four principles that should be monitored:

- (1) Learners need to be able to understand the structural properties of the model.
- (2) The mapping between model and concept to be learnt shall be unambiguous to the learners.
- (3) The attributes of objects of the model that are relevant to the concept to be learnt form a cohesive conceptual structure – other attributes shall be designed not to disturb this structure.
- (4) A model shall be applicable to a range of instances. (pp. 100–102)

While English and Halford’s work refers to visual models or physical manipulatives and Goguen’s to the late 1990s’ computer interfaces (screen plus separate input devices), more recent developments towards tangible computing are reflected in the guidance given by Tran et al. (2017) for “designing optimal embodied experiences” to support mathematical thinking. Such tools should (1) “cue movements that align with the mental model of the mathematical concept” and (2) “make movements visible and give opportunities for reflection” (p. 15). All this suggests that an epistemological analysis of the mathematical concept at hand is needed as a basis for a sensible design, and is thus provided in the next section.

“Learners may manipulate physical objects seemingly meaningfully, but that does not necessarily mean that the abstract concepts that these objects aim to convey are

understood” (Tran et al. 2017, p. 16). It is therefore necessary to design the means to explore exactly how the tools are adopted and integrated into the students’ understanding. As one measure, Tran and colleagues propose to “keep a record of the process of movements and get meaningful insights from it [...] For instance, certain types of movement patterns can illustrate that students have particular misconceptions” (p. 16). In the *Methods* section, we describe how we proceeded to do just that and present the results in the subsequent one.

Epistemological Grounding of the Division Gesture

Division in the context of algebra is epistemologically based on division in the domain of arithmetic. In algebra, it is extended from being an operation on numbers that results in a quotient, to an operation on equations that results in an equivalent equation. These epistemological considerations that guided the representation of division in the MAL project will be expanded in this section.

Algebra tiles represent known and unknown integers by the *amounts* of tiles standing for positive and negative units. Thus, it is the division of natural numbers that the model needs to build upon. Two models or interpretations of division, referred to as ‘partitive’ and ‘quotative’, are widely assumed to underlie students’ elementary understanding of division. In partitive division, an amount is divided into a number of equal parts. In quotative division, an amount is divided into parts of a given size (English and Halford 1995, p. 188). Other models of division exist, but these two are the most suitable for the initial presentation of division of natural numbers (Fischbein et al. 1985).

The partitive model, as it is initially used for natural number division, involves three ‘key notions’: equal partition, whole partition and maximal partition (Reid and Vallejo-Vargas 2017; Ordoñez 2014). These refer to the idea that the distribution of a set of objects into groups divides them into groups of equal size, that the objects remain whole and that all the objects are distributed. When fractions are introduced, the second requirement (whole partition) no longer applies, and in division with remainders the third requirement is modified to state that as many objects as possible are distributed. For algebra tiles, a partitive model of division applies. When dividing a quantity such as $4x$, it is clear how it can be partitioned into two or four groups, but it is not possible to form groups of a given size as the value of x is unknown.

Central to algebra is the concept of equivalence transformation: A transformation of an equation that preserves its solution set. Adding and subtracting equally on each side is an equivalence transformation. This fact has been used already in arithmetic in strategies such as solving $356 + 199$ by solving $356 + 200$ first and then subtracting 1 from the result. Algebraically, one could write $356 + 199 = x \Leftrightarrow 356 + 200 = x + 1$. Similarly, we can argue that dividing the terms on each side by the same number (unequal to zero) is an equivalence transformation. An arrangement of tiles as in Fig. 1 may make it ‘easy to see’ the solution.

We will expand this way of representing division as an equivalence transformation in a touch gesture that consistently relies on the three key notions of dividing as partitioning.

Design of the Division Gesture

Throughout the MAL system, there is a distinction made between numbers (both known and unknown, i.e. variables) and operations. Numbers are represented by tiles that vary in colour and size to indicate if they are positive or negative (differing colours) and whether they are known or unknown (a small square represents the number one and sets of these represent larger known numbers; a rectangle as wide as the small square but longer represents an unknown number). Operations are represented by actions. This is consistent with the design guidance from Tran et al. (2017), suggesting that actions which align with mental models of mathematical concepts allow “movements to influence thoughts in ways that improve mathematical cognition” (p. 15). As noted above, the operations of addition and subtraction are represented by the actions of placing and removing tiles. Division is represented by the division gesture, which consists of two stages: The first stage of the gesture is splitting the tiles on each side into n groups of equal size (see Fig. 2), thus using the partitive model.

The tiles themselves, by being unbreakable, force whole partitions. In order to partition the equations into partial equations, the gesture is then completed by drawing $n - 1$ successive lines, by moving a fingertip across the screen without losing contact. These touch gestures allow the separation of corresponding groups and are collectively referred to as a ‘separating gesture’, one which initiates a shift of attention from the equal groupings on each side to the equivalent sub-equations.

Feedback is provided that reinforces the key notions. If the grouping of tiles satisfies the notions of equal and maximal partition, the separation gesture produces outlines separating the groups (see Fig. 3). This makes the effect of the separating gesture visible, as Tran et al. (2017) recommend. The separating gesture has no effect if the groups on each side are not equal, and if the distribution is not maximal, one group will be larger than another. There is also more active feedback to signal that a problem has

