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"Discrete-Continuous Dynamic Choice Models: Identification and Conditional Choice Probability Estimation"

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Abstract

This paper develops a general framework for models, static or dynamic, in which agents simultaneously make both discrete and continuous choices. I show that such models are non-parametrically identified. Based on the constructive identification arguments, I build a novel two-step estimation method in the lineage of Hotz and Miller (1993) but extended to discrete and continuous choice models. The method is especially attractive for complex dynamic models because it significantly reduces the computational burden associated with their estimation. To illustrate my new method, I estimate a dynamic model of female labor supply and consumption.

Keywords: Discrete and continuous choice, dynamic model, identification, structural estimation, female labor supply.

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1 Introduction

Many economic problems involve joint discrete and continuous choices. For example, a firm can decide on a pricing scheme (per unit, flat rate) and the corresponding price level (Timmins, 2002). It can decide what to produce and the corresponding sales price (Crawford et al., 2019). Firms also decide whether to register their business and how many workers to hire (Ulyssea, 2018). Students select their majors and decide how much effort to exert into their study (Ahn et al., 2019). Consumers can decide what to buy and how much of it to consume (e.g. appliance choice and demand for energy, Dubin and McFadden, 1984). In housing, buyers decide their house size and housing tenure (Hanemann, 1984; Bajari et al., 2013). The buyer of a car selects a model and the mileage of the car (Bento et al., 2009). Individuals decide whether to retire or not and how much they plan to consume accordingly (Iskhakov et al., 2017). Similarly, labor force participation and consumption are joint choices for potential workers (Altuğ and Miller, 1998; Blundell et al., 2016).

In these cases, a rational agent makes both decisions *simultaneously*. Here 'simultaneous' means that given the information she has, the agent jointly make both choices. As a result, the discrete choice is endogenous with respect to the continuous choice and vice versa. To take a labor market example, if she works, a woman consumes differently than if she does not work: she has two different conditional consumption choices. And at the same time, her decision of whether to work or not is dependent on these two conditional continuous choices. Unfortunately, the identification of models with simultaneous choices is difficult (Matzkin, 2007). Indeed, there is a core *observability problem* because we only observe the continuous choice made in the selected discrete alternative, and we do not know the counterfactual choices the individual would have made in the other alternatives. Ideally, we would like to recover counterfactual continuous choices using the choices of individuals with similar characteristics but who chose another alternative. However, doing so is not possible if individuals also differ on factors which are unobserved by the econometrician and affect both continuous and the discrete choices. In this case, two identical individuals as measured by their observed covariates might still differ on the unobserved dimension. There is a problem of *selection on unobservables*, which prevents the identification of counterfactual continuous choices. To further pursue the example, if an econometrician observes that working individuals consume more than unemployed individuals, she cannot identify whether this is because the consumption choice conditional on working is truly higher or because individuals with an unobserved higher taste for consumption select themselves more into working.

This paper develops a general framework of *simultaneous discrete-continuous choice* models suited for static or dynamic problems. I provide minimal necessary conditions under which non-parametric identification of the model can be obtained, using an *instrument* for the *unobserved selection*. Then, building upon the identification, I provide an estimation method for these models. The method is attractive because it yields significant computational gains over the estimation of dynamic models, in the lineage of Hotz and Miller (1993). I also show how to apply this method to a dynamic discrete-continuous choice model of female labor force participation and consumption.

The first contribution of this paper is that I provide a constructive proof for the non-parametric identification of a general class of structural models in which agents simultaneously make a discrete and a continuous choices. To do so, I require an instrument that must be *relevant* for the selection into the discrete choice and *excluded* from the conditional continuous choices. For example, the previous discrete choice can be a good instrument in the presence of switching costs. Indeed, it impacts the current discrete decision through the switching cost. Conditional on the current discrete decision, it is excluded from the current continuous choice. In this way, observable differences in the distribution of the choices with respect to the instrument can be attributed to unobserved differences in selection and not differences in continuous choices. I show that, paired with restrictions on the effect of unobserved heterogeneity on the continuous choice (monotonicity, rank invariance), the instrument allows us to achieve non-parametric identification of the optimal discrete and continuous choices. Once the optimal choices are identified, it can be further shown that the rest of the model is identified, for example, by exploiting first-order conditions (in the spirit of Blundell et al., 1997).

The second contribution of the paper is in terms of estimation. I build a two-step estimation method, similar to Hotz and Miller (1993) but for discrete *and continuous* choices. In the first step, one estimates the policies, which I name after Hotz-Miller's CCPs: Conditional Continuous Choices (*CCCs*), and Conditional Choice Probabilities (*CCPs*). This step is built on the identification arguments. The policies are estimated directly from the data without solving the structural model. To this end, I propose a novel method that estimates the entire monotone continuous choice functions directly instead of proceeding pointwise. In the second step, one uses the estimated CCCs and CCPs to estimate the structure of the model. For example, I exploit the fact that within my general framework, the payoffs are related to optimal choices through the first-order conditions. My estimation method is attractive because it yields sizeable *computational gains*. Typical dynamic discrete *or* continuous choice models are difficult to estimate because they involve solving the theoretical model (either by backward recursion or fixed point algorithms). Dynamic discrete-continuous choices model are even more difficult to estimate because the mixed choices can introduce kinks and non-concavities in the value function (Iskhakov et al., 2017). Given that I can recover the CCCs and CCPs in the first stage, I can exploit them to estimate the rest of the model without having to compute the value function or solve the model. It yields computational gains comparable to the computational gains generated by Hotz and Miller (1993) in the dynamic discrete choice literature. The gains are so important that they not only reduce the time required to estimate the models, but also make it possible to estimate models that have thus far been deemed computationally intractable. In this respect, my method may facilitate the use of simultaneous discrete-continuous choice models.

Finally, I illustrate my method by building and estimating a dynamic life-cycle model of women's consumption and labor force participation, in the spirit of Blundell et al. (1997, 2016). This application has been implemented under a parametric framework for practical reasons. First, doing so avoids 'curse of dimensionality' concerns, and second, it makes my empirical findings comparable to the existing literature. I add to existing models a more flexible distribution of unobserved heterogeneity. Thanks to my method, I flexibly estimate the complete *distribution* of consumption choices and working probabilities at any given set of observed covariates (assets, earnings, family status, education, etc.). Hence, I can recover distributions such as that of the marginal propensity to consume when earnings or benefits increase for any individual. I use these estimated policies to estimate the parameters of the structural model. For example, I find a constant relative risk aversion of 1.63, close to the value of 1.56 in Blundell et al. (1994) and the value of 1.53 in Alan et al. (2009). All things considered, the method developed in this paper allows for more complete models in terms of unobserved heterogeneity, with a faster estimation and I still find estimates consistent with the existing literature. Therefore the method is very attractive in practice.

Related Literature:

There is a vast empirical literature that uses *dynamic discrete choice* models. For example, such works study labor market transition and career choice (Keane and Wolpin, 1997), fertility choice (Eckstein and Wolpin, 1989) and education choice (Arcidiacono, 2004). Starting from the bus replacement problem of Rust (1987), developments have been made in the estimation and identification of these models. They include non-exhaustively: Hotz and Miller (1993); Hotz et al.

(1994); Rust (1994); Magnac and Thesmar (2002); Aguirregabiria and Mira (2002, 2007); Kasahara and Shimotsu (2009); Arcidiacono and Miller (2011); Hu and Shum (2012); Arcidiacono and Miller (2019, 2020); Abbring and Daljord (2020). For a survey, see Aguirregabiria and Mira (2010) or Arcidiacono and Ellickson (2011).

Similarly, the literature on *dynamic continuous choice* models is also voluminous, especially concerning consumption/saving choices (Carroll, 2006) or investment choices (Hong and Shum, 2010). There are also methods such as Bajari et al. (2007) that can be applied to either dynamic discrete choice models or dynamic continuous choice models (but not both).

However, in many cases, economic problems involve several joint decisions, not only one discrete choice or only one continuous choice. For example, labor force participation is very much related to saving decisions. By focusing only on one of these two dimensions and ignoring the other endogenous choice, one might be missing something important. Unfortunately, empirical applications of the dynamic discrete-continuous choice framework are less popular, as there was no generally identified setup prior to this work. For example, Blundell et al. (1997) provide identification of such models once the optimal choices are identified but do not directly address the identification of the choices. The existing literature employs several tricks to overcome the unobserved selection problem. The most common is to have implicit or explicit assumptions about the unobserved selection process. For example, Dubin and McFadden (1984), Hanemann (1984) or Bento et al. (2009) have specific assumptions about their error disturbances (independence, measurement errors, known joint distribution), which generate a specific selection process. Blevins (2014) studies the non-parametric identification of dynamic discrete-continuous choice models but assumes a very specific timing in which the discrete choice takes place before the realization of the nonseparable shock and the continuous decision. Hence, unobserved selection is not allowed to depend on nonseparable shock realization. Similarly, Iskhakov et al. (2017) break the simultaneity issue by assuming that the discrete retirement choice is taken before and based on the expectations about the continuous consumption choice. Murphy (2018) also imposes that the two choices are taken sequentially. In his paper, parcel owners first decide whether to build or not, and only afterwards, a nonseparable price shock is realized and they decide on their house size accordingly if they chose to build in the first stage. The problem is that the sequentiality of the choices is a strong assumption, and it might lead to biased results if the true decision process is in fact simultaneous. For example, in Murphy (2018), it is likely that small price realization will increase both the house size *and* the probability of building a house. The imposed timing ignores this, as the discrete building choice is only based on expectations about the price shock and corresponding house size choice. Thus it might miss part of what is truly happening in the data. My general simultaneous choice framework nests these different timing assumptions, which have testable implications within the framework. Thus I can verify when the sequentiality assumption is reasonable. Another technique is to discretize the continuous choice such that the model becomes a dynamic discrete choice model. For example, De Groote and Verboven (2019) study the adoption of solar photovoltaic systems and discretize the continuous level of adopted capacities. This is appealing, as it allows the application of known techniques in the dynamic discrete choice literature. However, discretizing the continuous choice is implicitly equivalent to making an assumption about the unobserved selection process via the assumption on the distribution of the additive discrete error terms. I show in this paper that by exploiting the continuous nature of the choice, the unobserved selection process can be identified instead of being assumed. Therefore, one can focus on the true discrete-continuous choice problem without discretizing the continuous choice. Another solution is to completely abstract from the nonseparable shock, i.e., to assume that individuals with the same observed covariates will make the same continuous choice. A more convincing alternative is to reduce the level of unobserved heterogeneity, for example, by including only a finite number of unobserved types (Blundell et al., 2016). My approach is more general, as I allow for a more flexible distribution of unobserved heterogeneity.

The closest literature for the identification of simultaneous discrete-continuous choice can be found in static reduced-form identification analysis of non-parametric simultaneous equations (Matzkin, 2007, 2008; Imbens and Newey, 2009), nonseparable models (Chesher, 2003; Chernozhukov et al., 2020), the discrete-continuous Roy model (Newey, 2007), treatment effects with endogenous selection into treatment (Heckman and Vytlacil, 2005; Chernozhukov and Hansen, 2005, 2006, 2008), or in reduced-form identification analysis of dynamic treatment effects (Heckman and Navarro, 2007; Heckman et al., 2016). In this literature, the idea of using an instrument for non-parametric identification of simultaneous equations is frequent (Newey and Powell, 2003). However, my main contribution here is that I obtain identification under very weak and testable assumptions on the instrument. I only need a condition that the instrument is *relevant*, except at most at a finite set of points. This relevance yields a non-overlapping condition, similar to what Torgovitsky (2015) and D'Haultfœuille and Février (2015) employ in a different context with continuous treatment. Using the additional assumption that the optimal choice is monotone with respect to the unobserved nonseparable shock (as in quantile regression), the relevance is sufficient to recover identification. Indeed, I show that there exists a unique monotone function identified by the system, while if I had proceeded pointwise, uniqueness would have not held. By proceeding pointwise, other studies mentioned above need either stronger assumptions on the effect of the instrument (often regarding the rank of a matrix of the probabilities of selecting into treatment with respect to the instrument) or a different, less general setup for the selection mechanism (e.g., an additive process). To the best of my knowledge, Vuong and Xu (2017) are the only other authors to exploit the power of monotonicity in a similar fashion as I do for identification. However, they use it to relax strict monotonicity and still maintain a strong rank condition on the effect of the instrument on the selection process, while I am as general as possible with my mild condition of relevance. By developing a framework where the optimal choices take the form of a triangular simultaneous system of equations, I establish a connection and show how one can use the results from this literature on reduced-form identification to identify more general dynamic structural models.

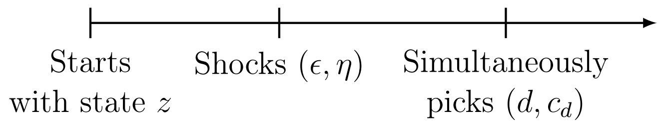
I also contribute to the literature on faster estimation methods that avoid the computation of the value function (Rust, 1987; Hotz and Miller, 1993; Hotz et al., 1994; Carroll, 2006; Arcidiacono and Miller, 2011; Iskhakov et al., 2017). I provide a faster alternative to indirect inference and the most recent developments of endogenous grid methods (Iskhakov et al., 2017). A comparison of different estimation methods can be found in section 6.

Finally, my application contributes to a large literature on labor market participation and consumption, focusing on women. For example, see, Heckman and Macurdy (1980); Blundell et al. (2016). Thanks to my method, I estimate the complete distribution of individual responses.

This paper is organized as follows. Section 2 describes a general *simultaneous discrete-continuous choice* framework. Section 3 discusses identification. Section 4 shows how dynamic models are embedded in the framework. Section 5 describes the estimation method built on the constructive identification arguments. Section 6 compares my novel method with existing methods using Monte-Carlo simulations. Section 7 estimates an empirical discrete-continuous choice model of women's labor supply and consumption. Section 8 concludes the paper.

2 Framework

I consider the general problem with the following timing where the agent:



The individual simultaneously selects a discrete action $d \in \mathcal{D} = \{0, 1\}$ and accordingly makes one continuous choice $c_d \in \mathcal{C}_d \subset \mathbb{R}$ to maximize his payoff. The simultaneous decision is made given some state $z \in \mathcal{Z}$ observed by the researcher, as well as two random preference shocks $\epsilon = (\epsilon_0, \epsilon_1) \in \mathcal{E} \subset \mathbb{R}^2$ and $\eta \in \mathcal{H} \subset \mathbb{R}$ that are unobserved by the econometrician. ϵ only affects the discrete choice d, while η impacts the continuous choice c and the discrete choice. Note that the *same* η impacts the continuous choice decision in both discrete states (c_0 and c_1). In other words, I have *rank invariance* (Heckman et al., 1997; Chernozhukov and Hansen, 2005), that is, η is not discrete-choice specific.[1] The payoffs of the agent are given by the function $\mathcal{V}_d(c_d, z, \eta, \epsilon)$. The agent simultaneously selects d and c_d to solve:

$$\max_{d, c_d} \quad \mathcal{V}_d(c_d, z, \eta, \epsilon) \tag{1}$$

I require additional assumptions for tractability and identification of the model.

Assumption 1 (Additive Separability) *The shock ϵ enters the payoff additively such that $\forall d \in \{0, 1\}$:*

$$\mathcal{V}_d(c_d, z, \eta, \epsilon) = \tilde{v}_d(c_d, z, \eta) + \epsilon_d.$$

The additive separability assumption is usual in the discrete choice model literature (Rust, 1987; Arcidiacono and Miller, 2011). It applies to ϵ, while η can still enter the payoff in a nonseparable manner. A consequence of Assumption 1 is that the optimal continuous policy functions will not

[1] In theory, the continuous choices could even represent different variables depending on the discrete option selected: for example, if d represents the choice between working and studying, c might represent the amount of time worked and the effort of the student respectively, hence the possibly different support. The main restriction is that even if they represent two different choices, these two continuous choices are impacted by the *same* unobserved shock η.

depend directly on ϵ. Indeed, c_d are defined as the (interior) solutions to the maximization of the conditional payoff. Here, because of the additivity, we have that

$$c_d = \underset{c}{argmax}\ (\tilde{v}_d(c, z, \eta) + \epsilon_d) \iff c_d = \underset{c}{argmax}\ \tilde{v}_d(c, z, \eta).$$

Assumption 2 (Instrument) *State z contains two kinds of variables $z = (x, w)$, where $x \in \mathcal{X}$ represent general state variables and w is an instrument such that $\forall d \in \{0, 1\}$:*

$$\tilde{v}_d(c_d, z, \eta) = \tilde{v}_d(c_d, x, w, \eta) = v_d(c_d, x, \eta) + m_d(x, w, \eta).$$

The support $\mathcal{W}$ of w contains two different values, as $\mathcal{D} = \{0, 1\}$.[2]

w is an 'instrument' to recover the optimal continuous policies. On the one hand, with the additive functional form of $m_d(x, w, \eta)$, w is *excluded* from the optimal continuous policy choice. Indeed,

$$c_d \ = \ \underset{c}{argmax}\ (v_d(c, x, \eta) + m_d(x, w, \eta) + \epsilon_d) \ = \ \underset{c}{argmax}\ v_d(c, x, \eta)$$

On the other hand, it might still be *relevant* and impact the discrete choice.

Assumption 3 (Monotonicity) *The conditional payoff functions are twice continuously differentiable such that $\forall d \in \{0, 1\}$*

$$\frac{\partial^2 v_d(c_d, x, \eta)}{\partial c_d \partial \eta} > 0.$$

Assumption 3 implies that, conditional on $(D = d, X = x)$, the conditional optimal policy function $c^*_d(\eta, x)$ is C^1 and strictly increases with respect to η.[3] It ensures that there will be a one-to-one mapping from $\eta \in \mathcal{H}$ to $c_d \in \mathcal{C}_d$ for all d and x. This kind of monotonicity condition has been widely used for identification (Chernozhukov and Hansen, 2005; Bajari et al., 2007; Hong and Shum, 2010). In a sense, it means that I only identify monotone effects of the unobserved

[2] w being binary is a minimal condition for identification when d is binary. You can have discrete or even continuous w, the identification proof follows the same line, the m_d objects to be identified are just slightly different.

[3] Note Assumption 3 is equivalent to $\partial^2 \tilde{v}_d(c_d, x, w, \eta)/(\partial c_d \partial \eta) > 0$. Indeed, because of additivity in Assumption 2:

$$\frac{\partial^2 \tilde{v}_d(c_d, x, w, \eta)}{\partial c_d \partial \eta} = \frac{\partial^2 v_d(c_d, x, \eta)}{\partial c_d \partial \eta} + \underbrace{\frac{\partial}{\partial \eta}\left(\frac{\partial m_d(x, w, \eta)}{\partial c_d}\right)}_{=0}.$$

nonseparable source of heterogeneity (η, here). A very important implication of Assumption 3 is that this framework applies to problems where we observe continuous choices in *each* discrete option. For example, it does not apply directly to the problem of an investor who decides whether to invest ($d = 1$) or not ($d = 0$) and the corresponding investment conditional on investing ($d = 1$) (Hong and Shum, 2010). Indeed, in this case, $c_0^*(h) = 0$ for all h and it is not strictly increasing. However, it would apply to a discrete choice of portfolio and corresponding conditional investment. Similarly, it does not apply directly to the house construction problem of Murphy (2018), where the agent only decides of his house size if he chooses to build one $d = 1$. However, this setup still applies to a slightly modified version of the building problem where the discrete decision would be to build ($d = 1$) or to buy (or rent) an existing house ($d = 0$), and c_d would be the corresponding house size/housing service.

Assumptions 1, 2 and 3 yield the following triangular structure for the optimal choices:

$$\begin{cases} C_d = c_d^*(X, \eta) \\ D = d^*(c_0, c_1, X, W, \eta, \epsilon) \end{cases} \implies \begin{cases} C_d = c_d^*(X, \eta) \\ D = d^*(X, W, \eta, \epsilon) \end{cases} .$$

With this triangular structure, there is a link between my general structure and (reduced-form) systems of simultaneous equations (Chesher, 2003; Matzkin, 2008; Imbens and Newey, 2009), as well as with the related literature on heterogeneous/quantile treatment effects (Chernozhukov and Hansen, 2005; Vuong and Xu, 2017). To identify the structure, one needs to first identify the choices. To identify the system of choices, I need additional assumptions on the shocks.

Assumption 4 (Independent of w) *Conditional on $X = x$, the pair of shocks (ϵ, η) is independent and identically distributed and is independent of W.*

Assumption 5 (Independent Shocks) *Conditional on $X = x$, the discrete choice-specific and the continuous choice-specific shocks are independent of one another: $\eta \perp \epsilon$*

Assumption 6 (Continuous choice shock distribution) *Conditional on $X = x$, the continuous choice-specific shock η has an atomless distribution.*

Normalization 1 (Continuous shock) *Conditional on $X = x$, η is distributed as $\mathcal{U}(0, 1)$.*

Assumption 7 (Discrete choice shock distribution) *Conditional on $X = x$, the discrete choice-specific shock ϵ has continuous support and is independent and identically distributed with continuous distribution $F_{\epsilon|X=x}(\epsilon)$ over the full support $\mathbb{R}$.*

Assumption 8 (Regularity) $\forall d \in \{0,1\}$,

$$\forall (x,w,\eta): \quad \max_c \; v_d(c,x,w,\eta) < \infty.$$

Assumption 4 is an independence assumption between the shocks and the instrument, conditional on the other observables X. Assumption 5 assumes independence between the two shocks. Both of these assumptions are not as restrictive as they may appear. Indeed, note that the additive term $m_d(x,w,\eta)$ can be interpreted in two different ways that we cannot identify separately. First, in Assumption 2, I describe m_d as an additive part of the payoff $\tilde{v}_d$. Second, it can also be interpreted as part of a more general additive discrete shock term $\tilde{\epsilon}_d$ where $\tilde{\epsilon}_d$ could depend on w, x and η, i.e., $\tilde{\epsilon}_d(x,w,\eta) = m_d(x,w,\eta) + \epsilon_d$. Therefore, in this sense, the independence assumptions 4 and 5 on ϵ_d are still general: we could have a general $\tilde{\epsilon}_d$ that is not independent of $z = (x,w)$ or η. Then, ϵ_d is the remaining part of the discrete shock that is independent of η and w.
The main restriction is the exclusion restriction that $\eta \perp W|X$. It is crucial for the identification to have the same distribution of η, regardless of the value of w.

