

# Converging Decision Processes with Multiplicative Reward System

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## CONVERGING DECISION PROCESSES WITH MULTIPLICATIVE REWARD SYSTEM

Toshiharu FUJITA

### Abstract

Converging decision process is a decision process model with converging transition, which is one of the nonserial branch systems proposed by Nemhauser. This paper deals with multiplicative reward system on a finite-stage deterministic converging decision process. The purpose of this work is to give a recursive method to solve our model by bidecision approach.

### 1. Introduction

Nonserial dynamic programming was introduced by Nemhauser [15] and has been widely discussed [1, 2, 3]. Nonserial dynamic systems are classified into the four structures: diverging branch systems, converging branch systems, feedback loop systems, and feedforward loop systems. We have also studied nonserial dynamic programming, especially diverging branch systems and converging branch systems. Nondeterministic dynamic programming [5, 11] and mutually dependent decision processes [4, 6, 9] are a type of dynamic programming model with diverging branch systems. Recently, we discussed some converging decision process models [7, 8]. In this paper, we introduce a converging decision process model with multiplicative reward system [10, 14]. We consider our model in a framework of bidecision processes [12, 13] and derive bicursive formula which consists of two interrelated recursive equations. These equations enable to solve our model recursively.

### 2. Notation and formulation

We introduce a finite-stage converging decision process model with a multiplicative reward system.

1.  $X$ , a nonempty finite set, is the state space. The states in the process are expressed by  $x_1, x_2, \dots, x_N \in X$ . The set of indexes that indicate the initial states is denoted by  $I_{\text{Init}} \subset \{1, 2, \dots, N-1\}$  and  $x_i$  ( $i \in I_{\text{Init}}$ ) are specified at the beginning of the process. Moreover we define two index sets as follows:

$$\overline{I_{\text{Init}}} = \{1, 2, \dots, N\} \setminus I_{\text{Init}}$$

and

$$\overline{I_{\text{Init}}^*} = \{1, 2, \dots, N-1\} \setminus I_{\text{Init}}.$$

The process progresses through states  $x_i$  ( $i \in \overline{I_{\text{init}}}$ ) according to converging branch system and is terminated at state  $x_N$ .

2.  $U$ , a nonempty finite set, is the decision space. Furthermore, we denote by  $U_n$  a point-to-set valued mapping from  $X$  to  $2^U \setminus \{\emptyset\}$ , where we denote the power set of  $U$  by  $2^U$ .  $U_n(x_n)$ , called the feasible decision space, represents the set of all feasible decisions in state  $x_n$ . The selected decision for state  $x_n$  ( $n = 1, 2, \dots, N-1$ ) is represented by  $u_n \in U_n(x_n)$  ( $n = 1, 2, \dots, N-1$ ).
3. The transition matrix  $E = (e_{ij}) \in \{0, 1\}^{N \times N}$  is defined by

$$e_{ij} = \begin{cases} 1 & (\text{if } x_j \text{ is the next state to } x_i) \\ 0 & (\text{otherwise}), \end{cases}$$

and let  $I_j = \{i \mid e_{ij} = 1\}$  ( $j = L+1, L+2, \dots, N$ ). We assume that the directed graph which represents the state transition does not contain a loop and that each state (node) has a unique path to the terminal state.

Let  $Gr(U_n)$  be the graph of  $U_n(\cdot)$ :

$$Gr(U_n) = \{(x_n, u_n) \mid u_n \in U_n(x_n), x_n \in X\}.$$

When an index set  $I = \{m_1, m_2, \dots, m_M\}$  ( $m_1 < m_2 < \dots < m_M$ ) is given, the corresponding sequence

$$x_{m_1}, u_{m_1}, x_{m_2}, u_{m_2}, \dots, x_{m_M}, u_{m_M}$$

is denoted by  $\langle x_m, u_m \rangle_{m \in I}$ . Similarly

$$x_{m_1}, x_{m_2}, \dots, x_{m_M} \quad \text{and} \quad u_{m_1}, u_{m_2}, \dots, u_{m_M}$$

are denoted by  $\langle x_m \rangle_{m \in I}$  and  $\langle u_m \rangle_{m \in I}$ , respectively.

4.  $r_n : Gr(U_n) \rightarrow \mathbf{R}$  ( $n = 1, 2, \dots, N-1$ ) are the reward functions, where  $\mathbf{R} = (-\infty, \infty)$ . A decision  $u_n$  selected in state  $x_n$  confers a reward  $r_n(x_n, u_n)$ . The function  $k : X \rightarrow \mathbf{R}$  is the terminal reward function.
5. The converging transition laws are given by

$$f_n : Gr(U_{m_1}) \times Gr(U_{m_2}) \times \dots \times Gr(U_{m_M}) \rightarrow X \quad (n \in \overline{I_{\text{init}}}),$$

where  $I_n = \{m_1, m_2, \dots, m_M\}$ . If a process in states  $\langle x_m \rangle_{m \in I_n}$  selects actions  $\langle u_m \rangle_{m \in I_n}$ , it proceeds deterministically to the next state  $f_n(\langle x_m, u_m \rangle_{m \in I_n})$ .

Then our model is formulated as follows:

$$\begin{aligned} (\text{P}) \quad \text{Max} \quad & r_1(x_1, u_1)r_2(x_2, u_2) \cdots r_{N-1}(x_{N-1}, u_{N-1})k(x_N) \\ \text{s.t.} \quad & x_n = f_n(\langle x_m, u_m \rangle_{m \in I_n}) \quad n \in \overline{I_{\text{init}}} \\ & u_n \in U_n(x_n) \quad n = 1, 2, \dots, N-1. \end{aligned}$$

**EXAMPLE 2.1.** Let  $N = 7$ ,  $I_{\text{init}} = \{1, 2, 3, 4\}$ , and  $e_{15} = e_{26} = e_{36} = e_{47} = e_{57} = e_{67} = 1$  ( $e_{ij} = 0$  for the other pairs  $(i, j)$ ). Then, for the given initial states  $x_1, x_2, x_3, x_4$ , the

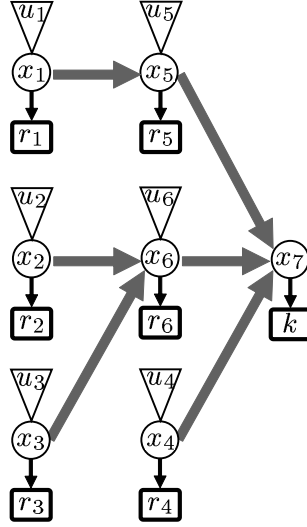


Figure 1. State transition tree for Example 2.1

other states  $x_5, x_6, x_7$  are determined by

$$x_5 = f_5(x_1, u_1), \quad x_1 \in X, u_1 \in U_1(x_1)$$

$$x_6 = f_6(x_2, u_2, x_3, u_3), \quad x_2, x_3 \in X, u_2 \in U_2(x_2), u_3 \in U_3(x_3)$$

$$x_7 = f_7(x_4, u_4, x_5, u_5, x_6, u_6), \quad x_4, x_5, x_6 \in X, u_4 \in U_4(x_4), u_5 \in U_5(x_5), u_6 \in U_6(x_6)$$

