Roskilde University

Grassmannian and boundary contribution to the z-determinant

Wojciechowski, K.P.; Scott, S.G.; Marchio, G.; Booss-Bavnbek, Bernhelm

Publication date:
1996

Document Version
Publisher's PDF, also known as Version of record

Citation for published version (APA):
Wojciechowski, K. P., Scott, S. G., Marchio, G., & Booss-Bavnbek, B. (1996). *Grassmannian and boundary contribution to the z-determinant*. Roskilde Universitet. Tekster fra IMFUFA No. 320
http://milne.ruc.dk/ImfufaTekster/

TEKST NR 320 1996

Grassmannian and Boundary Contribution to the ζ-Determinant

K. P. Wojciechowski et al.

TEKSTER fra

IMFUFA **ROSKILDE UNIVERSITETSCENTER**
INSTITUT FOR STUDIET AF MATEMATIK OG FYSIK SAMT DERES FUNKTIONER I UNDERVISNING, FORSKNING OG ANVENDELSER

IMFUFA, Roskilde University, P.O.Box 260, 4000 Roskilde, Denmark
DEPARTMENT OF STUDIES IN MATHEMATICS AND PHYSICS AND THEIR FUNCTIONS IN EDUCATION, RESEARCH, AND APPLICATIONS

Grassmannian and Boundary Contribution to the ζ-Determinant
K. P. Wojciechowski et al.

IMFUFA text no. 320 29 pages ISSN 0106-6242

This is a collection of three notes announcing recent progress made in understanding the ζ-determinant of self-adjoint global elliptic boundary problems for Dirac operators acting on sections of Clifford module bundles over compact Riemannian manifolds with boundary.

The **first note** is expository: We recall the concept of the ζ-regularized determinant for Dirac operators over closed manifolds, based on previous work by Ray and Singer, Hawking, and Singer; we introduce the smooth self-adjoint Grassmannian of self-adjoint global elliptic boundary conditions for (compatible) Dirac operators on compact odd-dimensional manifolds which differ from the Calderón projector by a smoothing operator; we construct the ζ-determinant over the smooth Grassmannian; we recall (and slightly modify) the algebraic construction of the 'canonical' determinant bundle due to Quillen, Segal, and Scott which is trivial when restricted to the smooth Grassmannian; and we announce the equality of the ζ-determinant and the canonical determinant up to a constant. In particular it follows that the phase of the canonical determinant is determined by the η-invariant and that pasting laws which are naturally formulated in the canonical, algebraic context remain valid for the analytically defined ζ-determinant.

The third section of the first note presents a different application of the Grassmannian, namely a discussion of the four-dimensional situation motivated by previous work of Morchio and Strocchi in Quantum Chromodynamics, i.e. the establishment of a gauge-invariant section of chiral symmetry in the γ_5-symmetric self-adjoint Grassmannian over the space of connections on $V \times \mathbb{C}^2$ (here V denotes a large 4-dimensional ball) which are pure gauge at the boundary. A special feature of our note is that a rigorous meaning is given to these various physical concepts.

The **second note** is devoted to studying the geometry of the determinant line bundle and presents a crucial component in the proof of the coincidence of the ζ-regularized determinant and the canonical (algebraically regularized) determinant over the smooth Grassmannian. The calculations are carried out for the odd-dimensional case.

The **third note** describes the even-dimensional case. Our starting point is the special situation of euclidean Quantum Chromodynamics discussed in the third section of our first note which yields a determinant with vanishing imaginary part as a consequence of the established chiral symmetry. Now we provide a further modification of the choice of gauge-invariant boundary conditions and establish the constance of the determinant on the connected components of the self-adjoint γ_5-invariant Grassmannian for the 4-ball for any fixed connection.

Applications of the new results are indicated briefly for new developments in the mathematical understanding of Quantum Field Theories as modified cohomology theories and for the conservation equation of the chiral current in Quantum Chromodynamics.

CONTENTS

Introduction

This is a collection of three research notes which deal with different aspects of the determinant theory of elliptic boundary value problems for Dirac operators on compact manifolds. The motivation for this work were recent studies in *Topological Quantum Field Theory* and in *Quantum Chromodynamics.* Let $D\!\!\!/ : C^\infty(M;S) \rightarrow C^\infty(M;S)$ denote a compatible Dirac operator acting on sections of a bundle S of Clifford modules over a compact manifold M with boundary Y. The first problem here is the correct definition of the determinant of $D\!\!\!/_P$, the Dirac operator $D\!\!\!/$ subject to the boundary condition P. We make a specific choice of the space of admissible boundary conditions which implies that the operator $D\!\!\!/_P$ has nice spectral properties. We use the ζ-function regularization to give an analytic definition of the determinant which we call the *ζ-determinant* of the operator $D\!\!\!/_P$ and denote by $det_\zeta D\!\!\!/_P$. Let us point out that in the case of a first-order operator we have to study the phase of the ζ-determinant. The phase is determined by the η-invariant of the operator $D\!\!\!/_P$ as will be explained in the beginning of the first paper.

We follow Quillen and Segal in order to give a geometric construction of the determinant. Quillen explained that the generalization of the algebraic determinant of an operator acting on a finite-dimensional space to the infinite-dimensional case of the Dirac operator does not yield a function, but a section of the 'determinant line bundle'. The determinant bundle over $Gr(D\!\!\!/)$, the Grassmannian of the generalized Atiyah-Patodi-Singer boundary conditions is a non-trivial line bundle. However, it becomes a trivial bundle when restricted to $Gr^*(D\!\!\!/)$, the subspace of self-adjoint conditions. There is a natural choice of the 'canonical trivialization', which defines the determinant as a function over $Gr^*(D\!\!\!/)$, which we call the *canonical determinant* and denote by $det_C D\!\!\!/_P$. The natural question here is if the equality

$$det_\zeta D\!\!\!/_P = det_C D\!\!\!/_P$$

holds for any $P \in Gr^*(D\!\!\!/)$.

We found that the phase of the canonical determinant is equal to the phase of the ζ-determinant. We shall work out the details of the proof of the equality separately. This topic and some further results on the geometry of the determinant line bundle are discussed in the first and the second paper. The results will play an important role in the analysis of the *Pasting Axiom* in *Quantum Field Theory.*

The last section of the first paper and the third paper deal with a different problem. We study the ζ-determinant of the Dirac operator with coefficients in $\mathbf{C}^2$ on a four-dimensional disc. Let $D\!\!\!/_A$ denote the Dirac operator lifted by means of an auxiliary connection A on the trivial bundle. It has the form

$$D\!\!\!/_A = \begin{pmatrix} 0 & D\!\!\!/_A^- = (D\!\!\!/_A^+)^* \\ D\!\!\!/_A^+ & 0 \end{pmatrix}.$$

We are looking for a map Φ from the space of connections which are pure gauge at the boundary to the space of elliptic self-adjoint boundary conditions for the operator $D\!\!\!/_A$, which satisfies certain additional conditions in order to describe the physical situation. It turns out that there is an obvious construction of such a map, which depends on the choice of a single boundary condition from the Grassmannian $Gr(D\!\!\!/_A^+)$. We show that the ζ-determinant of the operator $(D\!\!\!/_A)^2_{\Phi_A}$ does not depend on this particular choice.

Though we discuss problems which arose in different mathematical and physical situations, we use the same principle to solve them: We pick up a one-parameter family $\{P_r\}$ of boundary conditions, and we study the variations of the determinant under the change of the boundary condition. This is quite difficult in the general case, as the domain of the unbounded operator $D\!\!\!/_{P_r}$ is changing with the parameter. However, in the situations discussed above, we use our knowledge of the structure of the Grassmannians. It turns out that the choice of a family of boundary conditions is equivalent to the choice of a family $\{T_r\}$ of unitary pseudo-differential operators on Y of the form $Id + k_r$, where k_r is an operator with a smooth kernel. We use the family $\{T_r\}$ to construct a family $\{U_r\}$ of unitary transformations on the manifold M. The operator $D\!\!\!/_{P_r}$ is unitarily equivalent to the operator $\mathcal{D}_r = (U_r D\!\!\!/ U_r^{-1})_{P_0}$. The family $\{\mathcal{D}_r\}$ has a fixed domain and therefore we can use methods similar to the ones we used in the case of a closed manifold to study the variation of the determinant.

DETERMINANTS, MANIFOLDS WITH BOUNDARY AND DIRAC OPERATORS*

K. P. WOJCIECHOWSKI
Dept. Math., IUPUI (Indiana/Purdue)
Indianapolis IN 46202-3216, U.S.A.
e-mail: kwojciechowski@math.iupui.edu

S. G. SCOTT
Departamento de Matematicas
Universidad de los Andes, Bogota A.A. 4976, Colombia

G. MORCHIO
Dipartimento di Fisica dell'Università and INFN
I-56100 Pisa, Italy
e-mail: morchio@morchio.difi.unipi.it

and

B. BOOSS-BAVNBEK
Institut for matematik og fysik
Roskilde University
DK-4000 Roskilde, Denmark
e-mail: booss@mmf.ruc.dk

Abstract. We discuss two different applications of the theory of Grassmannians of elliptic boundary value problems to the theory of ζ–function determinants of Dirac operators over a manifold with boundary. Our work is motivated by constructions of the determinant in Topological Quantum Field Theory and in Quantum Chromodynamics.

Key words: Calderón projector, chiral symmetry, canonical determinant, Dirac operator, Grassmannian, ζ–determinant

1. The ζ–determinant on closed manifolds

Let $T : \mathbf{C}^N \to \mathbf{C}^N$ be an invertible, positive operator with eigenvalues $0 < \lambda_1 \leq \lambda_2 \leq .... \leq \lambda_N$. We have the equality

$$\begin{aligned} \det T &= \prod \lambda_j = \exp(\sum \ln \lambda_j e^{-s \ln \lambda_j}|_{s=0}) \\ &= \exp(-\frac{d}{ds}(\sum \lambda_j^{-s})|_{s=0}) = \exp(-\frac{d}{ds}\zeta_T(s)|_{s=0}) \,. \end{aligned}$$

If T is a positive definite self–adjoint elliptic operator acting on sections of a vector bundle over a closed manifold, then T has discrete spectrum λ_j . We define $\zeta_T(s)$ as above and this function is holomorphic for $\Re(s)$ sufficiently large. It has a meromorphic extension to the entire complex plane, which is holomorphic in the

* Fourth International Conference on *Clifford Algebras and Their Applications in Mathematical Physics*, Aachen (Germany), May 28–31, 1996. The talk was delivered by K.P. Wojciechowski.

neighborhood of $s=0$. It is natural then to define

$$\det T := \exp(-\frac{d}{ds}\zeta_T(s)|_{s=0}).$$

This was first observed by Ray and Singer in [8] who used this determinant in order to define analytic torsion. Hawking later observed that the $\zeta-$ determinant gave a natural renormalization of the path integral in curved space time in [5]. The ζ–determinant has been an important tool in quantum field theory since then.

Physicists have been particularly interested in the study of the determinant of the Dirac operator $\not{D}$. This is an elliptic self-adjoint differential operator of first order, which therefore has infinitely many discrete positive eigenvalues λ_j and infinitely many negative eigenvalues $-\mu_j$. Since $\not{D}$ is not positive, to make sense of the ζ-determinant in this case we choose a branch of $(-1)^s = e^{i\pi s}$ and define

$$\begin{aligned}\ln\det T &:= -d/ds\left(\sum\lambda_j^{-s} + \sum(-1)^{-s}\mu_j^{-s}\right)|_{s=0}\\ &= -d/ds\left(\frac{\zeta_{|T|}(s)+\eta_T(s)}{2} + e^{i\pi s}.\frac{\zeta_{|T|}(s)-\eta_T(s)}{2}\right)|_{s=0}.\end{aligned}$$

In the preceding formula $\zeta_{|T|}(s) = \sum\lambda_j^{-s} + \sum\mu_j^{-s}$ denotes the ζ–function of the operator $|T|$ and $\eta_T(s) = \sum\lambda_j^{-s} - \sum\mu_j^{-s}$ denotes the eta–function of the operator T which is a measure of spectral asymmetry. Once again $\eta_T(s)$ is a holomorphic function of s for $\Re(s)$ large with a meromorphic extension to the whole complex plane which is holomorphic in the neighborhood of $s=0$. We also know that $\zeta_{|T|}(0) = \zeta_{T^2}(0)$. This all gives the formula

$$\ln\det T = -\zeta'_{|T|}(0) + \frac{i\pi}{2}\left(\eta_T(0) - \zeta_{T^2}(0)\right).$$

We refer to the beautiful work of Singer [13] for a review of different aspects of the theory and applications of the ζ–determinant on closed manifolds. In this paper we discuss two aspects of the determinant theory for the Dirac operators on manifolds with boundary.

