Continuous-time frog model can spread arbitrary fast

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Abstract

The aim of the paper is to demonstrate that the continuous-time frog model can spread arbitrary fast. The set of sites visited by an active particle can become infinite in a finite time.

Mathematics subject classification: 60K35, 82C22

1 Introduction

At time $t = 0$ there are $\eta(x)$ particles at $x \in \mathbb{Z}^{\mathrm{d}}$, where $\{\eta(x)\}_{x\in\mathbb{Z}^{\mathrm{d}}}$ are independent and identically distributed according to a distribution μ on $\mathbb{N} \cup \{0\}$. The particles at the origin are active while all other particles are dormant. Active particles perform a simple continuous-time random walk independently of all other particles. Dormant particles stay put until the first arrival of an active particle to their site; upon arrival they become active and start their own simple random walks. The model was originally defined in discrete time $n = 0, 1, 2, \dots$ with particles performing a discrete-time simple random walk. In this paper we consider the continuous-time version.

In discrete time the frog model cannot spread faster than linearly, and the set of locations visited by active particles by the time $n \in \mathbb{N}$ is always contained in $n\mathcal{D}$, where $\mathcal{D} = \{(x_1, \dots, x_{\mathrm{d}}) : |x_1| + \dots + |x_{\mathrm{d}}| \le 1\}$. In [AMP02] and [AMPR01] the shape theorem for the discrete-time frog model was established, and in [AMPR01] it was also shown that if the tails of μ are sufficiently heavy, the limiting shape coincides with $\mathcal{D}$. For $\mu = \delta_1$ (delta measure concentrated at 1) the shape theorem for the continuous-time frog model was obtained in [RS04]. The frog model has been studied mostly in the discrete-time framework. Recent papers [DHL19] and [BFHM20] investigate respectively the coexistence in a two-type frog model and susceptibility properties on certain finite graphs, as well as provide an overview of other research on this model. The transitivity and recurrence properties of the frog model attract considerable attention [DGH+18, DP14, JJ16, HJJ17, HJJ16, GNR17].

In relation to the coexistence in two type continuous-time frog model the following question was raised in [DHL19].

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Question. Could the growth be superlinear in time in the continuous time frog model if $\eta(x)$ has a very heavy tail?

In this paper we give a positive answer to this question. Moreover, we show that in fact for distributions μ with very heavy tails the set of sites visited by active particles becomes infinite in a finite time. A precise formulation is given in Theorem 1.1.

Theorem 1.1. *There exists a distribution μ such that the time*

$$\tau := \inf\{t : \text{there are infinitely many active particles at } t\} \tag{1}$$

is a.s. finite.

We prove Theorem 1.1 in Section 2. In fact, in Section 2 we work with a more general model with time between the jumps of random walks following an arbitrary distribution rather than the unit exponential.

The speed of growth of stochastic particle systems has been an active field of research for about least half a century as the first studies go back at least to the seventies, see e.g. [Big76]. The superlinear speed for a branching random walk with polynomial tails was demonstrated in [Dur83]. The exact speed for the a branching random walk satisfying an exponential moment condition is given in [Big95, Big97]; further results and references can be found in [Big10]. More recently the spread rate and the maximal displacement of modified versions of the model came under investigation. A dispersion kernel with tails heavier than exponential but lighter than polynomial is treated in [Gan00]; the spread of the branching random walk with certain restrictions is the subject of [BM14, BDPKT]; in [FZ12, Mal15] the process evolves in a random environment.

In continuous-space settings we mention a model of growing sets introduced in [Dei03] whose speed of growth is further studied in [GM08], and the spatial birth process [BDPK+17]. The linear growth for a discrete-space two-type particle model is established in [KS05], see also [KS08]. The model in [KS05] is similar to the frog model, however, unlike in the frog model, particles of both types can move. We also mention here the first passage percolation as a stochastic growth model. There is an enormous amount of literature on the subject, and we refer to a recent survey [ADH17] for results and references.