(see Figure 1). In this case,

$$I_5 = \{1\}, \quad I_6 = \{2, 3\}, \quad I_7 = \{3, 5, 6\}$$

and the problem is described as follows:

$$\text{Max } r_1(x_1, u_1)r_2(x_2, u_2)r_3(x_3, u_3)r_4(x_4, u_4)r_5(x_5, u_5)r_6(x_6, u_6)k(x_7)$$

$$\text{s.t. } x_n = f_n(\langle x_m, u_m \rangle_{m \in I_n}) \quad n = 5, 6, 7$$

$$u_n \in U_n(x_n) \quad n = 1, 2, 3, 4, 5, 6. \quad \square$$

### 3. Bidecision approach

#### 3.1. Bidecision processes

On bidecision processes, both the family of maximum subproblems and the family of minimum subproblems are considered. In constructing the subproblems, we start

with the initial target state sequence  $Q = (x_N)$ . Then, we add  $x_n$  to  $Q$  in the order that coincides one of the node order for a depth-first search for a state transition tree with the root  $x_N$ . Without loss of generality, we regard that index order as

$$N \rightarrow N - 1 \rightarrow N - 2 \rightarrow \cdots \rightarrow 2 \rightarrow 1,$$

by renumbering the state index.

EXAMPLE 3.1. When we consider the decision process whose state transition tree is given by Figure 2, the sequence of the target state sequences becomes

$$(x_6) \rightarrow (x_5, x_6) \rightarrow (x_4, x_5, x_6) \rightarrow \cdots \rightarrow (x_1, x_2, x_3, x_4, x_5, x_6).$$

Then, we define the corresponding maximum subproblems and minimum subproblems as follows:

$$\begin{aligned} v^6(x_6) &= k(x_6) \\ v^5(x_5; x_2, u_2) &= \max_{u_5 \in U_5(x_5)} [r_5(x_5, u_5)k(x_6)] \\ v^4(x_4; x_2, u_2, x_3, u_3) &= \max_{u_m \in U_m(x_m) \ (4 \leq m \leq 5)} [r_4(x_4, u_4)r_5(x_5, u_5)k(x_6)] \\ v^3(x_3; x_2, u_2) &= \max_{u_m \in U_m(x_m) \ (3 \leq m \leq 5)} [r_3(x_3, u_3)r_4(x_4, u_4)r_5(x_5, u_5)k(x_6)] \\ v^2(x_2) &= \max_{u_m \in U_m(x_m) \ (2 \leq m \leq 5)} [r_2(x_2, u_2)r_3(x_3, u_3)r_4(x_4, u_4)r_5(x_5, u_5)k(x_6)] \\ v^1(x_1) &= \max_{u_m \in U_m(x_m) \ (1 \leq m \leq 5)} [r_1(x_1, u_1)r_2(x_2, u_2)r_3(x_3, u_3)r_4(x_4, u_4)r_5(x_5, u_5)k(x_6)] \\ w^6(x_6) &= k(x_6) \\ w^5(x_5; x_2, u_2) &= \min_{u_5 \in U_5(x_5)} [r_5(x_5, u_5)k(x_6)] \\ w^4(x_4; x_2, u_2, x_3, u_3) &= \min_{u_m \in U_m(x_m) \ (4 \leq m \leq 5)} [r_4(x_4, u_4)r_5(x_5, u_5)k(x_6)] \\ w^3(x_3; x_2, u_2) &= \min_{u_m \in U_m(x_m) \ (3 \leq m \leq 5)} [r_3(x_3, u_3)r_4(x_4, u_4)r_5(x_5, u_5)k(x_6)] \\ w^2(x_2) &= \min_{u_m \in U_m(x_m) \ (2 \leq m \leq 5)} [r_2(x_2, u_2)r_3(x_3, u_3)r_4(x_4, u_4)r_5(x_5, u_5)k(x_6)] \\ w^1(x_1) &= \min_{u_m \in U_m(x_m) \ (1 \leq m \leq 5)} [r_1(x_1, u_1)r_2(x_2, u_2)r_3(x_3, u_3)r_4(x_4, u_4)r_5(x_5, u_5)k(x_6)], \end{aligned}$$

where  $v^n$  and  $w^n$  ( $n = 1, 2, \dots, 6$ ) are the optimal value functions for maximum subproblems and minimum subproblems, respectively.  $\square$

We give the general form of subproblems.

$n = N$

Both of the bidecision subproblems corresponding to the state sequence  $(x_N)$  formed of the final state  $x_N$  are given by

$$v^N(x_N) = w^N(x_N) = k(x_N), \quad x_N \in X.$$

$1 \leq n < N$

The bidecision subproblems corresponding to the state sequence  $(x_n, x_{n+1}, \dots, x_N)$  with the initial state  $x_m$  ( $m \in \{v \mid v \geq n, I_v \cap \{n, n+1, \dots, N\} = \phi\}$ ) are given by

$$\begin{aligned} v^n(x_n; \langle x_m, u_m \rangle_{m \in J_n}) \\ = \max_{u_m \in U_m(x_m) \ (m=n, n+1, \dots, N-1)} [r_n(x_n, u_n) r_{n+1}(x_{n+1}, u_{n+1}) \cdots k(x_N)], \\ x_n \in X, x_m \in X, u_m \in U(x_m) \ (m \in J_n) \end{aligned}$$

and

$$\begin{aligned} w^n(x_n; \langle x_m, u_m \rangle_{m \in J_n}) \\ = \min_{u_m \in U_m(x_m) \ (m=n, n+1, \dots, N-1)} [r_n(x_n, u_n) r_{n+1}(x_{n+1}, u_{n+1}) \cdots k(x_N)], \\ x_n \in X, x_m \in X, u_m \in U(x_m) \ (m \in J_n), \end{aligned}$$

where

$$J_n = \bigcup_{l=n+1}^N \{j \in I_l \mid j < n\}.$$

The following proposition gives a method to get the sets  $J_n$  recursively.

**PROPOSITION 3.1.** *Put  $J_N = \phi$ , then,*

(i) *if  $n+1 \in \overline{I_{\text{init}}}$ ,*

$$J_n = J_{n+1} \cup \{j \in I_{n+1} \mid j < n\}.$$

(ii) *if  $n+1 \in I_{\text{init}} \setminus \{1\}$ ,*

$$J_n = J_{n+1} \setminus \{n\}.$$

For each  $n = 1, 2, \dots, N-1$ , we divide each feasible decision set  $U_n(x_n)$ ,  $x_n \in X$  into two disjoint subsets:

$$U_n^+(x_n) = \{u \in U_n(x_n) \mid r_n(x_n, u) \geq 0\}, \quad U_n^-(x_n) = \{u \in U_n(x_n) \mid r_n(x_n, u) < 0\}$$

that satisfy

$$U_n^+(x_n) \cap U_n^-(x_n) = \phi, \quad U_n^+(x_n) \cup U_n^-(x_n) = U_n(x_n).$$

Then we have the bicursive formula (system of two recursive equations) for the both subproblems:

**THEOREM 3.1.**

$$(1) \quad v^N(x_N) = k(x_N), \quad x_N \in X$$

$$(2) \quad v^n(x_n; \langle x_m, u_m \rangle_{m \in J_n}) \\ = \max_{u_n \in U_n^+(x_n)} [r_n(x_n, u_n) v^{n+1}(f_{n+1}(\langle x_m, u_m \rangle_{m \in I_{n+1}}); \langle x_m, u_m \rangle_{m \in J_{n+1}})] \\ \vee \max_{u_n \in U_n^-(x_n)} [r_n(x_n, u_n) w^{n+1}(f_{n+1}(\langle x_m, u_m \rangle_{m \in I_{n+1}}); \langle x_m, u_m \rangle_{m \in J_{n+1}})], \\ x_n \in X, n + 1 \in \overline{I_{\text{Init}}}$$

$$(3) \quad v^n(x_n; \langle x_m, u_m \rangle_{m \in J_n}) \\ = \max_{u_n \in U_n^+(x_n)} [r_n(x_n, u_n) v^{n+1}(x_{n+1}; \langle x_m, u_m \rangle_{m \in J_{n+1}})] \\ \vee \max_{u_n \in U_n^-(x_n)} [r_n(x_n, u_n) w^{n+1}(x_{n+1}; \langle x_m, u_m \rangle_{m \in J_{n+1}})], \\ x_n \in X, n + 1 \in I_{\text{Init}} \setminus \{1\}.$$

$$(4) \quad w^N(x_N) = k(x_N), \quad x_N \in X$$

$$(5) \quad w^n(x_n; \langle x_m, u_m \rangle_{m \in J_n}) \\ = \min_{u_n \in U_n^+(x_n)} [r_n(x_n, u_n) w^{n+1}(f_{n+1}(\langle x_m, u_m \rangle_{m \in I_{n+1}}); \langle x_m, u_m \rangle_{m \in J_{n+1}})] \\ \wedge \min_{u_n \in U_n^-(x_n)} [r_n(x_n, u_n) v^{n+1}(f_{n+1}(\langle x_m, u_m \rangle_{m \in I_{n+1}}); \langle x_m, u_m \rangle_{m \in J_{n+1}})], \\ x_n \in X, n + 1 \in \overline{I_{\text{Init}}}$$

$$(6) \quad w^n(x_n; \langle x_m, u_m \rangle_{m \in J_n}) \\ = \min_{u_n \in U_n^+(x_n)} [r_n(x_n, u_n) w^{n+1}(x_{n+1}; \langle x_m, u_m \rangle_{m \in J_{n+1}})] \\ \wedge \min_{u_n \in U_n^-(x_n)} [r_n(x_n, u_n) v^{n+1}(x_{n+1}; \langle x_m, u_m \rangle_{m \in J_{n+1}})], \\ x_n \in X, n + 1 \in I_{\text{Init}} \setminus \{1\},$$

where  $\vee(\wedge)$  is the maximum (minimum) operator:

$$a \vee b = \max(a, b) \quad (a \wedge b = \min(a, b)) \quad a, b \in \mathbf{R}.$$

PROOF. We consider the  $n$ th maximum subproblem with the given current state  $\bar{x}_n$  and parameters  $\langle \bar{x}_m, \bar{u}_m \rangle_{m \in J_n}$ . We note that  $m < n$  for  $\forall m \in J_n$ . Here, we show the first equation (2). (This statement also means that we suppose  $n + 1 \notin I_{\text{init}}$ .) By definition of the subproblems,

$$\begin{aligned} & v^n(\bar{x}_n; \langle \bar{x}_m, \bar{u}_m \rangle_{m \in J_n}) \\ &= \max_{u_m \in U_m(x_m)} \max_{(m=n, n+1, \dots, N-1)} [r_n(\bar{x}_n, u_n) r_{n+1}(x_{n+1}, u_{n+1}) \cdots r_{N-1}(x_{N-1}, u_{N-1}) k(x_N)]. \end{aligned}$$

From the existence of the maximum value, there exists an optimal decision sequence  $u_n^*, u_{n+1}^*, \dots, u_{N-1}^*$  satisfying

$$v^n(\bar{x}_n; \langle \bar{x}_m, \bar{u}_m \rangle_{m \in J_n}) = r_n(\bar{x}_n, u_n^*) r_{n+1}(x_{n+1}^*, u_{n+1}^*) \cdots r_{N-1}(x_{N-1}^*, u_{N-1}^*) k(x_N^*).$$

The corresponding state sequence  $x_{n+1}^*, \dots, x_{N-1}^*, x_N^*$  is sequentially determined by

$$x_l^* = \begin{cases} \text{(the given initial state)} & (l \in I_{\text{init}}) \\ f_l(\langle x_m^*, u_m^* \rangle_{m \in I_l}) & (l \notin I_{\text{init}}), \end{cases}$$

where

$$x_m^* = \begin{cases} \bar{x}_m & (m \leq n) \\ x_m^* & (m > n) \end{cases}, \quad u_m^* = \begin{cases} \bar{u}_m & (m < n) \\ u_m^* & (m \geq n). \end{cases}$$

If  $r_n(\bar{x}_n, u_n^*) \geq 0$ , then

$$\begin{aligned} & v^n(\bar{x}_n; \langle \bar{x}_m, \bar{u}_m \rangle_{m \in J_n}) \\ &= r_n(\bar{x}_n, u_n^*) \times [r_{n+1}(x_{n+1}^*, u_{n+1}^*) \cdots r_{N-1}(x_{N-1}^*, u_{N-1}^*) k(x_N^*)] \\ &\leq r_n(\bar{x}_n, u_n^*) \max_{u_m \in U_m(x_m)} \max_{(m=n+1, \dots, N-1)} [r_{n+1}(x_{n+1}^*, u_{n+1}^*) \cdots r_{N-1}(x_{N-1}^*, u_{N-1}^*) k(x_N^*)] \\ &= [r_n(\bar{x}_n, u_n^*) v^{n+1}(x_{n+1}^*; \langle x_m^*, u_m^* \rangle_{m \in J_{n+1}})] \\ &= [r_n(\bar{x}_n, u_n^*) v^{n+1}(f_{n+1}(\langle x_m^*, u_m^* \rangle_{m \in I_{n+1}}); \langle x_m^*, u_m^* \rangle_{m \in J_{n+1}})] \\ &= [r_n(\bar{x}_n, u_n^*) v^{n+1}(f_{n+1}(\bar{x}_n, u_n^*, \langle \bar{x}_m, \bar{u}_m \rangle_{m \in I_{n+1} \setminus \{n\}}); \langle \bar{x}_m, \bar{u}_m \rangle_{m \in J_{n+1}})] \\ &\leq \max_{\bar{u}_n \in U_n^+(\bar{x}_n)} [r_n(\bar{x}_n, \bar{u}_n) v^{n+1}(f_{n+1}(\langle \bar{x}_m, \bar{u}_m \rangle_{m \in I_{n+1}}); \langle \bar{x}_m, \bar{u}_m \rangle_{m \in J_{n+1}})]. \end{aligned}$$