2. The smooth Grassmannian of elliptic boundary problems on odd– dimensional manifolds

Let M be a compact smooth Riemannian manifold with boundary Y, and let $S \to M$ be a bundle of Clifford modules with compatible Clifford structure and connection. Let $\not{D}: C^\infty(M;S) \to C^\infty(M;S)$ denote the corresponding compatible Dirac operator acting on sections of S (see [4] for details). Let us assume now that $\dim M = n$ is odd. We also assume from now on that all metric structures involved are product in $N = [0,1]\times Y$ the collar neighborhood of the boundary. In this case $\not{D}$ takes the following form in N

$$\not{D} = \Gamma(\partial_u + B),$$

where u denotes the inward oriented normal coordinate and $\Gamma : S|_Y \to S|_Y$ is a unitary anti–involution, so $\Gamma^2 = -\mathrm{Id}$ and $\Gamma^* = -\Gamma$. The operator $B : C^\infty(Y; S|_Y) \to$

$C^\infty(Y; S|_Y)$ is the Dirac operator on the boundary Y, which anti-commutes with the tangential operator B. To simplify the exposition we assume that $\ker B = 0$. The results stated in this section hold also in the case of non-invertible tangential operator B. Contrary to the case of a closed manifold, the space

$$\ker \not{D} := \{s \in C^\infty(M; S) \mid \not{D}s = 0\}$$

of solutions of $\not{D}$ is an *infinite dimensional* subspace of $C^\infty(M; S)$. Furthermore one no longer has *regularity* of the solutions. To regain elliptic regularity we restrict the domain of the operator $\not{D}$ by imposing a boundary condition. Let Π denote the spectral projection of the operator B onto the subspace of $L^2(Y; S|_Y)$ spanned by the eigensections corresponding to the positive eigenvalues of B. The operator Π is a pseudodifferential operator, which allows us to apply analytical tools to the boundary problems defined below. We define

$$\begin{aligned} \not{D}_\Pi &:= \not{D} \qquad \text{with} \\ \operatorname{dom}(\not{D}_\Pi) &:= \{s \in H^1(M; S) \mid \Pi(s|_Y) = 0\}, \end{aligned}$$

where $H^1(M; S)$ denotes the first Sobolev space of sections of S on M. The operator $\not{D}_\Pi : \operatorname{dom} \not{D}_\Pi \to L^2(M, S)$ is a Fredholm operator and, moreover, $\ker \not{D}$ and $\operatorname{coker} \not{D}$ consist only of smooth sections of S. In fact it follows from Green's formula that $\not{D}_\Pi$ is a self-adjoint operator. We call $\not{D}_\Pi$ an *elliptic boundary problem* for $\not{D}$ and the operator Π an *elliptic boundary condition.*

The structure of the space of all elliptic boundary conditions for the operator $\not{D}$ is not yet known, and so we restrict our consideration to various subspaces built around the projection Π whose topology are well understood. We introduce the *Grassmannian* of pseudodifferential projections

$$\operatorname{Gr}(\not{D}) := \{P = P^2 \mid P = P^* \in \Psi_0 \text{ and } P - \Pi \in \Psi_{-1}\},$$

where Ψ_k denotes the spaces of pseudodiffential operators over Y of order k.

The second condition implies that the difference $P - \Pi$ is a compact operator in $L^2(Y; S|_Y)$. The operator $\not{D}_P : \operatorname{dom}(\not{D}_P) \to L^2(M; S)$ is a Fredholm operator, and its kernel and cokernel contain only smooth sections. In general, however, $\not{D}_P$ is not a self-adjoint operator as the number $\operatorname{index} \not{D}_P = \dim\ker \not{D}_P - \dim\operatorname{coker} \not{D}_P$ can take any integer value. The space $\operatorname{Gr}(\not{D})$ has infinitely many connected components and two projections P_1 and P_2 belong to the same connected component if and only if

$$\operatorname{index} \not{D}_{P_1} = \operatorname{index} \not{D}_{P_2}.$$

It follows from Green's formula that $\not{D}_P$ is a self-adjoint operator if and only if

$$-\Gamma P \Gamma = \mathrm{Id} - P.$$

Therefore we define the *self-adjoint Grassmannian* as follows

$$\operatorname{Gr}^*(\not{D}) := \{P \in \operatorname{Gr}(\not{D}) \mid -\Gamma P \Gamma = \mathrm{Id} - P\}.$$

Unfortunately at the moment we are not able to construct the ζ-determinant on $\operatorname{Gr}^*(\not{D})$.

We introduce the *smooth self-adjoint Grassmannian.* Let us point out that in fact we can identify the projections with their ranges which are closed infinite dimensional subspaces of $L^2(Y;S|_Y)$ with an infinite dimensional orthogonal complement. Let P_W denote the orthogonal projection onto $W \subset L^2(Y,S|_Y)$. We define

$$\mathrm{Gr}^*_\infty(\not{D}) := \{W \subset L^2(Y;S|_Y) \mid P_W \in \mathrm{Gr}^*\ (\not{D}) \text{ and } P_W - \Pi \in \Psi_{-\infty}\}.$$

This space was introduced and studied by Scott in [10]. The important analytical fact here is that the *Calderón projection* is also an element of the smooth Grassmannian. The Calderón projection $\mathcal{P}(\not{D})$ is defined as the orthogonal projection of $L^2(Y;S|_Y)$ onto the Cauchy data space

$$\mathcal{H}(\not{D}) = \overline{\{s|_Y \mid s \in C^\infty(M;S) \text{ and } \not{D}s = 0 \text{ in } M \setminus Y\}}^{L^2(Y;S|_Y)}.$$

We refer to [10] for the details.

In his recent work [16] Wojciechowski studied the η–function and the ζ–function of the operator $\not{D}_P$, where range(P) is an element of the smooth self–adjoint Grassmannian. It turns out that they share the properties of the corresponding functions on a closed manifold. Consequently we have the following result.

Theorem 2.1 *Let W be an element of the smooth, self–adjoint Grassmannian, then* $\det_\zeta(W) := \det(\not{D}_{P_W})$ *is well defined.*

Now we want to discuss the relation of the ζ–determinant to the canonical determinant introduced by Scott. Let H denote a separable Hilbert space and let $\mathcal{F}$ be the space of all (bounded) Fredholm operators on H. It was explained by Quillen in his famous paper [7] that it is impossible to construct the determinant as a function on $\mathcal{F}$. He showed that the determinant arises as a canonical section of the *determinant line bundle* over $\mathcal{F}$ and, using the ζ–function, constructed a natural metric whose curvature measures the local obstruction to triviality (see also [2] and [13]).

We describe briefly the Segal variant of the definition of this bundle. We restrict ourselves to $\mathcal{F}_0$, the connected component of the operators of index equal to 0. We fix an operator $T \in \mathcal{F}_0$ and define $\mathrm{Det}\,T$ the determinant line over T as follows. Let $\mathcal{F}_T := \{P \in \mathcal{F} \mid P - T \text{ is trace class}\}$. Then

$$\mathrm{Det}\,T := \mathcal{F}_T \times \mathbf{C}/_\sim\,,$$

where the equivalence is defined by $(Rg,z) \sim (R, \det(g)z)$ for $g \in \mathrm{End}(H_0)$ of the form $1 +$ trace–class. The determinant of T is then defined to be the canonical element $\det(T) := [(T,1)]$ of $\mathrm{Det}\,T$.

The bundle structure is given by introducing local trivializations in the following way. Let a denote an operator of trace class (possibly equal to 0) such that $T+a$ is an invertible operator. The local trivialization of the bundle over the open set $\mathcal{U}_a := \{R \in \mathcal{F}_0 \mid R + a \text{ is invertible}\}$ is defined by the section $R \mapsto [R+a, 1]$. The two local trivializations over $\mathcal{U}_a \cap \mathcal{U}_b$ are then patched together by the holomorphic transition function

$$g_{ab}(R) := \det((R+b)(R+a)^{-1}),$$

which is well defined since $(R+b)(R+a)^{-1} = 1 + (b-a)(R+a)^{-1}$ differs from the identity by a trace-class operator.

This yields a canonical holomorphic line bundle $\mathrm{Det}\,(\mathcal{F})$ over the space of Fredholm operators (see [10]) with a canonical global section $T \mapsto \det T$.

Through the embedding (identifying $\mathrm{dom}\, D\!\!\!/_{P_W}$ with L^2)

$$\begin{aligned} \mathrm{Gr}(D\!\!\!/) &\longrightarrow \mathcal{F} \\ W &\longmapsto D\!\!\!/_{P_W} \end{aligned}$$

we obtain by pull-back a holomorphic determinant line bundle Det over the parameter space of elliptic boundary value problems $\mathrm{Gr}(D\!\!\!/)$. This bundle is non–trivial, but its restriction to the sub–manifold $\mathrm{Gr}^*(D\!\!\!/)$, or to $\mathrm{Gr}^*_\infty(D\!\!\!/)$, is trivial. Scott has found a natural choice of trivialization of this bundle which canonically identifies the determinant section as function given by a nice explicit formula giving the value of this *canonical determinant* $\det_{\mathcal{C}}(W)$ at W. He also showed that

$$\det{}_\zeta(W) = \det{}_{\mathcal{C}}(W)$$

in the case $\dim M = 1$. The general case is open at the moment though Wojciechowski has proved the following result.

Theorem 2.2 *Up to a constant the phase of the ζ–determinant is equal to the phase of the canonical determinant.*

In other words, the phase of the canonical determinant is determined by the η–invariant. The details will be presented in the forthcoming work by Scott and Wojciechowski [11].

3. Grassmannian and chiral symmetry. The even dimensional case

Assume now that $\dim M = n$ is even. The new feature is that the bundle S splits into $S = S^+ \oplus S^-$ the direct sum of subbundles of spinors of positive and negative chirality. The Dirac operator has the following form on the collar N:

$$D\!\!\!/ = \begin{pmatrix} 0 & D\!\!\!/^- \\ D\!\!\!/^+ & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\Gamma^{-1} \\ \Gamma & 0 \end{pmatrix} \left(\partial_u + \begin{pmatrix} B & 0 \\ 0 & -\Gamma B \Gamma^{-1} \end{pmatrix} \right).$$

The operator $\Gamma : S^+|_Y \to S^-|_Y$ is a unitary isomorphism. The operator $D\!\!\!/^+ : C^\infty(M; S^+) \to C^\infty(M; S^-)$ has the following form on the collar

$$D\!\!\!/^+ = \Gamma(\partial_u + B),$$

where B is the Dirac operator on Y. Once again we define the Grassmannian $\mathrm{Gr}(D\!\!\!/^+)$ as in Section 2. The operator $D\!\!\!/^+_P$ is now a Fredholm operator with index given by the famous Atiyah–Patodi–Singer formula. In quantum chromodynamics one has to study the determinant of the total Dirac operator $D\!\!\!/$. There is natural choice of the boundary condition in this case

$$\mathcal{R}(\Pi) := \begin{pmatrix} \Pi & 0 \\ 0 & \Gamma(\mathrm{Id} - \Pi)\Gamma^{-1} \end{pmatrix}.$$

The operator $\not{D}_{\mathcal{R}(\Pi)}$ is a Fredholm operator. It is also a self-adjoint operator with discrete spectrum. However the $\zeta-$ determinant defined in Section 2 for such operators is usually equal to 0, because it is often the case that $\not{D}_{\mathcal{R}(\Pi)}$ has a non-trivial kernel. Therefore physicists came up with a regularization which took care of this problem in the case $n_+ = \ker \not{D}^+_{\Pi} = \ker \not{D}^-_{\Gamma(\mathrm{Id}-\Pi)\Gamma^{-1}} = n_-$. However, in the general case $n_+ \neq n_-$ the imaginary part of the determinant appears, unnatural in the case of an operator with symmetric spectrum.

We now describe a specific situation which appears in quantum field theory. We take as M the four dimensional disc of radius R. Our bundle S is now equal to $S \otimes \mathbf{C}^2$, the Clifford bundle of Euclidean spinors with coefficients in the trivial bundle $M \times \mathbf{C}^2$. The full Dirac operator is in this case the operator $\not{D}_A = \not{D} \otimes_A \mathrm{Id}$ the Dirac operator $\not{D}$ acting on sections of S lifted to sections of $S \otimes \mathbf{C}^2$ by means of the connection A acting on the trivial bundle. The detailed construction is described in [6], [13], or [15]. The connection A on the trivial bundle is of the form $d + \omega_A$, where ω_A is the one form (with coefficients in $\mathbf{C}^2$). Once again, to avoid technicalities, we deform the Euclidean metric in the collar neighborhood N of the boundary $Y = S^3$ to a product metric in N. We also restrict the class of the connection we admit. We assume that in N the connection A is of the form

$$d + h^{-1}dh$$

where $h : S^3 \to \mathrm{SU}(2)$ is a smooth map. Following the physics terminology we say that the connection A is *pure gauge at the boundary*. We denote by Conn_0 the space of connections on $M \times \mathbf{C}^2$, which are pure gauge on the boundary. The important feature is that under this assumption then the tangential operator $B_A = B_h$ corresponding to the partial Dirac operator $\not{D}^+_A$ takes the form:

$$B_h = (\mathrm{Id} \otimes h)(B \otimes \mathrm{Id}_{\mathbf{C}^2})(\mathrm{Id} \otimes h^{-1}), \tag{1}$$

where $B \otimes \mathrm{Id} = B \otimes_d \mathrm{Id} = B \oplus B$, and B is the Dirac operator on S^3, which is the tangential operator corresponding to $\not{D}^+$. We refer to [3] for all details. Now we want to impose a boundary condition on the operator $\not{D}_A$. More precisely we want to construct a continuous map of Conn_0 into the space of boundary conditions, which satisfies the set of assumptions given below. Of course once again, we have to restrict our choice of boundary condition. The natural choice here is:

$$\mathcal{R}(A) := \begin{pmatrix} \Pi_h & 0 \\ 0 & \Gamma(\mathrm{Id} - \Pi_h)\Gamma^{-1} \end{pmatrix},$$

where Π_h denotes the Atiyah-Patodi-Singer condition, the positive spectral projection of the operator B_h. Note that it follows from the formula (1) that the operator B (and also B_h) is an invertible operator with symmetric spectrum. For the calculations of the spectrum of B we refer to [9] (see also [3] and [14]). The map $\mathcal{R}(A)$ satisfies the following conditions:

1. For each A the operator $\not{D}_{A,\mathcal{R}(A)} := (\not{D}_A)_{\mathcal{R}(A)}$ is a self-adjoint operator with discrete spectrum and finite dimensional kernel, which consists only of smooth spinors.
2. The domain of the operator $\not{D}_{A,\mathcal{R}(A)}$ is gauge–invariant. This has the following meaning. Let $U : M \to \mathrm{SU}(2)$ denote a gauge transformation. Then the operator

$\not{D}_{UAU^{-1},\mathcal{R}(UAU^{-1})}$ is equal to the operator $(\mathrm{Id}\otimes U)\not{D}_A(\mathrm{Id}\otimes U^{-1})$ with domain $(\mathrm{Id}\otimes U)(\mathrm{dom}\,\not{D}_{A,\mathcal{R}(A)})$. Therefore the new operator is unitary equivalent to the operator $\not{D}_{A,\mathcal{R}(A)}$.
3. The operator $\not{D}_{A,\mathcal{R}(A)}$ has symmetric spectrum. In physics terminology this is expressed by saying that the boundary condition is γ_5 invariant. The involution $\gamma_5 : S = S^+ \cup S^- \to S$ is defined to be equal to the identity Id on S^+ and $-\mathrm{Id}$ on S^-. Clearly we have $\gamma_5 \not{D}_A \gamma_5 = -\not{D}_A$ and on the boundary

$$\gamma_5 \mathcal{R}(A) \gamma_5 = \mathcal{R}(A).$$

Now if $\not{D}_A s = \lambda s$ and s is an element of the domain of $\not{D}_{A,\mathcal{R}(A)}$, then $\gamma_5 s$ is in the domain of $\not{D}_{A,\mathcal{R}(A)}$ and

$$\not{D}_A(\gamma_5 s) = -\gamma_5 \not{D}_A s = -\lambda(\gamma_5 s).$$

The map $\mathcal{R}(A)$ satisfies all properties listed above. Ideally in physics one would like also to have the additional property
4. chiral symmetry: which means the equality $n_+ = n_-$.