The atomless Assumption 6 is made to obtain smooth conditional distributions of continuous choices. Here, I cannot identify the distribution of η separately from the rest of the problem. Therefore, as is standard in the literature (Blundell et al., 1997; Matzkin, 2003), I normalize it to a uniform distribution, which represents the quantiles of any atomless continuous distribution (conditional on X), in Normalization 1.

Assumption 7 is a regularity condition on the distribution of the discrete choice-specific shock. Along with Assumption 7, Assumption 8 is another regularity condition on the functional form that ensures that $0 < Pr(d|\eta, z) < 1$ for all d, η, z. Indeed:

$$Pr(D=0|\eta,z) = Pr\Big(\epsilon_0 - \epsilon_1 > (\max_c \; \tilde{v}_1(c,z,\eta)) - (\max_c \; \tilde{v}_0(c,z,\eta)) \mid \eta, z\Big).$$

By Assumption 7, $\epsilon_0 - \epsilon_1$ has full support $\mathbb{R}$, independent of η (Assumption 5). By Assumption 8, the payoff functions difference is bounded. Thus, $0 < Pr(D=0|\eta,z) < 1 \; \forall \eta, z$. Since $Pr(D = 1|\eta,z) = 1 - Pr(D=0|\eta,z)$, we have that:

$$\forall d, \eta, z \qquad 0 < Pr(d|\eta,z) < 1.$$

Similar to the distribution of η, the distribution of ϵ will not be identified in my setup. Thus, I need to assume that this distribution is known. Therefore, in practice, I will later follow the literature on (static or dynamic) discrete choice models (McFadden, 1980; Rust, 1987; Hotz and Miller, 1993; Matzkin, 1993; Magnac and Thesmar, 2002; Arcidiacono and Miller, 2011) and assume (generalized) extreme-value distributions. This family of distributions is convenient as it yields closed-form solutions linking the conditional value functions and the choice probabilities.[4] However, other distributions can be used (Chiong et al., 2016).

We need one last (testable) condition under which the framework is identified.

Assumption 9 (Instrument Relevance)

Assumption 9a *For any $x \in \mathcal{X}$, the additive terms of the payoff are such that there is, at most, a finite set of K (with $0 \leq K < \infty$) values h of η such that*

$$m_0(x, w = 0, h) - m_1(x, w = 0, h) = m_0(x, w = 1, h) - m_1(x, w = 1, h).$$

Assumption 9b *For any $x \in \mathcal{X}$, there exist two different values of w, denoted $w = 0$ and $w = 1$, for which the additive terms of the payoff are such that there is, at most, a finite set of K (with $0 \leq K < \infty$) values h of η such that*

$$Pr(D = 0|\eta = h, x, W = 1) - Pr(D = 0|\eta = h, x, W = 0) = 0.$$

Identification of the optimal policies requires that the instrument is *sufficiently relevant*. As stated in Assumption 9: there must be at most a finite set of values of η at which the instrument is *not* relevant for identification. In other words, $m_0(w = 0, x, h) - m_1(w = 0, x, h) \neq m_0(w = 1, x, h) - m_1(w = 1, x, h)$ or $Pr(D = 0|\eta = h, x, W = 1) \neq Pr(D = 0|\eta = h, x, W = 0)$ except at most at a finite set of values h. If this is the case, the instrument provides sufficient information to identify the continuous policies. This condition is fairly intuitive and is considerably less restrictive than *full rank* assumptions and other assumptions made for the identification of heterogeneous/quantile treatment effects (Newey and Powell, 2003; Chernozhukov and Hansen, 2005, 2006, 2008). As shown later in the identification proof, the idea is that by fully exploiting the monotonicity of the conditional continuous choices, full rank conditions on the selection process with respect to the instrument are more restrictive than necessary for their identification. A similar intuition about the power of monotonicity can be found in Vuong and Xu (2017). Note that Condition 9b has

[4]Notice that generalized extreme-value distributions implicitely eliminates the possible dependence of the distribution of ϵ on X. Dependence is allowed, but as the distribution is not identified anyway, one usually abstracts from it in practice.

testable implications for the observed reduced forms. It allows to test whether the structural model is identified, I will discuss this in the next section.

Lemma 1 (Equivalence) *Under Assumptions 2, 4 and 5, Assumptions 9a and 9b are equivalent.*

Proof. By construction:

$$\begin{aligned} Pr(D=0|\eta,x,w) = Pr\Big(\epsilon_0 - \epsilon_1 & \\ & > (\underset{c}{max}\ v_1(c,x,\eta)) - (\underset{c}{max}\ v_0(c,x,\eta)) \\ & \quad + \quad m_1(x,w,\eta) - m_0(x,w,\eta)\ \Big|\ \eta,x,w\Big). \end{aligned}$$

Since $(\underset{c}{max}\ v_1(c,x,\eta)) - (\underset{c}{max}\ v_0(c,x,\eta))$ is independent of w (Assumption 2) and since $\epsilon_d \perp (w,\eta)|x$ (Assumptions 4 and 5), we have that:

$$\begin{aligned} & Pr(D=0|\eta,x,w=0) \neq Pr(D=0|\eta,x,w=1) \\ \iff \quad & m_0(x,w=0,\eta) - m_1(x,w=0,\eta)\ \neq\ m_0(x,w=1,\eta) - m_1(x,w=1,\eta). \end{aligned}$$

Thus, Assumption 9a expressed in terms of structural forms is equivalent to Assumption 9b on the optimal conditional choice probabilities. ■

Summary of the setup:

Under the assumptions above, I consider the general problem where an individual selects (d, c_d) to maximize his payoff:

$$\underset{d,c_d}{max}\quad v_d(c_d,x,\eta)\ +\ m_d(x,w,\eta)\ +\ \epsilon_d.$$

The general setup described here can apply not only to a wide range of static but also *dynamic* discrete-continuous choice models. I provide one static example below, and I will describe how it embeds dynamic models in section 4. The idea is that, in the dynamic case, v_d represents the current conditional value functions, embedding the expectations about the future, as in Hotz and Miller (1993).

EXAMPLE 1: STATIC DEMAND FOR ENERGY

In the spirit of Dubin and McFadden (1984), consider the demand for energy with discrete appliance choice. The agent simultaneously decides between two energy sources $d = 0$ or 1 and the corresponding amount of energy she will consume (c_d). x contains observable information about the cost of each energy source and possibly the wealth or income of the agents. ϵ_d represents individual-specific unobserved preferences for each energy type. η could represent some other unobserved characteristics of the consumer impacting both her preference for the energy type and the amount of energy she wants to consume. The higher η is, the higher c_d for all d.

In practice, the greatest challenge is to find a good instrument w. Here, a good w could be some variable about the accessibility of each energy alternative. For example, the previous alternative selected by the individual might be a good instrument. First, conditional on the present alternative choice (d) and on current wealth (included in x), the past (w) should have no impact on the current energy consumption level (c_d). Thus, it would be an exogenous instrument. Moreover, changing alternatives is costly in terms of time, so individuals who were previously using energy 0 are less likely to use energy 1 now than their counterparts who were already using it. In this case, the agent incurs some disutility cost of switching from one alternative to the other and no cost if he does not switch. In other words, for all x and h, for alternative 1, $m_1(x, w = 0, h) < 0$ and $m_1(x, w = 1, h) = 0$, and for alternative 0 $m_0(x, w = 0, h) = 0$ and $m_0(x, w = 1, h) < 0$. In this case, $m_0(x, w = 0, h) - m_0(x, w = 1, h) > 0$ and $m_1(x, w = 0, h) - m_1(x, w = 1, h) < 0$, so they are different, and the instrument is relevant (Assumption 9).

Discussion of Simultaneity:

My general simultaneous choice framework nests the non-simultaneous timings where either the discrete or continuous choice is taken before and based on expectations about the other choice (and its shock realization). These two timings have *testable implications* for the optimal choices within the simultaneous framework:

- If the discrete choice is taken first, before the realization of η and the continuous choice, then the CCP $Pr(D = d|\eta, X, W)$ is *independent of* η. Indeed, η is not yet realized. The discrete choice is only based on expectations about η and the corresponding $c_d^*(x, \eta)$.

- Conversely, if the continuous choice is made first, before the discrete choice and the realization of ϵ, then the CCCs $c_d^*(x, \eta)$ are *independent of* d, i.e., $c_0^*(x, \eta) = c_1^*(x, \eta)\ \forall \eta$.

Since I identify the policy functions c_d^* and $Pr(D = 1|\eta, X, W)$ in the simultaneous framework, I can *test the timing* of the model.

In the next section, I study the identification of the discrete-continuous choice model.

3 Identification

I observe data on the variables (D, C_d, X, W). I only observe C_0 if $D = 0$ and C_1 if $D = 1$. For all (x, w, η) in $\mathcal{X} \times \mathcal{W} \times \mathcal{H}$, I study non-parametric identification of the following objects: the optimal *Conditional Continuous Choices* (CCCs) $c_d^*(\eta, x)$, the optimal *Conditional Choice Probabilities* (CCPs) $Pr(d|\eta, w, x)$ for $d = 0$ and $d = 1$, and the indirect payoff functions (taken at the optimal c) $\underset{c}{max}\ v_d(c, x, \eta)$ and $m_d(x, w, \eta)$. Without loss of generality, in this section, I focus on any given x value and omit x from what follows. This is not an issue because x is exogenous in this problem, and my assumptions about the distribution of the shocks are conditional on $X = x$. First, I characterize the reduced forms and constraints imposed by the structure. Then, I discuss the identification of the optimal policies (CCCs and CCPs) and of the payoffs.

3.1 Reduced forms and constraints

In the data, I observe (d, c_d, w). w is exogenous in the model while c_d and d are endogenous choices. There is a *fundamental observability problem*, as I only observe one value of c_d depending on the discrete choice selected:

$$c_d = c_0(1 - d) + c_1 d.$$

I do not observe both 'potential outcomes', only the selected one. Therefore, from the data, I recover the distribution of c conditional on d and w. I denote it $F_{C_d|d,w}(c_d) = Pr(C_d \leq c_d|D = d, W = w)$. I also recover the conditional probability of selecting d knowing w, denoted as $p_{d|w} = Pr(D = d|W = w)$. In other words, the data provide us with the following *reduced-form* functions, which exhaust all relevant information:

$$R = \{\{p_{d|w}\}_{(d,w)\in\{0,1\}\times\{0,1\}},\ \{F_{C_d|d,w}(c_d)\}_{\forall c_d\in\mathcal{C}_d,(d,w)\in\{0,1\}\times\{0,1\}}\}.$$

An important *vocabulary* remark is in order: in this paper, $Pr(D = d|W = w)$ is part of the *reduced forms*, while $Pr(D = d|\eta = h, W = w)$ is what I call the *conditional choice probabilities* (CCPs) or selection on unobservables (η) that I want to identify. This differs from the dynamic discrete choice literature, where $Pr(D = d|W = w)$ are actually called CCPs (Hotz and Miller, 1993; Arcidiacono and Miller, 2011). However, here, I have simultaneous choices and a nonseparable shock η, which affects both choices. Thus, the true counterparts to the usual CCPs are $Pr(D = d|\eta = h, W = w)$ for all d and not $Pr(D = d|W = w)$, hence the different terminology.

Now let us see the constraints implied by the structure on the reduced forms.

Lemma 2 *Under Assumptions 3-8 of the structural model, the distribution $F_{C_d|d,w}(c_d)$: $\mathcal{C}_d \to [0, 1]$ is C^1 and strictly increasing.*

Proof. The distribution of η is C^1 and strictly increasing (Assumption 6). As previously explained, under Assumptions 5, 7 and 8, the probability of selecting d knowing $\eta = h$ is different from zero (or one) for all h and for both w (i.e., $0 < Pr(d|h, w) < 1$). As a consequence, the distribution function of η conditional on d and w is also C^1 and strictly increasing. Now, note that, by the monotonicity Assumption 3, the distribution functions of c_d (conditional on w) are strictly monotone transformations of the distribution of $\eta|d$. In other words:

$$\underbrace{Pr(\eta \leq \ h \ |d, w)}_{=F_{\eta|d,w}(h)} = \underbrace{Pr(C_d \leq \ c^*_d(h) \ |d, w)}_{=F_{C_d|d,w}(c^*_d(h))} \quad \forall d, w.$$

Therefore, since $F_{\eta|d,w}(h)$ is C^1 and strictly increasing (with respect to h), $F_{C_d|d,w}(c^*_d(h))$ is also C^1 and strictly increasing (with respect to h). Now, since $c^*_d(h)$ are C^1 and strictly increase with respect to h (Assumption 3), $F_{C_d|d,w}(c_d)$ are also C^1 and strictly increase with respect to c_d for all d. ■

Lemma 2 provides some regularity conditions on the distributions generated by the structural form. The fact that $F_{C_d|d,w}(c_d)$ are C^1 is helpful for the testable conditions of our model provided in what follows.

Lemma 3 *Under Assumption 9b in which K is defined, there is the same finite number K of values of c_0 and c_1 such that*

$$\frac{d\Big(\overbrace{F_{C_d|d,W=1}(c_d)p_{d|1} - F_{C_d|d,W=0}(c_d)p_{d|0}}^{\Delta F_{C_d}(c_d)}\Big)}{dc_d} = 0 \qquad \forall d.$$

Proof. Appendix A ■

Under the relevance Assumption 9, there is only a finite number K of values h of η such that the instrument has no effect $Pr(d|\eta = h, w = 1) = Pr(d|\eta = h, w = 0)$. I will show that when this happens, we have $d(Pr(\eta \leq h|d, W = 1) - Pr(\eta \leq h|d, W = 0))/dh = 0$. Now, by the monotonicity of the optimal continuous choice, the observed conditional distributions of $C_d|d$ are transformations of the unobserved conditional distribution of $\eta|d$. Therefore, even if we do not observe the conditional distribution of $\eta|d$, we know that if the instrument is sufficiently relevant (Assumption 9), Lemma 3 will be fulfilled.

Lemma 3 yields observable and *testable* implications on the reduced forms. Indeed, the functions $\Delta F_{c_d}(c_d)$ are directly observable for all d, as is $d\Delta F_{c_d}(c_d)/dc_d$ (the derivative is well defined, cf Lemma 2). It can be used to test the relevance Assumption 9 that is crucial for identification. The idea is that if the function $\Delta F_{c_d}(c_d)$ is *flat* on a segment of values of c_d, then there is a segment of values of η such that the instrument is not relevant. In this case, the instrument has no differential impact on the conditional choice probabilities, so it does not help to identify the optimal continuous policy. If this is the case, the model is not point identified for this segment of η.

Lemmas 2 and 3 fully characterize the impact of my structure on the reduced forms. With these reduced forms, one would like to identify the structural form, i.e., the values of the payoffs $v_d(c_d^*(h), h)$ (taken at the optimal continuous choice) and $m_d(w, h)$.

The difficulty for the identification is that the shock η is unobserved and nonseparable. As a consequence, there is an unobserved variable that affects every structural object we would like to identify: the conditional payoffs $v_d(c_d^*(h), h)$ and $m_d(w, h)$, the optimal discrete choice $d^*(h, w, \epsilon)$, the corresponding conditional choice probabilities (CCPs) $Pr(d|h, w) = \mathbb{E}_\epsilon[d^*(h, w, \epsilon)|h, w]$ and the optimal conditional continuous choices $c_d^*(h)$ (CCCs) for all d. Thus, I first need to back out the value h of η. To do so, I will first identify the conditional continuous choices $c_d^*(h)$ from the reduced forms R by exploiting monotonicity, Bayes' law and the relevant instrument w. Then, I will use monotonicity to identify η from the data by inverting the monotone $c_d^*(h)$: $h = (c_d^*)^{-1}(c_d)$. Once I identify the values h of the shock η, I can identify the conditional choice probabilities (CCPs) of selecting alternative d knowing $\eta = h, w$: $Pr(d|h, w)$. Then, I use these $Pr(d|h, w)$ as in Hotz and Miller (1993) to identify the difference in payoffs between the two alternatives. Finally, I discuss identification of the payoffs under additional structural assumptions in the next section.

3.2 Identification of Conditional Continuous Choices (CCCs)

Difficulty: observability problem

As in the literature on continuous choices (Matzkin, 2003; Bajari et al., 2007; Hong and Shum, 2010), I would like to exploit the monotonicity Assumption 3 to identify the optimal continuous choices. For any value of w, by monotonicity, we have that

$$\begin{aligned} & Pr(\eta \leq \ h \ |d) = Pr(C_d \leq \ c_d^*(h) \ |d) \quad \forall d \\ \overset{\text{under Lemma 2}}{\Longleftrightarrow} \quad & c_d^*(h) = F_{C_d|d}^{-1}(Pr(\eta \leq h|d)) \quad \forall d. \end{aligned}$$

Thus, *if* we knew the distribution of η conditional on d, we could recover the optimal conditional continuous choices $c_d^*(h)$ by using the monotonicity of the conditional distribution of C_d knowing d to invert it. However, here we only know the unconditional distribution of η (by Assumption 6).[5] The conditional distributions of $\eta|d$ are unobserved. They depend on an unobserved selection mechanism: $Pr(\eta \leq h|d) = Pr(d|\eta \leq h)Pr(\eta \leq h)/Pr(d)$. Because of this selection with simultaneous discrete and continuous choices, we cannot use usual inversion methods based on monotonicity for identification.

Another way to see the problem would be the following. Knowing that η is uniform and independent of observables (Assumptions 4 and 6), we have:

$$\begin{aligned} Pr(\eta \leq \ h) &= \overbrace{Pr(C_d \leq c_d^*(h))}^{\text{unobserved}} \quad \forall d \\ &= Pr(C_d \leq c_d^*(h); (D = 0 \cup D = 1)) \\ &= \underbrace{Pr(C_0 \leq c_0^*(h);\ D = 0)}_{\text{observed}} + \underbrace{Pr(C_0 \leq c_0^*(h);\ D = 1)}_{\text{unobserved}} \\ &= \underbrace{Pr(C_1 \leq c_1^*(h);\ D = 0)}_{\text{unobserved}} + \underbrace{Pr(C_1 \leq c_1^*(h);\ D = 1)}_{\text{observed}}. \end{aligned}$$

Imagine that we observed both c_0 and c_1 for every individual, independently of the discrete choice d, i.e., if $D = 0$ or $D = 1$ is selected, we observe both c_0 and c_1. Then, we observe the unconditional distribution of $c_d^*(h)$: $Pr(C_d \leq c_d^*(h))$. In this case, knowing that η is uniform, one could exploit monotonicity to recover $c_d^*(h)$ by inverting its unconditional distribution: $c_d^*(h) = F_{C_d}^{-1}(Pr(\eta \leq h))$. However, here again, we observe c_0 if $D = 0$ and c_1 if $D = 1$. Because of this selection, we cannot

[5] Note that instead of normalizing the unconditional distribution, we could normalize the functional form of one of the conditional $\eta|d$ distributions. $\eta|_{D=0} \sim \mathcal{U}(0, 1)$ for example. However, the problem would be the same, as we still would not know the distribution of the other conditional shock $\eta|_{D=1}$.

identify the optimal continuous choice policies.

Identification via the instrument:

Instead, to identify $c_d^*(h)$, I use the properties of the instrument (Assumption 2) to obtain structural restrictions. Using Bayes' law we have, $\forall h \in [0,1]$:

$$\begin{aligned}
h = \quad & Pr(\eta \leq h) \\
= \quad & Pr(\eta \leq h|w) \\
= \quad & Pr(\eta \leq h \mid D=0, w)Pr(D=0|w) + Pr(\eta \leq h \mid D=1, w)Pr(D=1|w) \\
= \quad & Pr(c \leq c_0^*(h) \mid D=0, w)Pr(D=0|w) \\
& + Pr(c \leq c_1^*(h) \mid D=1, w)Pr(D=1|w) \\
= \quad & F_{C_0|D=0,w}(c_0^*(h))Pr(D=0|w) + F_{C_1|D=1,w}(c_1^*(h))Pr(D=1|w) \\
= \quad & F_{C_0|D=0,w}(c_0^*(h))p_{0|w} + F_{C_1|D=1,w}(c_1^*(h))p_{1|w},
\end{aligned}$$

where the first equality comes from the fact that $\eta \sim \mathcal{U}[0,1]$ by normalization. The second follows because $\eta \perp w$ by Assumption 4. The third equality comes from the law of total probability. The fourth equality comes from the monotonicity of $c_d^*(h)$. The fifth and sixth equalities are just changes in notation.