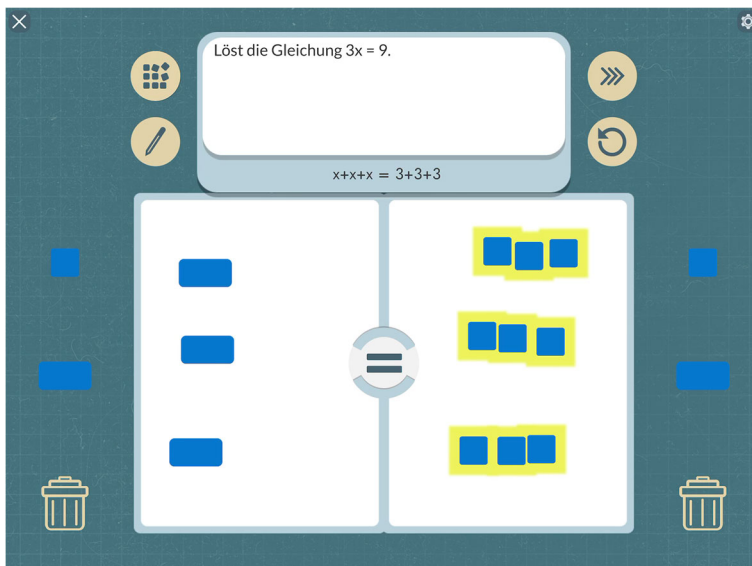


Fig. 2 Tiles split into equal groups at the first stage of the division gesture (Task 4)

occurred if a separating gesture is applied to unequal groups. In that case, the tiles wiggle briefly. The final feedback occurs when a correct grouping and separating has taken place – all sub-equations except one disappear, indicating that the focus may be reduced to one of them.

It is possible to implement a similar action with tangible tiles, albeit without the feedback the MAL system provides (see the preliminary study description that follows). In this context, the action of moving a finger across the mat to separate the corresponding groups is replaced by the action of placing a straw on the mat. This separates the corresponding groups, but has the disadvantage that the operation of division becomes associated with an object. In addition, there is no feedback available to indicate whether or not the groups on each side are equal and that the number of separations is correct, so it is possible to divide incorrectly. In the case of non-digital tiles, there is an absence of feedback about the correctness of the grouping and placing straws.

Methods of the Empirical Study

In this section, we first describe a preliminary study conducted prior to developing the touchscreen MAL system, which informed the design and which also provided some results that contrast with those from the main study reported here. We then describe the tasks and their implementations in the touchscreen environment, as well as our methods of data collection and analysis.

The Preliminary Study

In a paper-based prototyping study in 2017/2018, a ten-lesson teaching study with twelve Grade 9 students was conducted to gain experience about the teaching and

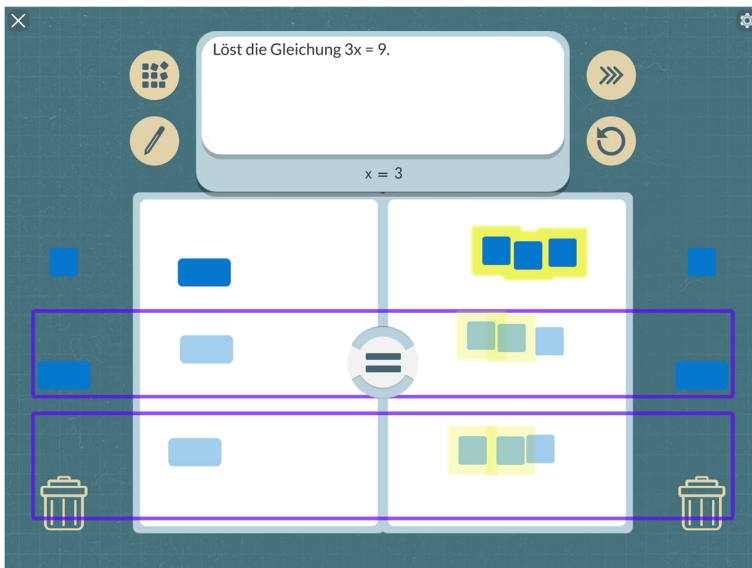


Fig. 3 Feedback after drawing lines separate equal groups in Task 4

learning of linear equations with physical, non-digital, algebra tiles. The students were chosen in order to bring about learning processes without interfering with the approach chosen by the teacher. They had been taught about solving linear equations in Grade 8 (as usual in most local school-internal curricula), but had persistent difficulties in doing so. The study was part of the initial design process of the app, which served as a comparison background for the follow-up data collection. As already noted, division was modelled here as first splitting the tiles into the same number of equal groups on each side of the equation and then separating corresponding groups with straws placed on the mat (see Fig. 4). The students were supplied with an extra sheet of paper to cover up all groups but one, to hide what they could ignore after division.

While for the most part the students divided using the material more or less as intended, there were also some relevant misuses of the division gesture. One was that most of the students did not use the paper to hide the groups they could now ignore, but instead removed those groups from the table. Recall that the action of removing tiles is associated with subtraction, so this could and, in fact, did in one case produce a misconception (see Fig. 5). Thus, English and Halford's second requirement of an unambiguous mapping between model and mathematics was clearly not met. Another misuse of the gesture was making unequal groups (see Fig. 6).

Of special interest is a task posed in Lesson 4, in which students were asked to solve two equations that had the potential for dividing 0 by a natural number: $3 \cdot (z + 2) = 6$ and $6n = 3n$. After removing 6 from each side of $3z + 6 = 6$, the students made remarks such as: "All the units cannot be gone, then there is no x ", suggesting that the absence of units no longer represents a number (a problem regarding the *subjective* applicability of the model demanded by English and Halford's fourth principle), and "Zero divided by three does not go", indicating that division of zero by a number and division of a number by zero were confused. We included such tasks in the study with the touchscreen app described in this article, to further investigate these confusions.

Task Design and Implementation

We developed fourteen tasks for the study (see Table 1). Tasks 1 through 5 were intended to familiarize the students with the app (supporting English and Halford's second principle). In Tasks 1 and 3, one of the interviewers used a screen-projected tablet to explain the general design of the app, the meaning of the left and right side of

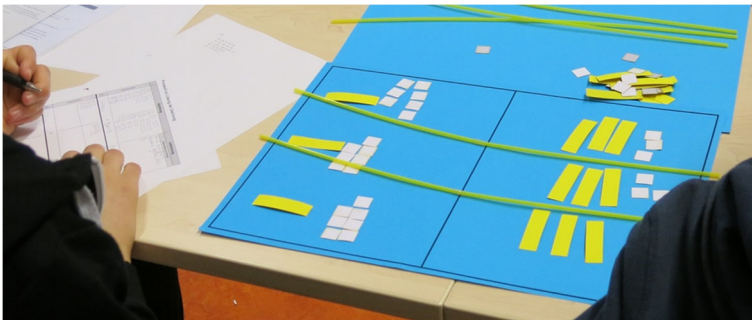


Fig. 4 The division gesture in the physical tile environment

Sym- bolisch	$12 = 6x$		-10 und $-5x$
anschaulich			Beschreibung in Worten
Sym- bolisch	$2 = x$		

Fig. 5 One pair of students repeatedly noted separate subtractions on each side after they had (correctly) performed division with the tiles and taken away all groups except one

the mat, the unit and the unknown tiles (supporting English and Halford's second principle), and how to proceed from the set-up mode to the solving mode (clarifying all signs on the screen, as recommended by Goguen). He showed (in Task 1) how subtraction was modelled and (in Task 3) how to divide by 2. Tasks 2 and 4 offered students opportunities to consolidate their interaction with the app in somewhat more complex situations. In particular, Task 4 required them to find out how to divide by 3.

