NONLINEAR AXISYMMETRIC ELASTIC ANALYSIS OF DEEP IMPERFECT SPHERICAL SHELLS

by

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ABSTRACT

The particular problem that is addressed in this thesis is the nonlinear elastic analysis of incomplete thin deep spherical shells, where the displacements on the boundary surface are fully restrained and the shell is loaded by a uniform pressure load. The shell may contain initial axisymmetric geometric or stress imperfections, and the initial response of the shell, the fundamental path, is axisymmetric. Solution techniques are developed and presented for the nonlinear fundamental path, and the nonlinear eigenvalue problem that yields the location of the points of the axisymmetric or periodic secondary paths that bifurcate from the axisymmetric fundamental path and the initial mode shapes of the secondary paths at the point of bifurcation of the secondary paths from the nonlinear axisymmetric fundamental path. Using the nonlinear elastic deep shell theory developed in this thesis, the behaviour of perfect and imperfect incomplete spherical shells with fully restrained boundary surface displacements under uniform pressure loading is examined.

This thesis also attempts to derive and present the equations and solution methods used in the nonlinear elastic analysis of thin spherical caps in as general a way as possible, in order that the solution methods developed in the thesis may be applied to shell structures composed of one or more incomplete thin elastic shells of arbitrary shape under arbitrary loading. By extending the derivation of the strain-displacement expressions for shells to include the terms that are quartic in displacements, the limitations present in linear thin shell theory may be lifted, and partial differential equations governing the nonlinear elastic behaviour of thin shells of arbitrary shape may be developed. The methods used to develop the total potential energy functional for pressure loaded spherical shells may be extended and used to express the total potential energy of shell structures composed of one or more incomplete shells of arbitrary shape under arbitrary loading.

NONLINEAR AXISYMMETRIC ELASTIC ANALYSIS OF DEEP SPHERICAL SHELLS

TABLE OF CONTENTS

CHAPTER ONE

INTRODUCTION

CHAPTER ONE

CONTENTS

INTRODUCTION

1.1 THE RESEARCH TOPIC

The theory of shells forms a part of the theory of elasticity concerned with the study of the deformation of elastic bodies under the influence of given loads. In this thesis it will be assumed that the material of the shell is isotropic and obeys Hooke's Law. There are two distinct classes of shells, thick shells and thin shells. A shell will be called thin if the maximum value of the ratio t/R, where t is the thickness of the shell and R is the radius of curvature of the middle surface, can be neglected in comparison with unity. Correspondingly, shells will be called thick whenever such terms cannot be neglected. If the displacements at a point are small in comparison with the thickness of the shell the governing differential equations are linear. If this limitation is not imposed the governing differential equations become nonlinear, and their solution becomes correspondingly more difficult. This thesis is concerned with the development and solution of the nonlinear differential equations that govern the nonlinear elastic behaviour of thin shells.

The method and notation used in deriving the nonlinear strain-displacement relations for thin shells of arbitrary shape, contained in Section 2.2 of Chapter Two, are based on the work of V.V. Novozhilov, Reference [1]. The two hypotheses of Kirchhoff, concerning the normals to the middle surface and the stresses normal to the middle surface, are used. This allows the study of the deformation of the shell to be expressed in terms of the deformation of the middle surface, in the same way as the hypothesis regarding plane sections in the bending of a beam reduces that problem to the study of the bending of the neutral axis. However, the derivation of the strain-displacement relations contained in this work has been extended, by comparison to that given by Novozhilov in Reference [1], to include the terms that are cubic and quartic in the displacements of the middle surface.

As a consequence of using the Kirchhoff assumptions we follow Novozhilov in expressing the opinion " ... that a simpler approach to the study of thin shells is impossible if one has in mind a theory which is to apply to all problems." On the basis of the Kirchhoff assumptions, with their inherent errors, one is thus justified in calling the theory developed in this work the Elastic Nonlinear Technical Theory of Thin Shells.

The nonlinear elastic analysis of thin shells will be illustrated using incomplete spherical shells as an example. In particular the equations and solution methods required for the nonlinear elastic analysis of deep spherical shells, with or without initial axisymmetric imperfections, under a uniform pressure load, and the solution of the linearised eigenvalue problem that yields the initial mode shapes and bifurcation loads of the secondary equilibrium paths which bifurcate from the axisymmetric fundamental equilibrium path are presented in this work. The

classical buckling and postbuckling analysis of complete spherical shells is presented in Chapter Three. Results for initially perfect incomplete spherical shells, with geometric slenderness values of between 3.5 and 60, and for initially imperfect spherical caps with slenderness values of 4, 6, 9, 12 and 30 are presented in Chapter Seven.

The introduction of those useful fictions of an incompressible weightless fluid and of an infinitely rigid material from which to construct a container allow the idealised 'system' under consideration to be represented in Figure 1.1.

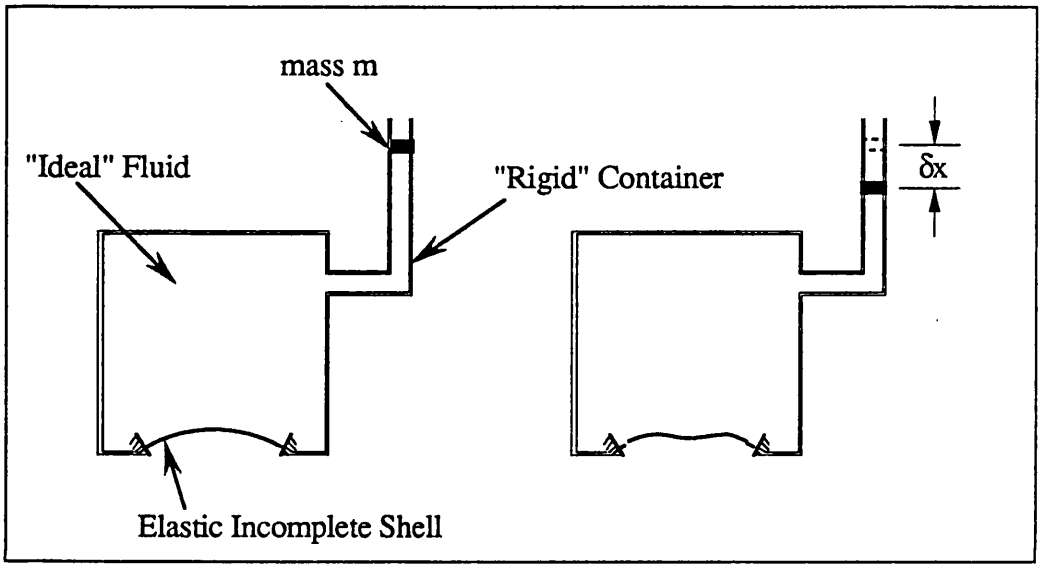


Figure 1.1 The Idealised System Under Consideration

1.2 THE PROBLEM

The particular problem that is addressed in this thesis is the nonlinear elastic analysis of incomplete thin deep spherical shells, where the displacements on the boundary surface are fully restrained and the shell is loaded by a uniform pressure load. The shell may contain initial axisymmetric geometric or stress imperfections, and the initial response of the shell, the fundamental path, is axisymmetric. Solution techniques are developed and presented for the nonlinear fundamental path, and the nonlinear eigenvalue problem that yields the location of the points of the axisymmetric or periodic secondary paths that bifurcate from the axisymmetric fundamental path and the initial mode shapes of the secondary paths at the point of bifurcation of the secondary paths from the nonlinear axisymmetric fundamental path. Using the nonlinear elastic deep shell theory developed in this thesis, the behaviour of perfect and imperfect incomplete spherical shells with fully restrained boundary surface displacements under uniform pressure loading is examined.

This thesis also attempts to derive and present the equations and solution methods used in the nonlinear elastic analysis of thin spherical caps in as general a way as possible, in order that the solution methods developed in the thesis may be applied to shell structures composed of one or more incomplete thin elastic shells of arbitrary shape under arbitrary loading. By extending the derivation of the strain-displacement expressions for shells to include the terms that are quartic in displacements, the limitations present in linear thin shell theory may be lifted, and partial differential equations governing the nonlinear elastic behaviour of thin shells of arbitrary shape may be developed. The methods used to develop the total potential energy functional for pressure loaded spherical shells may be extended and used to express the total potential energy of shell structures composed of one or more incomplete shells of arbitrary shape under arbitrary loading.

1.3 THE MATHEMATICS

To quote from Courant and Robbins, Reference [2]

> "The number continuum, whether it is accepted as a matter of course or only after a critical examination, has been the basis of mathematics – and in particular of analytic geometry and the calculus – since the seventeenth century."

The brief summary of the mathematical concepts implicitly used in the development of the nonlinear differential equations that govern the elastic behaviour of thin shells that is given below is taken, almost entirely, from Reference [2].

The Number Continuum

In passing from the adjective "infinite", meaning simply "without end", to the noun "infinity" we must not make the assumption that "infinity", usually expressed by the special symbol ∞, can be considered as though it were an ordinary number. The symbol ∞ cannot be included in the real number system and at the same time preserve the fundamental rules of arithmetic. Nevertheless the concept of the infinite pervades all of mathematics, since mathematical objects are usually studied not as individuals, but as members of classes or aggregates containing infinitely many objects of the same type, such as the totality of integers or of real numbers. For this reason it is necessary to analyze the mathematical infinite in a precise way. The modern theory of sets, created by Georg Cantor and his school at the end of the nineteenth century, has met this challenge with success. Cantor's theory of sets has influenced many fields of mathematics, and is of basic importance in the study of the logical and philosophical foundations of mathematics, Reference [2].

If the elements in two sets A and B may be paired with each other in such a way that there is a one to one correspondence between the elements of A and the elements of B, then the correspondence is said to be biunique, and A and B are said to be equivalent. Cantor's idea was to extend the concept of equivalence to infinite sets in order to define an "arithmetic" of infinities. Quoting from Courant and Robbins, Reference [2]

> "Cantor made the very significant discovery that *the set of all real numbers*, rational and irrational, *is not denumerable*. In other words, the totality of real numbers presents a radically different and, so to speak, higher type of infinity than that of the integers or of the rational numbers alone."

It may be imagined that the reason for the non-denumerability of the number continuum lies in the fact that the straight line is infinite in extent, and that a finite segment of the line would contain only a denumerable infinity of points. This is not the case, for it is easy to show that the entire number continuum is equivalent to any finite segment, say the segment from 0 to 1 with the endpoints excluded, and it follows that even a finite segment of the number axis

contains a non-denumerable infinity of points. It is this non-denumerable number continuum, say the segment from 0 to 1 with the endpoints excluded, that forms the mathematical basis for both the analytic geometry and the calculus.

Uses of Analytic Geometry and the Calculus

The mathematical tools used in deriving the strain-displacement relations for thin shells of arbitrary geometry are analytic geometry and the differential calculus. The introduction of the assumption that the material is isotropic and obeys Hooke's Law allows the stresses within the shell to be expressed in terms of the strains. Then the First Law of thermodynamics, or the principle that energy is conserved, provides the physical reason for considering the total potential energy of the 'system' and the integral calculus allows the total potential energy of the 'system' to be expressed in the form of an integral. The Second Law of thermodynamics provides the physical justification for studying the equilibrium states of the 'system', and the calculus of variations, in particular the Euler equation, provides the necessary and sufficient condition for equilibrium of the 'system'.