Similarly, if  $r_n(\bar{x}_n, u_n^*) < 0$ , then



$$\begin{aligned}
& v^n(\bar{x}_n; \langle \bar{x}_m, \bar{u}_m \rangle_{m \in J_n}) \\
& \leq r_n(\bar{x}_n, u_n^*) \min_{u_m \in U_m(x_m)} \min_{(m=n+1, \dots, N-1)} [r_{n+1}(x_{n+1}^*, u_{n+1}) \cdots r_{N-1}(x_{N-1}, u_{N-1}) k(x_N)] \\
& = [r_n(\bar{x}_n, u_n^*) w^{n+1}(x_{n+1}^*; \langle x_m^*, u_m^* \rangle_{m \in J_{n+1}})] \\
& \leq \max_{\bar{u}_n \in U_n^-(\bar{x}_n)} [r_n(\bar{x}_n, \bar{u}_n) w^{n+1}(f_{n+1}(\langle \bar{x}_m, \bar{u}_m \rangle_{m \in I_{n+1}}); \langle \bar{x}_m, \bar{u}_m \rangle_{m \in J_{n+1}})].
\end{aligned}$$

Therefore

$$\begin{aligned}
(7) \quad & v^n(\bar{x}_n; \langle \bar{x}_m, \bar{u}_m \rangle_{m \in J_n}) \\
& \leq \max_{\bar{u}_n \in U_n^+(\bar{x}_n)} [r_n(\bar{x}_n, \bar{u}_n) v^{n+1}(f_{n+1}(\langle \bar{x}_m, \bar{u}_m \rangle_{m \in I_{n+1}}); \langle \bar{x}_m, \bar{u}_m \rangle_{m \in J_{n+1}})] \\
& \quad \vee \max_{\bar{u}_n \in U_n^-(\bar{x}_n)} [r_n(\bar{x}_n, \bar{u}_n) w^{n+1}(f_{n+1}(\langle \bar{x}_m, \bar{u}_m \rangle_{m \in I_{n+1}}); \langle \bar{x}_m, \bar{u}_m \rangle_{m \in J_{n+1}})].
\end{aligned}$$

On the other hand, there exist optimal decisions  $u_n^+ \in U_n^+(\bar{x}_n)$ ,  $u_n^- \in U_n^-(\bar{x}_n)$  satisfying

$$\begin{aligned}
& \max_{\bar{u}_n \in U_n^+(\bar{x}_n)} [r_n(\bar{x}_n, \bar{u}_n) v^{n+1}(f_{n+1}(\langle \bar{x}_m, \bar{u}_m \rangle_{m \in I_{n+1}}); \langle \bar{x}_m, \bar{u}_m \rangle_{m \in J_{n+1}})] \\
& = r_n(\bar{x}_n, u_n^+) v^{n+1}(f_{n+1}(\bar{x}_n, u_n^+, \langle \bar{x}_m, \bar{u}_m \rangle_{m \in I_{n+1} \setminus \{n\}}); \langle \bar{x}_m, \bar{u}_m \rangle_{m \in J_{n+1}})
\end{aligned}$$

and

$$\begin{aligned}
& \max_{\bar{u}_n \in U_n^-(\bar{x}_n)} [r_n(\bar{x}_n, \bar{u}_n) w^{n+1}(f_{n+1}(\langle \bar{x}_m, \bar{u}_m \rangle_{m \in I_{n+1}}); \langle \bar{x}_m, \bar{u}_m \rangle_{m \in J_{n+1}})] \\
& = r_n(\bar{x}_n, u_n^-) w^{n+1}(f_{n+1}(\bar{x}_n, u_n^-, \langle \bar{x}_m, \bar{u}_m \rangle_{m \in I_{n+1} \setminus \{n\}}); \langle \bar{x}_m, \bar{u}_m \rangle_{m \in J_{n+1}}),
\end{aligned}$$

respectively. Let

$$x_{n+1}^+ = f_{n+1}(\bar{x}_n, u_n^+, \langle \bar{x}_m, \bar{u}_m \rangle_{m \in I_{n+1} \setminus \{n\}})$$

and

$$x_{n+1}^- = f_{n+1}(\bar{x}_n, u_n^-, \langle \bar{x}_m, \bar{u}_m \rangle_{m \in I_{n+1} \setminus \{n\}}).$$

Then there exist two optimal decision sequences  $u_{n+1}^+, \dots, u_{N-1}^+$  and  $u_{n+1}^-, \dots, u_{N-1}^-$  that meet the following equations:

$$\begin{aligned}
& v^{n+1}(f_{n+1}(\bar{x}_n, u_n^+, \langle \bar{x}_m, \bar{u}_m \rangle_{m \in I_{n+1} \setminus \{n\}}); \langle \bar{x}_m, \bar{u}_m \rangle_{m \in J_{n+1}}) \\
& = v^{n+1}(x_{n+1}^+; \langle \bar{x}_m, \bar{u}_m \rangle_{m \in J_{n+1}}) \\
& = r_{n+1}(x_{n+1}^+, u_{n+1}^+) r_{n+2}(x_{n+2}^+, u_{n+2}^+) \cdots r_{N-1}(x_{N-1}^+, u_{N-1}^+) k(x_N^+)
\end{aligned}$$

and

$$\begin{aligned}
 & w^{n+1}(f_{n+1}(\bar{x}_n, u_n^-, \langle \bar{x}_m, \bar{u}_m \rangle_{m \in I_{n+1} \setminus \{n\}}); \langle \bar{x}_m, \bar{u}_m \rangle_{m \in J_{n+1}}) \\
 &= w^{n+1}(x_{n+1}^-; \langle \bar{x}_m, \bar{u}_m \rangle_{m \in J_{n+1}}) \\
 &= r_{n+1}(x_{n+1}^-, u_{n+1}^-) r_{n+2}(x_{n+2}^-, u_{n+2}^-) \cdots r_{N-1}(x_{N-1}^-, u_{N-1}^-) k(x_N^-),
 \end{aligned}$$

respectively. The corresponding state sequences  $x_{n+2}^+, \dots, x_{N-1}^+, x_N^+$  and  $x_{n+2}^-, \dots, x_{N-1}^-$ ,  $x_N^-$  is sequentially determined by the following manner.

(i) For  $x_l^+$  ( $l = n+2, n+3, \dots, N$ ),

$$x_l^+ = \begin{cases} \text{(the given initial state)} & (l \in I_{\text{Init}}) \\ f_l(\langle x_m^+, u_m^+ \rangle_{m \in I_l}) & (l \notin I_{\text{Init}}), \end{cases}$$

where

$$x_m^+ = \begin{cases} \bar{x}_m & (m < n+1) \\ x_m^+ & (m \geq n+1), \end{cases} \quad u_m^+ = \begin{cases} \bar{u}_m & (m < n) \\ u_m^+ & (m \geq n). \end{cases}$$

(ii) For  $x_l^-$  ( $l = n+2, n+3, \dots, N$ ),

$$x_l^- = \begin{cases} \text{(the given initial state)} & (l \in I_{\text{Init}}) \\ f_l(\langle x_m^-, u_m^- \rangle_{m \in I_l}) & (l \notin I_{\text{Init}}), \end{cases}$$

where

$$x_m^- = \begin{cases} \bar{x}_m & (m < n+1) \\ x_m^- & (m \geq n+1), \end{cases} \quad u_m^- = \begin{cases} \bar{u}_m & (m < n) \\ u_m^- & (m \geq n). \end{cases}$$