2 The main result, proof, and further discussion

We prove our main result for a generalization of the frog model in which the particles perform not a simple continuous-time random walk, but a random walk with the exponential distribution of the waiting times between jumps replaced by an arbitrary distribution π on $(0,\infty)$. Let $\{(S_t^{(x,j)}, t \geq 0), x \in \mathbb{Z}^{\mathrm{d}}, j \in \mathbb{N}\}$ be the set of all random walks assigned to particles. For fixed t, x, and j, $S_t^{(x,j)}$ represents the position of j-th particle started at location x, t units of time after the particle was activated. For each realization of η, only the walks $(S_t^{(x,j)}, t \geq 0)$ with indices satisfying $j \leq \eta(x)$ are used. For fixed x, j, the jump times $\mathrm{j}_1, \mathrm{j}_2, \dots$ of $(S_t^{(x,j)}, t \geq 0)$ are such that $\mathrm{j}_{\mathrm{k}+1} - \mathrm{j}_{\mathrm{k}}$ are independent random variables distributed according to π, $k = 0, 1, \dots$ $(\mathrm{j}_0 = 0)$. In case of the standard continuous-time frog model, π is the unit exponential distribution.

In order not to exclude distributions with an atom at 0, we assume that there is at least one active particle at the beginning at the origin $\mathbf{0}_{\mathrm{d}}$. That is, for realizations of η with $\eta(\mathbf{0}_{\mathrm{d}}) = 0$ an active particle is added at the origin.

Let us introduce the model which can serve as a motivation for treating a more general model rather than only the standard frog model. Let $\mathrm{d}=1$. Imagine that we again have $\eta(x)$ particles at $x \in \mathbb{Z}$ at the beginning, but instead of the random walk the particles now move in the continuous space $\mathbb{R}$ according to independent standard Brownian motions. Other rules do not change - once some active particles reaches $y \in \mathbb{Z}$ for the first time, all $\eta(y)$ sleeping particles located at y activate and start their own Brownian motions. This model can be expressed in the discrete-space framework with π being the distribution of the time when the absolute value of a Brownian motion started at 0 hits 1.

We start from the following observation. If the activation of some of the sleeping particles upon coming into contact with an active particle is delayed or even suppressed entirely, the resulting process is going to spread slower than the frog model. This also applies to putting to sleep some active particle and removing (both sleeping and active) particles, because the spread can only be slowed down as a result. The slower spread here means that the set of sites visited by an active particle by time t for the slowed model is going to be a subset of the respective set for the original model.

Let $\mathcal{A}_t$ be the set of sites visited by an active particle by the time t. If for some $r>0$, $\pi((0,r])=0$, then for any distribution μ a.s. $\mathcal{A}_t \subset [-n,n]^{\mathrm{d}}$, where $n=\lceil \frac{t}{r} \rceil$, and hence $(\mathcal{A}_t, t \geq 0)$ grows at most linearly with time. Lemma 2.1 and Theorem 2.2 show that the reverse is also true.

Lemma 2.1. *Let the dimension* $\mathrm{d}=1$. *Assume that* $\pi((0,r])>0$ *for all* $r>0$. *Let* $\{A_n\}_{n\in\mathbb{N}}$ *and* $\{t_n\}_{n\in\mathbb{N}}$ *be increasing sequences of positive numbers,* $A_n \to \infty$, $t_n \to t_\infty \in (0,\infty]$. *Then there exists a distribution* μ *such that* $\mathbb{P}\{\sup \mathcal{A}_{t_n} \geq A_n \text{ for all } n \in \mathbb{N}\}>0$, *and, if* $t_\infty < \infty$, *the time* τ *defined in Theorem 1.1 is a.s. finite.*

Proof. Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence of positive numbers such that for all $n \in \mathbb{N}$,

$$\sum_{i=1}^{n} a_i \geq A_n,$$

and $a_n \geq n^2$. An example of such a sequence is given by $a_n = A_n \vee n^2$. Let $t_0=0$ and let $\Delta_n = t_{n+1}-t_n$ for $n \in \mathbb{N}$. Define

$$g(r,m) := 2^{-m-1}\left(\pi\left(\left(0,\frac{r}{m}\right]\right)\right)^m, \quad r,m>0,$$

and set $b_n = (g(\Delta_n, 2a_n))^{-1}\cdot n$ and let μ satisfy $\mu([b_{n+1},\infty)) \geq \frac{n}{a_n}$.