The stability of equilibrium states is not considered in this thesis, however it is of interest to note that stability will be intimately associated with the definition of the 'system'. In particular the inclusion in the 'system' of the fictitious materials is tantamount to the inclusion of infinity in the continuum of the fluid and material elasticities, and therefore in the mathematics describing the 'system'. In the recent past the mathematical proof of the necessary and sufficient conditions, in the calculus of variations, for the stability of an equilibrium state of a system has not been rigorously established, see for example Thompson and Hunt Reference [3]. It may be the case that the recent contribution by S.J.G. Gift, Reference [4], to the calculus of variations of the fifth necessary condition will allow the necessary and sufficient conditions for stability to be proved with mathematical rigour.

1.4 THE PHYSICS - THERMODYNAMICS

The First Law of thermodynamics is a general statement of the principle of conservation of energy. The following statement of the First Law of thermodynamics is taken from D.E. Atkinson, Reference [5].

> "A system may exchange energy with its surroundings in two ways: by gain or loss of heat or by doing work on the surroundings or having work done on it by the surroundings. In its thermodynamic sense, 'work' merely means any energy exchange except heat flow.
>
> The First Law of thermodynamics is usually stated by the equation
>
> $$\Delta E = q - w$$
>
> where ΔE is the change in energy of the system, q is heat transferred to the system from the surroundings, and w is work done by the system on the surroundings. This Law is simple and self-evident."

An adequate statement of the principle of conservation of energy for the frictionless adiabatic system considered in this thesis, Figure 1.1, could also be derived from Newton's Laws.

The Second Law of Thermodynamics is concerned with the equilibrium of the system, and may be stated in a number of ways, the following is one of the statements of Second Law given by Atkinson in Reference [5].

> "For any process that involves energy flow (or energy transduction) there is an equilibrium situation, and if the process proceeds spontaneously it must proceed in the direction of equilibrium."

Quoting from Reference [5] once more

> "In discussions of the Second Law, it is often made to seem that systems go toward equilibrium because they must "obey" the Law. But of course equilibrium states exist in themselves quite independently of any Law or other product of human thought. Thus the Second Law is at base merely a descriptive and intuitively obvious statement concerning equilibrium states. Anyone who has a common-sense feeling that equilibrium states exist, and that systems that are not at equilibrium tend to go toward equilibrium and cannot spontaneously move away from it understand as much of the conceptual basis of the Second Law as there is to understand."

When no heat is transferred to or from the 'system', an adiabatic process, the First Law of thermodynamics allows the energy of the 'system' to be expressed in terms of the potential energy (the elastic strain energy of the deformed shell and the potential energy of the load, i.e. of the mass m in a gravitational field) and the kinetic energy of the shell and the mass m. In this thesis we will consider only static adiabatic conditions, and as a result of imposing these limitations we need only consider the total potential energy of the 'system'. The Second Law of thermodynamics is a formal statement of Physics describing the obvious fact that the equilibrium states of the system are of particular interest.

1.5 THIN SHELL THEORY

In the theory of elasticity, the term shell is applied to bodies bounded by two curved surfaces when the distance between the surfaces is small in comparison with the other dimensions. The locus of points which lie at equal distances from these two surfaces defines the middle surface of the shell. The length intercepted between the two surfaces of the shell by a line perpendicular to the middle surface at any arbitrary point on the middle surface determines the thickness, t, of the shell at that point. The thickness may vary in magnitude, however, in the present work only shells of constant thickness are considered. If the shell has no boundary other than the two surfaces mentioned above it will be called complete, otherwise the shell will be referred to as incomplete. It will be assumed that each edge (also referred to as a boundary, e.g. boundary conditions) of an incomplete shell is a plane surface and that the edge surface and the middle surface of the shell intersect at right angles. The middle surface, thickness, and edges completely define the geometry of a shell.

As mentioned in Section 1.1, it will be assumed that the material of the shell is isotropic and obey Hooke's Law, and that the shell is thin, i.e. that the ratio of the radius of curvature of the shell to the thickness of the shell, t/R, may be neglected in comparison with unity. For most technical applications an admissible maximum value of the ratio of the thickness of the shell to the radius of curvature of the shell of less than 1/50, $t/R<1/50$, will result in sufficiently accurate calculations, and shells for which this inequality is violated will be regarded as thick shells.

The similarity between plates and shells was recognised toward the end of the last century, when Aron first presented a theory of shells based on the assumptions used by Kirchhoff in the theory of plates, Reference [1]. As a result of the similarities between plates and shells the development of both thick and thin shell theories is closely related to the development of the theory of plates. There are two fundamentally different methods for the solution of problems in the classical theory of plates. The first method, which was proposed by Cauchy and Poisson, is based on the expansion of the displacements and stresses in the plate in power series of z, the distance of a given point from the middle plane of the plate, and results in universal approach to the theory of plates. However, the conditions under which the series converge, and the number of boundary conditions and their formulation, gave rise to arguments, Reference [1]. The second method, which was proposed by Kirchhoff, introduced physical meaning into the theory of plates and remains in use today. Kirchhoff based his reasoning on several assumptions, analogous to those used in the theory of beams. The Kirchhoff assumptions used in the theory of plates may be stated as follows

- The straight fibres of a plate which are perpendicular to the middle surface before deformation remain so after deformation and do not change their length.

- The normal stresses acting on planes parallel to the middle surface may be neglected in comparison with the other stresses.

The method of Kirchhoff introduced into the basic theory of plates a simplification which had a definite physical interpretation and which was a completely obvious extension of the well proven theory of beams. The introduction of the concepts of internal forces and moments produced a further link between the theory of plates and the theory of beams. The development of a theory of plates or shells based on the Kirchhoff assumptions reduces the study of the three dimensional plate or shell to that of the two dimensional middle surface, and as a consequence the theory cannot be developed into a more accurate theory. In contrast the Cauchy-Poisson theory has the advantage that, in principle at least, an exact theory may be developed.

Inaccuracies in the thin shell theory developed by Aron were corrected by Love in his theory of thin shells, which is also based on the Kirchhoff theory of plates, References [1] and [6]. However, the development given by Love is inconsistent with regard to small terms, and as a consequence many different versions of the thin shell formulae have resulted.

The deficiencies and limitations that result from the use of the Kirchhoff assumptions as a basis for developing a thin shell theory have been studied by the Russian School in the theory of shells, and summarised by Novozhilov in Reference [1] from which the following quote is taken.

> "Although the works of Galerkin [93, 94, 128] relate to thick shells, he was able by his original methods to obtain all the formulae of the theory of shells from the equations of the general theory of elasticity. He thus played an important part in the development of a mathematically rigorous theory for thin shells. The equations of the theory of thin shells were for the first time deduced in this way by A.I. Lur'e [15, 17] who, however, was not able to give the necessary criterion for simplifying his formulae. This criterion was given in the papers by Novozhilov [30, 28] where the errors introduced by Kirchhoff into the theory of thin shells were elucidated and where it was proved that these errors are of the order t/R in comparison with unity. In this way it was established that the above mentioned assumptions are sufficient for the construction of the theory of thin shells."

The derivation of the strain-displacement relations contained in Chapter Two include the terms that are cubic and quartic in the displacements of the middle surface, and the limitations present in Reference [1], that displacements at a point are small in comparison with the thickness of the shell, may be lifted resulting in the Elastic Nonlinear Technical Theory of Thin Shells.

CHAPTER TWO

THE KIRCHHOFF THIN SHELL EQUATIONS

CHAPTER TWO

CONTENTS

THE KIRCHHOFF THIN SHELL EQUATIONS

2.1 INTRODUCTION

In this chapter nonlinear strain-displacement relations are derived for thin shells of arbitrary shape based on the two assumptions of Kirchhoff. The method and notation used in deriving the strain displacement relations is based on the work of V.V.Novozhilov, Reference [1]. However, by extending the derivations to include the terms that are cubic and quartic in displacements the limitation present in the work of Novozhilov,

> "that displacements at a point are small in comparison with the thickness of the shell"

may be lifted; and for all technical applications 'exact' partial differential equations governing the behaviour of elastic isotropic thin shells are available.

The nonlinear strain displacement relations for thin shells are developed in Section 2.2. These relations are then specialised for shells of spherical geometry. Then in Section 2.3 they are used to express the strain energy of the deformed shell, and integration across the thickness of the shell is performed. At this stage the middle surface strain expressions include cubic and quartic terms, while only linear and quadratic terms are maintained in the curvature expressions.

The load potential energy for uniform pressure loading is derived in Section 2.4, allowing the total potential energy functional to be expressed in terms of the middle surface displacements and their derivatives.

Equilibrium and stability of the system are considered in Section 2.5, and the Euler equation is introduced. Finally, in anticipation of the solution of the equilibrium equations, boundary conditions for spherical shells at the singular points (poles) and for clamped edges are given in Section 2.6.

2.2 STRAIN DISPLACEMENT RELATIONS

The nonlinear strain displacement relations for thin shells are developed in this Section. First some fundamental results from the theory of surfaces are presented in Section 2.2.1.

In Section 2.2.2 the first Kirchhoff assumption is introduced, allowing the displacements through the thickness of the shell (u_z, v_z, w_z) to be expressed in terms of the middle surface (reference surface) displacements and their derivatives.

Then, in Section 2.2.3, the conditions that are required at the boundary of incomplete shells in order that the displacements are continuous within the shell are briefly considered.

The nonlinear strain displacement relations are derived in Section 2.2.4. In Section 2.2.5 the notation that is to be used for spherical shells is introduced, and the nonlinear strain displacement relations for spherical shells are given.

2.2.1 Theory of Surfaces

Let α_1 and α_2 be the curvilinear coordinates of the surface describing lines of principal curvature and hence orthogonal. Then in general the surface of the shell may be described by,

$$\begin{aligned} x &= f_1(\alpha_1, \alpha_2) \\ y &= f_2(\alpha_1, \alpha_2) \\ z &= f_3(\alpha_1, \alpha_2) \end{aligned} \tag{2.1}$$

and in particular for spherical shells,

$$\begin{aligned} x &= r\cos\theta\sin\phi \\ y &= r\sin\theta\sin\phi \\ z &= r\cos\phi \end{aligned} \tag{2.2}$$

where $\phi=\alpha_1$, $\theta=\alpha_2$, see Figure 2.1.

The surface may also be defined by the vector equation,

$$\mathbf{r} = \mathbf{r}(\alpha_1, \alpha_2) \tag{2.3}$$

and introducing the following notation for derivatives:

$$\mathbf{r}_{\alpha_1} = \frac{\partial \mathbf{r}}{\partial \alpha_1}, \qquad \mathbf{r}_{\alpha_2} = \frac{\partial \mathbf{r}}{\partial \alpha_2} \tag{2.4}$$

For spherical shells these expressions reduce to,

$$\mathbf{r}_\phi = r\mathbf{e}_1, \qquad \mathbf{r}_\theta = r\sin\phi\,\mathbf{e}_2$$

where $\mathbf{e}_1$ and $\mathbf{e}_2$ are unit vectors parallel to the α_1 and α_2 (ϕ and θ) coordinate lines of the surface.