Thus:

$$h = F_{C_0|D=0,w}(c_0^*(h))p_{0|w} + F_{C_1|D=1,w}(c_1^*(h))p_{1|w} \qquad \forall h \in [0,1] \quad \forall w \in \{0,1\}. \tag{2}$$

Take equation 2 for both w, which yields the following system $\forall h$:

$$\begin{cases} h = F_{C_0|D=0,W=0}(c_0^*(h))p_{0|0} + F_{C_1|D=1,W=0}(c_1^*(h))p_{1|0} \\ h = F_{C_0|D=0,W=1}(c_0^*(h))p_{0|1} + F_{C_1|D=1,W=1}(c_1^*(h))p_{1|1} \end{cases}.$$

Thanks to the instrument, we have a system of two equations to identify two unknown increasing functions. The role of the instrument and Assumption 2 is now clearer. The instrument being exogenous to c_d is crucial here, otherwise, we would have two equations with four unknown functions: $c_0^*(h, W=0), c_0^*(h, W=1), c_1^*(h, W=0), c_1^*(h, W=1)$, which would not be identified. Similarly, without a relevant instrument (i.e., if $d \perp w$), the distributions conditional on w would be the same (i.e., $p_{0|0} = p_{0|1}$ and $F_{C_d|d,W=0}(c) = F_{C_d|d,W=1}(c)$), so the two equations would in fact contain exactly the same information.

Identification problem: Let the reduced form be described as:

$$R = \{\{p_{d|w}\}_{(d,w)\in\{0,1\}\times\{0,1\}},\ \{F_{C_d|d,w}(c_d)\}_{\forall c_d\in\mathcal{C}_d,(d,w)\in\{0,1\}\times\{0,1\}}\}$$

The conditional continuous choice (CCCs) policy functions are identified if and only if there exists a unique set of structural functions $\{c_d(h)\}_{\forall h\in[0,1],d\in\{0,1\}}$ strictly increasing with respect to h, which satisfies equation (2), and is compatible with R.

Theorem 1 (Identification) *For any reduced form drawn from the structural model, there exist unique conditional continuous choice (CCC) functions $c_d(h)$ (for $d=0$ and $d=1$) mapping $[0,1]$ into $\mathcal{C}_d$, which are strictly increasing and solve the system of equations (2):*

$$h = F_{C_0|D=0,w}(c_0(h))p_{0|w} + F_{C_1|D=1,w}(c_1(h))p_{1|w} \qquad \forall h\in[0,1] \quad \forall w\in\{0,1\}.$$

As a consequence, the optimal CCCs, $c_d^*(h)$ for $d=0$ and $d=1$, are point identified from the reduced form R as the unique increasing solutions to the identification problem.

Proof. The complete proof appears in Appendix B.

Sketch of the proof:

Existence of the solution is trivial: since the reduced forms are drawn from the structural model, the true $c_d^*(h)$ will be the solution to our system of equations (2) by construction.

What is more difficult to prove is the *uniqueness* of the solution. First, we show that the mapping between the conditional continuous choices, denoted $\tilde{c}_0(c_1)$, is identified from the reduced forms. Once we have it, using system (2), it is trivial to show that the continuous policies are also identified.

Combining the two equations in the system of equation (2), we have that:

$$\begin{aligned} & F_{C_0|D=0,W=0}(c_0^*(h))p_{0|0} + F_{C_1|D=1,W=0}(c_1^*(h))p_{1|0} = F_{C_0|D=0,W=1}(c_0^*(h))p_{0|1} + F_{C_1|D=1,W=1}(c_1^*(h))p_{1|1} \\ \iff\ & F_{C_0|D=0,W=1}(c_0^*(h))p_{0|1} - F_{C_0|D=0,W=0}(c_0^*(h))p_{0|0} = -\Big(F_{C_1|D=1,W=1}(c_1^*(h))p_{1|1} - F_{C_1|D=1,W=0}(c_1^*(h))p_{1|0}\Big) \\ \iff\ & \Delta F_{C_0}(c_0^*(h)) = -\Delta F_{C_1}(c_1^*(h)), \end{aligned}$$

where $\Delta F_{C_d}(c)$ are directly observed from the data, and are C^1 as a sum of C^1 functions (Lemma 2). However, the problem is that h is unobserved. Now, even without observing h, if two conditional choices $\tilde{c_0}$ and $\tilde{c_1}$ correspond to the same unobserved h, we will have: $\Delta F_{C_0}(\tilde{c_0}) = \Delta F_{C_1}(\tilde{c_1})$. Thus, for the true mapping $\tilde{c}_0(c_1)$ between the two continuous conditional choices we will have

$$\forall c_1 \qquad \Delta F_{C_0}(\tilde{c}_0(c_1)) = -\Delta F_{C_1}(c_1). \tag{3}$$

The mapping is identified if and only if there exists a unique function $\tilde{c_0}(c_1)$ solution to equation (3). What are these $\Delta F_{C_d}(c)$ functions? They are observable C^1 functions (Lemma 2). They are related to the unknown conditional choice probabilities as follows (cf proof of Lemma 3):

$$\forall h \quad \Delta F_{C_d}(c_d^*(h)) = \int_0^h \Big(Pr(D=d|\eta=\tilde{h}, W=1) - Pr(D=d|\eta=\tilde{h}, W=0)\Big) d\tilde{h}.$$

Moreover, since $Pr(D=1|\eta, W) = 1 - Pr(D=0|\eta, W)$, we have by construction that, $\forall h$:

$$\Delta F_{C_0}(c_0^*(h)) = \int_0^h \Big(Pr(D=0|\eta=\tilde{h}, W=1) - Pr(D=0|\eta=\tilde{h}, W=0)\Big) d\tilde{h} = -\Delta F_{C_1}(c_1^*(h)). \tag{4}$$

Which is what we had by rewriting system (2). However, it is very important: it means that $\Delta F_{C_0}(c)$ and $-\Delta F_{C_1}(c)$ are transformations (through unknown $c_d^*(h)$) of the same underlying object, which is based on the difference in conditional choice probabilities $Pr(D=0|\eta=h, W=1) - Pr(D=0|\eta=h, W=0)$. Thus, by construction, $\Delta F_{C_0}(c)$ and $-\Delta F_{C_1}(c)$ will go *'through the same values, in the same order'*, just not at the same 'speed'. The shape of ΔF_{C_d} is directly determined by the difference in conditional choice probabilities, hence the reason why we make our identification Assumption 9 on these probabilities directly.

Now, take the easier case where $Pr(D=0|\eta=h, W=1) > Pr(D=0|\eta=h, W=0)$ for all h.[6] In other words, the identification Assumption 9 is satisfied with $K=0$. Equation (4) implies that $\Delta F_{C_0}(c_0)$ and $-\Delta F_{C_1}(c_1)$ will be *strictly increasing* from $Pr(D=0|\eta=0, W=1) - Pr(D=0|\eta=0, W=0)$ at the minimum values of c_0 and c_1 (corresponding to $c_0^*(0)$ and $c_1^*(0)$) to $\int_0^1 \Big(Pr(D=0|\eta=\tilde{h}, W=1) - Pr(D=0|\eta=\tilde{h}, W=0)\Big) d\tilde{h}$ at the maximum values of c_0 and c_1 (corresponding to $c_0^*(1)$ and $c_1^*(1)$). $\Delta F_{C_d}(c_d)$ are thus C^1 and strictly monotone: they are invertible. In this case, the unique mapping between c_0 and c_1 is obtained by inverting equation (3):

$$\forall c_1 \qquad \tilde{c_0}(c_1) = \Delta F_{C_0}^{-1}\Big(-\Delta F_{C_1}(c_1)\Big).$$

The solution exists and is unique. Thus $\tilde{c_0}(c_1)$ is identified in this case.

Now, we can show that the continuous policies are still identified even if $\Delta F_{C_d}(c_d)$ are not strictly monotone but only *piecewise monotone*. This is the general case covered by our identification Assumption 9: if there exists a finite set of $K > 0$ (and $K < \infty$) values of h at which

[6] Obviously, the same reasoning applies in the reverse case where $Pr(D=0|\eta=h, W=1) < Pr(D=0|\eta=h, W=0)$ for all h.

$Pr(D=0|\eta=h,W=1)=Pr(D=0|\eta=h,W=0)$, then by equation (4), we can show that $\Delta F_{C_d}(c_d)$ are *piecewise monotone.* Piecewise monotonicity is not a problem for identification here. We are *not* solving equation (3) point by point, in which case there could exist several solutions for some values of c_1. Instead, we are solving for the entire *monotone* policy functions $c_d^*(h)$ directly. Therefore, even if pointwise there might exist several solutions, there exists a unique *monotone* function on the whole support of c_1 that solves equation (3). In practice, we first identify these K points at which $d\Delta F_{C_d}(c_d)/dc_d = 0$ (Lemma 3). We know that these points are increasingly matched together by construction. Then, we split the support of c_0 and c_1 accordingly. On the subsegments, $\Delta F_{C_d}(c_d)$ are strictly monotone and C^1, thus invertible. Therefore, we can recover the mapping piece by piece.

The only case in which identification does not hold is when ΔF_{C_d} are *flat* on some segment. This corresponds to the case where our identification assumption 9 is violated, and the instrument is not relevant to a set of nonnull masses. In this case, we only have *partial identification* of the policy functions: they are point identified everywhere outside of the flat segment (on which there is an infinite number of possible mappings between c_1 and c_0).

Once we identify the mapping $\tilde{c}_0(c_1)$, we can recover the policies using any equation of the initial system (2), as:

$$\forall c_1 \qquad h(c_1) = F_{C_0|D=0,W=0}(\tilde{c}_0(c_1))p_{0|0} + F_{C_1|D=1,W=0}(c_1)p_{1|0}.$$

Thus we have a unique increasing solution $(h(c_1), \tilde{c}_0(c_1)) \forall c_1 \in \mathcal{C}_1$. Since everything is increasing, we can simply change the arguments to obtain the unique solution $(c_0^*(h), c_1^*(h))\ \forall h \in [0,1]$. ■

One of the main take-aways from this the proof is that, with this setup, by exploiting *knowledge* about the monotonicity of the optimal continuous policies and directly solving for the complete function, I identify the policies with assumptions that are considerably less restrictive than what is usually imposed in related studies. For example, *full rank* assumptions on the effect of the instrument on the selection in identification of IV quantile treatment effects (Newey and Powell, 2003; Chernozhukov and Hansen, 2005, 2006, 2008) are too strong in this framework. In fact, even my subcase where $K=0$ was already less restrictive than full rank, for example. There is one notable exception of Vuong and Xu (2017), who are also solving for a complete function and not pointwise. However, they choose to use this method to relax strict monotonicity (and still impose some constraint on the conditional choice probabilities), while I use it to be as agnostic as possible about the conditional choice probabilities. My main identification requirement is to have a relevant instrument (Assumption 9), which seems fairly natural. Moreover, it is *testable* by observations of the $\Delta F_{C_d}(c_d)$ functions: as long as they are *not flat*, the policies are identified.

3.3 Identification of Conditional Choice Probabilities (CCPs)

Now that the conditional continuous choices (CCCs) are identified, I can directly identify the conditional choice probabilities (CCPs). Indeed, knowing the strictly monotone (and invertible) $(c_0^*(h), c_1^*(h))\forall h$, one can recover h from observing (d, c_d^{obs}). If $D = d$,

$$h = (c_d^*)^{-1}\big(c_d^{obs}\big).$$

From there, it is *as if* $\eta = h$ *were observed.* I observe (d, c_d, w, h) from the data. Thus, I can directly recover the conditional choice probabilities:

$$\forall(d, w, h) \in \{0,1\} \times \{0,1\} \times [0,1]: \qquad Pr(D = d|\eta = h, W = w).$$

Thus, the CCPs are identified once h is recovered from inverting the CCCs.

Inclusion of unobserved types in the model

The fact that η acts as an observed covariate once the CCCs are identified is crucial. Thanks to this, one can apply standard methods from the dynamic discrete choice literature where η would be among the *observed covariates*. This means that once η is identified, one could include *unobserved state variables/types* in the framework as in Arcidiacono and Miller (2011). The non-parametric identification is given by Kasahara and Shimotsu (2009) or Hu and Shum (2012).

3.4 Identification of the payoffs

Now that the optimal policy choices are identified, we can proceed to identify the structural model, i.e., the payoff functions $v_d(c_d^*(h), h)$ and $m_d(w, h)$. First, I focus on the identification of the differences in payoff between the discrete alternatives.

Identification of the differences in payoffs:

The conditional choice probabilities are identified in the data. We can use them with our structural assumptions to identify *difference in payoffs* in the model. We know that the CCPs are related to the structure of the model as follows:

$$\begin{aligned} Pr(D=0|\eta=h,w) &= Pr\Big(\epsilon_0-\epsilon_1 > (\underset{c}{max}\ v_1(c,h)+m_1(w,h)) - (\underset{c}{max}\ v_0(c,h)+m_0(w,h))\Big|\ h,w\Big) \\ &= Pr\Big(\epsilon_0-\epsilon_1 > v_1^*(h)+m_1(w,h) - \big(v_0^*(h)+m_0(w,h)\big)\ \Big|\ h,w\Big), \end{aligned}$$

where $v_d^*(h) \equiv v_d(c_d^*(h),h) = \underset{c}{max}\ v_1(c,h)$.

If the distribution of $\epsilon_0-\epsilon_1$ is known (and invertible), given that we know the CCPs, the difference in payoffs will also be identified. As is standard in the discrete choice literature, identification depends on the distribution of the difference in ϵ here.

For example, let us assume that ϵ follows a *Gumbel/extreme-value type-I distribution*(with location 0 and scale 1), as is commonly used in the discrete choice literature (McFadden, 1980; Hotz and Miller, 1993). In this case, we are in the logistic regression scenario and we have:

$$Pr(D=0|\eta=h,w) = \frac{1}{1+exp\Big(v_1^*(h)+m_1(w,h)-\big(v_0^*(h)+m_0(w,h)\big)\Big)}.$$

Thus we identify the *difference in payoffs* as:

$$v_1^*(h)+m_1(w,h)-\big(v_0^*(h)+m_0(w,h)\big) = log\left(\frac{1}{Pr(D=0|\eta=h,w)}-1\right).$$

Moreover, since $v_d^*(h)$ are independent of w by Assumption 2, we can also identify the *difference in the effect of the instrument*:

$$\begin{aligned} &m_1(w=1,h)-m_0(w=1,h)-(m_1(w=0,h)-m_0(w=0,h)) \\ &= log\left(\frac{1}{Pr(D=0|\eta=h,w=1)}-1\right) - log\left(\frac{1}{Pr(D=0|\eta=h,w=0)}-1\right). \end{aligned}$$

The differences in payoffs are also non-parametrically identified for other distributions of ϵ. Applications often use generalized-extreme value distributions as they yield easily tractable closed-form solutions (Arcidiacono and Miller, 2011), but other distributions are possible.

Identification of the payoffs:

To non-parametrically identify the *payoffs* directly using the CCPs and CCCs, one needs to add some structure to the problem. In other words, we need additional behavioural conditions to know how the agents behave. For example, by considering the framework applied to dynamic problems, I can use the identification power of the first-order conditions/Euler equation to non-parametrically directly identify the *payoffs* using the identified CCPs and CCCs. This is what I do in the next section by extending the framework to a dynamic setup.

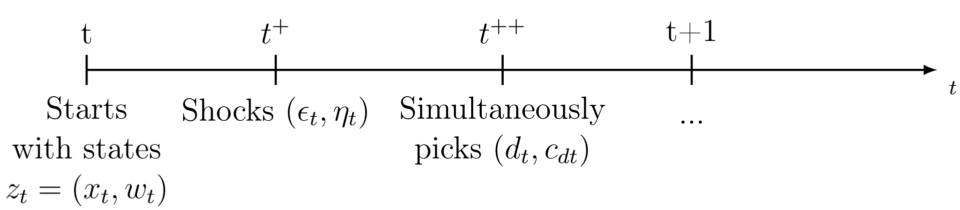
4 Extension to Dynamic models

The general framework that I developed embeds dynamic models: v_{dt} must simply be understood as current conditional value functions, embedding expectations about the future. Here, I show how general (non-stationary) dynamic models of agents enter the setup and are non-parametrically identified (in the spirit of Blundell et al., 1997). The model is very general and nests many life-cycle empirical applications of interest (e.g., Blundell et al., 2016; Iskhakov et al., 2017).

4.1 Dynamic Life-Cycle Framework of Labor and Consumption

In this section, I describe how a general dynamic model of labor and consumption choices enters the general framework described in section 2.

Each period t until T, the timing of the problem is as follows:



Current period utility:

The *current period conditional utility* for action (d, c_d) at time t is given by:

$$\mathcal{U}_{dt}(c_{dt}, x_t, w_t, \eta_t, \epsilon_t). \tag{5}$$

In this example, c_t is consumption, and c_{dt} are conditional consumptions, with $c_t = c_{0t}(1-d_t)+c_{1t}d_t$. d_t is the work decision (Blundell et al., 1997, 2016). x_t represents all the covariates. These include covariates impacting current utility such as age, education and other demographics. For notational convenience, x_t also include variables such as asset or income that do not necessarily directly impact preferences but still have an impact on consumption choice (and labor choice), notably through their transitions. w_t is again the instrumental variable that must fulfil some conditions I describe below.

I impose some conditions on *current utility* which are necessary (not sufficient) for the dynamic setup described here to fit into the structure described in section 2.

Assumption D**1** **(Additive Separability)** *The shock* ϵ_t *enters the payoff additively such that:*

$$\mathcal{U}_{dt}(c_{dt}, x_t, w_t, \eta_t, \epsilon_t) = \tilde{u}_{dt}(c_{dt}, x_t, w_t, \eta_t) + \epsilon_{dt}.$$

Assumption D**2** **(Instrument)** $w_t \in \mathcal{W} = \{0, 1\}$ *is an instrumental variable such that*

$$\tilde{u}_{dt}(c_{dt}, x_t, w_t, \eta_t) = u_{dt}(c_{dt}, x_t, \eta_t) + m_{dt}(x_t, w_t, \eta_t).$$

Assumption D**3** **(Monotonicity)** *The conditional current utility functions are twice continuously differentiable such that*

$$\frac{\partial^2 u_{dt}(c_{dt}, x_t, \eta_t)}{\partial c_{dt} \partial \eta_t} > 0 \qquad \forall d_t, c_{dt}, x_t, \eta_t$$

Transitions:

In a dynamic context, the individual chooses (d_t, c_{dt}) to maximize not only her current utility but also to maximize her *expected discounted sum of future payoffs*. She discounts the future period utilities at a rate β. In this context, the agent form *rational expectations* about the transition probabilities. These transitions from $(x_t, w_t, \epsilon_t, \eta_t)$ and the current choices (c_t, d_t) to $(x_{t+1}, w_{t+1}, \epsilon_{t+1}, \eta_{t+1})$ matter for the choices. In particular, how the current choices impact these transitions is especially important for the optimal choice decision. The impacts of the choices on the transitions are often expressed through a *budget constraint* like

$$a_{t+1} = (1 + r_t)a_t - c_t + y_t d_t.$$

For now I stay more general and simply assume the existence of general transitions of states and errors which depend on the choices:

$$f_t(x_{t+1}, w_{t+1}, \epsilon_{t+1}, \eta_{t+1} | c_t, d_t, x_t, w_t, \epsilon_t, \eta_t).$$

I need to make additional assumptions on these transitions for the setup to be identified (and to enter the general framework).

Assumption 10 (Conditional Independence) *For all $x_t \in \mathcal{X}$, $w_t \in \mathcal{W}$, $\epsilon_t \in \mathcal{E}$, $\eta_t \in \mathcal{H}$, we have:*

$$f_t(x_{t+1}, w_{t+1}, \epsilon_{t+1}, \eta_{t+1}|c_t, d_t, x_t, w_t, \epsilon_t, \eta_t) = f_t(x_{t+1}, w_{t+1}|c_t, d_t, x_t, w_t)\ f_\epsilon(\epsilon_{t+1})\ f_\eta(\eta_{t+1}).$$

Assumption 11 (Instrument Transition Exclusion) *For all $x_t \in \mathcal{X}$, $w_t \in \mathcal{W}$, the current instrument is excluded from the transitions, i.e.,*

$$f_t(x_{t+1}, w_{t+1}|c_t, d_t, w_t, x_t) = f_t(x_{t+1}, w_{t+1}|c_t, d_t, x_t).$$

Solution:

Knowing these transition probabilities, the individual chooses (d_t, c_{dt}) to sequentially maximize her expected discounted sum of payoffs. Let us define $V_t(z_t) = V_t(x_t, w_t)$ as the (ex ante) value function of this discounted sum of future payoffs at the beginning of t, just before the shocks (ϵ_t, η_t) are revealed and conditional on behaving according to the optimal decision rule:

$$V_t(z_t) \equiv \mathbb{E}\Big[\sum_{\tau=t}^{T} \beta^{\tau-t} \underset{d,c_{d\tau}}{max}[\ u_{d\tau}(c_{d\tau}, x_\tau, \eta_\tau) + m_d(x_\tau, w_\tau, \eta_\tau) + \epsilon_{d\tau}]\Big].$$

Given the state variable z_t and choice (d, c_{dt}) in period t, the expected value function in period $t+1$ is

$$\mathbb{E}_{z_{t+1}}[V_{t+1}(z_{t+1})|z_t, c_t, d_t] = \int_{z_{t+1}} V_{t+1}(z_{t+1}) f_t(z_{t+1}|z_t, c_t, d_t) dz_{t+1}.$$

By the conditional independence Assumption 10 and instrument exclusion from the transition (Assumption 11), we can remove w_t from the conditioning variables, which yields:

$$\mathbb{E}_{z_{t+1}}[V_{t+1}(z_{t+1})|z_t, c_t, d_t] = \mathbb{E}_{z_{t+1}}[V_{t+1}(z_{t+1})|x_t, c_t, d_t] = \int_{z_{t+1}} V_{t+1}(z_{t+1}) f_t(z_{t+1}|x_t, c_t, d_t) dz_{t+1}.$$

The ex ante value function can be written recursively:

$$V_t(z_t) = \mathbb{E}_{\epsilon,\eta}\left[\underset{d_t,c_{dt}}{max}\Big[\ u_{dt}(c_{dt}, x_t, \eta_t) + m_{dt}(x_t, w_t, \eta_t) + \epsilon_{dt} + \beta \mathbb{E}_{z_{t+1}}[V_{t+1}(z_{t+1})|x_t, c_{dt}, d_t]\Big]\ \right].$$

Thus, in each period, after observing (ϵ_t, η_t), the individual chooses d_t and c_{dt} to maximize her expected payoff:

$$\underset{d_t,c_{dt}}{max}\ u_{dt}(c_{dt}, x_t, \eta_t) + \beta \mathbb{E}_{z_{t+1}}[V_{t+1}(z_{t+1})|x_t, c_{dt}, d_t]\ + m_{dt}(x_t, w_t, \eta_t) + \epsilon_{dt}.$$

Denote the conditional value functions v_{dt} as:

$$v_{dt}(c_{dt}, x_t, \eta_t) \equiv u_{dt}(c_{dt}, x_t, \eta_t) + \beta \mathbb{E}_{z_{t+1}}[V_{t+1}(z_{t+1})|_{x_t, c_{dt}, d_t}]. \tag{6}$$

So that we return to our general setup. Indeed, the dynamic model can be interpreted as a static model, where in every period the agent selects d_t and c_{dt} to solve:

$$\underset{d_t, c_{dt}}{max} \quad v_{dt}(c_{dt}, x_t, \eta_t) + m_{dt}(x_t, w_t, \eta_t) + \epsilon_{dt}.$$

Lemma 4 (Dynamic Framework) *Under Assumptions D1, D2, D3, 10 and 11, Assumptions 1, 2 and 3 are satisfied for the conditional value functions defined in equation (6) in the dynamic setup.*

The other Assumptions 4, 5, 6, 7, 8 and 9 as well as Normalization 1 are imposed contemporaneously (with index t) and unconditionally on X_t (for simplicity).