Define a random sequence of sites $\{X_n^{(1)}\}_{n\in\mathbb{N}}$ consecutively as follows: set $X_0^{(1)}=0$, and for $n \in \mathbb{N}\cup\{0\}$ set $X_{n+1}^{(1)}=\infty$ if $X_n^{(1)}=\infty$, otherwise set

$$\begin{aligned} X_{n+1}^{(1)} = \min\Big\{ & k \in \mathbb{N} : a_{n+1} \leq k - X_n^{(1)} \leq 2a_{n+1}, \eta(k) \geq b_{n+1}, \\ & \max\{S_{\Delta_n}^{(X_n^{(1)},j)} : j=1,\dots,\eta(X_n^{(1)})\} \geq 2a_{n+1}\Big\}. \end{aligned} \tag{2}$$

Here and elsewhere we adopt the convention $\min\varnothing=\infty$. Let $\kappa_1 := \min\{k\in\mathbb{N} : X_k^{(1)}=\infty\}$, and define

$$\sigma_1 = \begin{cases} \min\{t \geq t_{\kappa_1} : \max\limits_{j=1,\dots,\eta(X_{\kappa_1-1}^{(1)})} S_{t-t_{\kappa_1-1}}^{(X_{\kappa_1-1}^{(1)},j)} \geq 2a_{\kappa_1}+1\}, & \text{on } \{\kappa_1<\infty, \kappa_1\neq 1\}, \\ \infty, & \text{on } \{\kappa_1=\infty\}, \\ \min\{t \geq t_1 : S_t^{(0,1)} \geq 2a_1+1\}, & \text{on } \{\kappa_1=1\}. \end{cases} \tag{3}$$

Note that a.s. $\{\sigma_1<\infty\}=\{\kappa_1<\infty\}$.

Having in mind the observation above, we slow down the spread in multiple ways as described throughout the proof. The first slowing rule is that at time $\sigma_0 = 0$ we remove every sleeping particle left of the origin and leave a single active particle at the origin. Further, from time σ_0 until σ_1 if a site with sleeping particles is visited by an active particle at time $\theta \in (t_{n-1}, t_n]$, then the sleeping particles at the site become active and start moving only after a delay at time t_n. Also, before time σ_1 we impose another slowing rule by restricting the activation of sleeping particles to the sites $X_1^{(1)}, X_2^{(1)}, \dots$. Denote by R_t the position of the rightmost active particle at time t.

On $\{\kappa_1 < \infty\}$ at time σ_1 we put to sleep every active particle keeping only one located at R_{σ_1}, and restart the process in the same fashion. (We note here that given $\{\sigma_1 < \infty\}$, the random variables $\eta(R_{\sigma_1}+1), \eta(R_{\sigma_1}+2), \dots$ are independent and distributed according to μ. Thus, the usage of the word 'restart' is justified as the restarted process is going to have the same distribution.)

Define the sequence $\{X_n^{(2)}\}_{n\in\mathbb{N}}$ by setting $X_0^{(2)} = R_{\sigma_1}$ on the event $\{\sigma_1 < \infty\}$ and $X_0^{(2)} = \infty$ on the complement $\{\sigma_1 < \infty\}^c = \{\sigma_1 = \infty\}$, and for $n \in \mathbb{N} \cup \{0\}$ by setting $X_{n+1}^{(2)} = \infty$ if $X_n^{(2)} = \infty$, and otherwise

$$
\begin{aligned}
X_{n+1}^{(2)} = \min\big\{&k \in \mathbb{N} : a_{n+1} \le k - X_n^{(2)} \le 2a_{n+1}, \eta(k) \ge b_{n+1}, \\
&\max\{S_{\Delta_n}^{(X_n^{(2)},j)} : j = 1, \dots, \eta(X_n^{(2)})\} \ge 2a_{n+1}\big\}. \qquad (4)
\end{aligned}
$$