Hence

$$\begin{aligned}
 (8) \quad & \max_{\hat{u}_n \in U_n^+(\bar{x}_n)} [r_n(\bar{x}_n, \bar{u}_n) v^{n+1}(f_{n+1}(\langle \bar{x}_m, \bar{u}_m \rangle_{m \in I_{n+1}}); \langle \bar{x}_m, \bar{u}_m \rangle_{m \in J_{n+1}})] \\
 & \quad \vee \max_{\hat{u}_n \in U_n^-(\bar{x}_n)} [r_n(\bar{x}_n, \bar{u}_n) w^{n+1}(f_{n+1}(\langle \bar{x}_m, \bar{u}_m \rangle_{m \in I_{n+1}}); \langle \bar{x}_m, \bar{u}_m \rangle_{m \in J_{n+1}})] \\
 &= [r_n(\bar{x}_n, u_n^+) r_{n+1}(x_{n+1}^+, u_{n+1}^+) r_{n+2}(x_{n+2}^+, u_{n+2}^+) \cdots r_{N-1}(x_{N-1}^+, u_{N-1}^+) k(x_N^+)] \\
 & \quad \vee [r_n(\bar{x}_n, u_n^-) r_{n+1}(x_{n+1}^-, u_{n+1}^-) r_{n+2}(x_{n+2}^-, u_{n+2}^-) \cdots r_{N-1}(x_{N-1}^-, u_{N-1}^-) k(x_N^-)] \\
 &\leq \max_{u_m \in U_m(x_m)} [r_n(\bar{x}_n, u_n) r_{n+1}(x_{n+1}, u_{n+1}) \cdots r_{N-1}(x_{N-1}, u_{N-1}) k(x_N)] \\
 &= v^n(\bar{x}_n; \langle \bar{x}_m, \bar{u}_m \rangle_{m \in J_n}).
 \end{aligned}$$

From Eqs. (7) and (8), we have Eq. (2).

The other recursive equations can be shown in a similar way.  $\square$

In case that non-negativity of all reward functions can be assumed, the following recursive equations which are the same type as those for additive reward system hold.

COROLLARY 3.1.

$$\begin{aligned}
v^N(x_N) &= k(x_N), & x_N &\in X \\
v^n(x_n; \langle x_m, u_m \rangle_{m \in J_n}) &= \max_{u_n \in U_n(x_n)} [r_n(x_n, u_n) v^{n+1}(f_{n+1}(\langle x_m, u_m \rangle_{m \in I_{n+1}}); \langle x_m, u_m \rangle_{m \in J_{n+1}})] \\
& & x_n &\in X, n+1 \in \overline{I_{\text{Init}}} \\
v^n(x_n; \langle x_m, u_m \rangle_{m \in J_n}) &= \max_{u_n \in U_n(x_n)} [r_n(x_n, u_n) v^{n+1}(x_{n+1}; \langle x_m, u_m \rangle_{m \in J_{n+1}})] \\
& & x_n &\in X, n+1 \in I_{\text{Init}} \setminus \{1\}.
\end{aligned}$$

PROOF. Since  $U_n^+(x_n) = U_n(x_n)$  and  $U_n^-(x_n) = \phi$ , it is clear from Theorem 3.1.  $\square$

#### 4. Example

EXAMPLE 4.1. We consider the converging decision process shown in Figure 2. By Proposition 3.1, we can get  $J_j$  ( $j = 6, 5, \dots, 1$ ) as follows:

$$J_6 = \phi.$$

Since, for  $n = 5$ ,  $n+1 = 6 \in \overline{I_{\text{Init}}}$ ,

$$J_5 = J_6 \cup \{j \in I_6 \mid j < 5\} = \phi \cup \{2\} = \{2\}.$$

Similarly,

$$\begin{aligned}
n = 4 : n+1 = 5 \notin I_{\text{Init}} &\implies J_4 = J_5 \cup \{j \in I_5 \mid j < 4\} = \{2\} \cup \{3\} = \{2, 3\}, \\
n = 3 : n+1 = 4 \in I_{\text{Init}} &\implies J_3 = J_4 \setminus \{3\} = \{2, 3\} \setminus \{3\} = \{2\}, \\
n = 2 : n+1 = 3 \in I_{\text{Init}} &\implies J_2 = J_3 \setminus \{2\} = \{2\} \setminus \{2\} = \phi,
\end{aligned}$$

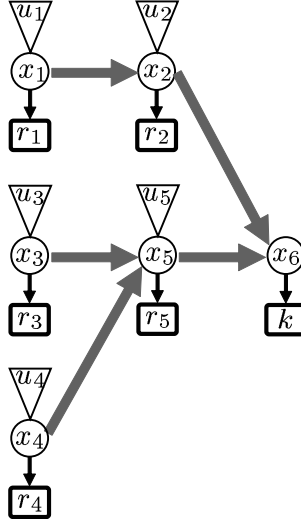


Figure 2. State transition tree for Example 4.1 and 4.2

$$n = 1 : n + 1 = 2 \notin I_{\text{init}}$$

$$\implies J_1 = J_2 \cup \{j \in I_2 \mid j < 1\} = \phi \cup \phi = \phi.$$

Then, we can get the bicursive formula by Theorem 3.1. First, let  $N = 6$ , we have

$$v^6(x_6) = w^6(x_6) = k(x_6).$$

Next, from  $6(= n + 1) \in \overline{I_{\text{init}}}$ , the system of recursive equations for  $n = 5$  is

$$\begin{aligned} v^5(x_5; \langle x_m, u_m \rangle_{m \in J_5}) &= \max_{u_5 \in U_5^+(x_5)} [r_5(x_5, u_5) v^6(f_6(\langle x_m, u_m \rangle_{m \in I_6}); \langle x_m, u_m \rangle_{m \in J_6})] \\ &\vee \max_{u_5 \in U_5^-(x_5)} [r_5(x_5, u_5) w^6(f_6(\langle x_m, u_m \rangle_{m \in I_6}); \langle x_m, u_m \rangle_{m \in J_6})] \\ w^5(x_5; \langle x_m, u_m \rangle_{m \in J_5}) &= \min_{u_5 \in U_5^+(x_5)} [r_5(x_5, u_5) w^6(f_6(\langle x_m, u_m \rangle_{m \in I_6}); \langle x_m, u_m \rangle_{m \in J_6})] \\ &\wedge \min_{u_5 \in U_5^-(x_5)} [r_5(x_5, u_5) v^6(f_6(\langle x_m, u_m \rangle_{m \in I_6}); \langle x_m, u_m \rangle_{m \in J_6})]. \end{aligned}$$

Since  $J_5 = \{2\}$ ,  $J_6 = \phi$ ,  $I_6 = \{2, 5\}$ , the above equations become

$$\begin{aligned} v^5(x_5; x_2, u_2) &= \max_{u_5 \in U_5^+(x_5)} [r_5(x_5, u_5) v^6(f_6(x_2, u_2, x_5, u_5))] \\ &\vee \max_{u_5 \in U_5^-(x_5)} [r_5(x_5, u_5) w^6(f_6(x_2, u_2, x_5, u_5))] \end{aligned}$$

and

$$\begin{aligned} w^5(x_5; x_2, u_2) &= \min_{u_5 \in U_5^+(x_5)} [r_5(x_5, u_5) w^6(f_6(x_2, u_2, x_5, u_5))] \\ &\quad \wedge \min_{u_5 \in U_5^-(x_5)} [r_5(x_5, u_5) v^6(f_6(x_2, u_2, x_5, u_5))], \end{aligned}$$

respectively.