The surface vector associated with a small change in α_1 and α_2 is given by,

$$\mathbf{ds} = \mathbf{r}_{\alpha_1} d\alpha_1 + \mathbf{r}_{\alpha_2} d\alpha_2 \tag{2.5}$$

and the square of its length is given by,

$$ds^2 = (\mathbf{r}_{\alpha_1}, \mathbf{r}_{\alpha_1})\, d\alpha_1^2 + 2(\mathbf{r}_{\alpha_1}, \mathbf{r}_{\alpha_2})\, d\alpha_1 d\alpha_2 + (\mathbf{r}_{\alpha_2}, \mathbf{r}_{\alpha_2}) d\alpha_2^2 \tag{2.6}$$

where $(\mathbf{r}_i, \mathbf{r}_j)$ denotes the scalar product, and as $\mathbf{r}_{\alpha 1}$ and $\mathbf{r}_{\alpha 2}$ are orthogonal $(\mathbf{r}_{\alpha 1}, \mathbf{r}_{\alpha 2}) \equiv 0$

The Lame parameters may now be defined as:

$$A_1^2 = (\mathbf{r}_{\alpha_1}, \mathbf{r}_{\alpha_1}) = \left(\frac{\partial x}{\partial \alpha_1}\right)^2 + \left(\frac{\partial y}{\partial \alpha_1}\right)^2 + \left(\frac{\partial z}{\partial \alpha_1}\right)^2 \tag{2.7}$$

$$A_2^2 = (\mathbf{r}_{\alpha_2}, \mathbf{r}_{\alpha_2}) = \left(\frac{\partial x}{\partial \alpha_2}\right)^2 + \left(\frac{\partial y}{\partial \alpha_2}\right)^2 + \left(\frac{\partial z}{\partial \alpha_2}\right)^2$$

Hence,

$$ds^2 = A_1^2 d\alpha_1^2 + A_2^2 d\alpha_2^2 \tag{2.8}$$

and when only α_1 or α_2 vary we have

$$ds_1 = A_1 d\alpha_1 ; \qquad ds_2 = A_2 d\alpha_2 \tag{2.9}$$

Defining the surface unit vectors as the right handed set $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_n)$ we have

$$\mathbf{e}_1 = \frac{1}{A_1}\mathbf{r}_{\alpha_1} \qquad \mathbf{e}_2 = \frac{1}{A_2}\mathbf{r}_{\alpha_2} \qquad \mathbf{e}_n = [\mathbf{e}_1 \bullet \mathbf{e}_2] \tag{2.10}$$

where $[\mathbf{e}_1 \bullet \mathbf{e}_2]$ is the vector product.

Differentiation of unit vectors, Figure 2.2, may be defined as follows:

$$\frac{|\Delta(\mathbf{e}_n)|}{|\mathbf{e}_n|} = \frac{M_1 M_2}{R_1} = \frac{(A_1 d\alpha_1)}{R_1}\mathbf{e}_1$$

Hence,

$$\frac{\partial \mathbf{e}_n}{\partial \alpha_1} = \frac{A_1}{R_1}\mathbf{e}_1 \tag{2.11}$$

similarly

$$\frac{\partial \mathbf{e}_n}{\partial \alpha_2} = \frac{A_2}{R_2}\mathbf{e}_2 \tag{2.12}$$

The expressions (2.11) and (2.12) are valid only when α_1 and α_2 are lines of principal curvature and therefore orthogonal, the proof of this is known as the 'Theorem of Rodriques' in the theory of surfaces.

To find $\frac{\partial \mathbf{e}_i}{\partial \alpha_j}$ where, $i = 1,2$ and $j = 1,2$ we note that,

$$\frac{\partial \mathbf{r}_{\alpha_1}}{\partial \alpha_2} = \frac{\partial \mathbf{r}_{\alpha_2}}{\partial \alpha_1} = \frac{\partial^2 \mathbf{r}}{\partial \alpha_1 \partial \alpha_2} \tag{2.13}$$

and using equation (2.10) we have

$$\frac{\partial A_1 \mathbf{e}_1}{\partial \alpha_2} = \frac{\partial A_2 \mathbf{e}_2}{\partial \alpha_1} \tag{2.14}$$

Consider next the projection of $\frac{\partial \mathbf{e}_1}{\partial \alpha_1}$ and $\frac{\partial \mathbf{e}_1}{\partial \alpha_2}$ on the $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_n$ axes, noting that

$$(\mathbf{e}_1, \frac{\partial \mathbf{e}_1}{\partial \alpha_1}) \equiv (\mathbf{e}_1, \frac{\partial \mathbf{e}_1}{\partial \alpha_2}) \equiv 0$$

and

$$\left(\mathbf{e}_2, \frac{\partial \mathbf{e}_1}{\partial \alpha_1}\right) = \frac{\partial (\mathbf{e}_2, \mathbf{e}_1)}{\partial \alpha_1} - \left(\mathbf{e}_1, \frac{\partial \mathbf{e}_2}{\partial \alpha_1}\right) = -\left(\mathbf{e}_1, \frac{\partial \mathbf{e}_2}{\partial \alpha_1}\right) \tag{2.15}$$

From (2.14) we have,

$$\frac{\partial \mathbf{e}_2}{\partial \alpha_1} = \frac{1}{A_2}\frac{\partial A_1 \mathbf{e}_1}{\partial \alpha_2} - \frac{1}{A_2}\frac{\partial A_2}{\partial \alpha_1}\mathbf{e}_2 \tag{2.16}$$

Hence:

$$\begin{aligned}\left(\mathbf{e}_2, \frac{\partial \mathbf{e}_1}{\partial \alpha_1}\right) &= -\frac{1}{A_2}\left(\mathbf{e}_1, \frac{\partial A_1 \mathbf{e}_1}{\partial \alpha_2}\right) + \frac{1}{A_2}\frac{\partial A_2}{\partial \alpha_1}(\mathbf{e}_1, \mathbf{e}_2) \\ &= -\frac{A_1}{A_2}\left(\mathbf{e}_1, \frac{\partial \mathbf{e}_1}{\partial \alpha_2}\right) - \frac{1}{A_2}\frac{\partial A_1}{\partial \alpha_2}(\mathbf{e}_1, \mathbf{e}_1) \\ &= -\frac{1}{A_2}\frac{\partial A_1}{\partial \alpha_2}(\mathbf{e}_1, \mathbf{e}_1)\end{aligned}$$

therefore

$$\left(\mathbf{e}_2, \frac{\partial \mathbf{e}_1}{\partial \alpha_1}\right) = -\frac{1}{A_2}\frac{\partial A_1}{\partial \alpha_2} \tag{2.17}$$

also

$$\left(\mathbf{e}_n, \frac{\partial \mathbf{e}_1}{\partial \alpha_1}\right) = \frac{\partial (\mathbf{e}_n, \mathbf{e}_1)}{\partial \alpha_1} - \left(\mathbf{e}_1, \frac{\partial \mathbf{e}_n}{\partial \alpha_1}\right) = -\left(\mathbf{e}_1, \frac{\partial \mathbf{e}_n}{\partial \alpha_1}\right)$$

and using equation (2.11)

$$\left(\mathbf{e}_n, \frac{\partial \mathbf{e}_1}{\partial \alpha_1}\right) = -(\mathbf{e}_1, \mathbf{e}_1)\frac{A_1}{R_1} = -\frac{A_1}{R_1} \tag{2.18}$$

Similarly (2.14) and (2.12) may be used to derive the projections of $\frac{d\mathbf{e}_1}{d\alpha_2}$ on $\mathbf{e}_2$ and $\mathbf{e}_n$ giving

$$\left(\mathbf{e}_2, \frac{\partial \mathbf{e}_1}{\partial \alpha_2}\right) = \frac{1}{A_1}\frac{\partial A_2}{\partial \alpha_1} \tag{2.19}$$

$$\left(\mathbf{e}_n, \frac{\partial \mathbf{e}_1}{\partial \alpha_2}\right) = 0 \tag{2.20}$$

The derivatives of $\mathbf{e}_2$ may be found in a similar fashion giving the results listed below.

$$\begin{aligned}
\frac{\partial \mathbf{e}_1}{\partial \alpha_1} &= -\frac{1}{A_2}\frac{\partial A_1}{\partial \alpha_2}\mathbf{e}_2 - \frac{A_1}{R_1}\mathbf{e}_n \\
\frac{\partial \mathbf{e}_1}{\partial \alpha_2} &= \frac{1}{A_1}\frac{\partial A_2}{\partial \alpha_1}\mathbf{e}_2 \\
\frac{\partial \mathbf{e}_2}{\partial \alpha_2} &= -\frac{1}{A_1}\frac{\partial A_2}{\partial \alpha_1}\mathbf{e}_1 - \frac{A_2}{R_2}\mathbf{e}_n \\
\frac{\partial \mathbf{e}_2}{\partial \alpha_1} &= \frac{1}{A_2}\frac{\partial A_1}{\partial \alpha_2}\mathbf{e}_1 \\
\frac{\partial \mathbf{e}_n}{\partial \alpha_1} &= \frac{A_1}{R_1}\mathbf{e}_1 \\
\frac{\partial \mathbf{e}_n}{\partial \alpha_2} &= \frac{A_2}{R_2}\mathbf{e}_2
\end{aligned} \tag{2.21}$$

Also the conditions of Codazzi and Gauss may be derived by considering the following identities:

$$\frac{\partial^2 \mathbf{e}_n}{\partial \alpha_1 \partial \alpha_2} = \frac{\partial^2 \mathbf{e}_n}{\partial \alpha_2 \partial \alpha_1} \quad \text{and} \quad \frac{\partial^2 \mathbf{e}_1}{\partial \alpha_1 \partial \alpha_2} = \frac{\partial^2 \mathbf{e}_1}{\partial \alpha_2 \partial \alpha_1}$$

Substituting from equation (2.21) into the first of the identities above, yields the condition of Codazzi

$$\frac{\partial}{\partial\alpha_1}\left(\frac{A_2}{R_2}\right)=\frac{1}{R_1}\frac{\partial A_2}{\partial\alpha_1}$$
$$\frac{\partial}{\partial\alpha_2}\left(\frac{A_1}{R_1}\right)=\frac{1}{R_2}\frac{\partial A_1}{\partial\alpha_2} \qquad (2.22)$$

and using equation (2.21) in the second identity results in the condition of Gauss.

$$\frac{\partial}{\partial\alpha_1}\left(\frac{1}{A_1}\frac{\partial A_2}{\partial\alpha_1}\right)+\frac{\partial}{\partial\alpha_2}\left(\frac{1}{A_2}\frac{\partial A_1}{\partial\alpha_2}\right)=-\frac{A_1A_2}{R_1R_2} \qquad (2.23)$$

We now have sufficient relations, in equations (2.21), (2.22), and (2.23), to derive the strain displacement expressions for the shell.