We then define $\kappa_2 := \min\{k \in \mathbb{N}\cup\{0\} : X_k^{(1)} = \infty\}$ and set

$$
\sigma_2 = \begin{cases} \min\{t \ge t_{\kappa_2} + \sigma_1 : \max\limits_{j=1,\dots,\eta(X^{(2)}_{\kappa_2-1})} S^{(X^{(2)}_{\kappa_2-1},j)}_{t-\sigma_1-t_{\kappa_2-1}} \ge 2a_{\kappa_2}+1\} & \text{on } \{1<\kappa_2<\infty\}, \\ \infty, & \text{on } \{\kappa_2 = \infty\}, \\ 1 \text{ (this value is arbitrary and does not affect anything)}, & \text{on } \{\kappa_2 = 0\}, \\ \min\{t \ge t_1 + \sigma_1 : S^{(X_0^{(2)},1)}_{t-\sigma_1} \ge 2a_1 + 1\}, & \text{on } \{\kappa_2 = 1\}. \end{cases} \qquad (5)
$$

Next define the sequences $\{X_n^{(3)}\}_{n\in\mathbb{N}}$, $\{X_n^{(4)}\}_{n\in\mathbb{N}}$, $\dots$, and the times κ_3, σ_3, $\dots$, consecutively in the same fashion.

On $\{\sigma_1 < \infty\}$, the same restrictions are introduced on the time interval $(\sigma_1, \sigma_2]$ as on $(\sigma_0, \sigma_1]$. Specifically, at σ_1 every sleeping particle left to $X_0^{(2)} = R_{\sigma_1}$ is removed. From time σ_1 until σ_2, the activation of sleeping particles at a site first visited by an active particles during the time interval $(\sigma_1 + t_{n-1}, \sigma_1 + t_n]$ takes place with a delay at $\sigma_1 + t_n$. The activation of sleeping particles is only allowed on sites $X_1^{(2)}, X_2^{(2)}, \dots$. On $\{\sigma_1 < \infty\} \cap \{\sigma_2 < \infty\}$, same restrictions are made during $(\sigma_2, \sigma_3]$, and so on.

For $n, m \in \mathbb{N}$ denote $Q_n^{(m)} = \{X_n^{(m)} < \infty\}$, and let $Q_\infty^{(m)} = \bigcap\limits_{n\in\mathbb{N}} Q_n^{(m)} = \lim\limits_{n\to\infty} Q_n^{(m)}$ be the event $\{X_k^{(m)} < \infty, k \in \mathbb{N}\} = \{\kappa_m = \infty\}$ that all elements of the sequence $\{X_n^{(m)}\}_{n\in\mathbb{N}}$ are finite. By construction $\eta(X_n^{(1)}) \ge b_n$ a.s. on $Q_n^{(1)}$, hence by Lemma 2.6

$$
\begin{aligned}
\mathbb{P}\left[\max\{S_{\Delta_n}^{(X_n^{(1)},j)} : j = 1, \dots, \eta(X_n^{(1)})\} \ge 2a_{n+1} \middle| Q_n^{(1)}\right] &\ge 1 - [1 - \mathbb{P}\{S_{\Delta_n} \ge 2a_n\}]^{b_n} \\
&\ge 1 - [1 - g(\Delta_n, 2a_n)]^{b_n} \ge 1 - [1 - g(\Delta_n, 2a_n)]^{(g(\Delta_n,2a_n))^{-1}\cdot n} \ge 1 - e^{-n}. \qquad (6)
\end{aligned}
$$

In (6) we used the inequality $\left(1 - \frac{1}{y}\right)^y < e^{-1}$ for $y > 1$. At the same time we have

$$
\begin{aligned}
\mathbb{P}\left[\eta(X_n^{(1)} + a_{n+1}) \vee \eta(X_n^{(1)} + a_{n+2}) \vee \dots \vee \eta(X_n^{(1)} + 2a_{n+1}) \ge b_{n+1} \middle| Q_n^{(1)}\right] &\ge 1 - [1 - \mu([b_{n+1}, \infty))]^{a_n} \\
&\ge 1 - \left[1 - \frac{n}{a_n}\right]^{a_n} \ge 1 - e^{-n}. \qquad (7)
\end{aligned}
$$