If Assumption $D1$ holds for the current utility function, Assumption 1 will hold for the conditional value functions by construction in equation (6). Assumptions $D2$ and $D3$ on the current utility do not translate directly into Assumptions 2 and 3 for the conditional value function. One needs additional assumptions about the transitions, i.e., Assumptions 10 and 11.

Conditional independence assumptions are standard for the identification and empirical tractability of dynamic discrete choice models (Rust, 1987; Blevins, 2014). Here, Assumption 10 implies that the transitions of the state variables are independent of the shocks (ϵ_t, η_t). Similarly, the shock transitions are independent of the variables here. There is *no time dependence* on the shocks, which are thus iid every period. Note that one can include some unobserved heterogeneous types correlated over time in the covariates following Arcidiacono and Miller (2011). This allows for some unobserved auto-correlation in the unobservables and attenuates the strength of the conditional independence.

Crucially, here, in addition to the standard conditional independence 10, Assumption 11 also implies that conditional on (d_t, c_t, x_t), the *transitions* are *independent* of the current instrument value w_t. In particular, the instrument is excluded from its own transition to future values, conditional on (d_t, c_t, x_t), i.e.,

$$w_{t+1} \perp w_t \mid c_t, d_t, x_t \quad \forall c_t, d_t, x_t$$
$$\text{or equivalently } f_w(w_{t+1}|c_t, d_t, x_t, w_t) = f_w(w_{t+1}|c_t, d_t, x_t).$$

It implies that instruments that are time independent $w_t = w$ for all t cannot be included. Assumption $D2$ combined with Assumption 11 will satisfy Assumption 2 on the conditional value as stated in Lemma 4 and as shown in the computation above. However, if the exclusion of the instrument from the transition (Assumption 11) does not hold, then w_t affects the expected future value function $\mathbb{E}_{z_{t+1}}[V_{t+1}(z_{t+1}|x_t, w_t, c_{dt}, d_t]$ and enters the conditional value functions v_{dt} in equation (6), which are rewritten as $v_{dt}(c_{dt}, x_t, w_t, \eta_t)$ for all d in this case. In this case, it is obvious that the original exclusion of the instrument from v_d in Assumption 2 is violated. Thus, the dynamic setup does fit into the general framework of section 2 without Assumption 11.

Similarly, Assumption $D3$ is just a necessary condition for Assumption 3 to hold. I also require the expectations about the future to be independent of current (η_t, ϵ_t). In this case, the monotonicity Assumption 3 in the general framework is also satisfied. Indeed, if the future is independent of the current η_t (Assumption 10), then we can write:

$$\frac{\partial v_{dt}(c_{dt}, x_t, \eta_t)}{\partial c_{dt} \partial \eta_t} = \frac{\partial u_{dt}(c_{dt}, x_t, \eta_t)}{\partial c_{dt} \partial \eta_t} + \underbrace{\frac{\partial \mathbb{E}_{z_{t+1}}[V_{t+1}(z_{t+1})|x_t, c_{dt}, d_t]}{\partial c_{dt} \partial \eta_t}}_{=0}.$$

Therefore, if the conditional independence and monotonicity of the current utility function hold (Assumptions 10 and $D3$), then the monotonicity of the conditional value functions v_{dt} (Assumption 3) also holds.

Instrument example:

The question that remains is, what could be a good instrument satisfying this restrictive conditional independence and exclusion from the transition in practice? In general, a good instrument would be to allow for *switching cost* and to use $w_t = d_{t-1}$ in this setup. Indeed, in this case, the exclusion assumption 11 is easily satisfied for the instrument: w_{t+1} is d_t. Therefore, conditional on the current d_t choice, w_{t+1} is directly known. $w_t = d_{t-1}$ does not provide any additional information, so it can be dropped from the conditioning variables in the transition. Moreover, it is unlikely that w_t provides any information about the other future covariates x_{t+1} after conditioning on the current d_t. Similarly, conditional on x_t, which could include for example, the experience of the individual, it is unlikely that d_{t-1} has an impact on u_{dt}. The exclusion restriction *D*2 is satisfied. Finally, we just need the instrument to be relevant (Assumption 9). This would be the case if one had some utility switching cost from entering or exiting the workforce for example.[7] In this case, we would have: $m_{0t}(x_t, w_t = 0, \eta_t) - m_{1t}(x_t, w_t = 0, \eta_t) \neq m_{0t}(x_t, w_t = 1, \eta_t) - m_{1t}(x_t, w_t = 1, \eta_t)$. And the instrument would be relevant.

Relaxing time independence of η:
One can loosen Assumption 10 and allow for first-order time dependence in η_t in this setup. In other words, I can have $f_\eta(\eta_t|\eta_{t-1})$. In fact, as I identify η_t separately for all t, I can identify these transitions, which are particularly interesting in some applications (e.g., if η represents some unobserved ability or productivity). The only problem is that it is more difficult to find a good instrument in practice in this case. Indeed, in the presence of auto-correlation in η_t, $w_t = d_{t-1}$ is no longer a good instrument, as it violates its independence from η_t in the initial period (Assumption 4). Indeed, in the initial period of the data, η_{-1} is not observed and is correlated with η_0. However, in this case, d_{-1} was a choice taken based on η_{-1} and thus correlated with η_{-1}. Therefore, in the first period, $w_0 = d_{-1}$ is correlated with η_{-1} and thus with η_0. The instrument w_0 is not independent of η_0, which violates Assumption 4. If we were able to condition on η_{t-1}, we could identify η_t: conditional on η_{t-1}, $w_t = d_{t-1} \perp \eta_t$. However, there is no way to recover η_{-1} which is outside the sample. Thus, I cannot allow for transition in η_t with $w_t = d_{t-1}$ as an instrument. Therefore, the best way to account for unobserved auto-correlation with $w_t = d_{t-1}$ as an instrument would be to include unobserved types à la Arcidiacono and Miller (2011) in the model and still impose conditional independence with an iid η_t.

If there exists another instrument satisfying Assumptions 4, *D*2 and 11, then one can allow for auto-correlated η_t. In fact, even if such an instrument is available only in one period t_0 (e.g., a unique unexpected event), then one can still allow for auto-correlation in η_t. Indeed, one can use the instrument in the period to identify the η_{t_0}. For all the following periods, $w_t = d_{t-1}$ can be used as a proper instrument if I include η_{t-1} in the covariates list.

[7] Alternatively, one could have some auto-correlation in d_t. Recall that $m_{dt}()$ can be interpreted as some observable part of the ϵ_t shocks. Thus, the relevance assumption with $w_t = d_{t-1}$ could also be interpreted as the existence of auto-correlation in a general $\tilde{\epsilon}_t$ term, with

$$\tilde{\epsilon}_{dt} = m_{dt}(x_t, w_t, \eta_t) + \epsilon_{dt}.$$

Thus, the assumption about no correlation in ϵ_t is less restrictive than it seems.

4.2 Identification of the dynamic model

First, I show how the CCCs and CCPs are identified in this dynamic model. Then, I show how the payoffs are also non-parametrically identified under additional assumptions.

4.2.1 Optimal choices: CCCs and CCPs

Under Lemma 4, the dynamic framework described in Section 4 fits into the general framework described in Section 2. Therefore the CCCs and CCPs are identified period by period following exactly the same proof I developed in the previous section. In other words, from data on (D_t, C_t, X_t, W_t, t), I recover reduced forms

$$R = \Big\{\{Pr(D_t = d|X_t = x, W_t = w, t)\}_{(d,w,x,t)\in\mathcal{D}\times\mathcal{W}\times\mathcal{X}_t\times\{0,\ldots,T\}},$$
$$\{F_{C_d|D_t=d,X_t=x,W_t=w,t}(c_d)\}_{\forall c_d\in\mathcal{C}_{dt},\ (d,w,x,t)\in\mathcal{D}\times\mathcal{W}\times\mathcal{X}_t\times\{0,\ldots,T\}}\Big\}.$$

From these reduced forms, following Section 3, I identify the CCCs and CCPs

$$c^*_{dt}(x, \eta = h) \text{ and } Pr(d|\eta = h, x, w, t) \qquad \forall d \in \{0,1\}, w \in \{0,1\}, h \in [0,1], x \in \mathcal{X}_t, t \in \{0, ..., T\}.$$

Special case: identification of the choices with terminal/absorbing actions

Imagine $d_t = 1$ is a *terminal action* or an *absorbing state*. For example $d_t = 1$ if the individual retires, $d_t = 0$ if she stays active. Assuming that an individual cannot go back to the working life, the retirement choice is absorbing (Iskhakov et al., 2017). In this case, assuming all the other modeling assumptions still hold, identification is more direct and simpler. I still use $w_t = d_{t-1}$ as the instrument, so the Assumption 11 on the transitions is still verified. Now, conditional on $w_t = 1$, an individual only has the choice to stay retired, i.e., $d_t = 1$. Thus, focus on previously retired individuals ($W_t = 1$), we have:

$$\begin{aligned}
h &= Pr(\eta_t \leq h|X_t, W_t = 1, t) \\
&= Pr(\eta_t \leq h|D_t = 1, X_t, W_t = 1, t) \\
&= Pr(c \leq c^*_{1t}(h, X_t)|D_t = 1, X_t, W_t = 1, t) \\
&= F_{C_1|D_t=1,X_t,W_t=1,t}(c^*_{1t}(h, X_t)) \qquad \forall X_t \in \mathcal{X}_t, h \in [0,1].
\end{aligned}$$

Since $F_{C_1|D_t=1,X_t,W_t=1,t}(c)$ are invertible (Lemma 2), we recover the continuous choices conditional on being retired as:

$$c^*_{1t}(h, X_t) = F^{-1}_{C_1|D_t=1,X_t,W_t=1,t}(h) \quad \forall X_t \in \mathcal{X}_t, h \in [0,1].$$

It remains to identify the other conditional continuous policy, and to do that one simply needs to take the equation (2) at $W_t = 0$, i.e., for individuals who did not select the absorbing state yet. It yields

$$\begin{aligned}
h = \quad & F_{C_0|D_t=0,X_t,W_t=0,t}(c^*_{0t}(h,X_t))Pr(D_t=0|X_t,W_t=0,t) \\
+ \quad & F_{C_1|D_t=1,X_t,W_t=0,t}(c^*_{1t}(h,X_t))Pr(D_t=1|X_t,W_t=0,t) \\
\iff \quad & F_{C_0|D_t=0,X_t,W_t=0,t}(c^*_{0t}(h,X_t)) = \frac{h - F_{C_1|D_t=1,X_t,W_t=0,t}(c^*_{1t}(h,X_t))Pr(D_t=1|X_t,W_t=0,t)}{Pr(D_t=0|X_t,W_t=0,t)} \\
\iff \quad & c^*_{0t}(h,X_t) = F^{-1}_{C_0|D_t=0,X_t,W_t=0,t}\left(\frac{h - F_{C_1|D_t=1,X_t,W_t=0,t}(c^*_{1t}(h,X_t))Pr(D_t=1|X_t,W_t=0,t)}{Pr(D_t=0|X_t,W_t=0,t)}\right).
\end{aligned}$$

Since everything on the right hand side of the equation is known (as I identified c^*1t previously), the other conditional policies $c^*_{0t}(h,X_t)$ are also identified $\forall X_t \in \mathcal{X}_t, h \in [0,1]$. Once the CCCs are identified, we proceed as usual to identify the CCPs.

4.2.2 Transitions

The transitions $f(x_{t+1}|c_t,d_t,x_t)$ are identified directly from the data by observing the conditional transitions of the variables between consecutive periods t and $t+1$. The transition of the instrument is known by construction if $w_{t+1} = d_t$. In other cases, it can also be recovered from the data (and one can test if it is indeed independent from w_t).

As standard in the dynamic model literature, I assume agents are *rational* so that the observed transitions are the same as the one expected by the agents. This way, the transitions recovered from the data can be used to build agents expectations at each time t, and help recover the primitives.

4.2.3 Payoff function

Once the CCCs, CCPs and transitions are identified, I can build upon existing literature to identiy the payoffs (Hotz and Miller, 1993; Blundell et al., 1997; Magnac and Thesmar, 2002; Escanciano et al., 2015). I need to introduce some additional structure to the dynamic model for non-parametric identification: I introduce additional structure on the covariates transition and current utility function.

Budget Constraint: let us introduce additional structure on the transitions via a *budget con-*

straint:

$$a_{t+1} = (1 + r_t)a_t - c_t + y_t d_t, \tag{7}$$

where a_t is the individual asset holdings, y_t is her income and r_t is the interest rate. The asset plays a different role than the other covariates. Indeed, its transition to a_{t+1} is directly impacted by the choice c_t through the budget constraint (7). Denote more generally all the covariates $x_t = (\tilde{x}_t, a_t)$ to emphasize the role of the asset.[8]

Assumption 12 (Asset exclusion) *The asset is excluded from the current period utility, i.e.,*

$$u_{dt}(c_{dt}, x_t, \eta_t) = u_{dt}(c_{dt}, \tilde{x}_t, a_t, \eta_t) = u_{dt}(c_{dt}, \tilde{x}_t, \eta_t).$$

Or equivalently, the exclusion Assumption 12 can be stated as:

$$\frac{\partial u_{dt}(c_{dt}, \tilde{x}_t, a_t, \eta_t)}{\partial a_t} = 0.$$

Assumption 13 (General Covariates Transitions) *For all $\tilde{x}_t \in \tilde{\mathcal{X}}, \forall d_t \in \mathcal{D}, \forall c_t \in \mathcal{C}$, c_t does not impact the $\tilde{x}_t$ and w_t transitions, i.e.,*

$$f_t(\tilde{x}_{t+1}, w_{t+1}|c_t, d_t, \tilde{x}_t) = f_t(\tilde{x}_{t+1}, w_{t+1}|d_t, \tilde{x}_t).$$

I also need some additional structure on the current period utility:

Assumption 14 (Stationary utility) *The current period utility is independent from time*

$$u_{dt}(c_{dt}, \tilde{x}_t, \eta_t) = u_d(c_{dt}, \tilde{x}_t, \eta_t)\ \forall t.$$

Assumption 15 (Monotonicity of c on the current utility) *The current period utility is monotone increasing with respect to c*

$$\frac{\partial u_{dt}(c_{dt}, \tilde{x}_t, \eta_t)}{\partial c} > 0 \qquad \forall d_t, c_{dt}, \tilde{x}_t, \eta_t.$$

Marginal utilities identification:

[8] Note that y_t and r_t are included in $\tilde{x}_t$. Even though, in most applications they will also be excluded from the current period utility. For notational simplicity and generality, I let them into the general $\tilde{x}_t$ term which enters in the current utility and represents all the covariates other than asset, i.e., all the covariates whose transitions are not impacted by c_t (Assumption 13).

Lemma 5 (**Escanciano et al. (2015)**) *Following Escanciano et al. (2015), under Assumptions D1-D3 and 4-15, the conditional marginal utilities at optimal continuous choices*

$$\frac{\partial}{\partial c_{dt}} u_d(c_{dt}, \tilde{x}_t, \eta_t)|_{c_{dt}=c^*_{dt}(\eta_t, x_t)}$$

are identified up to a scale by the Euler Equation for all d, x_t, η_t.

Proof.

Let us define

$$u'^*_d(x_t, \eta_t) = \frac{\partial}{\partial c_{dt}} u_d(c_{dt}, \tilde{x}_t, \eta_t)|_{c_{dt}=c^*_{dt}(\eta_t, x_t)},$$

i.e., the conditional marginal utilities at the optimal CCCs. Notice that these functions depend on $x_t = (\tilde{x}_t, a_t)$. In other words, the optimal conditional marginal utilities depend on the asset, through the optimal CCCs. Then, the *Euler equations* for all d can be rewritten as:

$$u'^*_d(x_t, \eta_t) = \beta(1 + r_t)\mathbb{E}_t\Big[u'^*_{d_{t+1}}(x_{t+1}, \eta_{t+1}) \;\Big| x_t, c_{dt} = c^*_{dt}(\eta_t, x_t), d_t = d\Big]. \tag{8}$$

We have a system of two equations with two unknown functions u'^*_0 and u'^*_1. Hence the importance of Assumption 14, otherwise we would have a different unknown function on each side of the equation. Now, under Assumption $D3$ and 15, I have

$$\partial u'^*_d(x_t, \eta_t)/\partial \eta_t > 0 \quad \forall d, x_t, \eta_t.$$

In this case, Escanciano et al. (2015) show that these functions are non-parametrically globally point identified by the system (8). ■

Conditional values:

Once the marginal utilities are identified through Lemma 5, I follow Blundell et al. (1997) to identify the conditional values.

Lemma 6 (**Blundell et al. (1997)**) *Under Assumptions D1-D3 and 4-15, the conditional value functions at optimal choices $v_{dt}(c^*_{dt}(x_t, \eta_t), x_t, \eta_t)$ are identified up to an unknown constant of integration K independent from the asset for all d, x_t, η_t. i.e.,*

$$v_{dt}(c^*_{dt}(\eta_t, x_t), x_t, \eta_t) = G_{dt}(\tilde{x}_t, a_t, \eta_t) + K_{dt}(\tilde{x}_t, \eta_t),$$

where G and K are defined in the proof.

Proof. Recall that $x_t = (\tilde{x}_t, a_t)$. We have the first order conditions, holding at optimal CCCs:

$$\forall d: \ \forall a_t \qquad \frac{\partial}{\partial a_t} v_{dt}(c_{dt}, \tilde{x}_t, a_t, \eta_t) = (1+r_t) \frac{\partial}{\partial c_{dt}} u_d(c_{dt}, \tilde{x}_t, \eta_t) \mid_{c_{dt}=c^*_{dt}(\eta_t, \tilde{x}_t, a_t)}. \tag{9}$$

Denote v^*_d is the conditional value taken at the optimal continuous choice, and similarly define u'^*_d as before. We can rewrite the FOC as:

$$\forall d: \ \forall a_t \qquad \frac{\partial}{\partial a_t} v^*_{dt}(\tilde{x}_t, a_t, \eta_t) = (1+r_t) \ u'^*_d(\tilde{x}_t, a_t, \eta_t). \tag{10}$$

Crucially, following Assumption 12, the asset is excluded from the current period utilities and marginal utilities. The identification strategy relies on this exclusion. Indeed, from this FOC (9) at the optimal CCCs, I can integrate with respect to the continuous asset and obtain

$$\forall d: \ \forall a_t \qquad v^*_{dt}(\tilde{x}_t, a_t, \eta_t) = \int_0^{a_t} (1+r_t) \ u'^*_d(\tilde{x}_t, a_t, \eta_t) \ da,$$

where the lower bound 0 is taken arbitrarily. Since u'^*_d are identified, we can identify the optimal conditional value functions non-parametrically as:

$$v^*_{dt}(\tilde{x}_t, a_t, \eta_t) \equiv G_{dt}(\tilde{x}_t, a_t, \eta_t) + K_{dt}(\tilde{x}_t, \eta_t).$$

Where the only remaining unknowns are $K_{dt}(\tilde{x}_t, \eta_t)$ which are unknown constant of integration, independent from a_t, and which depends on the arbitrary lower bound of integration. ■

Additive term $m_{dt}(x_t, w_t, \eta_t)$:

It remains to identify the additive terms. I identify the *differences* in total conditional values by relating them to the CCPs using an Hotz and Miller (1993)'s inversion, as in section 3.4. In other words,

$$\begin{aligned}\Delta v^*_t(\tilde{x}_t, a_t, \eta_t) + \Delta m_t(x_t, w_t, \eta_t) &= v^*_{1t}(\tilde{x}_t, a_t, \eta_t) - v^*_{0t}(\tilde{x}_t, a_t, \eta_t) + m_{1t}(x_t, w_t, \eta_t) - m_0(x_t, w_t, \eta_t) \\ &= G_{1t}(\tilde{x}_t, a_t, \eta_t) - G_{0t}(\tilde{x}_t, a_t, \eta_t) + K_{1t}(\tilde{x}_t, \eta_t) - K_{0t}(\tilde{x}_t, \eta_t) + m_{1t}(x_t, w_t, \eta_t) - m_0(x_t, w_t, \eta_t)\end{aligned}$$

are identified through the CCPs $Pr(d|\eta_t, x_t, w_t, t)$ for all d, x_t, η_t, w_t, t. Note that I cannot identify K_{dt} separately from m_{dt}. A natural *normalization* is to impose

$$K_{dt}(\tilde{x}_t, \eta_t) = 0 \qquad \forall d, \tilde{x}_t, \eta_t, t$$

Such that the only remaining *additive terms* are m_{dt}. Under this normalization, given that $v^*_{dt}(\tilde{x}_t, a_t, \eta_t) = G_{dt}(\tilde{x}_t, a_t, \eta_t)$ have been previously identified, it means that $\Delta m_t(x_t, w_t, \eta_t)$ are identified for all

d, x_t, η_t, w_t, t. Obviously, since these are identified through the discrete choice probabilities, one can only non-parametrically identify the differences in m_d, not their separate values at each d.