This is, however, not true in the case of the map $\mathcal{R}(A)$. The space Conn_0 has infintely many connected components classified by the topological invariant $\deg(h)$, the degree of the map h. Therefore we restrict $\mathcal{R}(A)$ to the fixed component of Conn_0 corresponding to the non-trivial number $\deg(h)$. In this case we can apply the Atiyah-Patodi-Singer Index Theorem [1] and obtain

$$n_+ - n_- = \mathrm{index}\,\not{D}^+_{\Pi(h)} = \deg(h).$$

We have to discuss different choices of mapping from the space of connections into a parameter space of boundary conditions. We restrict ourselves to the study of maps of the form $A \mapsto P(A)$, where $P(A) \in \mathrm{Gr}(\not{D}^+_A) = \mathrm{Gr}(\not{D}^+ \otimes \mathrm{Id}_{\mathbb{C}^2})$. The corresponding condition for $\not{D}_A$ is given by the formula

$$\begin{pmatrix} P(A) & 0 \\ 0 & \Gamma(\mathrm{Id} - P(A)\Gamma^{-1} \end{pmatrix} .$$

The natural choice here is $\mathcal{P}(\not{D}^+_A)$ the Calderón projection of the operator $\not{D}^+_A$. This is due to the result proved by Booss- Bavnbek and Wojciechowski

$$\mathcal{P}(\not{D}^-_A) = \Gamma(\mathrm{Id} - \mathcal{P}(\not{D}^+_A))\Gamma^{-1} ,$$

(see [4]). Therefore the map

$$A \mapsto \mathcal{P}(\not{D}_A) := \begin{pmatrix} \mathcal{P}(\not{D}^+_A) & 0 \\ 0 & \mathcal{P}(\not{D}^-_A)) \end{pmatrix}$$

satisfies the first three conditions. Moreover because of the choice of the boundary conditions we know that

$$n_+ = \dim\ker \not{D}^+_{A,\mathcal{P}(\not{D}^+_A)} = 0 = \dim\ker \not{D}^-_{A,\mathcal{P}(\not{D}^-_A)} = n_-$$

and chiral symmetry is preserved.

From the point of view of physics this solution is not completely satisfying because, unlike the Atiyah–Patodi–Singer condition, which depends only on the boundary data, the Calderón projection varies with change of the operator inside of the manifold. Therefore some alternative choices of the map $\mathcal{R}(A)$ have to be discussed.

Recently we have come up with a satisfying solution to this problem. This together with the discussion of the variation of the ζ-determinant under the change of the boundary conditions is a subject of a joint ungoing work of Booss–Bavnbek, Morchio, Strocchi, and Wojciechowski.

References

1. Atiyah, M.F., Patodi, V.K., and Singer, I.M.: 1975, 'Spectral asymmetry and Riemannian geometry. I', *Math. Proc. Cambridge Phil. Soc.* 77, 43-69.
2. Atiyah, M.F., and Singer, I.M.: 1984, 'Dirac operators coupled to vector potentials', *Proc. Nat. Acad. Sci. USA* 81, 2597–2600.
3. Booß–Bavnbek, B., Morchio, G., Strocchi, F., and Wojciechowski, K.P.: 1996, *Grassmannian and Chiral Anomaly*, preprint.
4. Booß–Bavnbek, B., and Wojciechowski, K.P.: 1993, *Elliptic Boundary Problems for Dirac Operators*, Birkhäuser, Boston.
5. Hawking, S.W.: 1977, 'Zeta function regularization of path integrals in curved spacetime', *Comm. Math. Phys.* 55, 133–148.
6. Palais, R.S. (ed.): 1965, *Seminar on the Atiyah–Singer Index Theorem*, Ann. of Math. Studies 57, Princeton University Press, Princeton.
7. Quillen, D.G.: 1985, 'Determinants of Cauchy–Riemann operators over a Riemann surface', *Funk. Anal. i ego Prilozhenya* 19, 37–41.
8. Ray, D., and Singer, I.M.: 1971, 'R–torsion and the Laplacian on Riemannian manifolds', *Adv. Math.* 7, 145–210.
9. Schmidt, J.R., and Bincer, A.M.: 1987, 'Chiral asymmetry and the Atiyah–Patodi–Singer index for the Dirac operator on a four–dimensional ball', *Phys. Rev. D (3)* 35, 3995-4000.
10. Scott, S.G.: 1995, 'Determinants of Dirac boundary value problems over odd–dimensional manifolds', *Comm. Math. Phys.* 173, 43–76.
11. Scott, S.G., and Wojciechowski, K.P.: 1996, *Abstract Determinant and ζ–Determinant on the Grassmannian*, preprint.
12. Segal, G.B.: *The Definition of Conformal Field Theory*, preprint.
13. Singer, I.M.: 1985, 'Families of Dirac operators with applications to physics', *Asterisque, hors série*, 323-340.
14. Trautman, A.: 1995, 'The Dirac operator on hypersurfaces', *Acta Phys. Polon. B* 26, 1283–1310.
15. Wojciechowski, K.P.: 1983, 'On the spectral flow and the index theorem for flat vector bundles', in: *Partial Differential Equations*, Banach Center Publications, vol. 10, PWN – Polish Scientific Publishers, Warsaw, pp. 413-422.
16. —: 1996, *Smooth, Self-Adjoint Grassmannian, the η–Invariant, and the ζ–Determinant of Boundary Problems*, preprint.

Determinants, Grassmannians and Elliptic Boundary Problems for the Dirac Operator.

Simon G. Scott
Departamento de Matematicas
Universidad de los Andes
Bogota A.A. 4976, Colombia

Krzysztof P. Wojciechowski
Department of Mathematics
IUPUI
Indianapolis, IN 46202, USA

Abstract: We study the relations between different determinants of the Dirac operator over a manifold with boundary considered as sections of a holomorphic line bundle over the Grassmannian of boundary conditions of Atiyah- Patodi-Singer type.
MOS number: 58G11, 58G25, 57R90, 35J55, 35S35.

§0 Introduction

Recent studies in Quantum Field Theory (see for instance [1], [15], [16], [18]) have stressed the importance of the correct definition of the renormalized determinant of the Dirac operator over a closed manifold. With new developments in the mathematical understanding of QFTs as modified cohomology theories [1], [18] the need to extend the study of the determinant of the Dirac operator to manifolds with boundary has become clear. In [12] Quillen gave a construction of the determinant line bundle over a space of Fredholm operators and explained that, without making further choices, the determinant has to be viewed as a canonical section of this bundle. For a family of $\overline{\partial}$ operators over a Riemann surface Quillen identified a canonical trivialization of the determinant bundle by defining a metric and holomorphic connection through ζ-function renormalization and computing the curvarture, thus identifying the determinant up to a phase with a specific holomorphic function. These constructions have been used in many different contexts since then (see for instance [4], [16], [17]).

This note announces recent progress made in understanding these constructions for the ζ-determinant over the Grassmannian of elliptic boundary conditions for the Dirac operator. This is especially important in view of recent results which show that Quillen's determinant satisfies a nice pasting law (see [13],[14]) which is naturally formulated in terms of a Fock space bilinear pairing associated to the Grassmannian and may explain the nature of the pasting axiom in Fermionic Field Theory.

In Section 1 we study the geometry of the determinant line bundle. We consider the inifinite Grassmannian $\mathcal{G}r_\infty(A)$ of elliptic boundary conditions to be the parameter space for a holomorphic family of first order elliptic differential operators. This follows the approach taken by Bismut and Freed (see [4]) who extended the results of [12] to a general smooth family of Dirac operators over a closed manifold and showed that the curvature of the ζ-metric is the 2-form component of the local family index density. The method we use is to link up the Quillen-Bismut-Freed analysis with the holomorphic geometry of the Grassmannian as elucidated by Booss-Bavnbek, Wojciechowski (see [5], [6]) and Segal (see [11], [15]). More precisely, we find that the ζ-metric determines the same geometry as the canonical metric on the fundamental holomorphic line bundle over $\mathcal{G}r_\infty(A)$. To do this we identify the ζ metric with a metric constructed by a natural algebraic regularization of the Laplacian determinant and calculate its curvature. This is naturally understood as a statement of the *Local Family Index Theorem* for $\mathcal{G}r_\infty(A)$. The case of a family of varying Dirac operators with fixed Atiyah-Patodi-Singer boundary condition has been studied by Bismut, Cheeger [3] and Melrose, Piazza [10]. By considering the dual situation of a fixed Dirac operator with varying boundary condition we are able to take advantage of the well-understood properties of $\mathcal{G}r_\infty(A)$ which effectively reduces the heat kernel analysis of the determinant to an algebraic problem. It is this reduction which allows the pasting to be formulated as a bilinear pairing on the Fermionic Fock space over $\mathcal{G}r_\infty(A)$. The details of the pasting will be presented in [14], but see also [13].

In Section 2 we present a crucial component in the proof of this theorem. We show using heat equation methods that the η-invariant, the phase of the ζ-determinant in the odd-dimensional case, is up to a constant equal to the phase of the determinant defined by the algebraic regularization.

Details of the proofs will be published in the forthcoming paper [14].

§1 Geometry of the determinant line bundle over the Grassmannian

Let M denote a compact odd-dimensional manifold with boundary Y . Let $A : C^\infty(S) \to C^\infty(S)$ denote a compatible Dirac operator acting on the space of sections of a bundle of Clifford modules S over M (see [6]). We discuss the case of a product metric structure in a neighborhood of the boundary. More precisely, we assume that the Riemannian metric on M and the Hermitian product on S are products in $N = [0,1] \times Y$, the collar neighborhood of Y in M . In this case, A has the form

$$A = \Gamma(\partial_u + B), \tag{1.1}$$

over N, where $\Gamma : S|Y \to S|Y$ is a unitary bundle automorphism (Clifford multiplication by the unit normal vector) and $B : C^\infty(Y; S|Y) \to C^\infty(Y; S|Y)$ is the tangential part of A on Y. Here B is the corresponding Dirac operator on Y and hence is a self-adjoint elliptic operator of first order. Furthermore, Γ and B do not depend on u and satisfy the following identities.

$$\Gamma^2 = -Id \quad and \quad \Gamma B = -B\Gamma. \tag{1.2}$$

In particular, $S|Y$ decomposes into the direct sum $S^+ \bigoplus S^-$ of subbundles of eigenvectors of Γ corresponding to the eigenvalues $\pm i$. With respect to this decomposition the operator B has the representation

$$B = \begin{bmatrix} 0 & B^- = (B^+)^* \\ B^+ & 0 \end{bmatrix}. \tag{1.3}$$

We assume that $ker\ B = \{0\}$ in order to avoid unnecessary technical details in the presentation. The obvious modification to the general case will be presented elsewhere. Let $\Pi_>$ denote the spectral projection of B onto the subspace of $L^2(Y; S|Y)$ spanned by the eigenvectors corresponding to the positive eigenvalues of B. It is well-known (see [2], [6]) that $\Pi_>$ is an elliptic boundary condition for the operator A , which means that the operator $A_{\Pi_>}$ defined by

$$\begin{cases} A_{\Pi_>} = A \\ dom\ A_{\Pi_>} = \{s \in H^1(M; S|M) : \Pi_>(s|Y) = 0\} \end{cases} \tag{1.4}$$

is an unbounded operator, such that $A_{\Pi_>} : dom(A_{\Pi_>}) \to L^2(M; S)$ is a Fredholm operator and the kernel of $A_{\Pi_>}$ and its cokernel consist of smooth sections of S . It is also well-known that

$$index\ A_{\Pi_>} = dim\ ker\ B^+ \quad ,$$

and hence in our case it is equal to 0 . The operator $A_{\Pi_>}$ is now a self-adjoint operator and it is a particular example from the class of self- adjoint boundary problems which appear naturally in this context.