2.2.2. Deformation of the Shell and the Nonlinear Variation of the Displacements Within the Thickness of the Shell

Let the arbitrary point O on the middle surface of the shell undergo the displacement Δ as a result of the shell's deformation, Figure 2.3. The point O_1 located on the normal to the middle surface passing through O, and at a distance z from O undergoes the displacement Δ_z. That is, during the process of deformation the point O moves to the position O' and the point O_1 to the position O'_1, Figure 2.3.

Making use of the first of the Kirchhoff assumptions:

- The straight fibres of the shell which are perpendicular to the middle surface before deformation remain so after deformation and do not change their length.

It follows that the segment $O'O'_1$ will be perpendicular to the deformed middle surface and that its length will be equal to that of OO_1, namely z. Hence we have the vector equations

$$z\mathbf{e}_n+\Delta_z=\Delta+z\mathbf{e}_n$$

$$\Delta_z=\Delta+\left(\mathbf{e}'_n-\mathbf{e}_n\right)z \qquad (2.24)$$

and

$$\mathbf{e}'_n=\left[\mathbf{e}'_1\bullet\mathbf{e}'_2\right]$$

where $\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_n$ are the unit vectors of the deformed middle surface.

Defining the projections of Δ and Δ_z on e_1, e_2, and e_n as u, v, w and u_z, v_z, w_z respectively, the vector equation of the deformed middle surface may be written as,

$$\mathbf{R} = \mathbf{r} + \Delta = \mathbf{r} + u\mathbf{e}_1 + v\mathbf{e}_2 + w\mathbf{e}_n \tag{2.25}$$

On differentiation of (2.25) and substitution from (2.21) we have

$$\frac{1}{A_1}\frac{\partial \mathbf{R}}{\partial \alpha_1} = (1+\varepsilon_1)\mathbf{e}_1 + \omega_1 \mathbf{e}_2 - \vartheta \mathbf{e}_n \tag{2.26a}$$

where

$$\varepsilon_1 = \frac{1}{A_1}\frac{\partial u}{\partial \alpha_1} + \frac{1}{A_1 A_2}\frac{\partial A_1}{\partial \alpha_2} v + \frac{w}{R_1} \tag{2.27a}$$

$$\omega_1 = \frac{1}{A_1}\frac{\partial v}{\partial \alpha_1} - \frac{1}{A_1 A_2}\frac{\partial A_1}{\partial \alpha_2} u$$

$$\vartheta = -\frac{1}{A_1}\frac{\partial w}{\partial \alpha_1} + \frac{u}{R_1}$$

Similarly

$$\frac{1}{A_2}\frac{\partial \mathbf{R}}{\partial \alpha_2} = \omega_2 \mathbf{e}_1 + (1+\varepsilon_2)\mathbf{e}_2 - \psi \mathbf{e}_n \tag{2.26b}$$

where

$$\varepsilon_2 = \frac{1}{A_2}\frac{\partial v}{\partial \alpha_2} + \frac{1}{A_1 A_2}\frac{\partial A_2}{\partial \alpha_1} u + \frac{w}{R_2} \tag{2.27b}$$

$$\omega_2 = \frac{1}{A_2}\frac{\partial u}{\partial \alpha_2} - \frac{1}{A_1 A_2}\frac{\partial A_2}{\partial \alpha_1} v$$

$$\psi = -\frac{1}{A_2}\frac{\partial w}{\partial \alpha_2} + \frac{v}{R_2}$$

The local coordinate frame of the deformed middle surface (e'_1, e'_2, e'_n) is given by (2.10) as

$$\mathbf{e}'_1 = \frac{1}{A'_1}\frac{\partial \mathbf{R}}{\partial \alpha_1}; \qquad \mathbf{e}'_2 = \frac{1}{A'_2}\frac{\partial \mathbf{R}}{\partial \alpha_2}; \qquad \mathbf{e}'_n = \left[\mathbf{e}'_1 \bullet \mathbf{e}'_2\right] \tag{2.28}$$

and A'_1 and A'_2 are the Lame parameters of the deformed middle surface given by equation (2.7) as

$$A'_1 = \left\{\left(\frac{\partial \mathbf{R}}{\partial \alpha_1}, \frac{\partial \mathbf{R}}{\partial \alpha_1}\right)\right\}^{\frac{1}{2}} \qquad A'_2 = \left\{\left(\frac{\partial \mathbf{R}}{\partial \alpha_2}, \frac{\partial \mathbf{R}}{\partial \alpha_2}\right)\right\}^{\frac{1}{2}}$$

Hence

$$\dot{A}_1 = A_1\left[(1+\varepsilon_1)^2 + \omega_1^2 + \vartheta^2\right]^{\frac{1}{2}} \tag{2.29}$$

$$\dot{A}_1 = A_1\left[1+\varepsilon_1 + \frac{1}{2}\left(\omega_1^2+\vartheta^2\right) - \frac{1}{2}\varepsilon_1\left(\omega_1^2+\vartheta^2\right) + \frac{1}{2}\varepsilon_1^2\left(\omega_1^2+\vartheta^2\right) - \frac{1}{8}\left(\omega_1^2+\vartheta^2\right)^2 + \text{O 5th}\right]$$

similarly

$$\dot{A}_2 = A_2\left[1+\varepsilon_2 + \frac{1}{2}\left(\omega_2^2+\psi^2\right) - \frac{1}{2}\varepsilon_2\left(\omega_2^2+\psi^2\right) + \frac{1}{2}\varepsilon_2^2\left(\omega_2^2+\psi^2\right) - \frac{1}{8}\left(\omega_2^2+\psi^2\right)^2 + \text{O 5th}\right]$$

and

$$\dot{\mathbf{e}}_1 = \frac{1}{\dot{A}_1}\frac{\partial \mathbf{R}}{\partial \alpha_1} = \frac{A_1}{\dot{A}_1}\left[(1+\varepsilon_1)\mathbf{e}_1 + \omega_1\mathbf{e}_2 - \vartheta\mathbf{e}_n\right]$$

$$\begin{aligned}\dot{\mathbf{e}}_1 = &\left[1 - \frac{1}{2}\left(\omega_1^2+\vartheta^2\right) + \varepsilon_1\left(\omega_1^2+\vartheta^2\right) - \frac{3}{2}\varepsilon_1^2\left(\omega_1^2+\vartheta^2\right) + \frac{3}{8}\left(\omega_1^2+\vartheta^2\right)^2 + \text{O 5th}\right]\mathbf{e}_1 \\ &+ \omega_1\left[1 - \varepsilon_1 - \frac{1}{2}\left(\omega_1^2+\vartheta^2\right) + \varepsilon_1^2 + \frac{3}{2}\varepsilon_1\left(\omega_1^2+\vartheta^2\right) - \varepsilon_1^3 + \text{O 4th}\right]\mathbf{e}_2 \\ &- \vartheta\left[1 - \varepsilon_1 - \frac{1}{2}\left(\omega_1^2+\vartheta^2\right) + \varepsilon_1^2 + \frac{3}{2}\varepsilon_1\left(\omega_1^2+\vartheta^2\right) - \varepsilon_1^3 + \text{O 4th}\right]\mathbf{e}_n\end{aligned} \tag{2.30}$$

Similarly

$$\begin{aligned}\dot{\mathbf{e}}_2 = &\ \omega_2\left[1 - \varepsilon_2 - \frac{1}{2}\left(\omega_2^2+\psi^2\right) + \varepsilon_2^2 + \frac{3}{2}\varepsilon_2\left(\omega_2^2+\psi^2\right) - \varepsilon_2^3 + \text{O 4th}\right]\mathbf{e}_1 \\ &+ \left[1 - \frac{1}{2}\left(\omega_2^2+\psi^2\right) + \varepsilon_2\left(\omega_2^2+\psi^2\right) - \frac{3}{2}\varepsilon_2^2\left(\omega_2^2+\psi^2\right) + \frac{3}{8}\left(\omega_2^2+\psi^2\right)^2 + \text{O 5th}\right]\mathbf{e}_2 \\ &- \psi\left[1 - \varepsilon_2 - \frac{1}{2}\left(\omega_2^2+\psi^2\right) + \varepsilon_2^2 + \frac{3}{2}\varepsilon_2\left(\omega_2^2+\psi^2\right) - \varepsilon_2^3 + \text{O 4th}\right]\mathbf{e}_n\end{aligned} \tag{2.31}$$

also

$$\dot{\mathbf{e}}_n = \left[\dot{\mathbf{e}}_1 \bullet \dot{\mathbf{e}}_2\right]$$

$$\begin{aligned}\dot{\mathbf{e}}_n = &\ [\ \vartheta - \varepsilon_1\vartheta - \omega_1\psi - \frac{1}{2}\vartheta\left(\omega_1^2+\vartheta^2+\omega_2^2+\psi^2-2\varepsilon_1^2\right) \\ &+ \omega_1\psi(\varepsilon_1+\varepsilon_2) + \frac{1}{2}\omega_1\psi\left(\omega_1^2+\vartheta^2+\omega_2^2+\psi^2-2\varepsilon_1^2-2\varepsilon_2^2-2\varepsilon_1\varepsilon_2\right) \\ &+ \vartheta\left(\omega_2^2+\psi^2\right)\left(\varepsilon_2+\frac{1}{2}\varepsilon_1\right) - \vartheta\varepsilon_1^3 + \frac{3}{2}\vartheta\varepsilon_1\left(\omega_1^2+\vartheta^2\right) + \text{O 5th}\ ]\ \mathbf{e}_1 \\ &+ [\ \psi - \varepsilon_2\psi - \vartheta\omega_2 - \frac{1}{2}\psi\left(\omega_1^2+\vartheta^2+\omega_2^2+\psi^2-2\varepsilon_2^2\right) \\ &\quad + \vartheta\omega_2(\varepsilon_1+\varepsilon_2) + \frac{1}{2}\omega_2\vartheta\left(\omega_1^2+\vartheta^2+\omega_2^2+\psi^2-2\varepsilon_1^2-2\varepsilon_2^2-2\varepsilon_1\varepsilon_2\right) \\ &\quad + \psi\left(\omega_1^2+\vartheta^2\right)\left(\varepsilon_1+\frac{1}{2}\varepsilon_2\right) - \psi\varepsilon_2^3 + \frac{3}{2}\psi\varepsilon_2\left(\omega_2^2+\psi^2\right) + \text{O 5th}\ ]\ \mathbf{e}_2 \\ &+ [\ 1 - \omega_1\omega_2 - \frac{1}{2}\left(\omega_1^2+\omega_2^2+\vartheta^2+\psi^2\right) + \omega_1\omega_2(\varepsilon_1+\varepsilon_2) + \varepsilon_1\left(\omega_1^2+\vartheta^2\right) \\ &\quad + \varepsilon_2\left(\omega_2^2+\psi^2\right) + \frac{1}{2}\omega_1\omega_2\left(\omega_1^2+\omega_2^2+\vartheta^2+\psi^2-2\varepsilon_1^2-2\varepsilon_2^2-2\varepsilon_1\varepsilon_2\right) \\ &\quad - \frac{3}{2}\varepsilon_1^2\left(\omega_1^2+\vartheta^2\right) - \frac{3}{2}\varepsilon_2^2\left(\omega_2^2+\psi^2\right) + \frac{3}{8}\left(\omega_1^2+\vartheta^2\right)^2 \\ &\quad + \frac{3}{8}\left(\omega_2^2+\psi^2\right)^2 + \frac{1}{4}\left(\omega_1^2+\vartheta^2\right)\left(\omega_2^2+\psi^2\right) + \text{O 5th}\ ]\ \mathbf{e}_n\end{aligned} \tag{2.32}$$