All subsequent tasks were solved in pairs without interaction with the whole group. Task 5 required both subtraction and division. As a result of the observed difficulties in the preliminary study, Task 6 was intended to make clear that the empty mat on one side represents 0. At the same time, the task leads to a boundary case (Hefendehl-Hebeker 2007) with regard to division as modelled in the app—because it is defined with regard to (virtual) objects, how would students deal with their absence? It led to some interesting situations, prompting us to pose a similar task at the very end of the data collection to be solved symbolically, the 'extra task'.

In Tasks 8 through 10, three equations were presented to the students, with a prompt to solve each of them in two different ways. We were interested to see if the students would make use of the possibility to divide first, indicating flexibility in strategy use

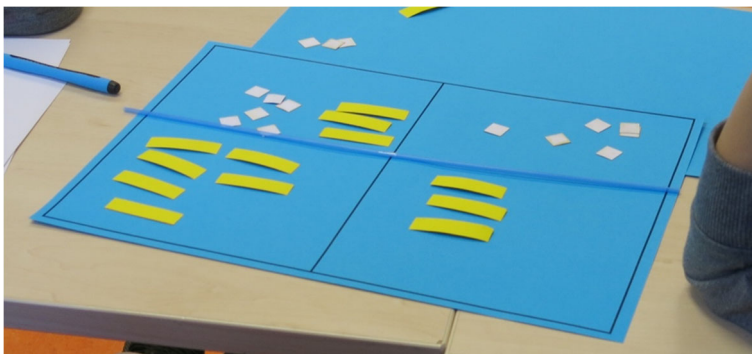
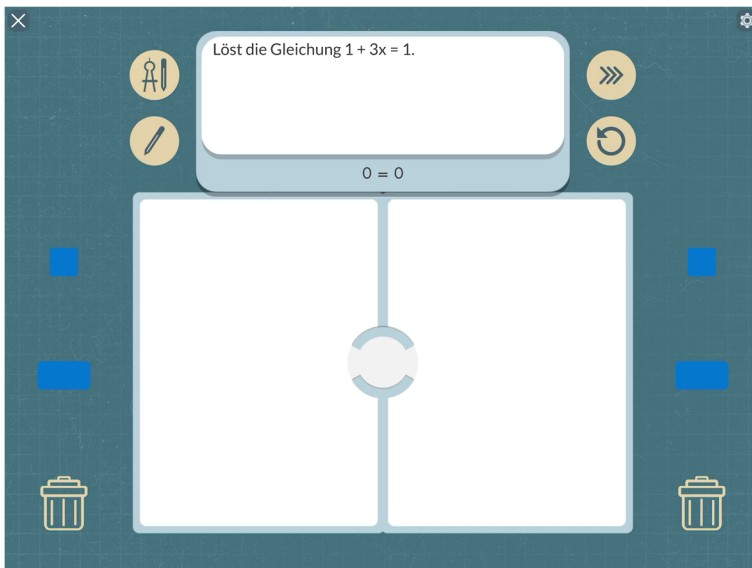


Fig. 6 An unequal grouping

Table 1 Division Gesture Tasks

Task #	Task description
1	Solve the equation $x + 2 = 5$.
2	Solve the equation $1 + 2x = x + 2$.
3	Solve the equation $2x = 4$.
4	Solve the equation $3x = 9$.
5	Solve the equation $2x + 1 = 7$.
6	Solve the equation $1 + 3x = 1$.
7	Look again at the equation $3x = 9$. This time deliberately make a mistake in the division and see what happens. Explain what the mistake is.
8	Vary the order of your transformations and solve the following equation twice in different ways: $2x + 2 = 6$.
9	Vary the order of your transformations and solve the following equation twice in different ways: $3 \cdot (2 + 2x) = 3 \cdot (x + 3)$.
10	Vary the order of your transformations and solve the following equation twice in different ways: $2x + 10 = 4x + 2$.
11, 12	<i>Reflection tasks on false solutions involving division</i>
Posttest	<i>Included 4 symbolic equations allowing different order of equivalence transformations</i>
Extra task	Solve the following equation without using the app: $2x + 3 = 3$.

(Star and Rittle-Johnson 2008). Tasks 1–10 were uploaded to the app and displayed as shown in Fig. 7. Tasks 11 and 12 were paper-based and asked the students to comment on false solutions of equations and how the app would have reacted to them.

**Fig. 7** Task 6 as it was presented in the app

In addition, a pre- and a post-test were designed to see if any marked change occurred after a brief use of the app. In the first part of the tests, the students were asked to solve five linear equations. Again, to test the students' flexibility, four of these equations allowed for the equivalence transformations to be performed in differing order without obtaining non-integer numbers (as in Tasks 8–10). The pre-test was administered two days before the task-based interviews; the post-test at the end of the tasks, just before the extra task.

Data Collection and Analysis

The design was implemented in April 2019 with eight students of a Grade 8 class in a upper-track private school in Germany. They had learnt about equations earlier in class and had been identified by their teacher as medium-achieving. The teacher also proposed the four pairs (two all-male, two all-female) who were placed at individual tables together with an interviewer each who would help on demand. This setting was chosen to encourage communication about the tasks and the system among students, and to give them the possibility to share responsibilities. Each pair was filmed and the screen recording was activated on the tablets (but partially failed). The interviewers were supplied with guidelines and asked to note the order of operations for Tasks 8 through 10.