Similarly, we have

$$\begin{aligned} v^4(x_4; x_2, u_2, x_3, u_3) &= \max_{u_4 \in U_4^+(x_4)} [r_4(x_4, u_4) v^5(f_5(x_3, u_3, x_4, u_4); x_2, u_2)] \\ &\quad \vee \max_{u_4 \in U_4^-(x_4)} [r_4(x_4, u_4) w^5(f_5(x_3, u_3, x_4, u_4); x_2, u_2)] \\ w^4(x_4; x_2, u_2, x_3, u_3) &= \min_{u_4 \in U_4^+(x_4)} [r_4(x_4, u_4) w^5(f_5(x_3, u_3, x_4, u_4); x_2, u_2)] \\ &\quad \wedge \min_{u_4 \in U_4^-(x_4)} [r_4(x_4, u_4) v^5(f_5(x_3, u_3, x_4, u_4); x_2, u_2)] \\ v^3(x_3; x_2, u_2) &= \max_{u_3 \in U_3^+(x_3)} [r_3(x_3, u_3) v^4(x_4; x_2, u_2, x_3, u_3)] \\ &\quad \vee \max_{u_3 \in U_3^-(x_3)} [r_3(x_3, u_3) w^4(x_4; x_2, u_2, x_3, u_3)] \\ w^3(x_3; x_2, u_2) &= \min_{u_3 \in U_3^+(x_3)} [r_3(x_3, u_3) w^4(x_4; x_2, u_2, x_3, u_3)] \\ &\quad \wedge \min_{u_3 \in U_3^-(x_3)} [r_3(x_3, u_3) v^4(x_4; x_2, u_2, x_3, u_3)] \\ v^2(x_2) &= \max_{u_2 \in U_2^+(x_2)} [r_2(x_2, u_2) v^3(x_3; x_2, u_2)] \vee \max_{u_2 \in U_2^-(x_2)} [r_2(x_2, u_2) w^3(x_3; x_2, u_2)] \\ w^2(x_2) &= \min_{u_2 \in U_2^+(x_2)} [r_2(x_2, u_2) w^3(x_3; x_2, u_2)] \wedge \min_{u_2 \in U_2^-(x_2)} [r_2(x_2, u_2) v^3(x_3; x_2, u_2)] \end{aligned}$$

and

$$\begin{aligned} v^1(x_1) &= \max_{u_1 \in U_1^+(x_1)} [r_1(x_1, u_1) v^2(f_2(x_1, u_1))] \vee \max_{u_1 \in U_1^-(x_1)} [r_1(x_1, u_1) w^2(f_2(x_1, u_1))] \\ w^1(x_1) &= \min_{u_1 \in U_1^+(x_1)} [r_1(x_1, u_1) w^2(f_2(x_1, u_1))] \wedge \min_{u_1 \in U_1^-(x_1)} [r_1(x_1, u_1) v^2(f_2(x_1, u_1))]. \quad \square \end{aligned}$$

EXAMPLE 4.2. Let  $N = 6$ ,  $I_{\text{init}} = \{1, 3, 4\}$ . We consider the converging decision process problem shown in Figure 2 with the following data:

$$X = \{s_1, s_2, s_3\}, \quad U_n(x) = U = \{a_1, a_2\} \quad (\forall n \in \{1, 2, 3, 4, 5\}, \forall x \in X)$$

$$x_1 = s_1, \quad x_3 = s_2, \quad x_4 = s_1$$

$f_2(x, u)$	$x \backslash u$	$a_1$	$a_2$	$f_5(x, u, y, v)$	$(x, u) \backslash (y, v)$	$(s_1, a_1)$	$(s_1, a_2)$
	$s_1$	$s_3$	$s_2$		$(s_2, a_1)$	$s_2$	$s_3$
	$s_1$	$s_3$	$s_2$		$(s_2, a_2)$	$s_1$	$s_2$
$f_6(x, u, y, v)$	$(x, u) \backslash (y, v)$	$(s_1, a_1)$	$(s_1, a_2)$	$(s_2, a_1)$	$(s_2, a_2)$	$(s_3, a_1)$	$(s_3, a_2)$
	$(s_2, a_1)$	$s_3$	$s_2$	$s_2$	$s_3$	$s_1$	$s_2$
	$(s_2, a_2)$	$s_1$	$s_3$	$s_3$	$s_1$	$s_2$	$s_3$
	$(s_3, a_1)$	$s_1$	$s_2$	$s_2$	$s_1$	$s_3$	$s_1$
	$(s_3, a_2)$	$s_3$	$s_1$	$s_3$	$s_2$	$s_2$	$s_2$

and

$(x, u)$	$(s_1, a_1)$	$(s_1, a_2)$	$(s_2, a_1)$	$(s_2, a_2)$	$(s_3, a_1)$	$(s_3, a_2)$	
$r_1(x, u)$	2	-1					
$r_2(x, u)$			-1	1	2	-1	
$r_3(x, u)$			3	-2			
$r_4(x, u)$	-1	2					
$r_5(x, u)$	-1	1	-2	1	2	1	

$x$	$s_1$	$s_2$	$s_3$
$k(x)$	2	-2	-1

(In the above tables, unnecessary values for the given initial states are omitted.)

We have already obtained the bicursive formula in Example 4.1. Hereafter, the calculation will proceed based on them.

First, for the terminal state  $x_6$ , we get

$$v^6(x_6) = w^6(x_6) = k(x_6) = \begin{cases} 2 & (x_6 = s_1) \\ -2 & (x_6 = s_2) \\ -1 & (x_6 = s_3) \end{cases}$$

Then, we calculate  $v^5(x_5; x_2, u_2)$ ,  $w^5(x_5; x_2, u_2)$  and the corresponding optimal decision functions  $\bar{\pi}_5^*(x_5; x_2, u_2)$ ,  $\underline{\pi}_5^*(x_5; x_2, u_2)$ . When  $x_5 = s_1$  and  $(x_2, u_2) = (s_2, a_1)$ , we have

$$\begin{aligned} v^5(s_1; s_2, a_1) &= \max_{u_5 \in \{a_2\}} [r_5(s_1, u_5) v^6(f_6(s_2, a_1, s_1, u_5))] \\ &\vee \max_{u_5 \in \{a_1\}} [r_5(s_1, u_5) w^6(f_6(s_2, a_1, s_1, u_5))] \\ &= [r_5(s_1, a_2) v^6(f_6(s_2, a_1, s_1, a_2))] \vee [r_5(s_1, a_1) w^6(f_6(s_2, a_1, s_1, a_1))] \end{aligned}$$