We define the Grassmannian of elliptic boundary value problems $\mathcal{G}r_\infty(A)$ as follows. The elements of $\mathcal{G}r_\infty(A)$ are pseudodifferential projections P acting on $C^\infty(Y; S|Y)$, such that they are orthogonal ($P = P^2 = P^*$), and such that the difference $P - \Pi_>$ is an operator with a smooth kernel. We can identify any projection P_W in the Grassmannian with its range $W \subset L^2(Y; S|Y)$. An important example of an element of $\mathcal{G}r_\infty(A)$ is provided by the Calderon projection $P(A)$ of the operator A . This is the orthogonal projection onto the subspace $\mathcal{H}(A)$ of $C^\infty(Y; S|Y)$ defined as

$$\mathcal{H}(A) = \{v \in C^\infty(Y; S|Y) \ : \exists_{s \in C^\infty(M;S)}\ As = 0\ and\ s|Y = v\}.$$

We refer to [6] (see also [5] and [13]) for more details on $P(A)$ and $\mathcal{H}(A)$.

It was explained in [13] that one can construct the determinant line bundle over $\mathcal{G}r_\infty(A)$ in many different ways. The Quillen determinant line bundle $\mathcal{L}$ is the holomorphic pullback of the determinant line bundle from a space of Fredholm operators under the map $W \to A_{P_W}$, where P_W denotes the orthogonal projection onto the subspace $W \subset L^2(Y; S|Y)$.

We can also use the construction of the determinant bundle due to Segal (see [15]). Let $T : H_0 \to H_1$ denote a Fredholm operator with index equal to 0 acting on separable Hilbert spaces H_0 and H_1 . Let $Fred_T$ denote the space of all Fredholm operators which differ from T by a trace class operator. We define

$$Det\ T = Fred_T \times\ C\ / \cong$$

where the relation is defined as follows. Let $S : H_0 \to H_1$ denote an invertible operator such that $S - T$ is an operator of trace class. Then any operator $Q \in Fred_T$ is of the form $Q = S(Id + q)$, where $q : H_0 \to H_0$ is a trace class operator. We identify

$$(S(Id + q), z) \cong (S, z \cdot det_F(Id + q))$$

where $det_F R$ denote the Fredholm determinant of the operator R . For a smooth family of such Fredholm operators the lines fit together to define a line bundle canonically isomorphic to $\mathcal{L}$. Under this isomorphism the canonical determinant section, defined over the index zero component by

$$T \to [(T, 1)]$$

maps to the canonical determinant section of $\mathcal{L}$.

In order to study the determinant bundle $\mathcal{L}$ associated to the family of elliptic boundary value problems A_W parameterized by the Grassmannian we use the operators

$$\rho_W : P_W P(A) : \mathcal{H}(A) \to W \tag{1.5}$$

The operator ρ_W is Fredholm with index equal to *index* A_W and with corresponding determinant line $Det\ \rho_W$. Globally we obtain a holomorphic line bundle Det isomorphic to $\mathcal{L}$ with determinant section canonically identified with that of $\mathcal{L}$ (see [13] for the details).

We now restrict our attention to the connected component $\mathcal{G}r^0_\infty(A)$ of $\mathcal{G}r_\infty(A)$ parameterizing subspaces W with *index* $A_W = 0$. Locally we may work over the open dense subset of $\mathcal{G}r^0_\infty(A)$ consisting of all those W which are the graphs of the invertible operators $T : L^2(Y; S^+) \to L^2(Y; S^-)$ such that $T - (B^+B^-)^{-\frac{1}{2}}B^+$ is an operator with a smooth kernel. The orthogonal projection onto $W = Graph(T)$ is given by

$$P_W = \begin{bmatrix} (Id + T^*T)^{-1} & (Id + T^*T)^{-1}T^* \\ T(Id + T^*T)^{-1} & T(Id + T^*T)^{-1}T^* \end{bmatrix} \tag{1.6}$$

Let K denote the operator such that $\mathcal{H}(A) = Graph(K)$. Then we have a canonical trivialization of Det over U_{Graph} defined by P_W with

$$W = Graph(T) \to [(\rho_W, det_F(TK^*))] \tag{1.7}$$

The isomorphism $Det \cong \mathcal{L}$ identifies $detA_W$ with $det\ \rho_W$ and hence a canonically renormalized determinant of A_W defined with respect to this local gauge by the formula $det_{\mathcal{C}}A_W = det_F(\rho_W)$. One computes

$$det_{\mathcal{C}}A_W = det_F(\frac{1}{2}(Id + KT^*)).$$

To connect this with the global geometry of the determinant bundle we define a Hermitian metric on $\mathcal{L}$ via the Laplacians

$$\Delta_{A_W} = A_W^* A_W \quad , \quad \Delta_{\rho_W} = \rho_W^* \rho_W \quad .$$

It is not difficult to see that there exists a natural isomorphism between the determinant lines $Det\ \Delta_{A_W}$ and $Det\ \Delta_{\rho_W}$ preserving the canonical sections. The Laplacian $\Delta_{\rho_W} : \mathcal{H}(A) \to \mathcal{H}(A)$ is an operator of the form $Id_{\mathcal{H}(A)} + smoothing\ operator$ and hence has a well-defined Fredholm determinant as a number in C . Therefore it is natural to define the regularized determinant of Δ_{A_W} by

$$det_{\mathcal{C}}\Delta_{A_W} = det_F \Delta_{\rho_W} \tag{1.8}$$

Proposition 1.1 There is a natural inner product on $\mathcal{L}$ given over the index 0 component of $\mathcal{G}r_\infty(A)$ by

$$\|det\ A_W\|_{\mathcal{C}}^2 = det_{\mathcal{C}}\Delta_{A_W} \tag{1.9}$$

where A_W is invertible and 0 otherwise.

One also has the usual Quillen norm $\|.\|_\zeta$ defined on $\mathcal{L}$ by $\|det\ A_W\|_\zeta^2 = det_\zeta \Delta_{A_W}$, where the right-hand side denotes the regularised zeta-function determinant, and it is important to know its relation to the $\mathcal{C}$ norm of Proposition 1.1. A holomorphic line bundle $\mathcal{L}$ with a Hermitian inner- product has a canonical connection compatible with the two structures whose curvature is the $(1,1)$ form equal to $\overline{\partial}\partial \log \|s\|^2$ for any holomorphic section s . The elliptic Grassmannian $\mathcal{G}r_\infty(A)$ is endowed with a preferred form of type $(1,1)$, namely the Kähler form

$$\omega = \frac{i}{2\pi} Tr\, PdP \wedge dP. \tag{1.10}$$

Since any $P \in \mathcal{G}r_\infty(A)$ is of the form $\Pi_> +$ *smoothing operator* the trace on the right side is well-defined.

Theorem 1.2: *The metrics $\|.\|_\zeta$ and $\|.\|_\mathcal{C}$ on the determinant line bundle have curvature equal to*

$$R = -2\pi i \omega.$$

The proof for the $\mathcal{C}$ metric follows from straightforward computations, the case of the Quillen metric uses heat kernel methods generalizing those presented in Section 2, details will be presented in [14]. On the other hand explicit calculations show that

Theorem 1.3: *Over U_{Graph} the following formula holds:*

$$det_F \Delta_{\rho W} = \frac{det_F \frac{1}{2}(Id + K^*T) \cdot det_F \frac{1}{2}(Id + T^*K)}{det_F \frac{1}{2}(Id + T^*T)},$$

or in different notation

$$det_\mathcal{C} \Delta_{A_W} = \frac{|det_\mathcal{C} A_W|^2}{det_F \frac{1}{2}(Id + T^*T)}.$$

Thus from Theorem 1.2 and Theorem 1.3 the canonical determinant of A_W on the set U_{Graph} is related to the global geometry of $\mathcal{L}$ by

$$det_\mathcal{C} \Delta_{A_W} = \exp(-k)|det_\mathcal{C} A_W|^2,$$

where k is the standard Kähler potential. Equivalently , by taking a section over U_{Graph} flat for the $\mathcal{C}$ metric connection, $det_\mathcal{C} A_W$ is the function defined relative to this trivialization up to a scalar of absolute one. We note that the local anomaly formula of Theorem 1.2. is a measure of the failure of the canonical determinant to be multiplicative. Thus the fact that $det_C \Delta_{A_W}$ is not of the form $|holomorphic\ function|^2$ is equivalent to the non-triviality of the determinant line bundle.

The local formula of Theorem 1.3 is also true for the Quillen metric up to a constant scale factor. The proof is modelled on the proof of theorem 2.1 in Section 2 (see [14] for the detailed exposition) . As an example, consider the simplest case of the operator $A = i\frac{d}{dx}$ on $M = [0,1]$ with boundary conditions parametrized by CP^1 . Then $W = Graph(a)$ for $a \in C \setminus \{0\}$ corresponds to the homogenous coordinate $[1, -\overline{a}^{-1}] \in CP^1$. The ζ-determinant of $\Delta_{i\frac{d}{dx}Graph(a)}$ is equal to

$$det_\zeta \Delta_{i\frac{d}{dx}Graph(a)} = 2\frac{|1-a|^2}{1+|a|^2}$$

which coincides with the result of Theorem 2 up to factor 4 .

§2 The phase of the determinant on the self-adjoint Grassmannian

In this section we restrict ourselves to the situation studied in [13] (see also [20]). We study the canonical determinant and the ζ-determinant on the real submanifold of $\mathcal{G}r_\infty(A)$ parameterizing self- adjoint generalized Atiyah-Patodi-Singer boundary conditions for A. Specifically we define

$$\mathcal{G}r^*_\infty(A) = \{P \in \mathcal{G}r_\infty(A)\ :\ P\ is\ orthogonal\ and\quad -\Gamma P \Gamma = Id - P\} \tag{2.1}$$

The second condition implies that the range of the projection in $L^2(Y; S|Y)$ is a Lagrangian subspace with respect to the symplectic structure defined on $L^2(Y; S|Y)$ by the involution Γ. The projection $P_>$ is an element of $\mathcal{G}r^*_\infty(A)$ (in the case of invertible operator B). Another important example is provided by $P(A)$, the Calderon projection of the operator A (see [13]).

In [20] it was proved for $P \in \mathcal{G}r^*_\infty(A)$ that $\eta_{A_P}(s)$ and $\zeta_{A_P^2}(s)$ behave exactly like the η-function and the ζ-function of the Dirac operator on a closed manifold. In particular $\eta_{A_P}(0)$, $\zeta_{A_P^2}(0)$ and $d/ds(\zeta_{A_P^2})|_{s=0}$ are well-defined, and hence the ζ-determinant of A_P equal to

$$det_\zeta A_P = e^{\frac{i\pi}{2}\eta_{A_P}(0)} . e^{-d/ds(\zeta_{A_P^2})|_{s=0}} \tag{2.2}$$

is well-defined.

It was explained in Section 1 that we have a canonical section of the determinant line bundle which over the set U_{Graph} , after the identification of the natural trivialization, allows us to define the determinant as the function which we call the canonical determinant. Actually $\mathcal{G}r^*_\infty(A) \subset U_{Graph}$ and so the determinant line bundle restricted to $\mathcal{G}r^*_\infty(A)$ is a canonically trivial line bundle. Now we make use of a result about the self-adjoint Grassmanian stronger than for U_{Graph} . Namely, there is a canonical one-to-one correspondence between elements of $Gr^*_\infty(A)$ and L^2-unitary isomorphisms $S : L^2(Y; S^+) \to L^2(Y; S^-)$ which differ from the operator $(B^+B^-)^{-\frac{1}{2}}B^+$ by a smoothing operator. The correspondence is given by

$$S \to P = \frac{1}{2}\begin{bmatrix} Id & S^{-1} \\ S & Id \end{bmatrix}. \tag{2.3}$$

It is obvious that $Ran(P) = Graph(S)$. The canonical determinant $det_{\mathcal{C}}(W)$ is given by

$$det_{\mathcal{C}}(W) = det_F \ \frac{1}{2}(Id + KS^{-1}) \tag{2.4}$$

Let us assume for the moment that the operator $KS^{-1} : L^2(Y; S^-) \to L^2(Y; S^-)$ is of the form $e^{i\alpha}$, where $\alpha : L^2(Y; S^-) \to L^2(Y; S^-)$ is a self-adjoint operator with a smooth kernel. This is always the case when KS^{-1} is close to the Id in $GL_\infty(S^-)$, the group of the invertible operators of the form Id plus *smoothing operator* acting on spinors of negative "chirality" on Y. The reason is that the space of such operators is a Lie algebra of $GL_\infty(S^-)$. This means

$$det \ \frac{1}{2}(Id + KS^{-1}) = det(e^{\frac{i\alpha}{2}} \cdot \frac{e^{\frac{i\alpha}{2}} + e^{-\frac{i\alpha}{2}}}{2}) = e^{\frac{i}{2}tr(\alpha)} \cdot det \ cos \ \frac{\alpha}{2}. \tag{2.5}$$

This formula explains the structure of the canonical determinant, which is similar to the structure of the ζ- determinant (see (2.2)). The latter has phase determined by the η-invariant and modulus equal to the exponent of $\zeta'_{A_W}(0)$. The main result of Section 2 is the following Theorem.