The case study drew on five data sources: written pre-tests and post-tests, observers' notes, screen recordings, video recordings and transcripts thereof. Each of these data sources was examined for anomalies, e.g. long pauses in the usage of the app, negative feedback unexpected by the students, surprising actions given the task design, repetitions and corrections. Episodes of interest identified this way were then examined in detail, making use of the other data sources and comparing them with similar episodes of other student pairs.

To investigate the students' adoption of the division gesture, we focused on their actions in Tasks 4–10. We looked first at all available data to determine general preferences regarding order of operations in solving equations, and then in more detail at episodes of interest. Task 4 offered an initial indication of the adoption of the gesture. Task 7 asked the students to revisit the same equation and then intentionally to make a mistake to see what happened. This gave us additional information on the students' understanding of the division gesture and how they interpreted the feedback both to correct division and to incorrect groupings. The students' activity on these two tasks was transcribed and augmented by still images from the screen recordings. We then subjected the transcripts to interpretative analysis.

For a complementary analysis, we began with other data sources. For example, all the (silent) screen recordings were watched to identify episodes of interest. This was a filter to select episodes for further analysis, using videos to observe gestures, and expressions and transcripts to gain more insight into details of phrasing. For example, one pair, when they were working on Task 7, did not interact with the touchscreen for an unusually long period. Examination of the video revealed that the students seemed confused by the feedback they were receiving from the MAL system. We then compared the actions of other students on that task, and of those two students on the pre- and post-test.

The analysis of the extra task began with an examination of the written work, as this was the only task where the students were asked to write down their thinking. With three of the four pairs, what was written suggested that further analysis was needed to understand the students' reasoning and the videos were watched and transcribed.

Empirical Results and Discussion

Table 2 summarizes the students' approaches to selected tasks in which it was necessary both to divide and to subtract to solve the equation. Some tasks were intended to encourage division as the first step and these are marked as 'Division' under 'Expected first approach'. Task 6 and the extra task (both marked *) could not be solved by dividing first within the MAL system, as this would involve a non-whole partition. They also have $x = 0$ as the solution.

In general, one can observe a strong tendency to subtract first. This was expected for some tasks, but proved somewhat surprising for Tasks 9 and 10, which were specifically formulated to invite division first as a more effective approach. In the extra task, where the solution is 0, the tendency was reversed. We discuss such tasks in detail in the second sub-section. But first, we consider evidence of the students' adoption of the division gesture, their understanding of the feedback and any signs that they might be prepared to go beyond the limitations of the touchscreen app.

Students' Adoption of the Division Gesture

Task 3 was the first task that involved use of the division gesture. The teacher presented it on the screen and the students divided by 2 by imitating his actions. Task 4 required a division by 3 instead of by 2. All four pairs quickly made three equal groups on each side. Furthermore, most immediately inferred that they had to make the separating gesture twice to separate all three groups. However, one pair (Lio and Miro) seemed to think that drawing one line should be enough to show their intentions and waited for a reaction from the app. The students apparently assumed that the app would 'know' the

Table 2 Summary of solution methods used for selected tasks

Task	Expected first approach	Lio & Miro	Leona & Ciara	Giselle & Petra	Luca & Martin
Task 6*	Subtraction	Subtraction (Lio suggests to divide at first)	Subtraction (after hesitation)	Subtraction (after hesitation)	Subtraction (after hesitation)
Task 8	Subtraction	Subtraction	Division	Subtraction	Subtraction
Task 9	Division	Subtraction	Subtraction	Subtraction	Subtraction
Task 10	Division	Subtraction	Subtraction	Subtraction	Subtraction
Extra task*	Subtraction	Both: subtraction	Ciara: subtraction Leona: division	Both: subtraction	Luca: subtraction Martin: division

grouping anyway and that the separating gesture would merely be the command to divide accordingly. The interviewer hinted that they could draw one again, and they did so, exclaiming “Ah!” when this action resulted in the app fading out the superfluous tiles. This episode was interesting as it showed that, for the students, the grouping of objects was already the main part of the division process, and one separating gesture completed it.

In contrast to the preliminary study, these students showed no confusion between division and subtraction. This may have been because, in the touchscreen context, subtraction and division are very different actions. In subtraction, the students removed tiles themselves and, in division, they separated the groups, but the app removed the tiles in the superfluous groups. There was also immediate feedback when groups of unequal size were separated. Task 7 was designed to ensure that the students encountered the wiggling tile feedback in this instance, but all the pairs had already seen it by accident at this point. In contrast, in the preliminary study, students could form unequal groups (as in Fig. 6) and, unless the teacher was watching and intervened, they could proceed without any indication that their manipulations were incorrect.

In relation to understanding the feedback, Tasks 4–7 showed that students had understood the basic idea that the groups needed to be equal in size. The feedback for incorrect groupings – the tiles wiggling and the drawn lines disappearing – were quickly accepted. In one case, however, the feedback did not meet one pair’s expectations. For Task 7, Ciara and Leona grouped the tiles they had so that the two groups did not have exactly the same tiles in each group and then a finger was moved once across the screen to make a separation line, as if they were dividing by two (see Fig. 8).

Their expectation for negative feedback at this point was clear, since they looked puzzled and waited for several seconds, and then tried to ‘force’ the negative feedback by pressing on the ‘I am ready’ button (which activates the feedback once the equation is set up) without success (see Fig. 9).