About the role of each assumptions:

Assumption 12 excludes the asset from the utility. Having an excluded asset is essential to recover the conditional values once the marginal current utilities are identified.

Assumption 13 implies that the only covariate whose transition is impacted by the choice c_t is the asset, through the budget constraint (7). This assumption is made to pin down a simpler Euler equation than with general transitions with several variables impacted by c_t.

To identify the marginal utility non-parametrically from the Euler Equation, one needs to impose some structure on the effect of time in the utility function. I impose that the current period utility is time independent through Assumption 14. Note that, in general, even if the current period utility is time independent, the conditional value functions are still time-dependent, because of a finite horizon, or because of time-dependent transitions. Also note that this assumption is only necessary for non-parametric identification. In parametric models, I can identify time-dependent utilities.

Assumption 15 is a slightly stronger monotonicity condition than the ones I imposed before. In most empirical applications it will be satisfied though.

5 Estimation

I build a two-step estimation process in the spirit of Hotz and Miller (1993); Arcidiacono and Miller (2011) in the discrete choice literature. In the first step, I estimate the conditional continuous choices (CCCs) and the conditional choice probabilities (CCPs) based on reduced forms directly estimated from the data. This step is data-driven and is independent from the model specification. In a second step, I use the estimated optimal policies to estimate the structural parameters. Therefore, my estimation method is an analogous to that of Hotz and Miller (1993) and Hotz et al. (1994) but extended to discrete-continuous choices. Its main desirable feature concerns computational gains. By estimating the optimal choices only once, the computational burden of the estimation is significantly reduced. Indeed, one does not need to solve for the value function or the likelihood for each new set of selected parameters. This allows us to estimate models that were previously computationally intractable. It does so at minimal efficiency costs (compared to simulated method of moments, for example). I expose the estimation method in this section, and I compare my estimator's performance with several alternatives in terms of speed and efficiency in the next section 6.

5.1 1st stage: conditional choices

Reduced forms:

I observe data about (D_t, C_t, X_t, W_t, t). Where $c_t = (1-d_t)c_{0t} + d_t c_{1t}$. From the data, I estimate the reduced forms:

$$R = \Big\{ \{Pr(D_t = d | X_t = x, W_t = w, t)\}_{(d,w,x,t) \in \mathcal{D} \times \mathcal{W} \times \mathcal{X}_t \times \{0,\dots,T\}}, \\ \{F_{C_d | D_t = d, X_t = x, W_t = w, t}(c_d)\}_{\forall c_d \in \mathcal{C}_{dt},\ (d,w,x,t) \in \mathcal{D} \times \mathcal{W} \times \mathcal{X}_t \times \{0,\dots,T\}} \Big\}.$$

This initial estimation of the reduced forms is crucial, as all the subsequent estimates are derived from it. The reduced forms probabilities $Pr(D_t = d|X_t = x, W_t = w, t)$ can be estimated non-parametrically by kernel or by Sieve logistic or probit regressions. Recall that these probabilities are not the CCPs, as the CCPs are also conditional on η_t.

The continuous choice conditional distributions can also be estimated with non-parametric kernel methods (e.g. Hayfield and Racine, 2008). Another alternative is to first estimate the quantile functions via non-crossing conditional quantile estimation (Muggeo, 2018; Lipsitz et al., 2017, for example), and then invert them to recover the conditional distributions.

In the dynamic setup, the reduced forms also include the *transition probabilities* from t to $t+1$: $f_t(x_{t+1}, w_{t+1}|c_t, d_t, x_t)$ is estimated as usual. Let us distinguish again the asset from other covariates: $x_t = (\tilde{x}_t, a_t)$. Under Assumption 13, $f_t(\tilde{x}_{t+1}, w_{t+1}|d_t, \tilde{x}_t)$ can be estimated using auto-regressive processes for the general covariates $\tilde{x}_t$. In the special case where $w_t = d_{t-1}$, then the transition of the instrument is given by construction. The asset plays a particular role and its transition is given by the budget constraint (7): $a_{t+1} = (1 + r_t)a_t - c_t + y_t d_t$.

Conditional Continuous Choices (CCCs):

I estimate the CCCs based on the identification proof. The idea is that we want to solve for the monotone functions $c_{dt}(h, x)$, which solves the empirical counterpart of system (2):

$$h = \widehat{F}_{C_{0t}|D_t=0,x_t,w_t}(c_{0t}(h, x_t))\ \widehat{Pr(D_t = 0|w_t, x_t)} + \widehat{F}_{C_{1t}|D_t=0,x_t,w_t}(c_{1t}(h, x_t))\ \widehat{Pr(D_t = 1|w_t, x_t)} \quad \forall w_t, h, x_t, t,$$

where I replaced the reduced forms by their empirical counterparts. In practice, solving for two functions c_0 and c_1 is not convenient. To simplify, I build upon the identification proof and I first estimate the monotone mapping $\widehat{c}_{0t}(c_{1t}, x_t)$ between the two consumptions. Then I will estimate $\widehat{h}(c_1)$. Consider the empirical counterpart to equation (3):

$$\widehat{\Delta F}_{C_{0t}|x_t}(c_{0t}(c_{1t}, x_t)) = -\widehat{\Delta F}_{C_{1t}|x_t}(c_{1t}) \qquad \forall c_{1t}$$

Thus, for any given x_t, I estimate the conditional consumption mapping $\widehat{c}_{0t}(c_{1t}, x_t)$ by solving for the *whole* monotone mapping functions $c_{0t}(c_{1t}, x_t)$ minimizing

$$\underset{c_{0t}(c_{1t},x_t)}{argmin} \int_{\mathcal{C}_1} \left(\widehat{\Delta F}_{C_{0t}|x_t}(c_{0t}(c_{1t}, x_t)) + \widehat{\Delta F}_{C_{1t}|x_t}(c_{1t})\right)^2 \text{weight}(c_{1t})\ dc_{1t}.$$

It gives a weighted minimum distance estimator to solve for the whole function, instead of proceeding pointwise c_1 by c_1.[9] On a practical note, I resort to *constrained optimization* to solve for the function: select a grid of c_1, and search for the corresponding c_0 by imposing the monotonicity constraint that if $c_1^a < c_1^b$, then $c_0(c_1^a) < c_0(c_1^b)$ for every point in the grid. I repeat this estimation procedure separately for several values of x_t.

Once $\widehat{c}_{0t}(c_{1t}, x_t)$ is estimated for all x_t, I can estimate $\widehat{h}_t(c_1, x_t)$ using any equation of system (2) (with $w_t = 0$ or $w_t = 1$) as:[10]

$$\widehat{h}_t(c_{1t}, x_t) = \widehat{F}_{C_{0t}|D_t=0,w_t,x_t}(\widehat{c}_{0t}(c_{1t}, x_t))\ \widehat{Pr(D_t = 0|w_t, x_t)} + \widehat{F}_{C_{1t}|D_t=0,w_t,x_t}(c_{1t})\ \widehat{Pr(D_t = 1|w_t, x_t)}.$$

Once I have estimated the monotone functions $(\widehat{h}_t(c_{1t}, x_t), \widehat{c}_{0t}(c_{1t}, x_t))$ for all $c_{1t} \in \mathcal{C}_1$ and for all x_t, I easily recover the CCCs:

$$\{\widehat{c}_{0t}(h, x_t), \widehat{c}_{1t}(h, x_t)\} \qquad \forall (h, x_t) \in [0, 1] \times \mathcal{X}$$

[9] For the weight(c_{1t}), I put uniform weights on the quantile of c_{1t}. More precisely, I do not solve for the optimal consumption mapping, but for the optimal quantile mapping between the two consumptions. I look for the quantile of γ_0 of c_0 which corresponds to a given quantile of c_1 denoted γ_1. Thus, following the notation in the Appendix B for the identification proof, the objective can in fact be written as:

$$\underset{\gamma_{0t}(\gamma_{1t},x_t)}{argmin} \int_0^1 \left(\widehat{\Delta F}_{0|x_t}(\gamma_{0t}(\gamma_{1t}, x_t)) + \widehat{\Delta F}_{1|x_t}(\gamma_{1t})\right)^2 \text{weight}(\gamma_{1t})\ d\gamma_{1t}.$$

In this case, weight$(\gamma_{1t}) = 1$ for all γ_{1t}. And the support of the integral is simply $[0, 1]$ since the quantiles are uniform.

[10] In practice, the two equations might not yield exactly the same results because of the noise in the estimation of the reduced forms. Therefore, I will estimate two different h with each equation (with $w = 0$ and $w = 1$) and obtain my final estimate by weighting the two estimates by the number of observations when $(X_t = x_t, W_t = 0)$ and $(X_t = x_t, W_t = 1)$.

by flipping the arguments (because everything is monotone).

Conditional Choice Probabilities (CCPs):

Once the monotone CCCs are estimated, I estimate h_t from observed (c_t, d_t, x_t) in the data, by inverting the CCCs.

$$\text{If } d_t = d: \quad \widehat{h}_t = \widehat{c}_{dt}^{-1}(c_t^{obs}, x_t).$$

Then, you can use $\widehat{h}_t$ as if it was *observed* (like a generated covariate), and estimate the Conditional Choice Probabilities

$$\widehat{Pr}(D_t = d | \eta_t = h, X_t = x_t, W_t = w_t).$$

Again, similarly to the reduced forms probabilities, this estimation can be done non-parametrically with kernel or by Sieve logistic or probit regressions.

Alternative methods:

One could resort to estimation methods proposed in the IV-quantile treatment effect literature and based on Chernozhukov and Hansen (2006, 2008), or that based on Vuong and Xu (2017), described in Feng et al. (2020). With respect to these methods, the advantage of the method developed here is that it is entirely based on the constructive identification proof and does not impose any additional assumptions. The estimation is more flexible, and does not require *full rank* or other assumptions on the conditional choice probabilities (as in the practical estimation paper of Feng et al. (2020)) to hold, for example.

Alternatively, the CCCs and CCPs coud be jointly estimated by Sieve, directly from the data, without estimating reduced forms beforehand. Indeed, for any CCC guess (which has to be monotone in η), one can recover the corresponding η from observing c_d in the data. Joint with a CCP guess, one can derive the likelihood of any data point. Therefore, the CCCs and CCPs can be estimated directly by Sieve maximum likelihood.

5.2 2nd stage: structural model

I provide an estimation method for *parametric* models here. I do so for practical reasons since this avoids the curse of dimensionality and because it fits most applications. Assume the model is parametrized by $\theta \in \Theta$. As I did not address the identification of β (Magnac and Thesmar, 2002), I do not estimate it either, so it does not enter θ. The parameters θ can be divided into two parts $\theta = (\theta_0, \theta_1)$: where θ_0 enters the marginal utility and θ_1 does not. In the setup, u is parametrized by θ_0, and denoted $u_d(c_{dt}, \tilde{x}_t, \eta_t, \theta_0)$. The additive term $m_{dt}(x_t, w_t, \eta_t, \theta_1)$ is parametrized by θ_1. More precisely, θ_1 impacts the difference $\Delta m_t(x_t, w_t, \eta_t, \theta_1)$ of $m_{1t}(x_t, w_t, \eta_t) - m_{0t}(x_t, w_t, \eta_t)$, since only the difference is identified by the discrete choices.

I want to estimate θ. To do so, I use the CCCs, the CCPs and the transition estimated in the first stage. My estimation method is based on the minimization of two different objectives identifying different parameters: one based on the Euler equation and the other based on the conditional choice probabilities.

Euler objective:

Recall the notation for the marginal utilities at the optimal CCCs:

$$u_d'^*(\tilde{x}_t, a_t, \eta_t, \theta_0) = \frac{\partial}{\partial c_{dt}} u_d(c_{dt}, \tilde{x}_t, \eta_t, \theta_0)|_{c_{dt}=c^*_{dt}(\eta_t, \tilde{x}_t, a_t)}.$$

Thus, we have the Euler equation:

$$\begin{aligned} & u_d'^*(\tilde{x}_t, a_t, \eta_t, \theta_0) = \beta(1+r_t)\mathbb{E}_t\left[u'^*_{d_{t+1}}(\tilde{x}_{t+1}, a_{t+1}, \eta_{t+1}, \theta_0) \,\middle|\, x_t, c_{dt} = c^*_{dt}(\eta_t, x_t), d_t = d\right] \\ \overset{def}{\Longleftrightarrow} \quad & q_1(t, d_t, \eta_t, x_t, \theta_0) \ = q_2(t, d_t, \eta_t, x_t, \theta_0), \end{aligned}$$

where $x_t = (\tilde{x}_t, a_t)$.

The CCCs and the CCPs have been estimated in the first stage for all d_t, x_t, η_t, t. I also estimated the transitions. Thus, I can estimate θ_0 as:

$$\min_{\theta_0} Q^{euler}(\theta_0) = \sum_i \left(q_1(t, d_t, \eta_t, x_t, \theta_0) - q_2(t, d_t, \eta_t, x_t, \theta_0)\right)^2.$$

In other words, $\hat{\theta}_0$ minimizes the differences between the two sides of the Euler equation for every observation i in the sample.[11] Now, q_1 is directly given as a function of θ_0 and of the observed

[11] In practice, it is better to rewrite the objective such that it is scale invariant and comparable for all the values of θ_0. For example, if ceteris paribus a specific parameter value in θ_0 scales down everything in q_1 and q_2, the errors will be small at this parameter value, regardless of whether or not this parameter is far from the truth. To avoid that, one need that, for any set of parameters θ_0, the Euler Equation errors are on a similar scale. Log-linearization can be used to achieve this, for example. In the next section the parametric model is such that I can isolate consumption on the left-hand side of the Euler Equation. In this case, the left-hand side of the equation is based on consumption data and is independent of the parameters. Thus I can compare the results between parameters on the same basis.

characteristics and choices. q_2, on the other hand, contains an expectation and can be computed in several ways.

The first way is to use individuals present for two consecutive periods and to estimate the expectation of future utility using all individuals with the same current states x_t, c_{dt}, d_t. Since c is continuous and x contains continuous covariates, this can be done parametrically or via non-parametric kernel mean regression. This method is the simplest, but it requires many observations. It is close to the idea of Euler-GMM estimation, as pioneered by Hansen and Singleton (1982). The problem is that when the marginal utilities are highly nonlinear, the expectation is poorly estimated and this type of GMM estimation does not work well and needs to be refined Alan et al. (2009).

Hence, I prefer to use an alternative approach based on *forward simulations*, in the spirit of (Hotz et al., 1994). The idea is to use the CCCs, the CCPs and the transition to estimate the expectation term via one-period-ahead simulation. This method is slightly longer but less affected by the nonlinearity problem. It requires to estimate the transitions consistently.

Probability objective:

Now, the Euler equation does not provide any information about the parameters impacting the differences of the additive term, θ_1. To estimate these parameters, I use the relation between the choice probabilities and the conditional value function (Hotz and Miller, 1993). In particular, if ϵ is extreme value type I, we have:

$$Pr(D=0|\eta_t, x_t, w_t, \theta) = \frac{1}{1+exp\left(v^*_{1t}(x_t,\eta_t,\theta)+m_{1t}(x_t,w_t,\eta_t,\theta_1)-(v^*_{0t}(x_t,\eta_t,\theta)+m_0(x_t,w_t,\eta_t,\theta_1))\right)} \quad (11)$$

Knowing θ, one can estimate the conditional optimal values $v^*_{dt}(x_t, \eta_t, \theta)$ by *forward simulation* of the life-cycle, for example (Hotz et al., 1994). Note that the value functions are parametrized by θ and not only θ_1. Thus, a way to estimate the parameters is to minimize the differences between the estimated CCPs and the theoretical probabilities (equation (11)) with respect to θ for all observations:

$$\underset{\theta}{min} \quad Q^{proba}(\theta) = \sum_i \left(Pr(D=0|\eta_t, x_t, w_t, \theta) - \widehat{Pr(D=0|\eta_t}, x_t, w_t)\right)^2.$$

Global objective:

There are two *consistent* ways to estimate θ. The faster one is to perform the estimation in two separate steps: (i) estimate θ_0 from the Euler equation and (ii) estimate the remaining θ_1 from the probability objective (taking $\hat{\theta}_0$ as given). This yields a consistent estimation of θ.

However, the probability objective also depends on (part of) θ_0 which is identified by the Euler equation. An efficient way to account for this information is to perform the estimation in one step and find the parameters θ that minimize a weighted sum of both objectives:

$$\hat{\theta} = \underset{\theta}{argmin} \quad \text{weight}^{euler} Q^{euler}(\theta_0) + \ \text{weight}^{proba} Q^{proba}(\theta),$$

where the optimal weights are to be determined. At the optimal weights, the one step method is consistent and more efficient than the two-step estimation.

6 Estimator Performance

I test my estimator's performance with Monte Carlo simulations of the estimation of a parametric toy model of simultaneous labor and consumption choices. This model is a simplified version of the application performed in the next section. I provide additional robustness checks in Appendix C.

6.1 Toy model

The agent chooses to work (d_t) and consume/save (c_t) from $t = 1$ to $t = T$. Then she retires for one period in $t = T + 1$. She dies in $t = T + 2$.

Working life:

In each period the agent obtains utility:

$$u_{dt}(c_t, x_t, \eta_t) + m_{dt}(x_t, w_t, \eta_t) + \epsilon_{dt} = \begin{cases} c_t^{1-\sigma}/(1-\sigma)\ \tilde{\eta}_t^0(\eta_t, \gamma_0, s_0) + \epsilon_{0t} & \text{if } d_t = 0 \\ c_t^{1-\sigma}/(1-\sigma)\ \tilde{\eta}_t^1(\eta_t, \gamma_1, s_1) + \alpha + \omega(1 - w_t) + \epsilon_{1t} & \text{if } d_t = 1 \end{cases}$$

subject to the budget constraint:

$$a_{t+1} = (1 + r)a_t + d_t y_t - c_t + (1 - d_t)b_t.$$

t is the age of the agent. c_t is the individual consumption. d_t is the labor choice, equal to 1 if she works. w_t is the instrument, equal to the past labor choice d_{t-1}. a_t is the asset holdings. b_t represents benefits earned by unemployed people. y_t represents the earnings. y_t take only two values, y_L and y_H, for low and high income. In this way, the asset is the only continuous covariate, and I can reduce the state space with only two values in the support of y. I observe the income for every individual, even when she does not work. The interest rate r is fixed and equal to 0.05. $\epsilon_t = (\epsilon_{0t}, \epsilon_{1t})$ are additive idiosyncratic shocks impacting preferences for work. They are extreme-value type I. $\tilde{\eta}^d$ are nonseparable taste shocks to utility. $\tilde{\eta}^d(\eta, \gamma_d, s_d)$ is the η^{th} quantile of a lognormal(γ_d, s_d) distribution. In other words, $\tilde{\eta}^d \sim \mathcal{LN}(\gamma_d, s_d)$, so that $\tilde{\eta}^d$ are labor-dependent monotone transformations of the uniform η. Having η as quantiles of some specific distribution is a convenient way of modelling unobserved taste shocks in this type of setup. Thus, $(\gamma_0, \gamma_1, s_0, s_1)$ capture the different effects of unobserved taste shocks on the utility depending on working choice. I normalize $\gamma_0 = 0, s_0 = 0.25$ to interpret the parameters of working individuals with respect to this reference.

The other parameters are more conventional: σ is the risk aversion or intertemporal elasticity of substitution, α is the utility cost of work, and ω is the cost of searching for a job when one was previously unemployed ($w_t = 0$). Thus, $\theta = (\overbrace{\sigma, \gamma_0, \gamma_1, s_1}^{\equiv\theta_0}, \overbrace{\alpha, \omega}^{\equiv\theta_1})$, where θ_1 only impacts the probability of working and not the consumption choices, and θ_0 impacts both.

Transitions:

The asset transition is given by the budget constraint.

In the income transitions, I model gains from working experience: $Pr(y_{t+1} = y_H|d_t = 1, y_t) > Pr(y_{t+1} = y_H|d_t = 0, y_t)\ \forall y_t$. Income is also persistent, so if one had a high income in t, one is more likely to obtain a high income in $t+1$: $Pr(y_{t+1} = y_H|d_t, y_t = y_H) > Pr(y_{t+1} = y_H|d_t, y_t = y_L)\ \forall d_t$. It yields the following transition matrix:

$$Pr(y_1 = y_H|d_0, y_0) = \Pi(d_0, y_0) = \begin{pmatrix} \pi_{0L} & \pi_{0H} \\ \pi_{1L} & \pi_{1H} \end{pmatrix}$$

where $\pi_{1L} > \pi_{0L}$, $\pi_{1H} > \pi_{0H}$, $\pi_{1H} > \pi_{1L}$, and $\pi_{0H} > \pi_{0L}$.

These four parameters are estimated directly from the data by estimating $Pr(y_t = y_H|y_{t-1}, d_{t-1})$ with a bin operator, i.e., by computing the number of observations with $y = y_H$ over the total of observations with each specific y_{t-1}, d_{t-1} combination.