$$\begin{aligned}
&= [1 \times v^6(s_2)] \vee [-1 \times w^6(s_3)] = [1 \times (-2)] \vee [-1 \times (-1)] \\
&= 1
\end{aligned}$$

$$\bar{\pi}_5^*(s_1; s_2, a_1) = a_1$$

and

$$\begin{aligned}
w^5(s_1; s_2, a_1) &= \max_{u_5 \in \{a_2\}} [r_5(s_1, u_5)w^6(f_6(s_2, a_1, s_1, u_5))] \\
&\quad \wedge \max_{u_5 \in \{a_1\}} [r_5(s_1, u_5)v^6(f_6(s_2, a_1, s_1, u_5))] \\
&= [r_5(s_1, a_2)w^6(f_6(s_2, a_1, s_1, a_2))] \wedge [r_5(s_1, a_1)v^6(f_6(s_2, a_1, s_1, a_1))] \\
&= [1 \times w^6(s_2)] \wedge [-1 \times v^6(s_3)] = [1 \times (-2)] \wedge [-1 \times (-1)] \\
&= -2
\end{aligned}$$

$$\underline{u}_5^*(s_1; s_2, a_1) = a_2.$$

Similarly, we get

$$\begin{aligned}
v^5(s_1; s_2, a_2) &= [r_5(s_1, a_2)v^6(s_3)] \vee [r_5(s_1, a_1)w^6(s_1)] = [1 \times (-1)] \vee [-1 \times 2] \\
&= -1,
\end{aligned}$$

$$\bar{\pi}_5^*(s_1; s_2, a_2) = a_2$$

$$\begin{aligned}
w^5(s_1; s_2, a_2) &= [r_5(s_1, a_2)w^6(s_3)] \wedge [r_5(s_1, a_1)v^6(s_1)] = [1 \times (-1)] \wedge [-1 \times 2] \\
&= -2,
\end{aligned}$$

$$\underline{u}_5^*(s_1; s_2, a_2) = a_1$$

$$\begin{aligned}
v^5(s_1; s_3, a_1) &= [r_5(s_1, a_2)v^6(s_2)] \vee [r_5(s_1, a_1)w^6(s_1)] = [1 \times (-2)] \vee [-1 \times 2] \\
&= -2,
\end{aligned}$$

$$\bar{\pi}_5^*(s_1; s_3, a_1) = a_1, a_2$$

$$\begin{aligned}
w^5(s_1; s_3, a_1) &= [r_5(s_1, a_2)w^6(s_2)] \wedge [r_5(s_1, a_1)v^6(s_1)] = [1 \times (-2)] \wedge [-1 \times 2] \\
&= -2,
\end{aligned}$$

$$\underline{u}_5^*(s_1; s_3, a_1) = a_1, a_2$$

$$\begin{aligned}
v^5(s_1; s_3, a_2) &= [r_5(s_1, a_2)v^6(s_1)] \vee [r_5(s_1, a_1)w^6(s_3)] = [1 \times 2] \vee [-1 \times (-1)] \\
&= 2,
\end{aligned}$$

$$\begin{aligned}
 \bar{\pi}_5^*(s_1; s_3, a_2) &= a_2 \\
 w^5(s_1; s_3, a_2) &= [r_5(s_1, a_2)w^6(s_1)] \wedge [r_5(s_1, a_1)v^6(s_3)] = [1 \times 2] \wedge [-1 \times (-1)] \\
 &= 1, \\
 \underline{\pi}_5^*(s_1; s_3, a_2) &= a_1.
 \end{aligned}$$

Other results required for  $x_5 = s_2, s_3$  are

$(x_2, u_2)$	$(s_2, a_1)$	$(s_2, a_2)$	$(s_3, a_1)$	$(s_3, a_2)$
$v^5(s_2; x_2, u_2)$	4	2	4	2
$\bar{\pi}_5^*(s_2; x_2, u_2)$	$a_1$	$a_1, a_2$	$a_1$	$a_1$
$w^5(s_2; x_2, u_2)$	-1	2	2	-2
$\underline{\pi}_5^*(s_2; x_2, u_2)$	$a_2$	$a_1, a_2$	$a_2$	$a_2$
$v^5(s_3; x_2, u_2)$	4	-1	2	-2
$\bar{\pi}_5^*(s_3; x_2, u_2)$	$a_1$	$a_2$	$a_2$	$a_2$
$w^5(s_3; x_2, u_2)$	-2	-4	-2	-4
$\underline{\pi}_5^*(s_3; x_2, u_2)$	$a_2$	$a_1$	$a_1$	$a_1$

(The results for  $x_2 = s_1$  are omitted as they are unnecessary for this example. The same applies below.)

Next, we calculate  $v^4(x_4; x_2, u_2, x_3, u_3)$ ,  $w^4(x_4; x_2, u_2, x_3, u_3)$  and the corresponding optimal decision functions  $\bar{\pi}_4^*(x_4; x_2, u_2, x_3, u_3)$ ,  $\underline{\pi}_4^*(x_4; x_2, u_2, x_3, u_3)$  for the given initial states  $x_4 = s_1$  and  $x_3 = s_2$ . When  $(x_2, u_2, u_3) = (s_2, a_1, a_1)$ , we have

$$\begin{aligned}
 v^4(s_1; s_2, a_1, s_2, a_1) &= \max_{u_4 \in \{a_2\}} [r_4(s_1, u_4)v^5(f_5(s_2, a_1, s_1, u_4); s_2, a_1)] \\
 &\quad \vee \max_{u_4 \in \{a_1\}} [r_4(s_1, u_4)w^5(f_5(s_2, a_1, s_1, u_4); s_2, a_1)] \\
 &= [r_4(s_1, a_2)v^5(f_5(s_2, a_1, s_1, a_2); s_2, a_1)] \\
 &\quad \vee [r_4(s_1, a_1)w^5(f_5(s_2, a_1, s_1, a_1); s_2, a_1)] \\
 &= [2 \times v^5(s_3; s_2, a_1)] \vee [-1 \times w^5(s_2; s_2, a_1)] \\
 &= [2 \times 4] \vee [-1 \times (-1)] = 8 \\
 \bar{\pi}_4^*(s_1; s_2, a_1, s_2, a_1) &= a_2.
 \end{aligned}$$

Through similar calculations, we get values of  $v^4(s_1; x_2, u_2, s_2, u_3)$  and  $w^4(s_1; x_2, u_2, s_2, u_3)$  as follows:



$(x_2, u_2)$	$(s_2, a_1)$	$(s_2, a_2)$	$(s_3, a_1)$	$(s_3, a_2)$
$v^4(s_1; x_2, u_2, s_2, a_1)$	8	-2	4	2
$\bar{\pi}_4^*(s_1; x_2, u_2, s_2, a_1)$	$a_2$	$a_1, a_2$	$a_2$	$a_1$
$w^4(s_1; x_2, u_2, s_2, a_1)$	-4	-8	-4	-8
$\underline{\pi}_4^*(s_1; x_2, u_2, s_2, a_1)$	$a_1, a_2$	$a_2$	$a_1, a_2$	$a_2$
$v^4(s_1; x_2, u_2, s_2, a_2)$	8	4	8	4
$\bar{\pi}_4^*(s_1; x_2, u_2, s_2, a_2)$	$a_2$	$a_2$	$a_2$	$a_2$
$w^4(s_1; x_2, u_2, s_2, a_2)$	-2	1	2	-4
$\underline{\pi}_4^*(s_1; x_2, u_2, s_2, a_2)$	$a_2$	$a_1$	$a_1$	$a_2$