This reveals an ambiguity in the feedback as designed. From the students’ perspective, they had made an error, as they had divided by two instead of dividing by three. The MAL system, however, interpreted their separating gesture as being only the first of two lines in a division by three. In the design, it is necessary to make one separation

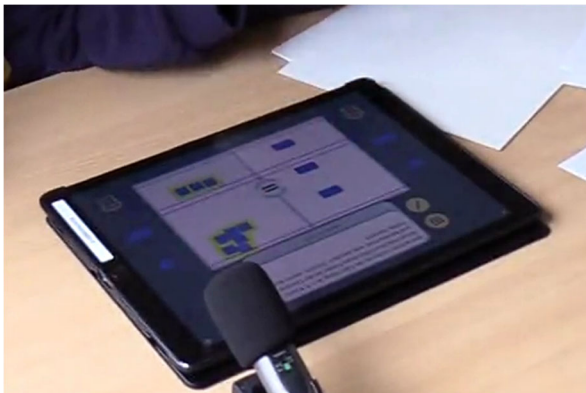


Fig. 8 Ciara and Leona try to make a mistake by dividing by 2

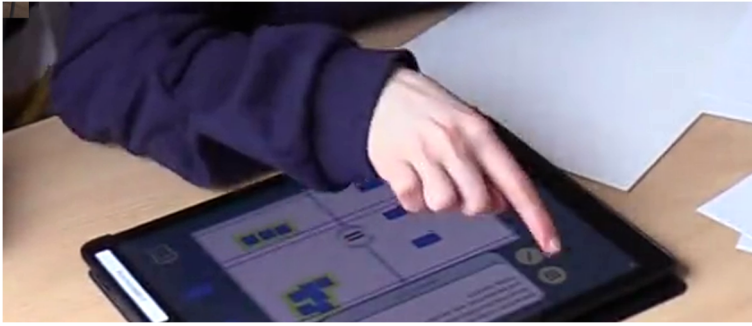


Fig. 9 Leona presses the ‘I am ready’ button to get feedback

gesture as a first step whenever dividing by any number; hence, no error is signalled provided a correct division is still possible. The app’s behaviour here indicates an incomplete division, because a division by three would still be possible. But the students’ intention was to show they were done: they even pressed a button to try to provoke feedback.

At first, Lio and Miro had assumed that drawing a single line signalled that the division was complete and that the number of groups indicated the divisor. In contrast, Leona and Ciara expected that drawing a single line would be interpreted as division by 2. The students’ confusion arose from a mismatch between the completed operation they had in mind (division by two) and the incomplete operation modelled by the system (division by three). This provided us with the insight that drawing lines was being used for two purposes: *to signal separating* and also *to signal that the division is complete*. However, in the operation of division, these are two stages in the division and should be handled separately in the interaction with the digital system. In the next section, we discuss some possible design changes to address this mismatch.

Leona and Ciara were the only pair who divided first when solving Task 8 ($2x + 2 = 6$). Figure 10 shows how they arranged the tiles and performed the separation gesture. Their first attempt to divide resulted in the wiggling tiles feedback. They then rearranged the tiles on the right side, so they were more lined up with the unknown tiles on the left side. Our suspicion is that they were intending to divide only $2x$ and 6 by 2, leaving the 2 on the left side out of the division, and violating the key notion of maximal partition. It may be that the two unit tiles are regarded as superfluous.

After they received the wiggly tiles feedback again, they noticed that they had not yet touched the ‘I am ready’ button (which showed a compass and pencil, as they were still constructing the equation, as far as it was concerned). They touched it, and then attempted the same separating gesture again, we assume because they thought that the previous feedback was due to their forgetting to touch the ‘I am ready’ button. They became confused, and the interviewer intervened, asking, “Are you trying to divide by two first?” After the girls discussed other ways to proceed, the interviewer asked, “What do you have to do to divide?” Leona answered, “Make a line across”, and the interviewer asked, “And before that?” Ciara then moved the two tiles on the left next to the two unknown tiles and made a successful separating gesture.

This was the first time they had tried to divide when a side contained both unknown tiles and unit tiles. As previously they had been successful grouping the variables on the left and the units of the right, they may have assumed dividing the units on the left was

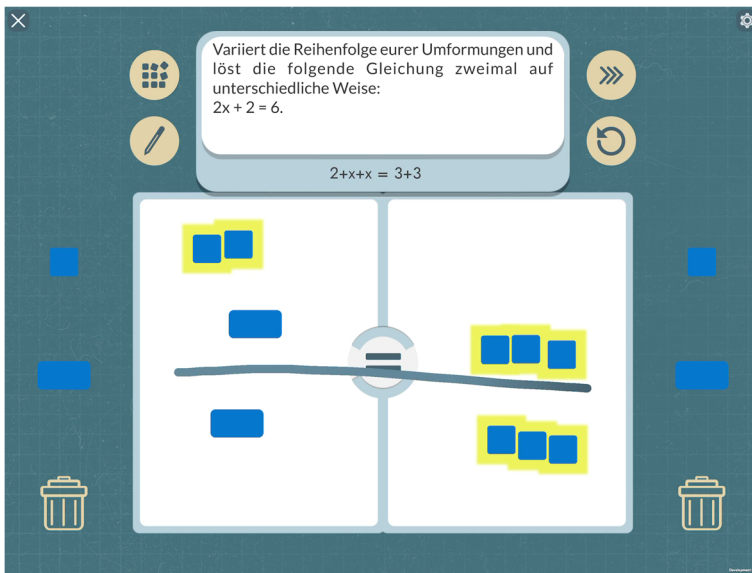


Fig. 10 Leona and Ciara's second attempt to divide by 2 in Task 8

unnecessary. Here, the feedback was useful in not allowing them to proceed (a second-order feedback, unlike cases in the preliminary study when students had made unequal groups and then proceeded with the division). However, it was not sufficient for them to work out what was wrong; they needed a hint to reflect on their movements (Tran et al. 2017, p.14).

In one case, students tested whether or not they were limited to whole partitions. In Task 12, they were supposed to check whether fictional student Marius' symbolic solution was correct or not, by using the app. Marius had divided by 2 in a situation where there were an odd number of unit tiles on the left side of the mat and an odd number of unknowns on the right side of the mat. Ciara and Leona formed two equal groups on each side, placing the leftover unit and unknown between the groups, and then tried to draw a separating line through those tiles (see Fig. 11).