From equation (2.24), $\Delta_z=\Delta+(e'_n-e_n)z$ where $\Delta=ue_1+ve_2+we_n$ and substituting for e'_n from equation (2.32) the variation of the displacement as a function of z is given by:

$$\Delta_z = u_z e_1 + v_z e_2 + w_z e_n$$

where

$$\begin{aligned} u_z = u + z\,[&\vartheta - \vartheta\varepsilon_1 - \omega_1\psi - \tfrac{1}{2}\vartheta\left(\omega_1^2+\vartheta^2+\omega_2^2+\psi^2-2\varepsilon_1^2\right) \\ &+ \omega_1\psi(\varepsilon_1+\varepsilon_2) + \tfrac{1}{2}\omega_1\psi\left(\omega_1^2+\vartheta^2+\omega_2^2+\psi^2-2\varepsilon_1^2-2\varepsilon_2^2-2\varepsilon_1\varepsilon_2\right) \\ &+ \vartheta\left(\omega_2^2+\psi^2\right)\left(\varepsilon_2+\tfrac{1}{2}\varepsilon_1\right) - \vartheta\varepsilon_1^3 + \tfrac{3}{2}\vartheta\varepsilon_1\left(\omega_1^2+\vartheta^2\right) + \text{O 5th}\,] \end{aligned} \tag{2.33}$$

$$\begin{aligned} v_z = v + z\,[&\psi - \psi\varepsilon_2 - \omega_2\vartheta - \tfrac{1}{2}\psi\left(\omega_1^2+\vartheta^2+\omega_2^2+\psi^2-2\varepsilon_2^2\right) \\ &+ \vartheta\omega_2(\varepsilon_1+\varepsilon_2) + \tfrac{1}{2}\omega_2\vartheta\left(\omega_1^2+\vartheta^2+\omega_2^2+\psi^2-2\varepsilon_1^2-2\varepsilon_2^2-2\varepsilon_1\varepsilon_2\right) \\ &+ \psi\left(\omega_1^2+\vartheta^2\right)\left(\varepsilon_1+\tfrac{1}{2}\varepsilon_2\right) - \psi\varepsilon_2^3 + \tfrac{3}{2}\psi\varepsilon_2\left(\omega_2^2+\psi^2\right) + \text{O 5th}\,] \end{aligned}$$

$$\begin{aligned} w_z = w - z\,[&\tfrac{1}{2}(\omega_1+\omega_2)^2 + \tfrac{1}{2}\left(\vartheta^2+\psi^2\right) - \omega_1\omega_2(\varepsilon_1+\varepsilon_2) \\ &- \varepsilon_1\left(\omega_1^2+\vartheta^2\right) - \varepsilon_2\left(\omega_2^2+\psi^2\right) + \tfrac{3}{2}\varepsilon_1^2\left(\omega_1^2+\vartheta^2\right) \\ &- \tfrac{1}{2}\omega_1\omega_2\left(\omega_1^2+\vartheta^2+\omega_2^2+\psi^2-2\varepsilon_1^2-2\varepsilon_2^2-2\varepsilon_1\varepsilon_2\right) \\ &+ \tfrac{3}{2}\varepsilon_2^2\left(\omega_2^2+\psi^2\right) - \tfrac{3}{8}\left(\omega_1^2+\vartheta^2\right)^2 - \tfrac{3}{8}\left(\omega_2^2+\psi^2\right)^2 \\ &- \tfrac{1}{4}\left(\omega_1^2+\vartheta^2\right)\left(\omega_2^2+\psi^2\right) + \text{O 5th}\,] \end{aligned}$$

2.2.3 Incomplete Shells and Continuity of Displacements

The intersection of a boundary surface and the middle surface of the shell may be written as,

$$g(\alpha_1,\alpha_2)=0 \qquad \text{or} \qquad \alpha_1 = f_1(\xi)\ ,\ \alpha_2 = f_2(\xi) \tag{2.34}$$

where α_1 and α_2 are the curvilinear coordinates of the shell middle surface describing the lines of principal curvature and therefore orthogonal, and ξ is the curvilinear coordinate, of the boundary, on the middle surface defined by $g(\alpha_1, \alpha_2) = 0$.

Hence at the boundary,

$$\left(\frac{d\alpha_1}{d\xi}\right)^2+\left(\frac{d\alpha_2}{d\xi}\right)^2 = 1 \tag{2.35}$$

The displacements of the shell u_z,v_z and w_z are given by equations (2.33) and (2.27) on the region $g(\alpha_1, \alpha_2) < 0$ (or $g(\alpha_1, \alpha_2) > 0$), where the use of the first Kirchhoff assumption has allowed u_z,v_z and w_z to be defined by the middle surface displacements (u,v,w), their

derivatives with respect to α_1 and α_2 ($\delta u/\delta\alpha_1$, $\delta u/\delta\alpha_2$, $\delta v/\delta\alpha_1$, $\delta v/\delta\alpha_2$, $\delta w/\delta\alpha_1$, $\delta w/\delta\alpha_2$) and the distance normal to the middle surface (z). The limitations of thin shell theory will therefore require the boundary surface to intersect the middle surface of the shell at right angles. Also in order that the derivatives of u, v, and w with respect to α_1 and α_2 are defined on the region $g(\alpha_1, \alpha_2)<0$ (or $g(\alpha_1, \alpha_2)>0$) u, v, and w must be continuous functions of α_1 and α_2 on the region $g(\alpha_1, \alpha_2)\leq 0$ (or $g(\alpha_1, \alpha_2)\geq 0$). And as the choice of the middle surface as the reference surface is arbitrary u_z,v_z and w_z are also required to be continuous on the region $g(\alpha_1, \alpha_2)\leq 0$ (or $g(\alpha_1, \alpha_2)\geq 0$).

When u, v, and w are continuous on only one side of the boundary $g(\alpha_1, \alpha_2) = 0$ (a one-sided neighbourhood of $g(\alpha_1, \alpha_2) = 0$), then the derivatives of u, v, and w with respect to α_1 and α_2 at the boundary $g(\alpha_1, \alpha_2) = 0$ are not defined. Consequently the derivatives of u, v, and w with respect to α_1 and α_2 at the boundary $g(\alpha_1, \alpha_2) = 0$ will be determined so that the displacements u_z,v_z and w_z are continuous on the region $g(\alpha_1, \alpha_2)\leq 0$ (or $g(\alpha_1, \alpha_2)\geq 0$).

Let $\bar{u}_z(\xi)$, $\bar{v}_z(\xi)$ and $\bar{w}_z(\xi)$ be the orthogonal set of displacements of the boundary surface $g(\alpha_1, \alpha_2) = 0$ such that,

$$\bar{u}_z = \bar{u}_0 + z\bar{u}_1 = u_z\frac{d\alpha_2}{d\xi} + v_z\frac{d\alpha_1}{d\xi} \tag{2.36}$$

$$\bar{v}_z = \bar{v}_0 + z\bar{v}_1 = v_z\frac{d\alpha_2}{d\xi} - u_z\frac{d\alpha_1}{d\xi}$$

$$\bar{w}_z = \bar{w}_0 + z\bar{w}_1 = w_z$$

Hence

$$u = \bar{u}_0\frac{d\alpha_2}{d\xi} - \bar{v}_0\frac{d\alpha_1}{d\xi} \tag{2.37}$$

$$v = \bar{v}_0\frac{d\alpha_2}{d\xi} + \bar{u}_0\frac{d\alpha_1}{d\xi}$$

$$w = \bar{w}_0$$

and

$$K_u = \bar{u}_1\frac{d\alpha_2}{d\xi} - \bar{v}_1\frac{d\alpha_1}{d\xi} \tag{2.38}$$

$$K_v = \bar{v}_1\frac{d\alpha_2}{d\xi} + \bar{u}_1\frac{d\alpha_1}{d\xi}$$

$$K_w = \bar{w}_1$$

where

$$K_u = \vartheta - \vartheta\varepsilon_1 - \omega_1\psi - \frac{1}{2}\vartheta\left(\omega_1^2 + \vartheta^2 + \omega_2^2 + \psi^2 - 2\varepsilon_1^2\right)$$
$$+ \omega_1\psi(\varepsilon_1 + \varepsilon_2) + \frac{1}{2}\omega_1\psi\left(\omega_1^2 + \vartheta^2 + \omega_2^2 + \psi^2 - 2\varepsilon_1^2 - 2\varepsilon_2^2 - 2\varepsilon_1\varepsilon_2\right)$$
$$+ \vartheta\left(\omega_2^2 + \psi^2\right)\left(\varepsilon_2 + \frac{1}{2}\varepsilon_1\right) - \vartheta\varepsilon_1^3 + \frac{3}{2}\vartheta\varepsilon_1\left(\omega_1^2 + \vartheta^2\right) + \text{O 5th}$$

$$K_v = \psi - \psi\varepsilon_2 - \omega_2\vartheta - \frac{1}{2}\psi\left(\omega_1^2 + \vartheta^2 + \omega_2^2 + \psi^2 - 2\varepsilon_2^2\right)$$
$$+ \vartheta\omega_2(\varepsilon_1 + \varepsilon_2) + \frac{1}{2}\omega_2\vartheta\left(\omega_1^2 + \vartheta^2 + \omega_2^2 + \psi^2 - 2\varepsilon_1^2 - 2\varepsilon_2^2 - 2\varepsilon_1\varepsilon_2\right)$$
$$+ \psi\left(\omega_1^2 + \vartheta^2\right)\left(\varepsilon_1 + \frac{1}{2}\varepsilon_2\right) - \psi\varepsilon_2^3 + \frac{3}{2}\psi\varepsilon_2\left(\omega_2^2 + \psi^2\right) + \text{O 5th}$$

$$K_w = -\frac{1}{2}(\omega_1 + \omega_2)^2 - \frac{1}{2}\left(\vartheta^2 + \psi^2\right) + \omega_1\omega_2(\varepsilon_1 + \varepsilon_2)$$
$$+ \varepsilon_1\left(\omega_1^2 + \vartheta^2\right) + \varepsilon_2\left(\omega_2^2 + \psi^2\right) - \frac{3}{2}\varepsilon_1^2\left(\omega_1^2 + \vartheta^2\right)$$
$$+ \frac{1}{2}\omega_1\omega_2\left(\omega_1^2 + \vartheta^2 + \omega_2^2 + \psi^2 - 2\varepsilon_1^2 - 2\varepsilon_2^2 - 2\varepsilon_1\varepsilon_2\right)$$
$$- \frac{3}{2}\varepsilon_2^2\left(\omega_2^2 + \psi^2\right) + \frac{3}{8}\left(\omega_1^2 + \vartheta^2\right)^2 + \frac{3}{8}\left(\omega_2^2 + \psi^2\right)^2$$
$$+ \frac{1}{4}\left(\omega_1^2 + \vartheta^2\right)\left(\omega_2^2 + \psi^2\right) + \text{O 5th}$$

and ε_1, ω_1, and ϑ and ε_2, ω_2, and ψ are given by equation (2.27).