Theorem 2.1: *Let W denote an element of $Gr^*_\infty(A)$. The phase of the canonical determinant is equal to the integral of the variation of the η- invariant. More precisely, let $\{W_r\}_{0 \le r \le 1}$ denote a one parameter family in $Gr^*_\infty(A)$ such that $W_0 = H(A)$ and $W_1 = W$ and let A_r denote the operator A_{W_r} , then the phase of the canonical determinant is equal to $\int_0^1 d/dr(\eta_{A_r})dr$.*

The proof of the Theorem consists of two parts. First we compute the variation of the η-invariant for a specific family of boundary conditions. Then we show that the result is independent of all choices and deformations made. We sketch the proof of the first part. Let

$$P = \frac{1}{2}\begin{bmatrix} Id & S^{-1} \\ S & Id \end{bmatrix} = \frac{1}{2}\begin{bmatrix} Id & 0 \\ 0 & SK^{-1} \end{bmatrix}\begin{bmatrix} Id & K^{-1} \\ K & Id \end{bmatrix}\begin{bmatrix} Id & 0 \\ 0 & KS^{-1} \end{bmatrix} =$$

$$\begin{bmatrix} Id & 0 \\ 0 & e^{-i\alpha} \end{bmatrix} P(A) \begin{bmatrix} Id & 0 \\ 0 & e^{i\alpha} \end{bmatrix}$$

We show that $\frac{1}{\pi}Tr\ \alpha$ is the integral of the variation of the η-invariant for some family of the boundary conditions. We define the operator $\mathcal{A}_r$ as A_{P_r} , where the projection P_r is given by the formula

$$P_r = \begin{bmatrix} Id & 0 \\ 0 & e^{-ir\alpha} \end{bmatrix} P(A) \begin{bmatrix} Id & 0 \\ 0 & e^{ir\alpha} \end{bmatrix} \tag{2.6}$$

We will show that in this case

$$\int_0^1 d/dr(\eta_{\mathcal{A}_r})dr = \frac{1}{\pi}Tr\ \alpha. \tag{2.7}$$

In fact, it is not difficult to see that we can replace Calderon projection by the spectral projection. We refer to [14] for the details. In the following we consider the family of projections defined by the formula

$$\Pi_r = \begin{bmatrix} Id & 0 \\ 0 & e^{-ir\alpha} \end{bmatrix} \Pi_> \begin{bmatrix} Id & 0 \\ 0 & e^{ir\alpha} \end{bmatrix}, \tag{2.8}$$

and we define the operator $\tilde{\mathcal{A}}_r$ as A_{Π_r} . Employing the method used in papers [7] (see Appendix 1) and [9], we perform a *Unitary Twist* on the operator $\tilde{\mathcal{A}}_r$. The operator we obtain has the same spectrum as $\tilde{\mathcal{A}}_r$ and moreover the new family has a fixed domain. We define a specific unitary transformation U_r : $L^2(M;S) \to L^2(M;S)$ as follows. First introduce a smooth nonnegative function $f : [0,1] \to [0,1]$ equal to 1 for $0 \leq u \leq \frac{1}{8}$ and equal to 0 for $\frac{7}{8} \leq u \leq 1$. We define

$$U_r = \begin{cases} Id & on\ \ M \setminus N \\ \begin{bmatrix} Id & 0 \\ 0 & e^{irf(u)\alpha} \end{bmatrix} & on\ \ \{u\} \times Y \end{cases} \tag{2.9}$$

It is obvious that the operator

$$\tilde{A}_r = A_{\Pi_r} = A_{\begin{bmatrix} Id & 0 \\ 0 & e^{-ir\alpha} \end{bmatrix}\Pi_> \begin{bmatrix} Id & 0 \\ 0 & e^{ir\alpha} \end{bmatrix}}$$

has the same spectrum as the operator $(U_r A U_r^{-1})_{\Pi_>}$ and so we can study the variation of the η-invariant of the family $\{(U_r A U_r^{-1})_{\Pi_>}\}$. We follow the strategy of the papers [7] and [9] (see also [8], [19], [20]) and show that the only contribution to the variation of the η-invariant comes from the cylinder. The operator $U_r A U_r^{-1}$ is given by the formula

$$U_r A U_r^{-1} = A +$$

$$+\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}\begin{bmatrix} 0 & B^-(e^{-irf(u)\alpha} - Id) \\ (e^{irf(u)\alpha} - Id)B^+ & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & -rf'(u)\alpha \end{bmatrix},$$

and the operator $\frac{d(U_r A U_r^{-1})}{dr}$ has the following form:

$$\frac{d(U_r A U_r^{-1})}{dr} = \begin{bmatrix} 0 & f(u)B^-\alpha e^{-irf(u)\alpha} \\ f(u)\alpha e^{irf(u)\alpha}B^+ & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & f'(u)\alpha \end{bmatrix}. \quad (2.10)$$

We use this representation in order to study:

$$\frac{d}{dr}\eta_{(U_r A U_r^{-1})_{\Pi_>}}|_{r=0} = -\frac{2}{\sqrt{\pi}}\cdot\lim_{\epsilon\to 0}\sqrt{\epsilon}\cdot Tr\ \frac{d}{dr}(U_r A U_r^{-1})|_{r=0}e^{-\epsilon((U_r A U_r^{-1})^2_{\Pi_>})|_{r=0}} \quad (2.11)$$

The contribution due to the first term on the right side of (2.10) is equal to 0 . This follows from (1.2). Hence we only have to study the trace of the operator

$$-f'(u)\begin{bmatrix} 0 & 0 \\ 0 & \alpha \end{bmatrix} e^{-\epsilon((U_r A U_r^{-1})^2_{\Pi_>})}|_{r=0} \ .$$

The function $f'(u)$ is non-zero only for $\frac{1}{8} \le u \le \frac{7}{8}$ and we can use *Duhamel's Principle* and replace the original heat kernel by the heat kernel of the Dirac operator $\Gamma(\partial_u + B)$ on the cylinder $(-\infty, +\infty) \times Y$. Now we obtain the formula

$$\frac{d}{dr}\eta_{(U_rAU_r^{-1})_{\Pi_>}}|_{r=0} = -\frac{2}{\sqrt{\pi}}\lim_{\epsilon\to 0}\sqrt{\epsilon}\cdot Tr\ \frac{d}{dr}(U_rAU_r^{-1})|_{r=0}e^{-\epsilon((U_rAU_r^{-1})^2_{\Pi_>})}|_{r=0} =$$

$$\frac{2}{\sqrt{\pi}}\lim_{\epsilon\to 0}\sqrt{\epsilon}\cdot Tr\ f'(u)\begin{bmatrix} 0 & 0 \\ 0 & \alpha \end{bmatrix} e^{-\epsilon(-\partial_u^2+B^2)} = \frac{2}{\sqrt{\pi}}\lim_{\epsilon\to 0}\sqrt{\epsilon}\frac{1}{\sqrt{4\pi\epsilon}}Tr\ f'(u)\alpha e^{-\epsilon B^+B^-} =$$

$$\frac{1}{\pi}\int_0^1 du\ \lim_{\epsilon\to 0}Tr\ f'(u)\alpha e^{-\epsilon B^+B^-} = \frac{1}{\pi}Tr\ \alpha. \tag{2.12}$$

In fact, using the unitary twist, we are able to show that $\frac{d}{dr}\eta_{(U_rAU_r^{-1})_{\Pi_>}(0)} = \frac{1}{\pi}Tr\ \alpha$ for any $0 \leq r \leq 1$. We use a similar argument to show that we can replace $\Pi_>$ by the Calderon projection and also to show that the integral depends only on the end-points of the family. Details will appear in [14].

Bibliography

[1] . Atiyah, M. F, Topological quantum field theories, *Publ. Math. I.H.E.S.*, 68 (1989), 175 –186.

[2] . Atiyah, M. F., Patodi, V. K. and Singer I. M., Spectral asymmetry and Riemannian geometry I, III, *Math. Proc. Camb. Phil. Soc.* 77 (1975), 43 –69; 79 (1976), 71 –99.

[3] . Bismut, J. -M. and Cheeger, J., Families index for manifolds with boundary, superconnections and cones I, II, *J. Funct. Anal.* 89 (1990), 313 - 363; 90(1990), 306–354.

[4] . Bismut, J.-M. and Freed, D., The analysis of elliptic families I, II, *Comm. Math. Phys.* 106 (1986), 159 –176; 107 (1986), 103 –163.

[5] . Booss-Bavnbek, B. and Wojciechowski, K. P., Pseudodifferential projections and the topology of certain spaces of elliptic boundary value problems, *Comm. Math. Phys.* 121 (1989), 1 –9.

[6] . Booss-Bavnbek, B. and Wojciechowski, K. P., **Elliptic Boundary Problems for Dirac Operators**, Birkhäuser, Boston, 1993.

[7] . Douglas, R. G. and Wojciechowski, K. P., Adiabatic limits of the η-invariants. The odd-dimensional Atiyah-Patodi-Singer problem, *Comm. Math. Phys.* 142 (1991), 139 –168.

[8] . Klimek, S. and Wojciechowski, K. P., Adiabatic cobordism theorems for analytic torsion and η-invariant, *J. Functional Analysis*, 136 (1996), 269 –293.

[9] . Lesch, M. and Wojciechowski, K. P., On the η-invariant of generalized Atiyah-Patodi-Singer problems, *Illinois J. Math.*, 40 (1996), 30 –46.

[10] . Melrose, R. B. and Piazza, P., *Families of Dirac operators, boundaries and the b-calculus*, Preprint, 1993.

[11] . Pressley, A. and Segal G. B., **Loop Groups**, Oxford University Press, Oxford, 1986.

[12] . Quillen, D., Determinants of Cauchy-Riemann operators over a Riemann surface, Functional Anal. Appl.19 (1985), 31 –34.

[13] . Scott, S. G., Determinants of Dirac boundary value problems over odd-dimensional manifolds, *Comm. Math. Phys.*, 173 (1995), 43 –76.

[14] . Scott, S. G. and Wojciechowski, K. P., *Determinants and Grassmannians*, in preparation.

[15] . Segal, G. B., *The definition of conformal field theory*, Preprint, 1993.

[16] . Singer, I. M., Families of Dirac operators with applications to physics, *Asterisque, hors serie, 1985*, 323 –340.

[17] . Witten, E., Quantum field theory, Grassmannians, and algebraic curves, *Comm. Math. Phys.*, 113 (1988), 529 –600.

[18] . Witten, E., Quantum field theory and the Jones polynomial, *Comm. Math. Phys.*, 121 (1988), 351 –399.

[19] . Wojciechowski, K. P., On the additivity of the η- invariant. The case of singular tangential operator, *Comm. Math. Phys.* 169 (1995), 315 –327.

[20] . Wojciechowski, K. P., *Smooth, self-adjoint Grassmannian, The η-invariant, and ζ-determinant of boundary problems*, Preprint, 1996.

GRASSMANNIAN AND BOUNDARY CONTRIBUTION TO THE ζ-DETERMINANT: INTRODUCTION INTO THE 4-DIMENSIONAL CASE*

B. BOOSS-BAVNBEK
Institut for matematik og fysik
Roskilde University
DK-4000 Roskilde, Denmark
e-mail: booss@mmf.ruc.dk

and

K. P. WOJCIECHOWSKI
Dept. Math., IUPUI (Indiana/Purdue)
Indianapolis IN 46202-3216, U.S.A.
e-mail: kwojciechowski@math.iupui.edu

Abstract. In this paper we discuss the boundary contribution to the ζ-determinant for a particular QCD model: For any fixed element P in the Grassmannian $\mathrm{Gr}(\not{D}^+)$ of the (half) Euclidean Dirac operator $\not{D}^+$ on the 4-ball $\mathcal{V}$ we define a smooth γ_5 and gauge-invariant mapping $A \mapsto \mathcal{R}_P(A)$ from the space of connections which are pure gauge at the boundary of $\mathcal{V}$ into the self-adjoint Grassmannian of the (total) Dirac operator $\not{D}_A$. The boundary condition $\mathcal{R}_P(A)$ depends solely on the data of A at the boundary and provides for chiral symmetry $n_+ - n_- = 0$. We show that for any fixed connection A and any smooth path $\{P_t\}$ in $\mathrm{Gr}(\not{D}^+)$ the variation

$$\frac{d}{dt}\left(-\ln \det{}_\zeta \not{D}_{A,\mathcal{R}_{P_t}(A)}\right)|_{t=0}$$

of the corresponding ζ-determinant vanishes. This yields that the ζ-determinant is constant on the connected components of the Grassmannian.

1. Euclidean Dirac Operators in Finite Volume

Let $\mathcal{V}$ denote the 4-dimensional ball of radius R and $\not{D}_A : C^\infty(\mathcal{V};\not{S}) \to C^\infty(\mathcal{V};\not{S})$ be the (total) Dirac operator acting on the bundle $\not{S} := \mathcal{V} \times (S \otimes \mathbf{C}^2)$ of spinors with coefficients in the trivial bundle $\mathcal{V} \times \mathbf{C}^2$, twisted according to a conncetion A for $\mathcal{V} \times \mathbf{C}^2$.

One fundamental problem here is the computation of the determinant $\det \not{D}_A$. Unfortunately, the operator $\not{D}_A$ has bad spectral properties. E.g., the space $\ker \not{D}_A$ of smooth solutions of $\not{D}_A$ is infinite-dimensional. Therefore it is necessary to impose suitable boundary conditions. A preliminary discussion of the various boundary conditions involved was provided in [2]. Here we shall be more specific and discuss a particular and universal choice of boundary conditions which seems to be most appropriate from the point of view of QCD.

To avoid technicalities, we deform the Euclidean metric of $\mathcal{V}$ and the Hermitian metric of S in a collar neighborhood N of the boundary $Y = S^3$ to product metrics.