Fig. 11 Ciara and Leona trying to divide a unit and an unknown by 2

The students engaged in a discussion about why this attempt was not possible when using the app (see Transcript 1) to divide by two $4x + 3$ on the left side, and $3x + 4$ on the right side of the mat, respectively. As they explained, they would need to solve the equation in a different way, since it is not possible to represent half a (unit or unknown) tile when using the app.

Transcript 1

Line number	Name	Utterance
1	Ciara	(Ciara draws a separation line through the leftover tile. The app provides negative feedback.) It is not possible.
2	Both	(laughter)
3	Leona	(Partly simultaneously) Nope- (laughs) Oh man.
4	Interviewer	All right. Sure and-
6	Ciara	So, you just have to solve it differently-
7	Leona	Yes. You can't divide directly.
8	Interviewer	(Partly simultaneously) M-hm- (nods slightly) exactly. And you also said why you just can't divide by two. At which point-? What prevents that? Which numbers allow you to ni-
9	Leona	The three.
10	Ciara	(simultaneously points to an equation on the task sheet)
11	Interviewer	Exactly the three-e-
12	Leona	(Partly simultaneously) Because you can't represent half a thing.

This is a relevant episode, since it shows the way the app allowed these students to extrapolate the ideas they already had for the case of exact divisions (when the objects on each side of the mat can be evenly split in an exact number of groups) to a more general case; namely, the case of fractions (when the objects cannot be evenly split into an exact number of groups, unless we consider to ‘cut’ some objects in equal portions), which, at least theoretically, is possible.

Dividing 0 as a Boundary Case

As described above, Task 6 was deliberately designed to lead to a situation where the division gesture was to be performed on the empty mat – representing 0 – on one side

$$\begin{aligned}
 12 + 12 + 12 + 12 &= 4x + 8 & 1:4 \\
 12 &= x + 2 & 1-2 \\
 10 &= x \\
 L &= \{10\}
 \end{aligned}$$

Fig. 12 Leona's post-test solution involving dividing first – the last line reads “ $L = \{10\}$ ”, with the double-struck L denoting the solution set (Lösungsmenge)

$2x + 3 = 3$	$ -3$	Problem: Wenn man $2x = 0$ hat und auf beiden Seiten durch 2 teilen möchte, weiß ich nicht wie ich das machen soll weil wenn man 0 durch 2 teilt kommt glaube ich wieder 0 als Ergebnis raus. Aber da bin ich mir nicht so sicher! :ü
$2x = 0$	$:2$	
$x = 0$		
	?	

Fig. 13 Giselle's note expressing uncertainty about whether or not her last equivalence transformation is legitimate

of the equation. Indeed, most of the students had problems accepting that 0 could be divided with the division gesture. Lio and Miro considered subtracting first, but hesitated as there would be no remaining tiles on the right side. Lio then suggested dividing first, but observed that this would involve the breaking of tiles. The interviewer reminded them that subtracting first is a possibility, and when they tried subtracting 1 and dividing by 3, they found the app gave the answer $x = 0$, which they had expected.

At first, Leona and Ciara said the empty right side (that resulted when they had subtracted 1) indicated that the equation has no solution. They then reinterpreted the empty side as 0 and Leona suggested the solution $x = 0$ without dividing. The interviewer prompted them to divide using the gesture, and they did so. Giselle and Petra at first saw the empty side as “not right”, and Giselle explicitly said, “you cannot divide by zero”. The interviewer pointed out that they were not dividing by zero. They used

The image shows three stages of handwritten work on a piece of paper:

- Top stage:** The equation $2x + 3 = 3$ is written and then crossed out with a large blue scribble. Below it, $2x = 0$ is written and also crossed out.
- Middle stage:** The equation $2x + 3 = 3$ is written again, with a circled '1' next to the right-hand side. This is also crossed out with a large blue scribble. Below it, $2x = 0$ is written and crossed out.
- Bottom stage:** A new solution is shown:

$$\begin{array}{rcl}
 2x + 3 & = & 3 \\
 x + 1,5 & = & 1,5 \\
 \underline{x} & = & 0
 \end{array}
 \qquad
 \begin{array}{l}
 | :2 \\
 | -1,5
 \end{array}$$

Fig. 14 Miro's solution to the equation $2x + 3 = 3$ (in Germany, a comma is used as the decimal separator, and a colon denotes division)

$$\begin{array}{l} \cancel{2x + 3 = 3} \quad | :3 \\ \cancel{2x = 0} \\ \cancel{x = 0} \end{array}$$

Fig. 15 Lio's first solution

the gesture to divide by 3 and expressed surprise that it worked. Petra said she always thought this was impossible. Luca and Martin initially said the empty right side “makes no sense”. The interviewer gave them the hint that zero was a possible solution, and they then divided using the gesture.

Based on our observation of the students' difficulties with dividing 0, an extra task was posed after the post-test to probe the extent to which their additional experience with the gesture would help the students to deal with such situations. The extra task was to solve $2x + 3 = 3$ on paper, independently. Of the eight students, three chose to divide first. This is noteworthy because, in Tasks 8 to 10 (which asked for two different ways for solving three equations), only Leona and Ciara in Task 8 chose to divide first (incorrectly). In the pre-test, all eight students consistently divided as the last step in the solving process; in the post-test (written before the extra task), only Leona deviated from this schema in one case (see Fig. 12). This was disappointing, as we had hoped that experiences with the division gesture would support flexibility.

We infer that these three students who divided first in the extra task aimed to avoid dividing 0. Unease is also expressed in a note added by one of the students, Giselle, who did so (see Fig. 13). A translation of what she wrote is: “Problem: When one has $2x = 0$ and one wants to divide by 2 on both sides, I don't know how I am supposed to do that, because when one divides 0 by 2 one gets 0 again as a result. But I am not so sure about that.”

$$\begin{array}{l} 2x + 3 = 3 \quad | :2 \\ x + 1,5 = 1,5 \quad | -1,5 \\ x = 0 \\ \hline 2x + 3 = 3 \quad | -3 \\ 2x = 0 \quad | :2 \\ x = 0 \end{array} \quad \left. \vphantom{\begin{array}{l} 2x + 3 = 3 \\ x + 1,5 = 1,5 \\ x = 0 \\ 2x + 3 = 3 \\ 2x = 0 \\ x = 0 \end{array}} \right\} 2 \text{ Optionen}$$

Fig. 16 Lio's two options

Lio and Miro discussed whether it was possible to subtract first extensively. They both had obtained the correct solution, but in different ways. Lio first subtracted 3 and then divided by 2. Miro started out twice with the subtraction of 3, but then deleted these approaches. He then chose to divide by 2 first (undeterred by the resulting decimals) and then subtracted 1.5 in the second step (see Fig. 14).