The shocks are iid and uncorrelated over time $\eta_{t+1} \perp \eta_t$ and $\epsilon_{t+1} \perp \epsilon_t$.

The agent discounts the future with discount factor β. I set it to 0.98 and do not estimate it.

Retirement:

At period $T+1$ the woman retires. She only consumes and can no longer work. She obtains the same period utility as when she was unemployed, without the additive ϵ shock. She obtains a $pension(y_T)$, which is a proportion set to 50% of her last income y_T. She lives for only one period in retirement and knows that she will die at $t = T + 2$.[12] There is no bequest motive. As a consequence, she will consume everything, i.e., $a_{T+2} = 0$. Thus, the last period consumption has a closed-form solution:

$$c^*_{T+1} = (1 + r)a_{T+1} + pension(y_T).$$

6.2 Comparison

I run Monte Carlo simulations of this toy model and estimate the parameters $\theta = (\sigma, \gamma_0, \gamma_1, s_1, \alpha, \omega)$ using my method. I compare my results with indirect inference Simulated Method of Moments (SMM) where the model is solved using Endogenous Grid Method (Iskhakov et al., 2017).

Results: (Table 1)
Table 1 shows the estimation results for a model with $T = 2$ periods.[13]

In terms of *speed*, my two-step method (*DCC* for Discrete-Continuous Choices) yields *sizeable computational gains*, even with respect to the state-of-the-art indirect inference method with endogenous grid (Carroll, 2006; Iskhakov et al., 2017). The idea is simple: I have a fixed computational cost of estimating the CCCs and CCPs in the first stage. However, thereafter, when solving for the optimal θ, I do not need to solve the model again, as I already have the optimal choices. I only need to perform some quick computations of the marginal utilities. Concerning the forward simulation, it is also a fixed cost, as the simulated path depends on first-stage CCCs, CCPs and transitions but not on θ. Thus, I only simulate forward once, and I retain the same path (as it is recommended to have the same basis for every set of parameters and to avoid adding some simulation noise to the estimation) for the computation of the expectations and conditional value functions for each tested set of parameters. The other methods, on the other hand, do not have my first-stage fixed cost, but they require considerably more computations in the second stage. If the model is very simple and the second stage is estimated quickly, these methods can perform quicker than mine in theory. However, as is well known, life-cycle models require a long time to solve, and the computational burden increases almost exponentially with the complexity of the model (more covariates, more

[12] This is a simplification; one could easily allow for a known length of retirement and solve the dynamic consumption problem of the retiree accordingly (as I do in the application). However, this does not deliver any particular insights in terms of estimator comparison, so I have agents live for only one period of retirement for simplicity.

[13] Coded in *R*, without parallelization here. Time results obtained from an Intel(R) Core(TM) i7-9750H CPU.

Table 1: $T = 2$ periods

	Truth	*Method* *DCC*	*SMM*
N		10,000	10,000
σ	1.60	1.6253	1.5924
		(0.0410)	(0.0156)
γ_1	0.00	0.0070	-0.0052
		(0.0238)	(0.0055)
s_1	0.40	0.4078	0.4001
		(0.0228)	(0.0071)
α	-0.50	-0.4727	-0.5023
		(0.0498)	(0.0348)
ω	-1.00	-0.9982	-0.9972
		(0.0581)	(0.0523)
Average Time taken:			
1st stage: CCPs and CCCs		118s	9s
2nd stage: Structural parameters		170s	14328s
Overall		288s	**14337s**

Other initializations: Number of Monte-Carlo $= 1,000$
$Pr(w_1 = 1) = 0.70$. $y_1 = y_H$ with probability 0.50. $a_1 \sim \mathcal{U}(0, 30)$. $r = 0.05$.

parameters, more periods). It can take several minutes to solve for one tested set of parameters, and finding the optimal parameters may require hundreds or even thousands of tests. Even here, in this very simple example with two periods, solving the model and estimating the moments for one set of parameters takes about 25 seconds with EGM. While with my method, computing the objective for one set of parameter takes less than a second. The more complex the model, the more this gap widens, as solving the model becomes even longer relatively speaking, while the fixed cost of computing the first stage policy only takes slighly longer. As a consequence, my two-stage method yields significant computational gains by reducing the burden of the second stage. Interestingly, *the more complex the model is*, *the more computation gains from my estimation method* relative to others. Obviously, having more complex models increases my computation time, but not in the same exponential manner as for the alternative methods.

In terms of statistical performances, my two-step method (DCC) estimates the parameters consistently and with small standard errors. As also shown in the simpler case of Appendix C where $T = 1$, Simulated Method of Moments (SMM) is consistent and more efficient for most parameters (if one uses a lot of moments). Both methods build upon the same initial estimation of the reduced forms conditional consumption distribution $F_{C_d|D_t,X_t,W_t,t}(c)$ and probability of working $Pr(D_t = 1|X_t, W_t, t)$. The moments are selected in these two objects.[14] And my first stage estimation of the CCCs is also built on these objects. Therefore, it is not surprising that both methods yield close results. I lose some efficiency due to the two-step nature of my method, similar to the efficiency loss of CCPs estimators. In theory, MLE is more efficient than both (see the $T = 1$ case in Appendix C), but it quickly becomes intractable to compute empirical likelihood with more periods.

Another advantage of my method is that if the model is misspecified, I still recover correct optimal choice estimates because the first stage is independent of the model assumptions (except for the choice of covariates to include). This is not the case for the alternative methods. Also, as I do not solve numerically for the optimal choices, I do not need to smooth potential kinks introduced by joint discrete-continuous choices, contrary to indirect inference with endogenous grid method (Iskhakov et al., 2017).

Overall, I have a method that is statistically consistent, with small standard errors, and considerably faster, by several orders of magnitude, than alternative state-of-the-art indirect inference with endogenous grid method. SMM built upon moments drawn from the reduced forms is also consistent and more efficient but the computational burden is too heavy for complex life-cycle models.

7 Application: women's labor and consumption

I illustrate the method developed in this paper with a parametric dynamic model of simultaneous employment and consumption choices for women over their life cycle. I choose a parametric

[14] If we do not pick moments in the reduced forms, i.e., moments conditioned by w, then the model is not identified and the SMM estimation will not be consistent.

application for practical reasons: to avoid the curse of dimensionality and to be able to compare my parameter estimates with the literature. This model especially matters for understanding how different benefit schedules affect the careers of women, particularly mothers, who are known to be the most responsive to incentives (Blundell and Macurdy, 1999; Blundell et al., 2016). It allows us to understand the mechanism underlying individual choices and thus to carry out counterfactual policy analysis in the long run.

My method is of particular interest for two main reasons here. First, life-cycle models such as that presented here are extremely computationally intensive to estimate, to the extent that one often needs to restrict the complexity of the model for the estimation to be tractable. By first estimating the optimal choices (CCCs and CCPs) and only then the structural parameters in a second step, I do not need to solve the model, and I am able to drastically reduce the computational cost, in the spirit of Hotz and Miller (1993), Hotz et al. (1994) and Arcidiacono and Miller (2011). Faster computation means that one can include more features in existing models, for example, more heterogeneity, observed or unobserved, and still be able to estimate them in a reasonable time. The complete estimation of this complex model only takes me a few hours here, while it could take weeks or months with alternative methods.

In addition to the speed increase, I also include more unobserved heterogeneity in the model with the η term. Thus, by construction, I estimate the *distribution* of consumption choices and working probabilities at any given set of observed covariates, and not only the average choices. This yields new insights in this literature.

7.1 Model

Overview:

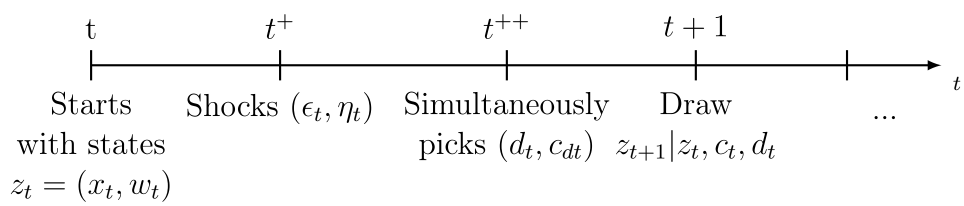
The parametric model enters the general dynamic framework described in section 4.1. It is a more realistic version of the toy model described previously, with the same key features. I model the annual consumption and labor supply choices of women from $t = 26$ to $t = 60$ years of age. Each period, women determine their household consumption c_t, and whether they work d_t.[15] At the age 60, they retire and live for 15 more years on their accumulated savings and their pension, which depends on their last income. Throughout their life, women may bear children. Fertility occurs

[15] I focus on the extensive margin, not on the number of hours worked. Individuals are assumed to either work full time or be unemployed. This might be restrictive, particularly for single mothers, who are known to resort more to part-time jobs.

randomly following the trend observed in the data, and is not explicitly modelled as a choice. Couples do not divorce, and new couples are not formed in the model. This is for simplicity to avoid dividing the assets or modelling individual husbands' assets. Women's productivities (and thus wages) evolve over their careers. Labor supply choice plays a key role in this evolution, as working experience increases expected future wages, while productivity can depreciate for unemployed women. Similarly, asset holdings evolves over the life cycle following a budget constraint that depends on previous asset holdings, consumption, women's productivities and labor choices (they are paid only if they work), their potential husband's annual income and the tax schedule to which they are subject. The benefit/tax schedule is simplified and estimated based on observed data. It differs depending on the individual's family situation, wealth and labor choice. Finally, women are subject to unobserved preference shocks η and ϵ. η is their unobserved taste shock for consumption, and ϵ represents their unobserved preference for work. With η, I can estimate heterogeneous consumption choices for individuals who are identical as measured by their covariates.

I now describe the model in greater detail.

Working life:



From age $t_0 = 26$ to age $T = 60$, a woman is in her working life. She makes her decision (d_t, c_t) to maximize her expected lifetime utility given her characteristics. These characteristics include her age (t), her income (y_t), her assets (a_t) and some demographics $\tilde{x}_t$: her number of children ($nchild_t$), whether she is in a couple ($couple_t$), and if so, her partner's annual income (y_t^p) and labor force participation d_t^p. All these covariates are included in $x_t = (a_t, y_t, \tilde{x}_t)$.[16] Her decision to work is also influenced by whether she worked before $w_t = d_{t-1}$, for which we observe $w_0 = d_{-1}$. w_t matters because of the utility cost of switching from being unemployed to employed. She also makes her decision based on two idiosyncratic shocks η_t and ϵ_t, unobserved by the econometrician. To satisfy

[16] A small detail: y_t is now excluded from $\tilde{x}_t$, at minimal risk of confusion. It is because y_t does not enter the current period utility directly.

the distributional Assumptions 4-7, we have $\eta_t \sim \mathcal{U}(0,1)$ iid over time, and ϵ_t is i.i.d. extreme-value type I.

Each period, the agent obtains utility:

$$u(c_t, d_t, w_t, x_t, \eta_t, \epsilon_t) = \begin{cases} (c_t/n_t)^{1-\sigma}/(1-\sigma)\ \tilde{\eta}_t^0(\eta_t, couple_t, nchild_t) + \epsilon_{0t} & \text{if } d_t = 0 \\ (c_t/n_t)^{1-\sigma}/(1-\sigma)\ \tilde{\eta}_t^1(\eta_t, couple_t, nchild_t) + \alpha + \omega(1-w_t) + \epsilon_{1t} & \text{if } d_t = 1 \end{cases}$$

$$\equiv \begin{cases} u_0(c_t, \tilde{x}_t, \eta_t) + \ \epsilon_{0t} \\ u_1(c_t, \tilde{x}_t, \eta_t) + \ \underbrace{\alpha + \omega(1-w_t)}_{=m_1(w_t)} + \epsilon_{1t} \end{cases}.$$

where c_t is the total household consumption over the period, n_t is an equivalence scale, which depends on the number of consumption units in the household, i.e., $n_t(couple_t, nchild_t)$, with $n_t(0,0) = 1$, $n_t(1,0) = 1.6$, $n_t(0, 1 \text{ or more}) = 1.4$, $n_t(1, 1 \text{ or more}) = 2$ (Blundell et al., 2016). Thus, c_t/n_t represents *individual consumption*. σ is the elasticity of intertemporal substitution/risk aversion parameter. The effect of the unobserved shock η_t varies depending on the work choice (d_t) and family situation ($couple_t, nchild_t$). $\tilde{\eta}_t^d$ are transformations of η_t, where $\tilde{\eta}_t^d \sim \mathcal{LN}(\gamma_d + \gamma_d^c couple_t + \gamma_d^n nchild_t, s_d)$. $\tilde{\eta}_t^d$ are the η^{th} quantiles of these distributions. This is a convenient way to include covariates in this setup. Since $\eta_t \sim \mathcal{U}(0,1)$, the transformation to $\tilde{\eta}_t^d$ allows for a wide range of effects of η_t. Note that with this functional form, the monotonicity Assumption *D*3 is satisfied. The parameters (γ_d, s_d) represent the baseline effect of unobserved heterogeneity depending on working behaviour, for single women without children. The parameters (γ_d^n, γ_d^c) determine the effect of the family situation. I set γ_0 to 0 and s_0 to 0.5 so that the other coefficients are interpreted with respect to this baseline. The agents incur a utility cost α from working. w_t is the instrument that corresponds to the past labor choice, $w_t = d_{t-1}$. The agents incur a an utility cost ω from searching for a job (if they were previously unemployed). Thus, $m_{dt}(w_t, x_t, \eta_t) = \alpha d_t + \omega(1-w_t)d_t$, and it is independent of x_t, η_t and t in this application (which is stronger than necessary for the identification). By the additivity of the instrument w_t, Assumption *D*2 is satisfied. Similarly, additive separability of ϵ_t (Assumption *D*1) is satisfied. Note that I have time independent current utility. However in the parametric framework I can identify time-varying utility (Assumption 14 is only necessary for non-parametric identification). Thus one could include and estimate time fixed effects for example.

Transition:

The woman makes her choice of d_t, c_t subject to the *household budget constraint*:

$$a_{t+1} = (1+r)a_t - c_t + d_t y_t + \text{couple}_t d_t^p y_t^p + T(d_t, x_t).$$

This budget constraint describes the asset transition over time. r is the real interest rate. If the woman works $d_t = 1$, she obtains a wage y_t. If she has a husband ($couple_t = 1$) who works ($d_t^p = 1$), the household also obtains his total income ($= 0$ if he does not work). $T(d_t, x_t)$ is the *benefit-tax* schedule. It is a function of the covariates and labor choice. I estimate it directly from the data.

Earnings y_t and husband's earnings y_t^p evolve over time according to an auto-regressive process:

$$\begin{aligned} y_{t+1} &= (\rho_y y_t + \rho_d d_t + \rho_{age} t) \times \text{educ}_t + u_t \\ y_{t+1}^p &= \rho_y^p y_t^p + \rho_d^p d_t^p + v_t, \end{aligned}$$

where u_t, v_t may be correlated. Working ($d_t = 1$) allows individuals to change their expected earnings and potentially increase them. Unemployment will decrease productivity if $\rho_y < 1$, i.e., human capital depreciates. Therefore, working is important not only for current consumption and savings but also for its impact on human capital accumulation. All these coefficients vary with the education level ($\leq$secondary, high school or university) of the woman, educ_t. I do not include the education of the partner to avoid having too many variables in the model, since I focus on the woman. The earning process is estimated directly from the data on observed transitions.

Auto-regressive processes are also estimated for fertility (having a new-born child) and for the husband's work decisions. These depend on past x_t only. d_t and c_t do not enter the transitions here.

Finally, by construction, the next value of the instrument $w_{t+1} = d_t$ and the other covariates than d_t are irrelevant for its transition. Since conditional on current (d_t, c_t, x_t), w_t does not enter the transition of the other variables, the conditional independence Assumption 10 is satisfied.

Retirement:

At age T, the woman retires, and can no longer decide to work. She obtains the same utility as when she did not work, with $d_t = 0$, without the additive ϵ shock.[17] She lives for another 15 years on her accumulated assets and receives a pension that is a proportion of her last income

[17] Instead, I can normalize the utility of retired women with parameters γ_r and s_r. In which case, I could estimate the γ_0 and s_0 of unemployed women with respect to the retirees baseline. However, my estimation would then be driven by the data comparison of the Euler equation at the retirement age, which represents only a small subset of my panel. Therefore, I prefer to set the utility of retirees equal to the utility of unemployed individuals.

y_T. I include no bequest motive in the model. One can easily solve the consumption problem of retirees, which depend on their last income and assets, to obtain the expected *retirement utility*: $R(x_T) = R(a_T, y_T, y_T^p, couple_T, nchild_T)$.

Life-cycle problem:

The working life decision problem is the one that interests us. Given the development above, at any age t during her working life, the woman decision problem can be written as:

$$V_t(z_t) \equiv \mathbb{E}\Big[\sum_{\tau=t}^{T} \beta^{\tau-t} \underset{d,c_{d\tau}}{max}[\ u_d(c_{d\tau}, \tilde{x}_\tau, \eta_\tau) + m_d(\tilde{x}_\tau, w_\tau, \eta_\tau) + \epsilon_{d\tau}] + \beta^{T-t} R(x_T)\Big],$$

where the future is discounted at a rate β.

For notational simplicity, denote $V_{T+1}(z_{T+1}) = R(x_T)$, a special form of the value function for the retirement period. We have already verified that the identification assumptions hold. Therefore, following computations performed in section 4.1, we return to the general setup, where the woman selects d_t and c_t at each age t to solve:

$$\underset{d_t,c_{dt}}{max} \quad v_d(c_{dt}, x_t, \eta_t) \ + m_d(\tilde{x}_t, w_t, \eta_t) + \epsilon_{dt}, \tag{12}$$

where the conditional value function is given by:

$$v_{dt}(c_{dt}, x_t, \eta_t) \ \equiv \ u_d(c_{dt}, \tilde{x}_t, \eta_t) + \beta \mathbb{E}_{z_{t+1}}[V_{t+1}(z_{t+1})|_{x_t, c_{dt}, d_t}].$$

The agent internalizes the effect of her choice on her future, discounting it at a rate β. Note that even here where the current utility has a known parametric form, the conditional value's form is generally more complex, with no closed form solution. It depends on complex transitions and expectations about the future. Therefore, the advantage of my method, i.e., that I am able to estimate both the optimal policies (CCCs and CCPs) and the parameters of interest without numerically solving for the conditional value, also applies to parametric dynamic models, hence the sizeable computational gains.

7.2 Data

To estimate the model, I use EU-SILC French survey data. It is a survey conducted every year and follows households from 2004 to 2015. The data contain information about the labor market status (income, job tenure), asset holdings (financial and housing), tax paid and benefits received, and personal characteristics of the individuals (family situation, education, etc.). Data are available for all the individuals within the household, which is why I also have detailed information about the partner.

Consumption is not directly available in the data, I reconstruct it for households present over two consecutive years based assets evolution and savings.

I set that a woman *works* ($d = 1$) if she worked more than 6 months during the year.

Income is only reported for employed women and husbands. I estimate y and y^p based on the income information of workers using the standard Heckman correction (Heckman, 1979) beforehand. For simplicity, I assume that income is observed for everyone using these estimations when I estimate the model.[18] I estimate this on the subsample of full-time working individuals so that I obtain a productivity y_t representing full-time equivalent productivity. In this estimation, I include covariates other than those used in the model, including education, experience, some parent background information, and zone and year fixed effects.

After cleaning the data for outliers and missing values, I end up with an unbalanced panel of $7,391$ women between 26 and 60 years of age, yielding a total of $21,945$ observations over 11 years.

I fix the real interest rate r at the average of the period ($= 0.05$), as given by the IMF French data.

Descriptive statistics:

Table 2 describes the sample of data I use; 76% of women work, with a strong auto-correlation in employment: if a woman worked before ($w_t = d_{t-1} = 1$), $Pr(D_t = 1|w_t = 1)$ is very high $= 0.96$, while if she did not, $Pr(D_t = 1|w_t = 0) = 0.14$ is low. This suggests that w should be a *relevant* instrument for d. On average, households consume $36k5$ euros per year. Observed consumption conditional on working ($c_{1t}|d_t = 1$) is higher than consumption conditional on being unemployed ($c_{0t}|d_t = 0$). However, we do not yet know how much of this is due to the selection: it is possible that women with high η_t select more into employment, boosting the average consumption conditional on working. Regarding the covariates, there is considerable variance in asset holdings. Most women (75%) are in couples, with a median number of 2 children. Their partner is generally working (93%), far more frequently than the women. The partner's income is also larger than

[18] Obviously, given that the probability of working is a key part of the model, ideally one would prefer to build our own Heckman correction within the model here, with some kind of nested iteration with the CCP, CCC and productivity estimation, in the spirit of Aguirregabiria and Mira (2002) for example. Another good way to deal with it would be to include *unobserved types*, as in Arcidiacono and Miller (2011), and have wage depend on these types.