---

* Working paper, *9 August 1996*. This is part of a joint project with G. Morchio and F. Strocchi, Pisa

Then the Euclidean Dirac operator $\not{D}: C^\infty(\mathcal{V};\not{S}) \to C^\infty(\mathcal{V},\not{S})$ splits over $N \cong I \times S^3$ into the following product form:

$$\not{D} = \begin{pmatrix} 0 & \not{D}^- \\ \not{D}^+ & 0 \end{pmatrix} = \Gamma(\partial_u + \mathcal{B}) = \begin{pmatrix} 0 & -V^{-1} \\ V & 0 \end{pmatrix} \left(\partial_u + \begin{pmatrix} B & 0 \\ 0 & -VBV^{-1} \end{pmatrix} \right), \quad (1)$$

where u denotes the inward oriented normal (radial) coordinate and $B : C^\infty(Y;S^+) \to C^\infty(Y;S^+)$ denotes the total Dirac operator on S^3. It is an invertible self-adjoint operator with discrete spectrum. Clifford multiplication by the (inward) normal vector provides for the unitary isomorphism $V : S^+|_{S^3} \to S^-|_{S^3}$.

Let $\Pi_>$ denote the spectral projection of B onto the direct sum of the eigenspaces of B for eigenvalues greater than 0. To begin with we consider the operator $\not{D}^+ : C^\infty(\mathcal{V};S^+) \to C^\infty(\mathcal{V};S^-)$ and define its realization corresponding to $\Pi_>$ as follows:

$$\not{D}^+_{\Pi_>} := \not{D}^+ \quad \text{with}$$
$$\operatorname{dom}(\not{D}^+_{\Pi_>}) := \{s \in H^1(\mathcal{V};S^+) \mid \Pi_>(s|_{S^3}) = 0\},$$

where $H^1(\mathcal{V};S^+)$ denotes the first Sobolev space of sections of S^+ on $\mathcal{V}$. The operator $\not{D}^+_{\Pi_>} : \operatorname{dom} \not{D}^+_{\Pi_>} \to L^2(\mathcal{V};S^-)$ is a Fredholm operator (with index equal to 0).

The adjoint operator is

$$(\not{D}^+_{\Pi_>})^* = \not{D}^-_{V(\mathrm{Id}-\Pi_>)V^{-1}}.$$

Hence the boundary condition

$$\Pi_>^{\#} := \begin{pmatrix} \Pi_> & 0 \\ 0 & V(\mathrm{Id}-\Pi_>)V^{-1} \end{pmatrix}$$

for the total Dirac operator makes the operator $\not{D}_{\Pi_>^{\#}}$ to a self-adjoint Fredholm operator with discrete spectrum.

In fact we can take any projection P belonging to the total Grassmannian $\mathrm{Gr}(\not{D}^+)$ of the half Dirac operator (i.e. a pseudo-differential projection with the same principal symbol like $\Pi_>$) and obtain a self-adjoint operator $\not{D}_{P^\#}$ by the preceding twisting construction. As shown in [2], any global, γ_5-invariant boundary condition providing for an operator with compact resolvent arises in this way. Then the nullspace of $\not{D}_{P^\#}$ splits into the zero modes of positive (resp. negative) chirality, two subspaces of dimension $n_\pm$ with $n_+ = \dim\ker \not{D}^+_P$ and $n_- = \dim\ker \not{D}^-_{V(\mathrm{Id}-P)V^{-1}}$. Moreover we have

$$n_+ - n_- = \operatorname{index} \not{D}^+_P = \operatorname{index} P\mathcal{P}(\not{D}^+), \quad (2)$$

where $P\mathcal{P}(\not{D}^+) : \mathcal{H}(\not{D}^+) \to \mathrm{range}(P)$ and $\mathcal{P}(\not{D}^+)$ denotes the Calderón projector, which is a projection of the L^2 sections over S^3 onto the Cauchy data space $\mathcal{H}(\not{D}^+)$ which is the L^2 closure of the space $\{s|_{S^3} \mid \not{D}^+ s = 0 \text{ in } \mathcal{V} \setminus \partial\mathcal{V}\}$.

Next we consider the extension of the preceding constructions to the twisted Dirac operator $\not{D}_A := \not{D} \otimes_A \mathrm{Id}_{\mathbf{C}^2}$ which is lifted from the Euclidean Dirac operator $\not{D}$ using a connection A on the bundle $\mathcal{V} \times \mathbf{C}^2$ (see [2] and references therein). We call a connection A pure gauge at the boundary, if there exists a collar neighbourhood

$N_A \subset N$ of the boundary such that $A|_{N_A} = h^{-1}dh$ for a suitable smooth gauge transformation $h: S^3 \to \mathrm{SU}(2)$.

Let Conn_0 denote the space of connections which are pure gauge at the boundary. This space has infinitely many connected components labelled by the degree $\deg(h)$ of the mapping h. We have (see [4], p. 320)

$$\deg h = -\frac{1}{24\pi^2}\int_{S^3} \mathrm{tr}(h^{-1}dh)^3 = -\frac{1}{32\pi^2}\int_{\mathcal{V}} \mathrm{tr} F \wedge F^* ,$$

where $F = F(A)$ is the curvature form of the connection A.

The tangential operator B of $\not{D}_A$ has the following form on the boundary (see [2]):

$$B = B_h = B \otimes_{h^{-1}dh} \mathrm{Id} = (\mathrm{Id} \otimes h)\ (B \otimes \mathrm{Id}_{\mathbf{C}^2})\ (\mathrm{Id} \otimes h)^{-1} \tag{3}$$

and the corresponding spectral projection is equal to

$$\Pi^h_> = (\mathrm{Id} \otimes h)(\Pi_> \otimes \mathrm{Id})(\mathrm{Id} \otimes h^{-1}).$$

We form the operator $\not{D}_{A,\Pi^h_>\#}$. It is a self–adjoint operator with symmetric spectrum and we want to compute its determinant. To begin with we discuss the determinant of the Dirac Laplacian $\mathcal{D}^2 := (\not{D}_{A,\Pi^h_>\#})^2$.

2. The ζ–Determinant

We use the concept of the ζ–determinant as it was introduced by Ray and Singer ([8], see also Schwarz's monograph [9]). Let us assume for the moment that $\not{D}_{A,\Pi^h_>\#}$ is invertible, hence its spectrum is $\{-\lambda_j, \lambda_j\}_{j\in\mathbf{N}}$ with $0 < \lambda_1 \le \lambda_2 \le \lambda_3 \le \dots$. Formally the determinant is the product of the eigenvalues and we have

$$\begin{aligned}\det \mathcal{D}^2 &= \prod_j (\lambda_j)^4 = \exp\left(2\sum \ln \lambda_j^2 e^{-s\ln\lambda_j^2}\right)|_{s=0} \\ &= \exp(-2\sum \frac{d}{ds}(e^{-s\ln\lambda_j^2})|_{s=0}) = \exp(-2\frac{d}{ds}(\sum(\lambda_j^2)^{-s})|_{s=0}) .\end{aligned}$$

To make it precise let us remind that the zeta function $\zeta_{\mathcal{D}^2}(s) = 2\sum(\lambda_j^2)^{-s}$ is well defined (see [1], [9], [10], [13]) and holomorphic for $\Re(s)$ large; it has a meromorphic extension to the whole complex plane; and $s = 0$ is not a pole. Thus $\zeta_{\mathcal{D}^2}(s)$ is holomorphic in a neighbourhood of 0 and we can define

$$\det \mathcal{D}^2 := e^{-\zeta'_{\mathcal{D}^2}(0)} .$$

This definition works well in the case of invertible $\mathcal{D}$, but unfortunately this is seldom the case. Let

$$n_+ := \dim\ker \not{D}^+_{A,\Pi^h_>} \text{ and } n_- := \dim\ker \not{D}^-_{A,V(\mathrm{Id}-\Pi^h_>)V^{-1}} .$$

We have

$$n_+ - n_- = \mathrm{index}\, \not{D}^+_{A,\Pi^h_>} = \deg(h) \tag{4}$$

by the Atiyah–Patodi–Singer theorem. If we replace the spectral projection by any arbitrary $P \in \mathrm{Gr}(\not{D}^+)$ we get correspondingly

$$n_+ - n_- = \operatorname{index} \not{D}^+_{A,P} = \deg(h) + \mathbf{i}(P, \Pi^h_>) ,$$

where $\mathbf{i}(P, \Pi^h_>)$ is the virtual index (see [2]).

Equation (4) implies that

$$\dim \ker \not{D}_{A,\Pi^{h\#}_>} = n_+ + n_-$$

in general is not equal to 0. A standard convention here is to replace $\det(\not{D}_{A,\Pi^{h\#}_>})^2$ by $\det'(\not{D}_{A,\Pi^{h\#}_>})^2$ defined by the removal of the zero eigenvalues. Because of the renormalization procedure, however, it is much more convenient to add a fermion mass term to $\not{D}_A$. Let γ_5 denote the involution on $\not{S}$ equal to ± 1 on $S^\pm \oplus \mathbf{C}^2$. We define

$$M = M(m, \theta_m) := m e^{i\gamma_5\theta_m} : \ker \not{D}_{A,\Pi^{h\#}_>} \to \ker \not{D}_{A,\Pi^{h\#}_>} .$$

The determinant of M on $\ker \not{D}_{A,\Pi^{h\#}_>}$ is equal to

$$m^{n_+ + n_-} \cdot e^{i(n_+ - n_-)\theta_m} .$$

Hence we may define

$$\operatorname{Det} \not{D}^2_{A,\Pi^{h\#}_>} := \det M(m, \theta_m) \cdot \det{}' \not{D}^2_{A,\Pi^{h\#}_>} .$$

This definition, however, yields a determinant of a self-adjoint operator (with symmetric spectrum) with a non-trivial imaginary part. There are also strong physical arguments against a complex determinant. It is therefore, the condition $n_+ = n_-$ becomes an important issue here.

3. A Universal Section in the Grassmannian

More generally, what we want to do is to construct a smooth map

$$\mathcal{R} : \mathrm{Conn}_0(V \times \mathbf{C}^2) \ni A \mapsto \mathcal{R}(A) \in \mathrm{Gr}(\not{D}_A)$$

which satisfies the following conditions (see [2]):
(a) $\not{D}_{A,\mathcal{R}(A)}$ is self-adjoint with compact resolvent;
(b) $\mathcal{R}(A)$ is γ_5-invariant;
(c) the domain $\operatorname{dom} \not{D}_{A,\mathcal{R}(A)}$ is gauge-invariant; and
(d) the index $n_+(\mathcal{R}_A) - n_-(\mathcal{R}_A)$ vanishes.

One example of such a map built on the Calderón projector was discussed in [2]. Though it provides a perfect solution from the mathematical point of view, physical constraints make us to look for another choice. We shall describe a construction suggested by Morchio and Strocchi in [7] and already discussed in our previous paper ([2], Alternative 4.2):

The task here is that we have to compensate for the degree of the gauge transformation h attached to the connection A by introducing a term of reversed degree into the boundary condition. We fix the generator map

$$f : S^3 \xrightarrow{\mathrm{Id}} \mathrm{SU}(2)$$
$$\{|z_1|^2 + |z_2|^2 = 1\} \ni (z_1, z_2) \mapsto f(z_1, z_2) := \begin{pmatrix} z_1 & \overline{z_2} \\ -z_2 & \overline{z_1} \end{pmatrix},$$

of degree 1 and define

$$\mathcal{R}_{\Pi_>}(A) := \{(\mathrm{Id} \otimes h)(\mathrm{Id} \otimes f^{-k})(\Pi_> \otimes \mathrm{Id}_{\mathbb{C}^2})(\mathrm{Id} \otimes f^k)(\mathrm{Id} \otimes h^{-1})\}^\#$$

for any $A \in \mathrm{Conn}_0$ with $A|_{N_A} = h^{-1}dh$ and $\deg h = k$.

The mapping $A \mapsto \mathcal{R}_{\Pi_>}(A)$ provides a choice of boundary conditions which satisfy (a)-(d). Moreover, $\mathcal{R}_{\Pi_>}(A)$ depends only on the boundary data and not on the form of the connection A inside of the manifold.

Let us remark that in fact we can use any projection in the Grassmannian to produce a corresponding section

$$\mathcal{R}_P : A \mapsto \mathcal{R}_P(A)$$

satisfying conditions (a)-(d). This leaves us with a universal mapping

$$\mathcal{R} : (A, P) \ni \mathrm{Conn}_0 \times \mathrm{Gr}(\not{D}^+) \mapsto \mathcal{R}_P(A) \in \mathrm{Gr}(\not{D}_A).$$

4. The Main Theorem

In the following we discuss the dependence of the determinant $\det \not{D}^2_{A,\mathcal{R}_P(A)}$ on the choice of the base projection P. Surprisingly we find:

Theorem 4.1 *Let $\{P_r\}_{0 \le r \le 1}$ be a smooth path of projections such that $P_1 - P_0$ is a smoothing operator. Then the variation of the ζ-determinant along the path is equal to 0.*

The starting point for the proof of the preceding theorem is the fact that

$$-\ln \det \not{D}^2_{A,\mathcal{R}_r(A)} = \zeta'_{(\not{D}_{A,\mathcal{R}_r})^2} == \int_0^\infty t^{-1} \mathrm{Tr}\, e^{-t(\not{D}_{A,\mathcal{R}_r(A)})^2} dt$$

where we write $\mathcal{R}_r$ for $\mathcal{R}_{P_r}$ and where on the right side we have the regularized integral (see [12], [5]).