Equations (2.37) and (2.38) are the boundary conditions that must be satisfied if the displacements u_z,v_z and w_z are to be defined by the thin shell equations (2.33) and (2.27) on the region $g(\alpha_1, \alpha_2)<0$ (or $g(\alpha_1, \alpha_2)>0$), and be continuous on the region $g(\alpha_1, \alpha_2)\leq 0$ (or $g(\alpha_1, \alpha_2)\geq 0$). The homogeneous boundary condition, all displacements on the boundary are zero, is of particular interest in its own right and in the idealized modelling of clamped shells and will be considered briefly below.

For a fully restrained boundary $\bar{u}_z(\xi) = \bar{v}_z(\xi) = \bar{w}_z(\xi) = 0$ for all ξ, hence

$$\bar{u}_0 = \bar{u}_1 = \frac{d\bar{u}_o}{d\xi} = 0 \qquad (2.39)$$

$$\bar{v}_0 = \bar{v}_1 = \frac{d\bar{v}_o}{d\xi} = 0$$

$$\bar{w}_0 = \bar{w}_1 = \frac{d\bar{w}_o}{d\xi} = 0$$

When $d\alpha_1/d\xi = 0$ on the boundary ($\alpha_1 = f_1(\xi)$ = constant, $\alpha_2 = \xi$) equations (2.37), (2.38) and (2.39) are satisfied when the displacements of the middle surface and their derivatives with respect to α_1 and α_2 at the boundary $g(\alpha_1, \alpha_2) = 0$ are given by,

$$u = 0 \quad \Rightarrow \quad \frac{\partial u}{\partial \alpha_2} = 0 \qquad (2.40)$$

$$v = 0 \quad \Rightarrow \quad \frac{\partial v}{\partial \alpha_2} = 0$$

$$w = 0 \quad \Rightarrow \quad \frac{\partial w}{\partial \alpha_2} = 0$$

$$\frac{\partial w}{\partial \alpha_1} = 0$$

$$\frac{\partial u}{\partial \alpha_1} = a \quad \Rightarrow \quad \frac{\partial v}{\partial \alpha_1} = 0 \text{ or } \frac{\partial v}{\partial \alpha_1} = \pm\sqrt{\frac{12a^2 - 8A_1 a + 4A_1^2}{3}}$$

Similarly when $d\alpha_2/d\xi = 0$ on the boundary ($\alpha_1 = \xi$, $\alpha_2 = f_2(\xi)$ = constant) the middle surface displacements and their derivatives at the boundary are given by,

$$u = 0 \quad \Rightarrow \quad \frac{\partial u}{\partial \alpha_1} = 0 \tag{2.41}$$
$$v = 0 \quad \Rightarrow \quad \frac{\partial v}{\partial \alpha_1} = 0$$
$$w = 0 \quad \Rightarrow \quad \frac{\partial w}{\partial \alpha_1} = 0$$
$$\frac{\partial w}{\partial \alpha_2} = 0$$
$$\frac{\partial v}{\partial \alpha_2} = a \quad \Rightarrow \quad \frac{\partial u}{\partial \alpha_2} = 0 \text{ or } \frac{\partial u}{\partial \alpha_2} = \pm \sqrt{\frac{12a^2 - 8A_2 a + 4A_2^2}{3}}$$

When both $d\alpha_1/d\xi$ and $d\alpha_2/d\xi$ are non zero it may be shown that equations (2.37), (2.38) and (2.39) are satisfied when,

$$u = 0 \tag{2.42a}$$
$$v = 0$$
$$w = 0$$
$$\frac{\partial w}{\partial \alpha_1} = 0 \quad \Rightarrow \quad \frac{\partial w}{\partial \alpha_2} = 0$$

and the four first derivatives of u and v with respect to α_1 and α_2 at the boundary are chosen to satisfy the following three equations,

$$\frac{\partial u}{\partial \alpha_1}\frac{d\alpha_1}{d\xi} + \frac{\partial u}{\partial \alpha_2}\frac{d\alpha_2}{d\xi} = 0 \tag{2.42b}$$
$$\frac{\partial v}{\partial \alpha_1}\frac{d\alpha_1}{d\xi} + \frac{\partial v}{\partial \alpha_2}\frac{d\alpha_2}{d\xi} = 0$$

$$\tfrac{1}{2}(\omega_1 + \omega_2)^2 - (\omega_1 + \omega_2)(\omega_1\varepsilon_1 + \omega_2\varepsilon_2) - \tfrac{1}{8}(\omega_1 + \omega_2)^2(3\omega_1^2 - 2\omega_1\omega_2 + 3\omega_2^2)$$
$$+ \tfrac{1}{2}(\omega_1 + \omega_2)(3\omega_1\varepsilon_1^2 + 3\omega_2\varepsilon_2^2) - \tfrac{1}{2}\omega_1\omega_2(\varepsilon_1 - \varepsilon_2)^2 + O'5th = 0$$

in which ε_1, ω_1, ε_2, and ω_2, are given in terms of the derivatives of u and v with respect to α_1 and α_2 by equation (2.27). Three of the four derivatives of u and v with respect to α_1 and α_2 may be expressed in terms of the fourth, which may be assigned arbitrarily. In particular when cubic expressions are used for the through thickness displacement relations of the shell, equations (2.42b) are satisfied by,

$$\frac{1}{A_1}\frac{\partial v}{\partial \alpha_1}\frac{d\alpha_1}{d\xi} + \frac{1}{A_1}\frac{\partial u}{\partial \alpha_2}\frac{d\alpha_2}{d\xi} = 0 \tag{2.43}$$
$$\frac{\partial u}{\partial \alpha_1}\frac{d\alpha_1}{d\xi} + \frac{\partial u}{\partial \alpha_2}\frac{d\alpha_2}{d\xi} = 0$$
$$\frac{\partial v}{\partial \alpha_1}\frac{d\alpha_1}{d\xi} + \frac{\partial v}{\partial \alpha_2}\frac{d\alpha_2}{d\xi} = 0$$

which has the solution

$$\frac{\partial u}{\partial \alpha_1} = a \quad \Rightarrow \quad \frac{\partial v}{\partial \alpha_1} = \frac{A_1}{A_2} a \ , \ \frac{\partial u}{\partial \alpha_2} = -a \frac{d\alpha_1}{d\alpha_2}\bigg|_\xi \ , \ \frac{\partial v}{\partial \alpha_2} = -\frac{A_1}{A_2} a \frac{d\alpha_1}{d\alpha_2}\bigg|_\xi \tag{2.44}$$

where a is an arbitrary constant.

In general thin shell theory does not allow the middle surface displacements and their derivatives with respect to α_1 and α_2 to be uniquely determined at the boundary by the displacements of the boundary surface. A fifth condition involving either $\delta u/\delta\alpha_1$ or $\delta v/\delta\alpha_2$ at the boundary is required for the unique definition of the boundary values of the middle surface displacements and their derivatives with respect to α_1 and α_2.

2.2.4 Nonlinear Strain Displacement Relations

The strains, ε, arising from the deformation of the shell's middle surface are given by,

$$\varepsilon = \frac{\dot{ds} - ds}{ds}$$

where

$$ds = A d\alpha, \qquad \dot{ds} = \dot{A} d\alpha$$

Making use of equation (2.29) to substitute for the Lame parameters A, $\dot{A}$ we have

$$\varepsilon_{\alpha_1} = \frac{\dot{A}_1 - A_1}{A_1} = \varepsilon_1 + \frac{1}{2}(\omega_1^2 + \vartheta^2) - \frac{1}{2}\varepsilon_1(\omega_1^2 + \vartheta^2) + \frac{1}{2}\varepsilon_1^2(\omega_1^2 + \vartheta^2) - \frac{1}{8}(\omega_1^2 + \vartheta^2)^2 + O\,5th \tag{2.45}$$

$$\varepsilon_{\alpha_2} = \frac{\dot{A}_2 - A_2}{A_2} = \varepsilon_2 + \frac{1}{2}(\omega_2^2 + \psi^2) - \frac{1}{2}\varepsilon_2(\omega_2^2 + \psi^2) + \frac{1}{2}\varepsilon_2^2(\omega_2^2 + \psi^2) - \frac{1}{8}(\omega_2^2 + \psi^2)^2 + O\,5th \tag{2.46}$$

The shear strain, ω, is given by

$$\sin(\omega) = \left(\dot{\mathbf{e}}_1, \dot{\mathbf{e}}_2\right)$$

and substituting from equations (2.30) and (2.31) we have

$$\begin{aligned}\sin(\omega) = {} & \omega_1 + \omega_2 + (\vartheta\psi - \varepsilon_2\omega_2 - \varepsilon_1\omega_1) \\ & - \frac{1}{2}(\omega_1 + \omega_2)(\omega_1^2 + \vartheta^2 + \omega_2^2 + \psi^2) + \omega_2\varepsilon_2^2 + \omega_1\varepsilon_1^2 \\ & - \vartheta\psi(\varepsilon_1 + \varepsilon_2) + (\omega_1^2 + \vartheta^2)\left(\frac{1}{2}\varepsilon_2\omega_2 + \frac{3}{2}\varepsilon_1\omega_1 + \varepsilon_1\omega_2\right) \\ & + (\omega_2^2 + \psi^2)\left(\frac{1}{2}\varepsilon_1\omega_1 + \frac{3}{2}\varepsilon_2\omega_2 + \varepsilon_2\omega_1\right) - \varepsilon_2^3\omega_2 - \varepsilon_1^3\omega_1 \\ & + O\,5th\end{aligned}$$

Making use of the expansion for x in terms of sin(x)

$$x = \sin x + \frac{1}{6}\sin^3 x + \frac{9}{120}\sin^5 x + \ldots$$

the shear strain may be expressed as follows:

$$\begin{aligned}\omega = {} & \omega_1 + \omega_2 + (\vartheta\psi - \varepsilon_2\omega_2 - \varepsilon_1\omega_1) \\ & - \tfrac{1}{2}(\omega_1 + \omega_2)\left(\omega_1^2 + \vartheta^2 + \omega_2^2 + \psi^2\right) + \tfrac{1}{6}(\omega_1 + \omega_2)^3 \\ & + \omega_2\varepsilon_2^2 + \omega_1\varepsilon_1^2 - \vartheta\psi(\varepsilon_1 + \varepsilon_2) - \varepsilon_2^3\omega_2 - \varepsilon_1^3\omega_1 \\ & + \left(\omega_1^2 + \vartheta^2\right)\left(\tfrac{1}{2}\varepsilon_2\omega_2 + \tfrac{3}{2}\varepsilon_1\omega_1 + \varepsilon_1\omega_2\right) \\ & + \left(\omega_2^2 + \psi^2\right)\left(\tfrac{1}{2}\varepsilon_1\omega_1 + \tfrac{3}{2}\varepsilon_2\omega_2 + \varepsilon_2\omega_1\right) \\ & + \tfrac{1}{2}(\omega_1 + \omega_2)^2(\vartheta\psi - \varepsilon_2\omega_2 - \varepsilon_1\omega_1) + \text{O 5th}\end{aligned} \tag{2.47}$$