Since

$$Q_{n+1}^{(1)} = Q_n^{(1)} \cap \{\max\{S_{\Delta_n}^{(X_n^{(1)},j)} : j = 1, \ldots, \eta(X_n^{(1)})\} \geq 2a_{n+1}\} \\ \cap \{\eta(X_n^{(1)} + a_{n+1}^{(1)}) \vee \eta(X_n^{(1)} + a_{n+2}^{(1)}) \vee \ldots \vee \eta(X_n^{(1)} + 2a_{n+1}) \geq b_{n+1}\}, \tag{8}$$

by (6) and (7)

$$\mathbb{P}\left[Q_{n+1}^{(1)} \middle| Q_n^{(1)}\right] \geq 1 - 2e^{-n}. \tag{9}$$

Hence

$$\mathbb{P}\left\{Q_\infty^{(1)}\right\} = \lim_{n\to\infty} \mathbb{P}\left\{Q_n^{(1)}\right\} = \mathbb{P}\left\{Q_1^{(1)}\right\} \prod_{n=1}^{\infty} \mathbb{P}\left[Q_{n+1}^{(1)} \middle| Q_n^{(1)}\right] \geq \mathbb{P}\left\{Q_1^{(1)}\right\} \prod_{n=1}^{\infty} (1 - 2e^{-n}) > 0. \tag{10}$$

A.s. on $Q_\infty^{(1)}$, $\sup \mathcal{A}_{t_n} \geq X_n \geq \sum_{i=1}^{n} a_i \geq A_n$, so the first statement of the lemma is proven.

Let $Q^\infty = \bigcup_{m=1}^{\infty} (Q_\infty^{(m)}) = \{\kappa_m = \infty \text{ for some } m \in \mathbb{N}\}$ be the event that for some $m \in \mathbb{N}$, all elements of the sequence $\{X_n^{(m)}\}_{n\in\mathbb{N}}$ are finite. Now we can use a standard restart argument to show that $\mathbb{P}\{(Q^\infty)^c\} = 0$, that is $\mathbb{P}\{Q^\infty\} = 1$. Because of the independence of the random walks, the distribution of $\{X_n^{(m+1)} - R_{\sigma_m}\}_{n\in\mathbb{N}}$ given $\bigcap_{i=1}^{m} (Q_\infty^{(i)})^c$ coincides with the (unconditional) distribution of $\{X_n^{(1)} - R_{\sigma_0}\}_{n\in\mathbb{N}} = \{X_n^{(1)}\}_{n\in\mathbb{N}}$. Hence by (10)

$$\begin{aligned} \mathbb{P}\{(Q^\infty)^c\} = \mathbb{P}\left\{\bigcap_{m=1}^{\infty} (Q_\infty^{(m)})^c\right\} &= \mathbb{P}\left\{(Q_\infty^{(1)})^c\right\} \prod_{m=1}^{\infty} \mathbb{P}\left[(Q_\infty^{(m+1)})^c \middle| \bigcap_{i=1}^{m} (Q_\infty^{(i)})^c\right] \\ &= \mathbb{P}\left\{(Q_\infty^{(1)})^c\right\} \prod_{m=1}^{\infty} \left[1 - \mathbb{P}\left\{Q_\infty^{(1)}\right\}\right] = 0. \end{aligned} \tag{11}$$

Thus $\mathbb{P}\{(Q^\infty)\} = 1$, consequently a.s. there exists $m \in \mathbb{N}$ such that the elements of the sequence $\{X_n^{(m)}\}_{n\in\mathbb{N}}$ are all finite and $\kappa_m = \infty$. Note that this implies that a.s. $\kappa_1, \ldots, \kappa_{m-1} < \infty$ if $m > 1$. In particular, a.s. on $\{m > 1\}$ we have $\sigma_{m-1} < \infty$. By construction the sites $X_1^{(m)}$, $X_2^{(m)}$, ..., are occupied at the time $\sigma_{m-1} + t_1$, $\sigma_{m-1} + t_2$, ..., respectively, and $X_{n+1}^{(m)} - X_n^{(m)} \geq a_{n+1}$, $n \in \mathbb{N} \cup \{0\}$. Thus an infinite number of sites have been visited by an active particle by the time $\sigma_{m-1} + t_\infty$, which is a.s. finite if $t_\infty < \infty$. □

Theorem 2.2. *Assume that $\pi((0, r]) > 0$ for all $r > 0$. Then there exists a distribution μ such that the time τ defined in Theorem 1.1 is a.s. finite.*

Proof. The one-dimensional projections of the particles of the d-dimensional model perform a random walk whose times between jumps are distributed according to $\pi^{(1)} = \sum_{n=1}^{\infty} \frac{1}{\mathrm{d}}(\frac{\mathrm{d}-1}{\mathrm{d}})^{n-1}\pi^{*n}$. Hence the projection of a continuous-time d-dimensional frog model on an axis dominates a continuous-time one-dimensional frog model having $\pi^{(1)}$ as the distribution between jumps of random walks and the same initial sleeping particles distribution μ.