In the following first we assume that all operators $\mathcal{D}_r := \not{D}_{A,\mathcal{R}_r(A)}$ are invertible. Then we have

$$\frac{d}{dr}\left(\int_0^\infty t^{-1} \mathrm{Tr}\, e^{-t\mathcal{D}_r^2} dt\right)|_{r=0} = 2 \lim_{\varepsilon \to 0} \mathrm{Tr}\, \dot{\mathcal{D}}_0 \mathcal{D}_0 \mathcal{D}_0^{-2} e^{-\varepsilon \mathcal{D}_0^2}.$$

We first show that

$$\lim_{\varepsilon \to 0} \mathrm{Tr}\, \dot{\mathcal{D}}_0 \mathcal{D}_0 e^{-\varepsilon \mathcal{D}_0^2} = 0$$

The argument works in the case of

$$\lim_{\varepsilon\to 0} \operatorname{Tr} \dot{\mathcal{D}}_0 \mathcal{D}_0 \mathcal{D}_0^{-2} e^{-\varepsilon \mathcal{D}_0^2}$$

as well.

We employ the method of the unitary twist first used in [3], Appendix A (see also [6], [13]). Hence we can assume that $\mathcal{R}_r(A)$ is of the form $P_r^{\#}$ with $P_0 = \Pi_>$ and close to P_0

$$P_r = e^{r\alpha} P_0 e^{-r\alpha}$$

where α is an operator with smooth kernel. Note that the operators with smooth kernel form the Lie algebra of the group

$$\mathrm{GL}_\infty := \{g \text{ invertible} \mid g = \mathrm{Id} + \text{ operator with smooth kernel}\}.$$

We introduce a smooth non-negative function χ equal to 0 close to $u = 1$ and 1 near $u = 0$ and define

$$U_r := \begin{cases} \mathrm{Id} & \text{on } \mathcal{V} \setminus N \\ \begin{pmatrix} e^{r\chi(u)\alpha} & 0 \\ 0 & V e^{r\chi(u)\alpha} V^{-1} \end{pmatrix} & \text{on } N = [0,1] \times S^3 \end{cases}$$

Then the operator $\not{D}_{A,P_r^{\#}}$ is unitary equivalent to the operator

$$(U_r^{-1} \not{D}_A U_r)_{P_0} .$$

On the collar N we get:

$$\begin{aligned} U_r^{-1} \not{D}_A U_r &= \not{D}_A \\ &+ \begin{pmatrix} 0 & -V^{-1} \\ V & 0 \end{pmatrix} \begin{pmatrix} e^{r\chi(u)\alpha}[B, e^{r\chi(u)\alpha}] & 0 \\ 0 & -V e^{-r\chi(u)\alpha}[B, e^{r\chi(u)\alpha}] V^{-1} \end{pmatrix} \\ &+ \begin{pmatrix} 0 & -V^{-1} \\ V & 0 \end{pmatrix} \begin{pmatrix} r\chi'(u)\alpha & 0 \\ 0 & rV\chi'(u)\alpha V^{-1} \end{pmatrix} . \end{aligned}$$

Next we compute the first derivate of the twisted family and get on the collar N:

$$\begin{aligned} \frac{d}{dr} \left(U_r^{-1} \not{D}_A U_r\right)|_{r=0} &= \begin{pmatrix} 0 & -V^{-1} \\ V & 0 \end{pmatrix} \begin{pmatrix} \chi(u)[B,\alpha] & 0 \\ 0 & -\chi(u) V [B,\alpha] V^{-1} \end{pmatrix} \\ &+ \begin{pmatrix} 0 & -V^{-1} \\ V & 0 \end{pmatrix} \begin{pmatrix} \chi'(u)\alpha & 0 \\ 0 & V\chi'(u)\alpha V^{-1} \end{pmatrix} . \end{aligned}$$

A further calculation shows that $\dot{\not{D}}_0 \not{D}_0$ can be written as the sum of four terms:

$$\begin{aligned} \frac{d}{dr} \left(U_r^{-1} \not{D}_A U_r\right)|_{r=0} \not{D}_A &= \begin{pmatrix} \chi(u)[B,\alpha] & 0 \\ 0 & -\chi(u) V [B,\alpha] V^{-1} \end{pmatrix} \partial_u \\ &+ \begin{pmatrix} \chi(u)[B,\alpha]B & 0 \\ 0 & \chi(u) V [B,\alpha] B V^{-1} \end{pmatrix} \\ &- \begin{pmatrix} \chi'(u)\alpha & 0 \\ 0 & -\chi'(u) V \alpha V^{-1} \end{pmatrix} \partial_u \\ &- \begin{pmatrix} \chi'(u)\alpha B & 0 \\ 0 & \chi'(u) V \alpha B V^{-1} \end{pmatrix} . \end{aligned}$$

We obtain

$$\lim_{\varepsilon \to 0} \operatorname{Tr} \dot{\mathcal{D}}_0 \mathcal{D}_0 e^{-\varepsilon \mathcal{D}_0^2} = 0$$

from Duhamel's principle which shows the disappearance of the corresponding trace for each of the four summands appearing in the preceding formula for $\dot{\mathcal{D}}_0 \mathcal{D}_0$ due to the antisymmetry of the matrices.

The details of the proof will be published elsewhere separately.

References

1. Atiyah, M.F., Patodi, V.K., and Singer, I.M.: 1975, 'Spectral asymmetry and Riemannian geometry. I', *Math. Proc. Cambridge Phil. Soc.* 77, 43-69.
2. Booß–Bavnbek, B., Morchio, G., Strocchi, F., and Wojciechowski, K.P.: 1996, *Grassmannian and Chiral Anomaly*, preprint.
3. Douglas, R.G., and Wojciechowski, K.P.: 1991, 'Adiabatic limits of the η-invariants. The odd-dimensional Atiyah-Patodi-Singer problem', *Comm. Math. Phys.* **142**, 139-168.
4. Gilkey, P.B.: 1995, *Invariance Theory, the Heat Equation, and the Atiyah–Singer Index Theory* (Second Edition), CRC Press, Boca Raton, Florida.
5. Klimek, S., and Wojciechowski, K.P.: 1996, 'Adiabatic cobordism theorems for analytic torsion and η–invariant', *J. Funct. Anal.* **136**, 269-293.
6. Lesch, M., and Wojciechowski, K.P.: 1996, 'On the η–invariant of generalized Atiyah–Patodi–Singer problems', *Illinois J. Math.* 40, 30–46.
7. Morchio, G., and Strocchi, F.: 1990, 'Boundary terms, long range effects, and chiral symmetry breaking', in: Mitter, H., and Schweifer, W. (eds), *Fields and Particles*, Proceedings Schladming, Austria, 1990, Springer–Verlag, Berlin–Heidelberg–New York, pp. 171-214.
8. Ray, D., and Singer, I.M.: 1971, 'R–torsion and the Laplacian on Riemannian manifolds', *Adv. Math.* 7, 145–210.
9. Schwarz, A.S.: 1993, *Quantum Field Theory and Topology*, Springer, Berlin–Heidelberg–New York. *(Russian original: Kvantovaya teoriya polya i topologiya, Nauka, Moscow 1989).*
10. Scott, S.G.: 1995, 'Determinants of Dirac boundary value problems over odd–dimensional manifolds', *Comm. Math. Phys.* **173**, 43–76.
11. Scott, S.G., and Wojciechowski, K.P.: 1996, *Abstract Determinant and ζ–Determinant on the Grassmannian*, preprint.
12. Singer, I.M.: 1985, 'Families of Dirac operators with applications to physics', *Asterisque, hors série*, 323-340.
13. Wojciechowski, K.P.: 1996, *Smooth, Self-Adjoint Grassmannian, the η-Invariant, and the ζ-Determinant of Boundary Problems*, preprint.

Liste over tidligere udkomne tekster tilsendes gerne. Henvendelse herom kan ske til IMFUFA's sekretariat tlf. 46 75 77 11 lokal 2263

217/92 "Two papers on APPLICATIONS AND MODELLING IN THE MATHEMATICS CURRICULUM"
by: Mogens Niss

218/92 "A Three-Square Theorem"
by: Lars Kadison

219/92 "RUPNOK - stationær strømning i elastiske rør"
af: Anja Boisen, Karen Birkelund, Mette Olufsen
Vejleder: Jesper Larsen

220/92 "Automatisk diagnosticering i digitale kredsløb"
af: Bjørn Christensen, Ole Møller Nielsen
Vejleder: Stig Andur Pedersen

221/92 "A BUNDLE VALUED RADON TRANSFORM, WITH APPLICATIONS TO INVARIANT WAVE EQUATIONS"
by: Thomas P. Branson, Gestur Olafsson and Henrik Schlichtkrull

222/92 On the Representations of some Infinite Dimensional Groups and Algebras Related to Quantum Physics
by: Johnny T. Ottesen

223/92 THE FUNCTIONAL DETERMINANT
by: Thomas P. Branson

224/92 UNIVERSAL AC CONDUCTIVITY OF NON-METALLIC SOLIDS AT LOW TEMPERATURES
by: Jeppe C. Dyre

225/92 "HATMODELLEN" Impedansspektroskopi i ultrarent en-krystallinsk silicium
af: Anja Boisen, Anders Gorm Larsen, Jesper Varmer, Johannes K. Nielsen, Kit R. Hansen, Peter Bøggild og Thomas Hougaard
Vejleder: Petr Viscor

226/92 "METHODS AND MODELS FOR ESTIMATING THE GLOBAL CIRCULATION OF SELECTED EMISSIONS FROM ENERGY CONVERSION"
by: Bent Sørensen

227/92 "Computersimulering og fysik"
af: Per M.Hansen, Steffen Holm, Peter Maibom, Mads K. Dall Petersen, Pernille Postgaard, Thomas B.Schrøder, Ivar P. Zeck
Vejleder: Peder Voetmann Christiansen

228/92 "Teknologi og historie"
Fire artikler af:
Mogens Niss, Jens Høyrup, Ib Thiersen, Hans Hedal

229/92 "Masser af information uden betydning"
En diskussion af informationsteorien i Tor Nørretranders' "Mærk Verden" og en skitse til et alternativ baseret på andenordens kybernetik og semiotik.
af: Søren Brier

230/92 "Vinklens tredeling - et klassisk problem"
et matematisk projekt af
Karen Birkelund, Bjørn Christensen
Vejleder: Johnny Ottesen

231A/92 "Elektrondiffusion i silicium - en matematisk model"
af: Jesper Voetmann, Karen Birkelund, Mette Olufsen, Ole Møller Nielsen
Vejledere: Johnny Ottesen, H.B.Hansen

231B/92 "Elektrondiffusion i silicium - en matematisk model" Kildetekster
af: Jesper Voetmann, Karen Birkelund, Mette Olufsen, Ole Møller Nielsen
Vejledere: Johnny Ottesen, H.B.Hansen

232/92 "Undersøgelse om den simultane opdagelse af energiens bevarelse og isærdeles om de af Mayer, Colding, Joule og Helmholtz udførte arbejder"
af: L.Arleth, G.I.Dybkjær, M.T.Østergård
Vejleder: Dorthe Posselt

233/92 "The effect of age-dependent host mortality on the dynamics of an endemic disease and
Instability in an SIR-model with age-dependent susceptibility
by: Viggo Andreasen

234/92 "THE FUNCTIONAL DETERMINANT OF A FOUR-DIMENSIONAL BOUNDARY VALUE PROBLEM"
by: Thomas P. Branson and Peter B. Gilkey

235/92 OVERFLADESTRUKTUR OG POREUDVIKLING AF KOKS
- Modul 3 fysik projekt -
af: Thomas Jessen

236a/93 INTRODUKTION TIL KVANTE HALL EFFEKTEN

af: Anja Boisen, Peter Bøggild

Vejleder: Peder Voetmann Christiansen
Erland Brun Hansen

236b/93 STRØMSSAMMENBRUD AF KVANTE HALL EFFEKTEN

af: Anja Boisen, Peter Bøggild

Vejleder: Peder Voetmann Christiansen
Erland Brun Hansen

237/93 The Wedderburn principal theorem and Shukla cohomology

af: Lars Kadison

238/93 SEMIOTIK OG SYSTEMEGENSKABER (2)

Vektorbånd og tensorer

af: Peder Voetmann Christiansen

239/93 Valgsystemer - Modelbygning og analyse

Matematik 2. modul

af: Charlotte Gjerrild, Jane Hansen, Maria Hermannsson, Allan Jørgensen, Ragna Clauson-Kaas, Poul Lützen

Vejleder: Mogens Niss

240/93 Patologiske eksempler. Om sære matematiske fisks betydning for den matematiske udvikling

af: Claus Dræby, Jørn Skov Hansen, Runa Ulsøe Johansen, Peter Meibom, Johannes Kristoffer Nielsen

Vejleder: Mogens Niss

241/93 FOTOVOLTAISK STATUSNOTAT 1

af: Bent Sørensen

242/93 Brovedligeholdelse - bevar mig vel

Analyse af Vejdirektoratets model for optimering af broreparationer

af: Linda Kyndlev, Kare Fundal, Kamma Tulinius, Ivar Zeck

Vejleder: Jesper Larsen

243/93 TANKEEKSPERIMENTER I FYSIKKEN

Et 1.modul fysikprojekt

af: Karen Birkelund, Stine Sofia Korremann

Vejleder: Dorthe Posselt

244/93 RADONTRANSFORMATIONEN og dens anvendelse i CT-scanning

Projektrapport

af: Trine Andreasen, Tine Guldager Christiansen, Nina Skov Hansen og Christine Iversen