To determine the curvature of the shell consider a surface parallel to the middle surface, at a distance z from the middle surface, see Figure 2.4, then we have

$$\begin{aligned} R_1^{(z)} &= R_1 + z \\ R_2^{(z)} &= R_2 + z \end{aligned}$$

and

$$\begin{aligned} ds_1^{(z)} &= A_1\left(1 + \frac{z}{R_1}\right)d\alpha_1 \\ ds_2^{(z)} &= A_2\left(1 + \frac{z}{R_2}\right)d\alpha_2 \end{aligned}$$

hence

$$\begin{aligned} A_1^{(z)} &= A_1\left(1 + \frac{z}{R_1}\right) \\ A_2^{(z)} &= A_2\left(1 + \frac{z}{R_2}\right) \end{aligned} \tag{2.48}$$

We may now express the strain $\varepsilon_{\alpha_1}^{(z)}$ as a function of z by using equation (2.45) as follows :

$$\begin{aligned}\varepsilon_{\alpha_1}^{(z)} = {} & \varepsilon_1^{(z)} + \tfrac{1}{2}\left(\omega_1^{(z)^2} + \vartheta^{(z)^2}\right) - \tfrac{1}{2}\varepsilon_1^{(z)}\left(\omega_1^{(z)^2} + \vartheta^{(z)^2}\right) \\ & + \tfrac{1}{2}\varepsilon_1^{(z)^2}\left(\omega_1^{(z)^2} + \vartheta^{(z)^2}\right) - \tfrac{1}{8}\left(\omega_1^{(z)^2} + \vartheta^{(z)^2}\right)^2 + \text{O 5th}\end{aligned} \tag{2.49}$$

With $\varepsilon_1^{(z)}$, $\omega_1^{(z)}$ and $\vartheta^{(z)}$ given by equations (2.27a) with A_1, A_2, R_1, R_2 and u, v, w replaced by $A_1^{(z)}$, $A_2^{(z)}$, $R_1^{(z)}$, $R_2^{(z)}$, and u_z, v_z, w_z. Performing these substitutions and using the conditions of Coddazi results in

$$\varepsilon_1^{(z)} = \frac{1}{1+\frac{z}{R_1}}\left[\frac{1}{A_1}\frac{\partial u_z}{\partial \alpha_1} + \frac{1}{A_1 A_2}\frac{\partial A_1}{\partial \alpha_2} v_z + \frac{w_z}{R_1}\right] \tag{2.50}$$

$$\omega_1^{(z)} = \frac{1}{1+\frac{z}{R_1}}\left[\frac{1}{A_1}\frac{\partial v_z}{\partial \alpha_1} - \frac{1}{A_1 A_2}\frac{\partial A_1}{\partial \alpha_2} u_z\right]$$

$$\vartheta^{(z)} = \frac{1}{1+\frac{z}{R_1}}\left[-\frac{1}{A_1}\frac{\partial w_z}{\partial \alpha_1} + \frac{u_z}{R_1}\right]$$

Similarly, the strain $\varepsilon_{\alpha_2}^{(z)}$ may be written as,

$$\begin{aligned}\varepsilon_{\alpha_2}^{(z)} &= \varepsilon_2^{(z)} + \frac{1}{2}\left(\omega_2^{(z)^2} + \psi^{(z)^2}\right) - \frac{1}{2}\varepsilon_2^{(z)}\left(\omega_2^{(z)^2} + \psi^{(z)^2}\right) \\ &+ \frac{1}{2}\varepsilon_2^{(z)^2}\left(\omega_2^{(z)^2} + \psi^{(z)^2}\right) - \frac{1}{8}\left(\omega_2^{(z)^2} + \psi^{(z)^2}\right)^2 + O\,5th\end{aligned} \tag{2.51}$$

with

$$\varepsilon_2^{(z)} = \frac{1}{1+\frac{z}{R_2}}\left[\frac{1}{A_2}\frac{\partial v_z}{\partial \alpha_2} + \frac{1}{A_1 A_2}\frac{\partial A_2}{\partial \alpha_1} u_z + \frac{w_z}{R_2}\right] \tag{2.52}$$

$$\omega_2^{(z)} = \frac{1}{1+\frac{z}{R_2}}\left[\frac{1}{A_2}\frac{\partial u_z}{\partial \alpha_2} - \frac{1}{A_1 A_2}\frac{\partial A_2}{\partial \alpha_1} v_z\right]$$

$$\psi^{(z)} = \frac{1}{1+\frac{z}{R_2}}\left[-\frac{1}{A_2}\frac{\partial w_z}{\partial \alpha_2} + \frac{v_z}{R_2}\right]$$

Finally the shear strain, $\omega^{(z)}$, is given by

$$\begin{aligned}\omega^{(z)} &= \omega_1^{(z)} + \omega_2^{(z)} + \left(\vartheta^{(z)}\psi^{(z)} - \varepsilon_2^{(z)}\omega_2^{(z)} - \varepsilon_1^{(z)}\omega_1^{(z)}\right) \\ &- \frac{1}{2}\left(\omega_1^{(z)} + \omega_2^{(z)}\right)\left(\omega_1^{(z)^2} + \vartheta^{(z)^2} + \omega_2^{(z)^2} + \psi^{(z)^2}\right) \\ &+ \frac{1}{6}\left(\omega_1^{(z)} + \omega_2^{(z)}\right)^3 + \omega_2^{(z)}\varepsilon_2^{(z)^2} + \omega_1^{(z)}\varepsilon_1^{(z)^2} \\ &- \vartheta^{(z)}\psi^{(z)}\left(\varepsilon_1^{(z)} + \varepsilon_2^{(z)}\right) - \varepsilon_2^{(z)^3}\omega_2^{(z)} - \varepsilon_1^{(z)^3}\omega_1^{(z)} \\ &+ \left(\omega^{(z)^2} + \vartheta^{(z)^2}\right)\left(\frac{1}{2}\varepsilon_2^{(z)}\omega_2^{(z)} + \frac{3}{2}\varepsilon_1^{(z)}\omega_1^{(z)} + \varepsilon_1^{(z)}\omega_2^{(z)}\right) \\ &+ \left(\omega_2^{(z)^2} + \psi^{(z)^2}\right)\left(\frac{1}{2}\varepsilon_1^{(z)}\omega_1^{(z)} + \frac{3}{2}\omega_2^{(z)}\varepsilon_2^{(z)} + \varepsilon_2^{(z)}\omega_1^{(z)}\right) \\ &+ \frac{1}{2}\left(\omega_1^{(z)} + \omega_2^{(z)}\right)^2\left(\vartheta^{(z)}\psi^{(z)} - \varepsilon_2^{(z)}\omega_2^{(z)} - \varepsilon_1^{(z)}\omega_1^{(z)}\right) + O5th\end{aligned} \tag{2.53}$$

The substitution of equations (2.33) into (2.50) and (2.52) will then allow the strains, equations (2.49), (2.51), (2.53), to be expressed in terms of the displacements of the middle surface u, v, and w and the distance, z, from the middle surface. The derivation of the nonlinear strains (as a function of z) presented above may be applied to any thin shell defined by equations (2.1) or (2.3).

2.2.5 Nonlinear Strain-Displacement Relations Specialised for Spherical Shells.

At this point in the derivation of the strain-displacement relations the formulae will be specialised for shells of spherical geometry. This will reduce the number of terms carried through successive substitutions.

The following notation will be used for shells of spherical geometry.

$$\phi \equiv \alpha_1 \quad , \quad \theta \equiv \alpha_2 \tag{2.54}$$

$$a \equiv R_1 \equiv R_2 \equiv r$$

$$A_1 \equiv a \quad , \quad A_2 \equiv a \sin\phi$$

and differentiation with respect to ϕ and θ will be denoted by,

$$\dot{u} \equiv \frac{\partial u}{\partial \phi} \quad , \quad \ddot{u} \equiv \frac{\partial^2 u}{\partial \phi^2} \quad , \quad \ldots$$

$$u' \equiv \frac{\partial u}{\partial \theta} \quad , \quad u'' \equiv \frac{\partial^2 u}{\partial \theta^2} \quad , \quad \ldots$$

The following notation will be used for the strain components of equation (2.27).

$$\varepsilon_1 = \frac{1}{a}\left(\dot{u} + w\right) \tag{2.55}$$

$$\varepsilon_2 = \frac{1}{a}\left(w + u\cot\phi + \dot{v}/\sin\phi\right)$$

$$\alpha = \omega_1 = \frac{1}{a}\dot{v}$$

$$\beta = \vartheta = \frac{1}{a}\left(u - w'\right)$$

$$\gamma = \omega_2 = \frac{1}{a}\left(\dot{u}/\sin\phi - v\cot\phi\right)$$

$$\delta = \psi = \frac{1}{a}\left(v - w'/\sin\phi\right)$$

On substitution of (2.33), (2.50) and (2.52) into (2.49), (2.51) and (2.53) we have,

$$\varepsilon_{\phi}^{(z)} = \varepsilon_{\alpha_1}^{(z)} = \frac{1}{1+\frac{z}{a}} \left[\left\{ \varepsilon_1 + \frac{1}{2}(\alpha^2+\beta^2) - \frac{1}{2}\varepsilon_1(\alpha^2+\beta^2) + \frac{1}{2}\varepsilon_1^2(\alpha^2+\beta^2) - \frac{1}{8}(\alpha^2+\beta^2)^2 + O5th \right\} \right.$$
$$\left. + \frac{z}{a}\left\{ \dot{\beta} - \dot{\beta}\varepsilon_1 - \beta\dot{\varepsilon}_1 - \dot{\alpha}\delta - \frac{1}{2}(\alpha+\gamma)^2 - \frac{1}{2}\alpha^2 - \frac{1}{2}\delta^2 + O3rd \right\} \right] \qquad (2.56)$$

$$\varepsilon_{\theta}^{(z)} = \varepsilon_{\alpha_2}^{(z)} = \frac{1}{1+\frac{z}{a}} \left[\left\{ \varepsilon_2 + \frac{1}{2}(\gamma^2+\delta^2) - \frac{1}{2}\varepsilon_2(\gamma^2+\delta^2) + \frac{1}{2}\varepsilon_2^2(\gamma^2+\delta^2) - \frac{1}{8}(\gamma^2+\delta^2)^2 + O5th \right\} \right.$$
$$+ \frac{z}{a} \{ \dot{\delta}/\sin\phi + \beta\cot\phi - \left(\dot{\delta}\varepsilon_2 + \delta\dot{\varepsilon}_2 + \dot{\gamma}\beta \right)/\sin\phi \qquad (2.57)$$
$$\left. - (\beta\varepsilon_1 + \alpha\delta + \gamma\delta)\cot\phi - \frac{1}{2}(\alpha+\gamma)^2 - \frac{1}{2}\gamma^2 - \frac{1}{2}\beta^2 + O3rd \} \right]$$