Table 2: EU-SILC unbalanced panel, 2004 − 2015, 7391 women

Statistic	N	Mean	St. Dev.	Min	Median	Max
Choices:						
Annual household c (k euros)	21,945	36.58	20.99	3.88	32.54	211.54
$c\|d = 0$	5,330	30.04	19.32	4.02	25.58	204.48
$c\|d = 1$	16,615	38.67	21.07	3.88	34.78	211.54
d	21,945	0.76	0.43	0	1	1
w	21,945	0.76	0.43	0	1	1
$d\|w = 0$	5,354	0.14	0.35	0	0	1
$d\|w = 1$	16,591	0.96	0.20	0	1	1
Covariates:						
Age	21,945	42.37	9.39	26	42	60
Annual Income y (Heckman)	21,945	19.74	5.29	8.10	19.07	43.32
Asset	21,945	108.29	118.55	−32	69.0	528
Nb of children	21,945	1.71	1.09	0	2	4
Couple	21,945	0.75	0.43	0	1	1
Working partner|Couple	16,442	0.93	0.25	0	1	1
Partner's income y^p |Couple	16,442	26.41	13.21	4.02	23.20	147.54
Completed Education	21,945					
≤ Secondary	5,240	0.24				
High School	9,999	0.46				
University	6,706	0.30				
Other:						
Receives Benefits	21,945	0.66	0.47	0	1	1
Benefits|Benefits > 0	14,478	5.16	4.46	0.002	3.60	23.07

c, y, y^p, asset and benefits expressed in real terms (base 2010) and in thousands of euros.

the woman's annual income. Finally, approximately 66% of the households received some kind of benefit. These benefits include not only unemployment benefits, but also family benefits, for example. This is why there are more people receiving benefits than the number of unemployed people.

7.3 1st stage: Optimal Choices Estimation

7.3.1 Estimation

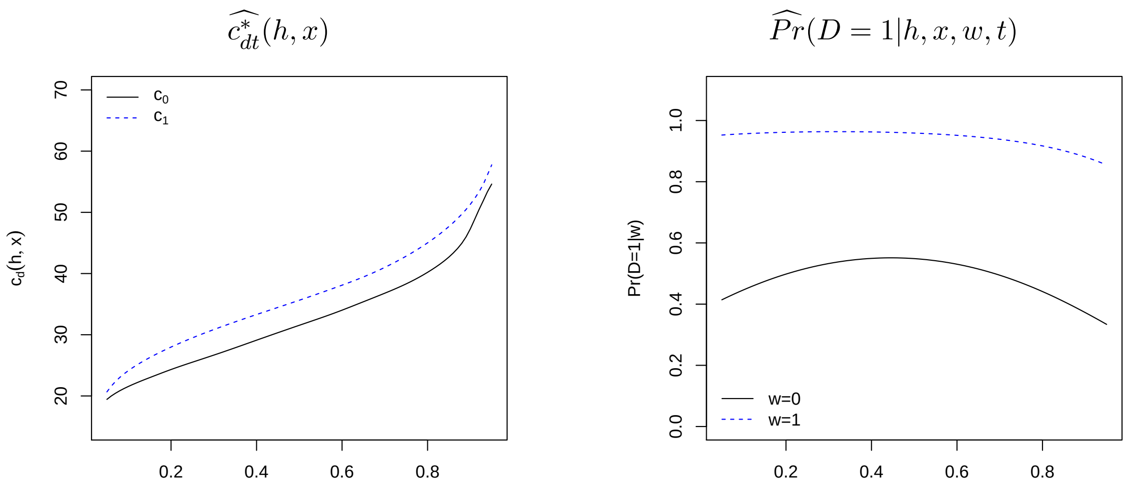
I follow the procedure described in section 5. First, I estimate the reduced-form probabilities and conditional distributions. The probabilities are estimated with a sieve logistic regression. The conditional distributions are estimated via non-parametric kernel methods (R package *np*, Hayfield and Racine, 2008).

Then, I estimate the CCCs $c_{dt}(h_t, x_t)$ for all h_t, x_t accordingly. Once I have the CCCs, I estimate $\hat{h}_t$ from the observed (c_t, d_t, x_t). I then recover the CCPs $\widehat{Pr}(D_t = d|\eta_t = h, X_t = x, W_t = w_t)$ using a sieve logistic regression.

I also estimate the transitions to $X_{t+1}|X_t, D_t, C_t$ according to the description provided previously.

7.3.2 Results

Figure 1: CCCs and CCPs estimates



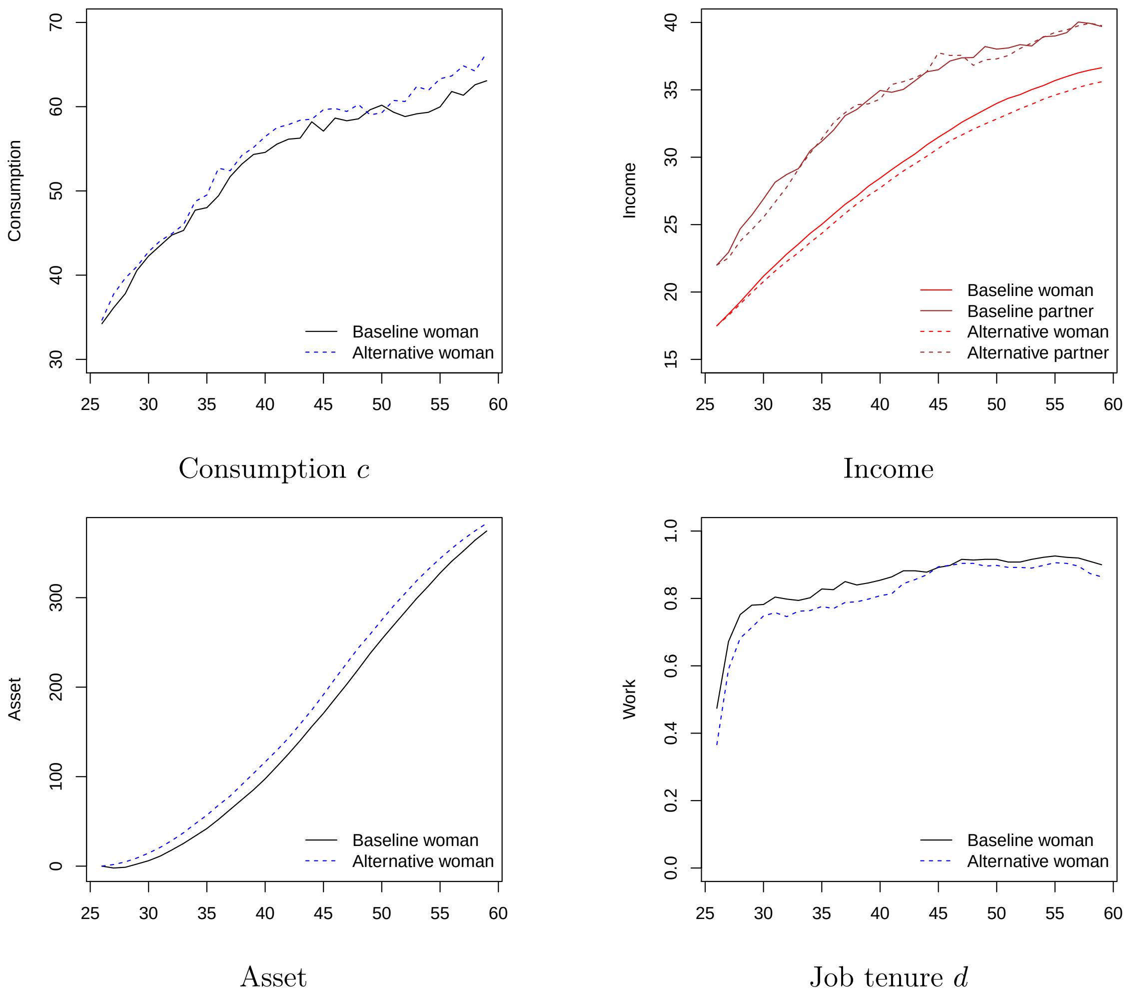
Average evolution of a *baseline woman* with median characteristics: $26y.o.$ woman with income of $17k5$ euros, no assets, in a couple, no children, with a partner earning $22k$ euros.

Optimal Choices:

Figure 1 shows the optimal choice estimates for the median 26 year-old woman. Potential consumption when working is always greater than alternative consumption when unemployed, with an average difference of approximately $3,930$ euros of consumption per year. By construction, these functions are monotone with respect to the taste shock h.

The conditional choice probabilities are more complicated. First, note that the probability of

Figure 2: Life-cycle simulations



Average evolution of a *baseline woman*: $26y.o.$ working ($w = 0$) woman with income of $17k5$ euros, no assets, in a couple, no children, with a partner earning $22k$ euros.
And an *alternative woman* with the same characteristics but 2 children.

working is always greater and close to 1 for individuals who were employed previously. If the median woman was previously unemployed, however, her probability of working today is less than 50%. The relation between the employment decision and the taste shock is complex. By working, the woman will obtain an income that she will be able to consume. However, at the same time, she will have less leisure time. There is a trade-off between a substitution and wealth effect. If she was previously employed ($w_t = 1$), the higher her taste shock (η_t) was, the less likely the median woman was to work. The substitution effect dominates. Note that even is she is less likely to work, in any case, the higher η_t is, the more she consumes. If she was previously unemployed, the case is more complicated. Up to approximately the median taste shock, the wealth effect dominates, and she will choose to work more to consume more. After this threshold, it decreases, and the more she wants to consume, the less she will work.

Life-cycle simulation:

Figure 2 shows the average results over $1,000$ life-cycle simulation for the median 26 year-old woman and for an alternative woman with the same characteristics but two children. First, consumption, income and asset all increase throughout the life cycle. Her partner's income also increases in a similar fashion. Once they enter the labor force, women are increasingly likely to work until retirement. By having two children, the alternative woman is less likely to work initially, and this persists throughout her life cycle. As a consequence, she on average has an income disadvantage, while her husband seems to suffer no particular penalty. However, with two children she will initially obtain more benefits and be able to accumulate slightly more assets. The households with two children consume only slightly more, which suggests that they obtain considerably lower utility from their individual consumption.

7.4 2nd stage: Structural model Estimation

7.4.1 Estimation

Now, I want to estimate the set of structural parameters: $\theta = (\underbrace{\sigma, \gamma_0^c, s_0^c, \gamma_0^n, s_0^n, \gamma_1^c, s_1^c, \gamma_1^n, s_1^n}_{\equiv\,\theta_0}, \underbrace{\alpha, \omega}_{\equiv\,\theta_1})$.

Following section 4.2.3, denote the marginal utilities at optimal CCCs as:

$$u_d'^*(\tilde{x}_t, a_t, \eta_t) = \frac{\partial}{\partial c_{dt}} u_d(c_{dt}, \tilde{x}_t, \eta_t)|_{c_{dt}=c_{dt}^*(\eta_t, \tilde{x}_t, a_t)}.$$

Thus I have the Euler equation for all d:

$$u_d'^*(\tilde{x}_t, a_t, \eta_t, \theta_0) = \beta(1+r_t)\mathbb{E}_t\Big[u_{d_{t+1}}'^*(\tilde{x}_{t+1}, a_{t+1}, \eta_{t+1}, \theta_0)\ \Big| x_t, c_{dt}, d_t\Big].$$

Here, the functional form of $\partial u_d(c_d, \tilde{x}, \eta)/\partial c_d$ is known and depends on the parameter θ_0. Thus, θ_0 are estimated in a first step by minimizing the differences between the two sides of the equation. For the left-hand side of the equation, I use every observation in the data, including the estimated $\hat{\eta}$ as if it was observed and the corresponding observed c_{dt}. For the right-hand side, one can either take the empirical expectation about the future, or simulate it using the estimated CCCs, the CCPs and the transitions. Given the small number of observations I have, I prefer to use the former solution in this application.

The other parameters α and ω (in γ_1) additively enter the utility, so they are not in the Euler equation. They are identified via the CCPs. To recover α and ω, I choose to simulate complete life cycles for each set of parameters θ using the reduced forms. In this way, I obtain estimates of the conditional value functions $v_d()$, and using extreme-value type-1 form of ϵ, I can recover the theoretical $Pr(D_t = 1|\eta_t, X_t, W_t)$ and compare them to the CCPs. The optimal parameters θ minimize these differences. I run the two-stage estimates, so I estimate $\widehat{\theta}_1$ by minimizing the difference in probabilities with respect to θ_1 with θ_0 fixed to the $\widehat{\theta}_0$ estimated in the first stage.

7.4.2 Results

Table 3: Structural parameter estimates

	Parameter estimates	
	Parameter	Estimate
Discount factor	β	0.98
		(fixed)
Constant Relative Risk Aversion	σ	1.63
Effect of η by family...		
... when unemployed:		
$\mathcal{LN}(\gamma_0^c couple + \gamma_0^n nchild,\ s_0)$	γ_0	0
		(fixed)
	γ_0^c	-1.80
	γ_0^n	-0.31
	s_0	0.50
		(fixed)
... when employed:	γ_1	-1.04
$\mathcal{LN}(\gamma_1 + \gamma_1^c couple + \gamma_1^n nchild,\ s_1)$	γ_1^c	-0.65
	γ_1^n	-0.10
	s_1	0.54
Additive terms:		
Utility cost of working	α	-0.04
Utility cost of search	ω	-2.14

Structural Parameters: (Table 3)

I find a coefficient for risk aversion (and the elasticity of intertemporal substitution) similar to the literature: 1.63 versus 1.56 in Blundell et al. (1994), or 1.53 in Alan et al. (2009). It suggests my method yields consistent estimations, with more complex model and faster computation.

As expected, the utility cost of searching for a job when previously unemployed is high. In comparison, the utility cost of working is almost null. For the effect of the taste for consumption, note first that the smaller the coefficient is, the higher the utility since $1 - \sigma < 0$. Thus, note that, for a given consumption level, single working women without children have higher utility (on average) than if they were unemployed ($\gamma_1 = -1.04 < \gamma_0 = 0$). When they are in couples without children, their utility is similar ($-1.04 - 0.65$ versus -1.80). Additional children yield more disutility for employed women ($-0.10 > -0.31$). The variances are similar ($s_1 = 0.54$ close to the fixed $s_0 = 0.50$). e

One could use this estimated model to perform counterfactual analysis and study the effect of different labor market reforms on women's consumption and career choice, such as, the effect of increasing the age of retirement or changing the benefits given to single mothers.

8 Conclusion

This paper develops a general class of discrete-continuous choice models and provides a list of conditions to achieve non-parametric identification. The identification proof is original as it solves for a unique monotone function instead of proceeding pointwise, which allows identification under weaker relevance conditions than in the existing literature.

Given the identification, I provide a new estimation procedure yielding sizeable computational gains with respect to the existing alternatives for the estimation of dynamic models. The gains are so large that they should facilitate the use of complex dynamic discrete-continuous models in the future and offer greater latitude to researchers to test for several model specifications. This will allow us to find new results in several fields where my methodology can be applied: labor, housing, education, industrial organization, etc.

On a final note, part of the method described here applies more broadly to discrete-continuous dynamic *processes*, choices or not. This yields additional interesting dynamic applications.

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A Proof: Lemma 3

I prove Lemma 3.

Proof. First, let us relate $Pr(d|\eta = h, W = 1) - Pr(d|\eta = h, W = 0)$ to the distributions/quantiles. Recall that using Bayes and $\eta \perp w$, we have $\forall w$:

$$\begin{aligned}
h = &\quad Pr(\eta \leq h) \\
= &\quad Pr(\eta \leq h|w) \\
= &\quad Pr(\eta \leq h \mid D = 0, w)Pr(D = 0|w) + Pr(\eta \leq h \mid D = 1, w)Pr(D = 1|w) \\
= &\quad F_{\eta|D=0,w}(h)p_{0|w} + F_{\eta|D=1,w}(h)p_{1|w}.
\end{aligned} \tag{13}$$

Then, combining (13) at $w = 1$ and $w = 0$, we obtain $\forall h$:

$$\begin{aligned}
F_{\eta|D=0,W=1}(h)p_{0|1} - F_{\eta|D=0,W=0}(h)p_{0|0} &= -\Big(F_{\eta|D=1,W=1}(h)p_{1|1} - F_{\eta|D=1,W=0}(h)p_{1|0}\Big) \\
\overset{def}{\Longleftrightarrow} \Delta F_{\eta_0}(h) &= \Delta F_{\eta_1}(h).
\end{aligned} \tag{14}$$

Moreover, notice that we can rewrite $F_{\eta_d|w}(h)$:

$$\begin{aligned}
F_{\eta|d,w}(h) &= Pr(\eta \leq h|d, w) \\
&= Pr(\eta \leq h,\ d|w)/Pr(d|w) \\
&= Pr(d|\eta \leq h, w)Pr(\eta \leq h|w)/Pr(d|w).
\end{aligned} \tag{15}$$

Let us focus on the choice $D = 0$ (by symmetry it will be the same for $D = 1$) and rewrite (14) by plugging (15) into it:

$$\begin{aligned}
\Delta F_{\eta_0}(h) = &\big[Pr(D = 0|\eta \leq h, W = 1)Pr(\eta \leq h|W = 1)/\cancel{Pr(D = 0|W = 1)}\big]\cancel{Pr(D = 0|W = 1)} \\
&- \big[Pr(D = 0|\eta \leq h, W = 0)Pr(\eta \leq h|W = 0)/\cancel{Pr(D = 0|W = 0)}\big]\cancel{Pr(D = 0|W = 0)}.
\end{aligned}$$

Moreover, since $W \perp \eta$: $Pr(\eta \leq h|W = 1) = Pr(\eta \leq h|W = 0) = Pr(\eta \leq h) = h$, we have:

$$\Delta F_{\eta_0}(h) = \big[\ Pr(D = 0|\eta \leq h, W = 1) - Pr(D = 0|\eta \leq h, W = 0)\ \big]h. \tag{16}$$

Now, note that if $\eta \sim \mathcal{U}(0, 1)$:

$$Pr(D = 0|\eta \leq h_0, w) = \int_0^{h_0} Pr(D = 0|\eta = h, w)/F(h_0)dh = \int_0^{h_0} Pr(D = 0|\eta = h, w)/h_0 dh. \tag{17}$$

Thus, we can rewrite (16) $\forall h_0$ as:

$$\begin{aligned}
\Delta F_{\eta_0}(h_0) &= \Big[\int_0^{h_0} Pr(D=0|\eta=h, W=1)/h_0 dh - \int_0^{h_0} Pr(D=0|\eta=h, W=0)/h_0 dh \Big] h_0 \\
&= \int_0^{h_0} \Big(Pr(D=0|\eta=h, W=1) - Pr(D=0|\eta=h, W=0) \Big) dh. \qquad (18)
\end{aligned}$$

Which leads to:

$$Pr(D=0|\eta=h, W=1) - Pr(D=0|\eta=h, W=0) = \frac{d\Delta F_{\eta_0}(h)}{dh}.$$

Moreover, if $c_d(h)$ is a strictly monotone solution to our problem, we have $F_{\eta|d,w}(h) = F_{C_d|d,w}(c_d(h))$, and thus $\forall h$:

$$\begin{aligned}
\Delta F_{\eta_0}(h) &= F_{\eta|D=0,1}(h)p_{0|1} - F_{\eta|D=0,0}(h)p_{0|0} \\
&= F_{C_0|D=0,W=1}(c_0(h))p_{0|1} - F_{C_0|D=0,W=0}(c_0(h))p_{0|0} \\
&\overset{def}{=} \Delta F_{C_0}(c_0(h)).
\end{aligned}$$

Thus:

$$\begin{aligned}
Pr(D=0|\eta=h, W=1) - Pr(D=0|\eta=h, W=0) &= \frac{d\Delta F_{\eta_0}(h)}{dh} \\
&= \frac{d\Delta F_{C_0}(c_0(h))}{dh} \\
&= \frac{d\Delta F_{C_0}(c_0)}{dc_0} \underbrace{\frac{dc_0(h)}{dh}}_{>0}.
\end{aligned}$$

So, under assumption 9b, there is a finite set of K points $h \in [0,1]$ such that $Pr(D=0|\eta = h, W=1) - Pr(D=0|\eta=h, W=0) = 0$. Then there is a finite set of K points $c_0(h_k)$ such that $d\Delta F_{C_0}(c_0(h))/dh = 0$. And since $c_0'(h) > 0$ because of the monotonicity of the quantiles, it implies that there is a finite set of K points such that $d\Delta F_{C_0}(c_0)/dc_0 = 0$.

We can follow exactly the same reasoning for $D = 1$. We obtain that, if there is a finite set of K points $h \in [0,1]$ such that $Pr(D=1|\eta=h, W=1) - Pr(D=1|\eta=h, W=0) = 0$, then there is a finite set of K points $c_1(h_k)$ such that $d\Delta F_{C_1}(c_1(h))/dh = 0$. Since $c_1'(h) > 0$ because of the monotonicity of the quantiles, it implies that there is a finite set of K points such that $d\Delta F_{C_1}(c_1)/dc_1 = 0$. ∎

B Proof of Identification Theorem 1

I develop the complete proof of Theorem 1 about the identification of the continuous choice policies.

Proof. For a given increasing solution $c_d(h)$, let us first introduce the notation:

$$p_{d|w} \equiv Pr(D = d|W = w)$$

$$\gamma_d(h) \equiv F_{C_d|D=d,W=0}(c_d(h))$$

$$\Psi_{d1}(\gamma_d(h)) \equiv F_{C_d|D=d,W=1}(c_d(h)) = \underbrace{F_{C_d|D=d,W=1}(F^{-1}_{C_d|D=d,W=0}(}_{\equiv\Psi_{d1}()}\underbrace{F_{C_d|D=d,W=0}(c_d(h))}_{\equiv\gamma_d(h)})).$$

Recall from Lemma 2 that $F_{C_d|d,w}(c)$: $\mathcal{C}_d \to [0,1]$ is C^1 and strictly increasing function of c. Now, under assumption 3, a solution $c_d(h)$ is also a strictly increasing and C^1 function of h. Thus $\gamma_d(h)$ are C^1 and strictly increasing functions of h, from $\gamma_d(0) = 0$ to $\gamma_d(1) = 1$. The mappings $\Psi_{d1}(\gamma_d)$ give us the mapping from the quantiles of c_d with instrument value $W = 0$ to their counterpart with instrument value $W = 1$. Similarly, given Lemma 2, we have that Ψ_{d1} are also C^1 and strictly increasing functions of γ_d from $\Psi_{d1}(0) = 0$ to $\Psi_{d1}(1) = 1$. These mappings are directly reconstructed from the data since the data identifies $F_{C_d|d,w}$ $\forall d, w$. So, from data on (c_d, d, w) we now recover: $\forall(d, w)$ $p_{d|w}$ and $\forall d$ $\Psi_{d1}(\gamma_d)$.