Vejledere: Gestur Olafsson og Jesper Larsen

245a+b /93 Time-Of-Flight målinger på krystallinske halvledere
Specialerapport

af: Linda Szkotak Jensen og Lise Odgaard Gade

Vejledere: Petr Viscor og Niels Boye Olsen

246/93 HVERDAGSVIDEN OG MATEMATIK - LÆREPROCESSER I SKOLEN

af: Lena Lindenskov, Statens Humanistiske Forskningsråd, RUC, IMFUFA

247/93 UNIVERSAL LOW TEMPERATURE AC CONDUCTIVITY OF MACROSCOPICALLY DISORDERED NON-METALS

by: Jeppe C. Dyre

248/93 DIRAC OPERATORS AND MANIFOLDS WITH BOUNDARY

by: B. Booss-Bavnbek, K.P.Wojciechowski

249/93 Perspectives on Teichmüller and the Jahresbericht Addendum to Schappacher, Scholz, et al.

by: B. Booss-Bavnbek

With comments by W.Abikoff, L.Ahlfors, J.Cerf, P.J.Davis, W.Fuchs, F.P.Gardiner, J.Jost, J.-P.Kahane, R.Lohan, L.Lorch, J.Radkau and T.Söderqvist

250/93 *EULER OG BOLZANO - MATEMATISK ANALYSE SET I ET VIDENSKABSTEORETISK PERSPEKTIV*

Projektrapport af: Anja Juul, Lone Michelsen, Tomas Højgård Jensen

Vejleder: Stig Andur Pedersen

251|93 ***Genotypic Proportions in Hybrid Zones***

by: Freddy Bugge Christiansen, Viggo Andreasen and Ebbe Thue Poulsen

252|93 ***MODELLERING AF TILFÆLDIGE FÆNOMENER***

Projektrapport af: Birthe Friis, Lisbeth Helmgaard Kristina Charlotte Jakobsen, Marina Mosbæk Johannessen, Lotte Ludvigsen, Mette Hass Nielsen

253/93 ***Kuglepakning***

Teori og model

af: Lise Arleth, Kåre Fundal, Nils Kruse

Vejleder: Mogens Niss

254/93 ***Regressionsanalyse***

Materiale til et statistikkursus

af: Jørgen Larsen

255/93 ***TID & BETINGET UAFHÆNGIGHED***

af: Peter Harremoës

256/93 ***Determination of the Frequency Dependent Bulk Modulus of Liquids Using a Piezoelectric Spherical Shell (Preprint)***

by: T. Christensen and N.B.Olsen

257/93 ***Modellering af dispersion i piezoelektriske keramikker***

af: Pernille Postgaard, Jannik Rasmussen, Christina Specht, Mikko Østergård

Vejleder: Tage Christensen

258/93 ***Supplerende kursusmateriale til***

"Lineære strukturer fra algebra og analyse"

af: Mogens Brun Heefelt

259/93 ***STUDIES OF AC HOPPING CONDUCTION AT LOW TEMPERATURES***

by: Jeppe C. Dyre

260/93 ***PARTITIONED MANIFOLDS AND INVARIANTS IN DIMENSIONS 2, 3, AND 4***

by: B. Booss-Bavnbek, K.P.Wojciechowski

261/93 OPGAVESAMLING
Bredde-kursus i Fysik
Eksamensopgaver fra 1976-93

262/93 Separability and the Jones Polynomial
by: Lars Kadison

263/93 Supplerende kursusmateriale til "Lineære strukturer fra algebra og analyse" II
af: Mogens Brun Heefelt

264/93 FOTOVOLTAISK STATUSNOTAT 2
af: Bent Sørensen

265/94 SPHERICAL FUNCTIONS ON ORDERED SYMMETRIC SPACES
To Sigurdur Helgason on his sixtyfifth birthday
by: Jacques Faraut, Joachim Hilgert and Gestur Olafsson

266/94 Kommensurabilitets-oscillationer i laterale supergitre
Fysikspeciale af: Anja Boisen, Peter Bøggild, Karen Birkelund
Vejledere: Rafael Taboryski, Poul Erik Lindelof, Peder Voetmann Christiansen

267/94 Kom til kort med matematik på Eksperimentarium - Et forslag til en opstilling
af: Charlotte Gjerrild, Jane Hansen
Vejleder: Bernhelm Booss-Bavnbek

268/94 Life is like a sewer ...
Et projekt om modellering af aorta via en model for strømning i kloakrør
af: Anders Marcussen, Anne C. Nilsson, Lone Michelsen, Per M. Hansen
Vejleder: Jesper Larsen

269/94 Dimensionsanalyse en introduktion metaprojekt, fysik
af: Tine Guldager Christiansen, Ken Andersen, Nikolaj Hermann, Jannik Rasmussen
Vejleder: Jens Højgaard Jensen

270/94 THE IMAGE OF THE ENVELOPING ALGEBRA AND IRREDUCIBILITY OF INDUCED REPRESENTATIONS OF EXPONENTIAL LIE GROUPS
by: Jacob Jacobsen

271/94 Matematikken i Fysikken.
Opdaget eller opfundet
NAT-BAS-projekt
vejleder: Jens Højgaard Jensen

272/94 Tradition og fornyelse
Det praktiske elevarbejde i gymnasiets fysikundervisning, 1907-1988
af: Kristian Hoppe og Jeppe Guldager
Vejledning: Karin Beyer og Nils Hybel

273/94 Model for kort- og mellemdistanceløb
Verifikation af model
af: Lise Fabricius Christensen, Helle Pilemann, Bettina Sørensen
Vejleder: Mette Olufsen

274/94 MODEL 10 - en matematisk model af intravenøse anæstetikas farmakokinetik
3. modul matematik, forår 1994
af: Trine Andreasen, Bjørn Christensen, Christine Green, Anja Skjoldborg Hansen. Lisbeth Helmgaard
Vejledere: Viggo Andreasen & Jesper Larsen

275/94 Perspectives on Teichmüller and the Jahresbericht
2nd Edition
by: Bernhelm Booss-Bavnbek

276/94 Dispersionsmodellering
Projektrapport 1. modul
af: Gitte Andersen, Rehannah Borup, Lisbeth Friis, Per Gregersen, Kristina Vejrø
Vejleder: Bernhelm Booss-Bavnbek

277/94 PROJEKTARBEJDSPÆDAGOGIK - Om tre tolkninger af problemorienteret projektarbejde
af: Claus Flensted Behrens, Frederik Voetmann Christiansen, Jørn Skov Hansen, Thomas Thingstrup
Vejleder: Jens Højgaard Jensen

278/94 The Models Underlying the Anaesthesia Simulator Sophus
by: Mette Olufsen(Math-Tech), Finn Nielsen (RISØ National Laboratory), Per Føge Jensen (Herlev University Hospital), Stig Andur Pedersen (Roskilde University)

279/94 Description of a method of measuring the shear modulus of supercooled liquids and a comparison of their thermal and mechanical response functions.
af: Tage Christensen

280/94 A Course in Projective Geometry
by Lars Kadison and Matthias T. Kromann

281/94 Modellering af Det Cardiovaskulære System med Neural Pulskontrol
Projektrapport udarbejdet af:
Stefan Frello, Runa Ulsøe Johansen, Michael Poul Curt Hansen, Klaus Dahl Jensen
Vejleder: Viggo Andreasen

282/94 Parallelle algoritmer
af: Erwin Dan Nielsen, Jan Danielsen, Niels Bo Johansen

283/94 Grænser for tilfældighed
(en kaotisk talgenerator)

af: Erwin Dan Nielsen og Niels Bo Johansen

284/94 Det er ikke til at se det, hvis man ikke lige ve' det!
Gymnasiematematikkens begrundelsesproblem

En specialerapport af Peter Hauge Jensen og Linda Kyndlev

Vejleder: Mogens Niss

285/94 Slow coevolution of a viral pathogen and its diploid host

by: Viggo Andreasen and Freddy B. Christiansen

286/94 The energy master equation: A low-temperature approximation to Bässler's random walk model

by: Jeppe C. Dyre

287/94 A Statistical Mechanical Approximation for the Calculation of Time Auto-Correlation Functions

by: Jeppe C. Dyre

288/95 PROGRESS IN WIND ENERGY UTILIZATION

by: Bent Sørensen

289/95 Universal Time-Dependence of the Mean-Square Displacement in Extremely Rugged Energy Landscapes with Equal Minima

by: Jeppe C. Dyre and Jacob Jacobsen

290/95 Modellering af uregelmæssige bølger
Et 3.modul matematik projekt

af: Anders Marcussen, Anne Charlotte Nilsson, Lone Michelsen, Per Mørkegaard Hansen

Vejleder: Jesper Larsen

291/95 1st Annual Report from the project
LIFE-CYCLE ANALYSIS OF THE TOTAL DANISH ENERGY SYSTEM

an example of using methods developed for the OECD/IEA and the US/EU fuel cycle externality study

by: Bent Sørensen

292/95 Fotovoltaisk Statusnotat 3

af: Bent Sørensen

293/95 Geometridiskussionen - hvor blev den af?

af: Lotte Ludvigsen & Jens Frandsen

Vejleder: Anders Madsen

294/95 Universets udvidelse - et metaprojekt

Af: Jesper Duelund og Birthe Friis

Vejleder: Ib Lundgaard Rasmussen

295/95 A Review of Mathematical Modeling of the Controled Cardiovascular System

By: Johnny T. Ottesen

296/95 RETIKULER den klassiske mekanik

af: Peder Voetmann Christiansen

297/95 A fluid-dynamical model of the aorta with bifurcations

by: Mette Olufsen and Johnny Ottesen

298/95 Mordet på Schrödingers kat - et metaprojekt om to fortolkninger af kvantemekanikken

af: Maria Hermannsson, Sebastian Horst, Christina Specht

Vejledere: Jeppe Dyre og Peder Voetmann Christiansen

299/95 ADAM under figenbladet - et kig på en samfunds-videnskabelig matematisk model

Et matematisk modelprojekt

af: Claus Dræby, Michael Hansen, Tomas Højgård Jensen

Vejleder: Jørgen Larsen

300/95 Scenarios for Greenhouse Warming Mitigation

by: Bent Sørensen

301/95 TOK Modellering af træers vækst under påvirkning af ozon

af: Glenn Møller-Holst, Marina Johannessen, Birthe Nielsen og Bettina Sørensen

Vejleder: Jesper Larsen

302/95 KOMPRESSORER - Analyse af en matematisk model for aksialkompressorer

Projektrapport sf: Stine Bøggild, Jakob Hilmer, Pernille Postgaard

Vejleder: Viggo Andreasen

303/95 Masterlignings-modeller af Glasovergangen
Termisk-Mekanisk Relaksation
Specialerapport udarbejdet af:

Johannes K. Nielsen, Klaus Dahl Jensen

Vejledere: Jeppe C. Dyre, Jørgen Larsen

304a/95 STATISTIKNOTER Simple binomialfordelingsmodeller

af: Jørgen Larsen

304b/95 STATISTIKNOTER Simple normalfordelingsmodeller

af: Jørgen Larsen

304c/95 STATISTIKNOTER Simple Poissonfordelingsmodeller

af: Jørgen Larsen

304d/95 STATISTIKNOTER Simple multinomialfordelingsmodeller

af: Jørgen Larsen

304e/95 STATISTIKNOTER Mindre matematisk-statistisk opslagsvær
indeholdende bl.a. ordforklaringer, resuméer og tabeller

af: Jørgen Larsen

305/95 The Maslov Index: A Functional Analytical Definition And The Spectral Flow Formula

By: B. Booss-Bavnbek, K. Furutani

306/95 Goals of mathematics teaching

Preprint of a chapter for the forth-comming International Handbook of Mathematics Education (Alan J.Bishop, ed)

By: Mogens Niss

307/95 Habit Formation and the Thirdness of Signs

Presented at the semiotic symposium

The Emergence of Codes and Intensions as a Basis of Sign Processes

By: Peder Voetmann Christiansen

308/95 Metaforer i Fysikken

af: Marianne Wilcken Bjerregaard, Frederik Voetmann Christiansen, Jørn Skov Hansen, Klaus Dahl Jensen Ole Schmidt

Vejledere: Peder Voetmann Christiansen og Petr Viscor

309/95 Tiden og Tanken
En undersøgelse af begrebsverdenen Matematik udført ved hjælp af en analogi med tid

af: Anita Stark og Randi Petersen

Vejleder: Bernhelm Booss-Bavnbek

310/96 Kursusmateriale til "Lineære strukturer fra algebra og analyse" (E1)

af: Mogens Brun Heefelt

311/96 2nd Annual Report from the project LIFE-CYCLE ANALYSIS OF THE TOTAL DANISH ENERGY SYSTEM

by: Hélène Connor-Lajambe, Bernd Kuemmel, Stefan Krüger Nielsen, Bent Sørensen

312/96 Grassmannian and Chiral Anomaly

by: B. Booss-Bavnbek, K.P.Wojciechowski

313/96 THE IRREDUCIBILITY OF CHANCE AND THE OPENNESS OF THE FUTURE
The Logical Function of Idealism in Peirce's Philosophy of Nature

By: Helmut Pape, University of Hannover

314/96 Feedback Regulation of Mammalian Cardiovascular System

By: Johnny T. Ottesen

315/96 a + b "Rejsen til tidens indre" - Udarbejdelse af et manuskript til en fjernsynsudsendelse + manuskript

af: Gunhild Hune og Karina Goyle

Vejledere: Peder Voetmann Christiansen og Bruno Ingemann

316/96 Plasmaoscillation i natriumklynger

Specialerapport af: Peter Meibom, Mikko Østergård

Vejledere: Jeppe Dyre & Jørn Borggreen

317/96 Poincaré og symplektiske algoritmer

af: Ulla Rasmussen

Vejleder: Anders Madsen

318/96 Modelling the Respiratory System

by: Tine Guldager Christiansen, Claus Dræby

Supervisors: Viggo Andreasen, Michael Danielsen

319/96 Externality Estimation of Greenhouse Warming Impacts

by: Bent Sørensen