Moreover, the remaining results are given by the following:

$(x_2, u_2)$	$(s_2, a_1)$	$(s_2, a_2)$	$(s_3, a_1)$	$(s_3, a_2)$
$v^3(s_2; x_2, u_2)$	24	-2	12	8
$\bar{\pi}_3^*(s_2; x_2, u_2)$	$a_1$	$a_2$	$a_1$	$a_2$
$w^3(s_2; x_2, u_2)$	-16	-24	-16	-24
$\underline{\pi}_3^*(s_2; x_2, u_2)$	$a_2$	$a_1$	$a_2$	$a_1$

$$\begin{aligned}
v^2(s_2) &= 16, & \bar{\pi}_2^*(s_2) &= a_1 \\
w^2(s_2) &= -24, & \underline{\pi}_2^*(s_2) &= a_1, a_2 \\
v^2(s_3) &= 24, & \bar{\pi}_2^*(s_3) &= a_1, a_2 \\
w^2(s_3) &= -32, & \underline{\pi}_2^*(s_3) &= a_1 \\
v^1(s_1) &= 48, & \bar{\pi}_2^*(s_1) &= a_1 \\
(w^1(s_1) &= -64, & \underline{\pi}_2^*(s_1) &= a_1).
\end{aligned}$$

Thus, the optimal value is  $v^1(s_1) = 48$  and the optimal state-decision sequences are obtained as follows.

First, set  $x_1 = s_1$  ( $1 \in I_{\text{init}}$ ), then

$$\begin{aligned}
x_1 = s_1 &\longrightarrow u_1^* = \bar{\pi}_2^*(s_1) = a_1 \\
\longrightarrow x_2 = f_2(s_1, a_1) = s_3, & \quad r_1(s_1, a_1) = 2 \geq 0 \longrightarrow u_2^* = \bar{\pi}_2^*(s_3) = a_1, a_2.
\end{aligned}$$

(i) If we take  $u_2^* = a_1$ ,

$$\begin{aligned}
\longrightarrow x_3 = s_2 \quad (3 \in I_{\text{init}}), & \quad r_2(s_3, a_1) = 2 \geq 0 \longrightarrow u_3^* = \bar{\pi}_3^*(s_2; s_3, a_1) = a_1 \\
\longrightarrow x_4 = s_1 \quad (4 \in I_{\text{init}}), & \quad r_3(s_2, a_1) = 3 \geq 0 \longrightarrow u_4^* = \bar{\pi}_4^*(s_1; s_3, a_1, s_2, a_1) = a_2
\end{aligned}$$

$$\longrightarrow x_5 = f_5^*(s_2, a_1, s_1, a_2) = s_3, \quad r_4(s_1, a_2) = 2 \geq 0 \quad \longrightarrow \quad u_5^* = \bar{\pi}_5^*(s_3; s_3, a_1) = a_2$$

$$\longrightarrow x_6 = f_6^*(s_3, a_1, s_3, a_2) = s_1.$$

(ii) If we take  $u_2^* = a_2$ ,

$$\longrightarrow x_3 = s_2 \quad (3 \in I_{\text{Init}}), \quad r_2(s_3, a_2) = -1 < 0 \quad \longrightarrow \quad u_3^* = \underline{\pi}_3^*(s_2; s_3, a_2) = a_1$$

$$\longrightarrow x_4 = s_1 \quad (4 \in I_{\text{Init}}), \quad r_3(s_2, a_1) = 3 \geq 0 \quad \longrightarrow \quad u_4^* = \underline{\pi}_4^*(s_1; s_3, a_2, s_2, a_1) = a_2$$

$$\longrightarrow x_5 = f_5^*(s_2, a_1, s_1, a_2) = s_3, \quad r_4(s_1, a_2) = 2 \geq 0 \quad \longrightarrow \quad u_5^* = \underline{\pi}_5^*(s_3; s_3, a_2) = a_1$$

$$\longrightarrow x_6 = f_6^*(s_3, a_2, s_3, a_1) = s_2.$$

Finally, we have two optimal decision sequences:

$$(u_1^*, u_2^*, u_3^*, u_4^*, u_5^*) = (a_1, a_1, a_1, a_2, a_2), (a_1, a_2, a_1, a_2, a_1),$$

which attain the optimal value.  $\square$

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## References

- [1] U. Bertelé and F. Brioschi, *Nonserial Dynamic Programming*, Academic Press, New York, 1972.
- [2] A. O. Esogbue and N. A. Warsi, A High-level Computing Algorithm for Diverging and Converging Branch Nonserial Dynamic Programming Systems, *Computers and Mathematics with Applications*, **12** (1986), 719–732.
- [3] A. O. Esogbue, A High Level Dynamic Programming Algorithm for Processing Nonserial Looped System, *Computers and Mathematics with Applications*, **16** (1988), 801–813.
- [4] T. Fujita, Mutually Dependent Markov Decision Processes, *Journal of Advanced Computational Intelligence and Intelligent Informatics*, **18** (2014), 992–998.
- [5] T. Fujita, Sure Ways to win a Game—Nondeterministic Dynamic Programming Approach—, *Bulletin of the Kyushu Institute of Technology*, **62** (2015), 1–14.
- [6] T. Fujita, Mutually Dependent Decision Processes Models, *Bulletin of the Kyushu Institute of Technology*, **63** (2016), 15–26.
- [7] T. Fujita, Three Recursive Approaches for Decision Processes with a Converging Branch System, *Bulletin of the Kyushu Institute of Technology*, **65** (2018), 1–21.
- [8] T. Fujita, On Markov Decision Process with Converging Branch System (in Japanese), *RIMS Kokyuroku (Kyoto Univ.)*, **2158** (2020), 184–194.
- [9] T. Fujita and A. Kira, Associative Criteria in Mutually Dependent Markov Decision Processes, *Proceedings of IIAI International Conference on Advanced Applied Informatics* (2014), 147–150.
- [10] T. Fujita and K. Tsurusaki, Stochastic Optimization of Multiplicative Functions with Negative Value, *Journal of Operations Research Society of Japan*, **41** (1998), 351–373.
- [11] T. Fujita, T. Ueno, and S. Iwamoto, A Nondeterministic Dynamic Programming Model, *Lecture Notes in Artificial Intelligence*, **3214** (2004), 1208–1214.

- [12] S. Iwamoto, From Dynamic Programming to Bynamic Programming, *Journal of Mathematical Analysis and Applications*, **177** (1993), 56–74.
- [13] S. Iwamoto, On Bidecision Processes, *Journal of Mathematical Analysis and Applications*, **187** (1994), 676–699.
- [14] S. Iwamoto, Associative Dynamic Programs, *Journal of Mathematical Analysis and Applications*, **201** (1996), 195–211.
- [15] G. L. Nemhauser, *Introduction to Dynamic Programming*, Wiley, New York, 1966.

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