Under this reparametrization, the system described in equation (2) rewrites, $\forall h$, with increasing $\gamma_d(h)$:

$$\begin{cases} h = \gamma_0(h)p_{0|0} \quad + \quad \gamma_1(h)p_{1|0} \\ h = \Psi_{01}(\gamma_0(h))p_{0|1} \; + \; \Psi_{11}(\gamma_1(h))p_{1|1} \end{cases} . \tag{19}$$

The conditional distribution functions $F_{C_d|d,w=0}$ are known, strictly increasing and invertible (Lemma 2). So if there is a unique solution $\{\gamma_d(h)\}_{d\in\{0,1\}}$ to system (19), there is a unique solution $\{c_d(h)\}_{d\in\{0,1\}}$ to the original system (2). Thus, we first show uniqueness of $\{\gamma_d(h)\}_{d\in\{0,1\}}$ to system (19), and then we will come back to $\{c_d(h)\}_{d\in\{0,1\}}$.

Lemma 7 (Identification) *(in the reparametrized problem)*
Under assumption 9, there exists a unique strictly increasing $\gamma_d(h)$ solution to system (19) starting from $(\gamma_0(0), \gamma_1(0)) = (0, 0)$ to $(\gamma_0(1), \gamma_1(1)) = (1, 1)$.

Proof of Lemma 7.

We prove that there exists a unique increasing solution to system (19).

Existence: *existence* is straightforward. Indeed, we are only focusing on images of the structural model. So by construction, with the true optimal policies, $\gamma_d^*(h) = F_{C_d|D=d,W=0}(c_d^*(h))$ are solutions to the system.

Uniqueness: we need to show this is the *unique* strictly increasing solution to this problem. To do this, we procede in two-steps. First we show that there is a unique increasing mapping between the two conditional quantiles, denoted $\gamma_1^*(\gamma_0)$, which solves the system. Then this mapping yields a unique increasing solutions $\gamma_d^*(h)$. The idea is that, in the end, we want to identify which conditional quantiles γ_0 and γ_1 corresponds to a given h. But to do that, we will first recover the *conditional quantile mapping*, i.e., which γ_1 corresponds to a given quantile γ_0 in choice 0. And then we recover to which h they both corresponds.

<u>*Step 1:*</u> Let us recover the *conditional quantile mapping*: $\tilde{\gamma_1}(\gamma_0)$, i.e., which γ_1 corresponds to a given γ_0, without knowing to which h they correspond. We want to show that there exist a *unique* conditional quantile mapping solution to our problem under assumption 9. First, note that $\gamma_0(h)$ and $\gamma_1(h)$ are C^1 and strictly increasing functions of h. Thus, a higher γ_0 corresponds to a higher h and thus to a higher γ_1. As a consequence, the mapping $\tilde{\gamma_1}(\gamma_0)$ will also be C^1 and strictly increasing function of γ_0 starting from $\tilde{\gamma_1}(0) = 0$ (since $\gamma_1(0) = \gamma_0(0) = 0$) to $\tilde{\gamma_1}(1) = 1$ (since $\gamma_1(1) = \gamma_0(1) = 1$). As a consequence, we need to show that there exists a unique *increasing* mapping $\tilde{\gamma_1}(\gamma_0) : [0,1] \to [0,1]$, with $\tilde{\gamma_1}(0) = 0$ and $\tilde{\gamma_1}(1) = 1$, solution to our system of equations (19).

We can get restrictions on our mapping using our structural system (19). Subtract the first line from the second line in the system of equations (19).

$$\begin{array}{rcl} \Psi_{01}(\gamma_0(h))p_{0|1} - \gamma_0(h)p_{0|0} & = & -\left(\Psi_{11}(\gamma_1(h))p_{1|1} - \gamma_1(h)p_{1|0} \right) \\ \overset{def}{\Longleftrightarrow} \qquad \Delta F_0(\gamma_0(h)) & = & \Delta F_1(\gamma_1(h)) \qquad \forall h \end{array} \tag{20}$$

Notice that the functions ΔF_0 and ΔF_1 are directly observed from the data as both Ψ_{d1} and $p_{d|w}$ are known $\forall d, w$. We also know they are C^1 functions of γ_d as the sum of C^1 functions (since $\Psi_{d1}(\gamma_d)$

are C^1).

Now, even if we do not observe h, if γ_1 and γ_0 correspond to the same unobserved h, we have: $\Delta F_0(\gamma_0) = \Delta F_1(\gamma_1)$ by equation (20). As a consequence, a conditional quantile mapping $\tilde{\gamma_1}(\gamma_0)$ solution to the system (19) must solve the equation:

$$\Delta F_0(\gamma_0) \ = \Delta F_1(\tilde{\gamma_1}(\gamma_0)) \qquad\qquad \forall \gamma_0 \in [0,1] \tag{21}$$

Now we show that there exists a unique *increasing* mapping $\tilde{\gamma_1}(\gamma_0) : [0,1] \rightarrow [0,1]$, with $\tilde{\gamma_1}(0) = 0$ and $\tilde{\gamma_1}(1) = 1$, solution to this equation (21) under assumption 9b.

First, let us see the implications of Lemma 3 in our reparametrized problem.

Lemma 3 bis: There is the *same finite number K of values of γ_0 and γ_1* such that

$$\frac{d\Delta F_d(\gamma_d)}{d\gamma_d} = 0 \quad \forall d$$

Proof. This is just a consequence of Lemma 3. First, notice that using our reparametrization:

$$\begin{aligned}
\Delta F_d(\gamma_d(h)) &= \Psi_{d1}(\gamma_d(h))p_{d|1} \ - \ \gamma_d(h)p_{d|0} \\
&= F_{C_d|D=d,W=1}(c_d(h))p_{d|1} - F_{C_d|D=d,W=0}(c_d(h))p_{d|0} \\
&= \Delta F_{C_d}(c_d(h))
\end{aligned}$$

So:

$$\begin{aligned}
& \frac{d\Delta F_{C_d}(c_d(h))}{dh} = \frac{d\Delta F_d(\gamma_d(h))}{dh} \\
\Longleftrightarrow \ & \frac{d\Delta F_{C_d}(c_d)}{dc_d}\ \frac{dc_d(h)}{dh} = \frac{d\Delta F_d(\gamma_d)}{d\gamma_d}\ \frac{d\gamma_d(h)}{dh}
\end{aligned}$$

Now, $dc_d(h)/dh > 0$ and $d\gamma_d(h)/dh > 0$ by strict monotonicity of the solution. So, if $d\Delta F_{C_d}(c_d)/dc_d = 0$ then $d\Delta F_d(\gamma_d)/d\gamma_d = 0$. As a consequence, Lemma 3 implies Lemma 3 bis in our reparametrized problem. ∎

<u>Case $K = 0$:</u> in the particular case where $K = 0$, there exists no point such that $\frac{d\Delta F_d(\gamma_d)}{d\gamma_d} = 0 \ \forall d$. ΔF_d are C^1 with no points at which the derivative is zero, so they are monotone and invertible. As a consequence, we can easily recover the unique quantile mapping by inverting ΔF_1 in system (21). We have:

$$\tilde{\gamma_1}(\gamma_0) \ = \ (\Delta F_1)^{-1}(\Delta F_0(\gamma_0)) \quad \forall \gamma_0 \in [0,1]$$

General case $K > 0$: There is a finite number $K < \infty$ of γ_0 and γ_1 such that $d\Delta F_d(\gamma_d)/d\gamma_d = 0$. Let us denote $\gamma_0^1 < \gamma_0^2 < ... < \gamma_0^K$ the ordered K γ_0 such that $d\Delta F_0(\gamma_0)/d\gamma_0 = 0$. And similarly, denote γ_1^k from $k = 1, ..., K$ the ordered K γ_1 such that $d\Delta F_1(\gamma_1)/d\gamma_1 = 0$.

First, we want to show that if the mapping $\tilde{\gamma_1}(\gamma_0)$ solves the system, then $\gamma_1(\gamma_0^k) = \gamma_1^k$. Let us take the derivative version of the system (21):

$$\frac{d\Delta F_0(\gamma_0)}{d\gamma_0} = \frac{d\Delta F_1(\tilde{\gamma_1}(\gamma_0))}{d\gamma_0} = \frac{d\Delta F_1(\gamma_1)}{d\gamma_1}\frac{d\tilde{\gamma_1}(\gamma_0)}{d\gamma_0} \qquad \forall\gamma_0$$

Since the mapping $\tilde{\gamma_1}(\gamma_0)$ must be increasing, we have: $d\tilde{\gamma_1}(\gamma_0)/d\gamma_0 > 0$. As a consequence, if a mapping $\tilde{\gamma_1}(\gamma_0)$ is a solution, then when γ_0 is such that $d\Delta F_0(\gamma_0)/d\gamma_0 = 0$, the mapped γ_1 must also have a null derivative, i.e., $d\Delta F_1(\tilde{\gamma_1}(\gamma_0))/d\gamma_1 = 0$. So, in a solution, the K points such that $d\Delta F_0(\gamma_0)/d\gamma_0 = 0$ are mapped to the K points such that $d\Delta F_1(\gamma_1)/d\gamma_1 = 0$.
Moreover, since we are looking for an *increasing* solution mapping $\tilde{\gamma_1}(\gamma_0)$), we necessarily have that these K points are sorted, i.e.:

$$\tilde{\gamma_1}(\gamma_0^k) = \gamma_1^k \qquad \forall k \in \{1, ..., K\}$$

If this was not the case, the $\tilde{\gamma_1}(\gamma_0)$ would not be increasing. Thus we have a unique solution for the K γ_0^k points.

Now, we show that $\tilde{\gamma_1}(\gamma_0)$ is also uniquely defined at other points than the γ_0^k. We use that the function ΔF_d are piecewise monotone and invertible (because C^1 with finite number of points with null derivatives) between the points of null derivative. It is similar to the $K = 0$ case, except that here we can only use piecewise monotonicity and partition the set accordingly.
Formally we procede as follows:

- Split the compact set $[0, 1]$ of γ_0 into $K + 1$ sub-intervals Γ_0^k:
  $\Gamma_0^1 = [0, \gamma_0^1]$, $\Gamma_0^2 = [\gamma_0^1, \gamma_0^2]$, ... , $\Gamma_0^{K+1} = [\gamma_0^K, 1]$ such that $[0, 1] = \underset{k\in\{1,...,K+1\}}{\cup}\Gamma_0^k$
  We denote $\mathcal{S}_0^k$ the image of those subsets by ΔF_0. We have $\Delta F_0 : \Gamma_0^k \to \mathcal{S}_0^k$.

- Do the same with the set of γ_1: split the compact set $[0, 1]$ of γ_0 into $K + 1$ sub-intervals Γ_1^k:
  $\Gamma_1^1 = [0, \gamma_1^1]$, $\Gamma_1^2 = [\gamma_1^1, \gamma_1^2]$, ... , $\Gamma_1^{K+1} = [\gamma_1^K, 1]$ such that $[0, 1] = \underset{k\in\{1,...,K+1\}}{\cup}\Gamma_1^k$.
  We denote $\mathcal{S}_1^k$ the image of those subsets by ΔF_1. We have $\Delta F_1 : \Gamma_1^k \to \mathcal{S}_1^k$.

- Since ΔF_d are C^1, in between the points of null derivative, ΔF_d are strictly monotone and invertible. It implies that $\mathcal{S}_d^k$ are compact sets, as image of compact sets by strictly monotone functions. Moreover, $\Psi_{d1}(0) = 0$ for all d. Thus, $\Delta F_d(0) = 0$ for all d. We also have $\Psi_{d1}(1) = 1$ for all d. Thus, $\Delta F_0(1) = p_{0|1} - p_{0|0} = (1-p_{1|1}) - (1-p_{1|0}) = -(p_{1|1} - p_{1|0}) = \Delta F_1(1)$. Moreover, since we showed that a solution must have $\tilde{\gamma_1}(\gamma_0^k) = \gamma_1^k$, and given that we know a solution exists, then the K points must satisfy our original equation (21). Which means that $\Delta F_0(\gamma_0^k) = \Delta F_1(\gamma_1^k)\ \forall k$. It implies that $\mathcal{S}_0^k = \mathcal{S}_1^k$ and we denote them $\mathcal{S}^k$ for all $k \in \{1, ..., K+1\}$. We have: $\mathcal{S}^0 = [0, \Delta F_0(\gamma_0^1)]$, $\mathcal{S}^1 = [\Delta F_0(\gamma_0^1), \Delta F_0(\gamma_0^2)]$, ..., $\mathcal{S}^{K+1} = [\Delta F_0(\gamma_0^K), \Delta F_0(1)]$. Notice that we could have defined the image sets based on ΔF_1 instead of ΔF_0, as $\Delta F_0(\gamma_0^k) = \Delta F_1(\gamma_1^k)$ $\forall k$.

Now, we are looking for an increasing mapping solution to the system. By *monotonicity*, we know that for a solution $\gamma_1 : \Gamma_0^k \to \Gamma_1^k$ since the upper bounds (γ_d^k) of these sets are image of each other. On each subintervals Γ_d^k, the corresponding function ΔF_d is strictly monotone and $C^1\ \forall d$. And we have that $\Delta F_0 : \Gamma_0^k \to S^k$ and $\Delta F_1 : \Gamma_1^k \to S^k$. So we can invert it segment by segment and get for any given k:

$$\tilde{\gamma_1}(\gamma_0) = (\Delta F_1)^{-1}(\Delta F_0(\gamma_0)) \quad \forall \gamma_0 \in \Gamma_0^k$$

This uniquely define the solution $\tilde{\gamma_1}(\gamma_0)$ on Γ_0^k.

- We repeat this $\forall k \in \{1, ..., K+1\}$ to obtain a unique mapping $\tilde{\gamma_1}(\gamma_0)$ covering the whole set of γ_0, i.e., $\underset{k\in\{1,\dots,K+1\}}{\cup} \Gamma_0^k = [0, 1]$.

So, we have a *unique mapping* $\tilde{\gamma_1}(\gamma_0)$ solution to equation (21).

Step 2: From this unique mapping between the conditional quantiles, we would like to recover the unique quantile functions $\gamma_d(h)$. To recover the functions $\gamma_d(h)$, we just need to use any equations of our original system (19) (the first one, for example) to obtain the $h(\gamma_0)$ corresponding to a given $(\gamma_0, \tilde{\gamma_1}(\gamma_0))$ as

$$h(\gamma_0) = \gamma_0 p_{0|0} + \tilde{\gamma_1}(\gamma_0) p_{1|0}$$

So we have a unique increasing solution $(\tilde{\gamma_1}(\gamma_0), h(\gamma_0))\ \forall \gamma_0 \in [0, 1]$. By changing the arguments it means that there exists a unique increasing solution $(\gamma_0(h), \gamma_1(h))$ to the system (19), starting from $(\gamma_0(0), \gamma_1(0)) = (0, 0)$ to $(\gamma_0(1), \gamma_1(1)) = (1, 1)$. We denote this unique solution $\gamma_d^*(h)$.
Therefore, we proved Lemma 7. ■

Thus, we have a unique increasing solution $\gamma_d^*(h)$ to system (19). Now recall that $\gamma_d(h) = F_{C_d|d,w=0}(c_d(h))$. By Lemma 2, $F_{C_d|d,w=0}(c_d)$ are strictly increasing and C^1, thus invertible. As a consequence:

$$c_d^*(h) = F^{-1}_{C_d|d,w=0}(\gamma_d^*(h))$$

So, if there exists a unique set of solution $\{\gamma_d(h)\}_{d\in\{0,1\}}$ to the rewritten system (19), there exists a unique set of increasing functions $\{c_d(h)\}_{d\in\{0,1\}}$ solution to the original system (2).
This unique set of functions identify the optimal CCCs $c_d^*(h)$ from the data (c_d, d, w). ■

C Appendix of Estimator Performance

C.1 $T = 1$ Special case

Let us focus on the one-shot decision problem with $T = 1$. This case is interesting because I can obtain closed-form solutions to the problem, and easily compare true Maximum Likelihood Estimator with my estimator. Obviously, because of the existence of this closed-form solution, the time comparison between the methods is irrelevant. But this $T = 1$ example is useful for efficiency comparison with maximum likelihood.

Closed form solution:
The agent works in $t = 1$, retires in $t = 2$ and dies in $t = 3$. The retiree consumes everything she has left, to obtain $a_3 = 0$. Thus the consumption of the retiree is $c_2 = (1+r)a_2 + pension(y_2)$, and is independent from η_2. Moreover, by the budget constraint, $a_2 = (1+r)a_1 + y_1 d_1 - c_{d1} + (1-d_1)b_1$. I set the benefits b_1 equal to 0 in this example. Thus in period $t = 1$, the only period of her working life, conditional on d, the agent solves:

$$\begin{aligned}\underset{c_{d1}}{max} \quad & \frac{c_{d1}^{1-\sigma}}{1-\sigma}\tilde{\eta}^d(\eta_1, \gamma_d, s_d) + \alpha d_1 + \omega(1-w)d_1 + \epsilon_{d1} \\ & + \beta\mathbb{E}\Big[\frac{((1+r)^2 a_1 + (1+r)y_1 d_1 - (1+r)c_{d1} + pension(y_2))^{1-\sigma}}{1-\sigma}\ \tilde{\eta}^d(\eta_2, \gamma_0, s_0)\Big].\end{aligned}$$

where $\mathbb{E}[\tilde{\eta}^d(\eta_2, \gamma_0, s_0)] = e^{\gamma_0 + s_0^2/2}$, also there is no ϵ_2 shock in the retirement period, and the retirement utility is the same as the utility when unemployed.

It yields the closed form solution for the conditional consumption in $t=1$:

$$c_{d1} = \frac{1}{(1+r) + \left(\beta(1+r)e^{\gamma_0 + s_0^2/2}/\tilde{\eta}^d(\eta_1, \gamma_d, s_d)\right)^{1/\sigma}} \Big((1+r)^2 a + (1+r)y_1 d + pension(y_2)\Big).$$

The agent consumes a share of available income which depends on the decision. Since the retiring utility is the same as the unemployed one, I only identify γ_1, s_1 with respect to γ_0, s_0 and β.[19] Thus, the parameters to estimate are $\theta = (\sigma, \gamma_1, s_1, \alpha, \omega)$. β is fixed at 0.98, $\gamma_0 = 0$, $s_0 = 0.25$.

Table 4: Comparison of the estimators when $T = 1$

	Truth	*Method* *DCC*		*MLE*		*SMM*	
		1,000	10,000	1,000	10,000	1,000	10,000
σ	1.60	1.5806	1.5782	1.6042	1.5992	1.6135	1.5970
		(0.1759)	(0.0827)	(0.0444)	(0.0137)	(0.0560)	(0.0211)
γ_1	0.00	0.0071	0.0040	-0.0061	0.0007	-0.0269	-0.0009
		(0.0714)	(0.0286)	(0.0205)	(0.0072)	(0.0213)	(0.0078)
s_1	0.40	0.4246	0.4043	0.4005	0.4001	0.3926	0.3857
		(0.0747)	(0.0366)	(0.0187)	(0.0060)	(0.0245)	(0.0073)
α	-0.50	-0.4782	-0.5092	-0.4928	-0.5000	-0.4986	-0.4850
		(0.3266)	(0.1016)	(0.0852)	(0.0268)	(0.0989)	(0.0401)
ω	-1.00	-1.0689	-1.0044	-1.0115	-0.9931	-1.0308	-1.0029
		(0.1715)	(0.0484)	(0.1577)	(0.0441)	(0.2919)	(0.0665)
Avg Time taken:		16s	32s	1s	9s	16s	50s

Other initializations: Number of Monte-Carlo $= 1,000$.
$Pr(w_1 = 1) = 0.7$. y_H is set to 20 and in this case I impose $Pr(y = y_H) = 1$.
$a_1 = 12.5$ for everyone here. Benefits $b = 0$. Pension Percentage of income $= 50\%$.
Fixed parameters: $\gamma_0 = 0$, $s_0 = 0.25$, $\beta = 0.98$, $r = 0.05$.

Results: (Table 4)

First, the speed comparison is irrelevant here. Indeed with one period one does not need to solve

[19] Notice that if I had a specific retirement utility, different from the unemployment one, I could also identify parameters of unemployed with respect to retirees.

for the value function so it is considerably easier, especially since we also have closed-form solutions to simulate the model and compute the likelihood. On average here SMM already takes longer than my DCC method but only because it requires to test more set of parameters to find the optimum, as the objective are different between the two functions. It could be the reverse, and one could expect both methods to go at relatively similar speed when $T = 1$ in general. The real benefits of my method are when it allows to avoid solving for the value function, i.e., as soon as $T > 1$.

Concerning the statistical efficiency, as expected when you have a closed form solution for the likelihood, MLE is always the most efficient. It is also the quickest as I do not need to estimate any reduced form in a first stage and I am using a known closed form solution in this $T = 1$ case. Obviously once I go to more period, MLE becomes the longest method, and is becoming untractable.

My method (DCC) is consistent and relatively efficient, but less than the MLE benchmark. Simulated Method of Moments (SMM) with moments drawn from the reduced forms conditional distributions of $c_d|d, w, x$ and conditional probabilities $p_{d|w,x}$ is also consistent. It is also more efficient (except for the additive parameter ω). I lose efficiency because of the two-step nature of my method, since I'm always computing the second step using first step optimal choices estimates. But overall, the efficiency loss is largely compensated by time gains in more complicated models.