Miro then offered an objection to Lio's solution, "You cannot divide zero by two, can you?" and Lio reacted by suggesting, "so really it cannot be solved". Lio then solved the equation by dividing first, writing down exactly what Miro had written. Miro then offered an additional objection to Lio's first solution, which implied $2x = x$, that for them seemed to be wrong. In response, Lio acknowledged that his first solution was wrong and crossed it out (see Fig. 15).

At this point, the interviewer reminded them of the experience they had had when solving Task 6 with the app. Lio remembered how they solved the equation $3x = 0$ with the app and recognized that it was similar to what he had done in his first solution: "And that's exactly what that is. It is the same, so I also did that." The interviewer asked Lio why he had crossed out his first solution and Lio said, "Our math teacher said that you can't divide anything by, you cannot divide anything by zero and then you should just leave it". The interviewer pointed out that Lio did not actually divide *by* 0, "But what have you done here?" Miro noticed immediately, "You have zero divided by something", which made Lio wonder whether his first solution was correct, "Oh, I did not, ah, I divided by two. Zero divided by two. Then maybe that's right (he laughs)." Even though Lio did not appear to be very confident, Miro seemed to be and pointed out that "then there are two solutions". Lio wrote out his first solution again and labeled the two solutions as "2 options" (see Fig. 16).

The tension between the students' customary approach to solving equations, by subtracting first, and their reluctance to divide when 0 is involved, led to a deeper exploration of the process of solving equations. This was the intent of Tasks 8–10, but those seem to have been less effective than tasks involving dividing zero. Tasks 8–10 may have shown the students that they *could* divide first, but they did not communicate that sometimes one *should* divide first, for example for reasons of efficiency.

Conclusions

Returning to our research questions –

- In what ways was the division gesture adopted by the students when solving equations?
- How did the epistemological potential of the division gesture unfold in the boundary case where the dividend is 0?

– we can report that the students quickly understood the division gesture and seemed not to confuse division and subtraction in the way some students in the preliminary study had done. Initial confusion about the number of times the separation gesture must be made were easily addressed by the interviewers, but suggest design changes to allow more independent use of the app in classroom contexts. The boundary case where the dividend is zero turned out to be important in encouraging flexibility in solution

approaches. Tasks that to us seemed well suited to dividing first were solved, for the most part, by subtracting first, and it was only in cases where subtracting first produced a situation where the dividend was zero that the students felt the need to divide first.

In the remainder of this final section, we build on some of the insights from the study (a) to illuminate known problems of learning to solve linear equations and (b) to bring forward the greater field of the design and investigation of digital and physical tools for the teaching and learning of mathematics.

Lessons for the Didactics of Linear Equations

In the third section of this article, the epistemological soundness of the designed division gesture was established. However, while the gesture is justified for the designers, who are familiar with the algebraic principles behind it, for the students it may seem arbitrary. In fact, some aspects of the gesture are – it could just as well be activated by separating one group from the others, as Lio and Miro attempted. The use of multiple separations is not strictly necessary to reflect algebraic principles. The students have no way to know that dividing zero being possible *is* a choice the designers made to reflect algebraic principles. Hence, they might question it, especially as it might conflict with their intuitions.

Dividing zero into equal parts seems impossible, as there is nothing from which to make the parts. There is also a difference between the division situation and subtraction. The action of removing equal units from both sides corresponds to an experience the students have had with objects, one they know preserves equality. However, the division gesture is unlike an action on objects: objects do not vanish when equal groups are formed. This divergence from experience may create a feeling of uncertainty or arbitrariness about the gesture.

These issues suggest that learning to solve linear equations through actions in an environment like the MAL system involves a shift in awareness or intuition concerning the nature of the rules. Furthermore, it may require an ontological shift concerning the nature of nothingness, from an absence, to a special object, namely 0, that can be divided. Understanding the division gesture may also require a shift in understanding of the nature of solving an equation, as it involves an operation that affects the equation as a whole, not just one part of it.

Lessons for the Design and Development Of Digital and Physical Tools

Regarding the design and development of digital and physical tools for the teaching and learning of mathematics, we observe the importance of feedback and the different kinds of feedback afforded by physical and digital tools, we observe the need for both physical and digital models and the modes of interaction they specify to correspond closely to the mathematical objects and operations they model, and we see the advantages didactical studies bring to the design process, and vice versa.

Feedback

Algebra tiles and manipulatives in general have long been advocated in mathematics education, because they make abstract mathematical objects and operations visible and

tangible. An important advantage they bring is the feedback they give. For example, the equality $3 + 4 = 7$ is visually evident when modelled with counters. Our ability to compare small quantities means that visible amounts provide more feedback than abstract representations of those amounts. In our study, we observe that physical tiles, by being whole objects, provide feedback that enforced the key notion of whole partition, while, in the touchscreen environment, this feedback was less strong. Leona and Ciara felt it might be possible to split tiles to divide an odd number by 2. Physical tiles, however, do not provide feedback regarding maximal and equal partitions.

In the preliminary study, students made unequal partitions and, unless the teacher noticed, proceeded unchecked. When students using the touchscreen app made unequal partitions, however, they encountered the wiggling tiles feedback and were not able to proceed until they made the groups equal. In addition, there is second-order feedback, when students expect a particular feedback, but it does not occur, for instance when Lio and Miro expected a single separation line to complete a division by 3. That nothing happened when they pressed the ‘I am ready’ button was itself a kind of feedback, but one that gave them no information. Second-order feedback forces reflection on the previous movements.