Specifically, recall that $\mathcal{A}_t$ is the set of sites visited by an active particle by time t for the d-dimensional frog models with time intervals between jumps distributed according to π, and let $\mathcal{A}_t^{(1)}$ be the sets of sites visited by an active particle by time t for the one-dimensional frog models with intervals between jumps distributed according to $\pi^{(1)}$. Then $(\mathcal{A}_t, t \geq 0)$ and $(\mathcal{A}_t^{(1)}, t \geq 0)$ can be coupled in such a way that a.s. $\Pi_1 \mathcal{A}_t \supset \mathcal{A}_t^{(1)}$ for all $t \geq 0$, where Π_1 is the projection on the first coordinate. Since $\pi^{(1)}$ satisfies conditions of Lemma 2.1 if π satisfies conditions of Theorem 2.2, by Lemma 2.1 the set $\mathcal{A}_s^{(1)}$ is infinite for some $s \in (0, \infty)$. Hence so is $\mathcal{A}_s$. $\square$

Theorem 1.1 for the standard frog model is a particular case of Theorem 2.2.

Remark 2.3. If the dimension $\mathrm{d} = 1$, then $\tau < \infty$ a.s. implies by symmetry that the time when every site has been visited by an active particle, i.e. the time there are no sleeping particles left, is also a.s. finite. In the terminology of [BFHM20] the model is susceptible, despite the underlying graph being infinite. Extending this to higher dimensions and other graphs would require additional arguments.

Remark 2.4. It follows from the proof of Lemma 2.1 that for every $\varepsilon > 0$, μ can be chosen in such a way that

$$\mathbb{P}\{\tau > \varepsilon\} \le \varepsilon. \tag{12}$$

Remark 2.5. Taking π to be the unit exponential distribution, $a_n = n^2$, $\Delta_n = \frac{1}{n^2}$, and $b_n = 2^{4n^2+1}n^{8n^2+1}$, we see that the conditions in the proof of Lemma 2.1 are satisfied and $t_\infty < \infty$. Thus, an example of an explicit condition on μ implying $\tau < \infty$ is given by $\mu([2^{4n^2+1}n^{8n^2+1}, \infty)) \ge \frac{1}{n-1}$, $n \ge 2$.

The next lemma provides a lower estimate of the tails of a random walk performed by an active particle. It is used in the proof of Lemma 2.1.

Lemma 2.6. *Let $(S_t, t \ge 0)$ be a continuous-time random walk on $\mathbb{Z}$, $S_0 = 0$, with times between jumps distributed according to π, and let $r > 0$. Then*

$$\mathbb{P}\{S_r \ge m\} \ge 2^{-m-1}\left(\pi\left(\left(0, \frac{r}{m}\right]\right)\right)^m. \tag{13}$$

Proof. Let $\mathrm{j_k}$ be the time of the k-th jump of $(S_t, t \ge 0)$. Since the direction and the timing of each jump are independent,

$$\begin{aligned}\mathbb{P}\{S_r \ge m\} \ge \mathbb{P}&\left\{\text{at least one jump occurs within intervals } \left(0, \frac{r}{m}\right], \left(\frac{r}{m}, \frac{2r}{m}\right], \dots, \left(\frac{(m-1)r}{m}, \frac{r}{m}\right]\right\} \\ &\times \mathbb{P}\{\text{first } m \text{ jumps are all to the right}\} \times \mathbb{P}\{S_r - S_{\mathrm{j_m}} \ge 0\} \\ &\ge \left(\pi\left(\left(0, \frac{r}{m}\right]\right)\right)^m 2^{-m}\frac{1}{2} = 2^{-m-1}\left(\pi\left(\left(0, \frac{r}{m}\right]\right)\right)^m.\end{aligned}$$

$\square$

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