Another aspect of the feedback we have not discussed here is the symbolic representation of the tiles on each side that the app provides. We saw no evidence that the students paid much attention to this feedback, but it may have been a factor in their choice to multiply out each side in Task 9: $3 \cdot (2 + 2x) = 3 \cdot (x + 3)$. If they had immediately placed three groups of $(x + 3)$ on the right side, the symbolic feedback would have been $(x + 3) + (x + 3) + (x + 3)$, not $3 \cdot (x + 3)$, as the app does not yet make the difference between repeated addition and multiplication (see Figs. 2 and 10 for illustration) – and it is an open design question if and how it should do so.

Models and the Modelled

As Goguen (1999) points out, the strength of a representation, as a special kind of semiotic morphism, is the tightness of the similarity between the representation and the system represented. In our case, the touchscreen app offers a representation of linear equations, which are made up of two groups of numbers and variables related by the equal sign. These are modelled by objects, the tiles, and by spaces, the two sides of the mat separated by the equality line. Equations are transformed by equivalence transformations, operations carried out on each side of the equation that result in a new equation with the same solution set, hence, is logically equivalent to the previous one.

These operations are modelled as actions on the tiles and at the core of the MAL system is the idea of feedback that indicates whether or not the action represents an equivalence transformation. The fit of these elements – (virtual) objects, (virtual) spaces and actions, including (touch) gestures – should be considered in the design of any physical, digital or hybrid tool for the teaching and learning of mathematics.

One obvious weakness of algebra tiles is their limitation to integers, making whole division a requirement of the division gesture. This limitation, however, can also be seen as a potential. When students notice that a model does not account for all instances, they can feel invited to invent extensions. These may, at first, consist of changes to the model, like Leona and Ciara’s idea to cut the tiles. In the end, however, students will become aware that no model is ever perfect (Goguen 1999). This, in turn, may help them appreciate the abstract unity of mathematics.

Manipulatives Teaching Researchers Lessons

The division gesture served as a means of unearthing the problem that students have with dividing 0. That is, it revealed a didactically interesting phenomenon that seems not to have been extensively researched and, furthermore, showed that it goes beyond a simple confusion with the division $by\ 0$. The MAL system provides a specific sign system where actions have to follow rules, thus defining relations between the elements of the sign system. This sign system allows students to represent linear equations with material-like, virtual objects, which can be manipulated by the division gesture (among other possibilities).

This approach becomes problematic with the absence of (virtual) objects, when one side of the mat is empty. The mat without any tiles, representing 0, is not only a boundary case of division, but also one of the possibilities of “the material engagement of the children’s [touch] gestures” (Sinclair and de Freitas 2014, p. 371). With no tiles to be touched and perceived, the students must learn to ‘see’ the empty side is a sign that they can act on.

Thus, as our analysis shows, the offered bodily experience is not self-evident. Nevertheless, it may provide a context in which the students’ unease with dividing 0 (whatever its roots may be in the particular case) might be addressed by a teacher, as was done with Lio and Miro. The question how operating with zero in different contexts (for a list, see Ruwisch 2008) can and should be designed by means of physical and digital tools deserves further investigation that can benefit from the interpretation offered with regard to the particular case described in this article.

Further, the confusion that Lio and Miro, and Leona and Ciara, experienced allows us to see that the process of dividing does not consist of two actions, grouping and separating (as we thought), but rather of three. We can now extract the steps needed to be followed by the students:

- group each side of the equation into the same number of equal groups (keeping the tiles whole and using them all);
- associate each group with a corresponding group on the other side using a gesture to separate all these pairs;
- indicate that the division is finished.

In the preliminary study, we had observed that some students were confused because the act of removing the superfluous groups was the same as the act of removing when subtracting. Our suggestion that the superfluous groups be covered, rather than removed, was not taken up, perhaps because it adds yet another object (not corresponding to a mathematical object) to an already cluttered tabletop. In the touchscreen environment, we were able to have the app automatically cover and remove (fade out) the superfluous groups. But this, then, collapsed the pairing of the groups and the completion of the division. As soon as the final separation is done, the app fades out the superfluous groups. In the preliminary study, with physical tiles, we considered the final act of removing the superfluous groups to be a limitation of the model and the automatic removal of the tiles in the touchscreen app to be a design improvement. The students’ confusion, however, has taught us that our didactical analysis of division was flawed. This awareness will, in turn, inform the next stage of the design.

Outlook

In the previous section, we have hinted at some topics of relevance, first regarding the didactics of linear equations and second regarding the greater field of the design and investigation of digital and physical tools for the teaching and learning of mathematics. In the outlook, future steps regarding the representation of equations with such tools are outlined.

The present study's scope was limited to a small section of possible situations of teaching and learning, covering only eight students working for a short time on a specific mathematical topic, with a digital tool which still is to be refined further for practical use in school. An in-depth elaboration of feedback functions is still lacking, e.g. the function to monitor actions (the steps of a division gesture) or to question expectations (as in second-order feedback).

Both the preliminary study and the study using the touchscreen app occurred as part of the larger MAL project, focused on developing smart objects: physical tiles that provide the same feedback as is available in the touchscreen app. The insights from these studies will feed the design of these smart objects. For example, we will have to consider how the three steps in dividing are best modelled. Grouping is fairly straightforward, but it is not clear to us whether separating should become a single action (one separation gesture with multiple fingers instead of drawing several successive lines) or if the drawing of successive lines should be followed by an explicit confirmation that the division is complete that then triggers feedback as to the correctness of the separations. This would have the advantage that the separating itself gets some more attention.

We are also interested in the possibilities of the touchscreen app for extending the algebra tile model beyond integers, by incorporating the splitting of tiles attempted by Leona and Ciara. Physical tiles without electronics, as used in the preliminary study, could be cut into (a limited number of) pieces and then be grouped. A virtual tile app might also allow this. But with the exception of a division by 2, this division of tiles (representing the expansion of a fraction) always would have to take place *before* division as an equivalence transformation by means of the separating gesture. The didactical merits of this segmentation in the students' learning processes would require further investigation.

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Compliance with Ethical Standards

Conflict of Interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

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