$$\varepsilon_{\theta\phi}^{(z)} = \frac{1}{2}\omega^{(z)} = \frac{1}{1+\frac{z}{a}} \left[\frac{1}{2} \{ \alpha+\gamma+(\beta\delta - \varepsilon_2\gamma - \varepsilon_1\alpha) - \frac{1}{2}(\alpha+\gamma)(\alpha^2+\beta^2+\gamma^2+\delta^2) \right. \qquad (2.58)$$
$$+ \frac{1}{6}(\alpha+\gamma)^3 + \gamma\varepsilon_2^2 + \alpha\varepsilon_1^2 - \beta\delta(\varepsilon_1+\varepsilon_2) - \varepsilon_2^3\gamma - \varepsilon_1^3\alpha$$
$$+ (\alpha^2+\beta^2)\left(\frac{1}{2}\varepsilon_2\gamma + \frac{3}{2}\varepsilon_1\alpha + \varepsilon_1\gamma\right) + (\gamma^2+\delta^2)\left(\frac{1}{2}\varepsilon_1\alpha + \frac{3}{2}\varepsilon_2\gamma + \varepsilon_2\alpha\right)$$
$$\left. + \frac{1}{2}(\alpha+\gamma)^2(\beta\delta - \varepsilon_2\gamma - \varepsilon_1\alpha) + O5th \} \right.$$
$$+ \frac{z}{2a} \{ \dot{\delta} + \dot{\beta}/\sin\phi - \delta\cot\phi - \dot{\delta}\varepsilon_2 - \delta\dot{\varepsilon}_2 - \dot{\gamma}\beta - \delta\dot{\beta}$$
$$- \left(\dot{\beta}\varepsilon_1 + \beta\dot{\varepsilon}_1 + \dot{\alpha}\delta + \alpha\dot{\delta} \right)/\sin\phi + (\delta\varepsilon_2 + \gamma\beta)\cot\phi + \delta\beta$$
$$- \alpha\dot{\beta} - \varepsilon_2\left(\dot{\beta}/\sin\phi - \delta\cot\phi - \gamma \right) - \varepsilon_1(\dot{\delta} - \dot{\alpha})$$
$$\left. - \gamma\left(\dot{\delta}/\sin\phi + \beta\cot\phi \right) + O3rd \} \right]$$

The nonlinear strain displacement relations for thin spherical shells are given by (2.56), (2.57) and (2.58), with the notation defined by (2.54) and linear strain components given by (2.55).

2.3 THE STRAIN ENERGY FOR A DEFORMED SPHERICAL SHELL

The second Kirchhoff assumption may be stated as follows:

- The normal stresses acting on planes parallel to the middle surface may be neglected in comparison with the other stresses.

This assumption allows the stress-strain relations for an isotropic linear elastic material (Hooke's Law) with modulus of elasticity E, and Poisson's ratio μ to be written as,

$$\sigma_{\phi}^{(z)} = \frac{E}{1-\mu^2}\left(\varepsilon_{\phi}^{(z)} + \mu\varepsilon_{\theta}^{(z)}\right) \tag{2.59}$$

$$\sigma_{\theta}^{(z)} = \frac{E}{1-\mu^2}\left(\varepsilon_{\theta}^{(z)} + \mu\varepsilon_{\phi}^{(z)}\right)$$

$$\sigma_{\phi\theta}^{(z)} = \frac{E}{1+\mu}\left(\varepsilon_{\theta\phi}^{(z)}\right)$$

The elastic strain energy, U, of the spherical shell may then be written as

$$U = \frac{1}{2}\int\int\int\left[\sigma_{\phi}^{(z)}\varepsilon_{\phi}^{(z)} + \sigma_{\theta}^{(z)}\varepsilon_{\theta}^{(z)} + 2\sigma_{\theta\phi}^{(z)}\varepsilon_{\theta\phi}^{(z)}\right]a^2\left(1+\frac{z}{a}\right)^2 \sin\phi \, d\theta d\phi dz \tag{2.60}$$

with the integration being performed over the entire volume of the shell.

Writing the nonlinear strain displacement relations (2.56), (2.57), (2.58) using the notation below,

$$\varepsilon_{\phi}^{(z)} = \frac{1}{1+\frac{z}{a}}\left(\varepsilon_{\phi} + z\chi_{\phi}\right)$$

$$\varepsilon_{\theta}^{(z)} = \frac{1}{1+\frac{z}{a}}\left(\varepsilon_{\theta} + z\chi_{\theta}\right) \tag{2.61}$$

$$\varepsilon_{\theta\phi}^{(z)} = \frac{1}{1+\frac{z}{a}}\left(\varepsilon_{\theta\phi} + z\chi_{\theta\phi}\right)$$

allows the strain energy, U, to be expressed as

$$\begin{aligned} U = \frac{1}{2}\int\int\int\{ & \frac{E}{1-\mu^2}\left[\varepsilon_{\phi} + z\chi_{\phi} + \mu(\varepsilon_{\theta} + z\chi_{\theta})\right]\left[\varepsilon_{\phi} + z\chi_{\phi}\right] \\ & + \frac{E}{1-\mu^2}\left[\varepsilon_{\theta} + z\chi_{\theta} + \mu(\varepsilon_{\phi} + z\chi_{\phi})\right]\left[\varepsilon_{\theta} + z\chi_{\theta}\right] \\ & + \frac{2E}{1+\mu}\left[\varepsilon_{\theta\phi} + z\chi_{\theta\phi}\right]^2\} \, a^2 \sin\phi \, d\theta d\phi dz \end{aligned} \tag{2.62}$$

Performing the integration with respect to z from -t/2 to +t/2 results in

$$U = \frac{1}{2}\int\int \frac{Et}{1-\mu^2}\left[\varepsilon_\phi(\varepsilon_\phi+\mu\varepsilon_\theta)+\varepsilon_\theta(\varepsilon_\theta+\mu\varepsilon_\phi)+2(1-\mu)\varepsilon_{\theta\phi}^2\right]a^2\sin\phi\,d\theta d\phi$$

$$+\frac{1}{2}\int\int \frac{Et^3}{12(1-\mu^2)}\left[\chi_\phi(\chi_\phi+\mu\chi_\theta)+\chi_\theta(\chi_\theta+\mu\chi_\phi)+2(1-\mu)\chi_{\theta\phi}^2\right]a^2\sin\phi\,d\theta d\phi \tag{2.63}$$

Which may also be expressed as,

$$U = \frac{1}{2}\int\int\left[N_\phi\varepsilon_\phi+N_\theta\varepsilon_\theta+2N_{\theta\phi}\varepsilon_{\theta\phi}\right]a^2\sin\phi\,d\theta d\phi$$

$$+\frac{1}{2}\int\int\left[M_\phi\chi_\phi+M_\theta\varepsilon_\theta+2M_{\theta\phi}\chi_{\theta\phi}\right]a^2\sin\phi\,d\theta d\phi \tag{2.64}$$

where

$$N_\phi = \frac{Et}{1-\mu^2}(\varepsilon_\phi+\mu\varepsilon_\theta)\,, \qquad M_\phi = \frac{Et^3}{12(1-\mu^2)}(\chi_\phi+\mu\chi_\theta)$$

$$N_\theta = \frac{Et}{1-\mu^2}(\varepsilon_\theta+\mu\varepsilon_\phi)\,, \qquad M_\theta = \frac{Et^3}{12(1-\mu^2)}(\chi_\theta+\mu\chi_\phi) \tag{2.65}$$

$$N_{\theta\phi} = \frac{Et}{1+\mu}\varepsilon_{\theta\phi}\,, \qquad M_{\theta\phi} = \frac{Et^3}{12(1+\mu)}\chi_{\theta\phi}$$

The nonlinear middle surface strains and curvatures first introduced in (2.61) are given by,

$$\varepsilon_\phi = \varepsilon_1+\frac{1}{2}(\alpha^2+\beta^2)-\frac{1}{2}\varepsilon_1(\alpha^2+\beta^2)+\frac{1}{2}\varepsilon_1^2(\alpha^2+\beta^2)-\frac{1}{8}(\alpha^2+\beta^2)^2+O5th$$

$$\varepsilon_\theta = \varepsilon_2+\frac{1}{2}(\gamma^2+\delta^2)-\frac{1}{2}\varepsilon_2(\gamma^2+\delta^2)+\frac{1}{2}\varepsilon_2^2(\gamma^2+\delta^2)-\frac{1}{8}(\gamma^2+\delta^2)^2+O5th \tag{2.66}$$

$$\begin{aligned}\varepsilon_{\theta\phi} = \frac{1}{2}[&(\alpha+\gamma)+(\beta\delta-\varepsilon_2\gamma-\varepsilon_1\alpha)-\frac{1}{2}(\alpha+\gamma)(\alpha^2+\beta^2+\gamma^2+\delta^2)\\&+\frac{1}{6}(\alpha+\gamma)^3+\gamma\varepsilon_2^2+\alpha\varepsilon_1^2-\beta\delta(\varepsilon_1+\varepsilon_2)-\varepsilon_2^3\gamma-\varepsilon_1^3\alpha\\&+(\alpha^2+\beta^2)(\frac{1}{2}\varepsilon_2\gamma+\frac{3}{2}\varepsilon_1\alpha+\varepsilon_1\gamma)+(\gamma^2+\delta^2)(\frac{1}{2}\varepsilon_1\alpha+\frac{3}{2}\varepsilon_2\gamma+\varepsilon_2\alpha)\\&+\frac{1}{2}(\alpha+\gamma)^2(\beta\delta-\varepsilon_2\gamma-\varepsilon_1\alpha)+O5th]\end{aligned}$$

$$\chi_\phi = \frac{1}{a}\left[\dot{\beta}-\beta\dot{\varepsilon}_1-\beta\varepsilon_1\dot{}-\alpha\delta-\frac{1}{2}(\alpha+\gamma)^2-\frac{1}{2}\alpha^2-\frac{1}{2}\delta^2+O3rd\right]$$

$$\begin{aligned}\chi_\theta = \frac{1}{a}[&\dot{\delta}/\sin\phi+\beta\cot\phi-\left(\dot{\delta}\varepsilon_2+\delta\dot{\varepsilon}_2+\dot{\gamma}\beta\right)/\sin\phi-\frac{1}{2}\beta^2\\&-\frac{1}{2}\gamma^2-\frac{1}{2}(\alpha+\gamma)^2-(\beta\varepsilon_1+\alpha\delta+\gamma\delta)\cot\phi+O3rd]\end{aligned}$$

(2.66 contd.)

$$\chi_{\theta\phi} = \frac{1}{2a}[\dot{\delta} + \dot{\beta}/\sin\phi - \delta\cot\phi - \dot{\delta}\varepsilon_2 - \delta\dot{\varepsilon}_2 - \dot{\gamma}\beta - \gamma\dot{\beta} - \dot{\alpha}\beta - \alpha\dot{\beta} + \dot{\delta}\beta$$
$$- \left(\dot{\beta}\varepsilon_1 + \beta\dot{\varepsilon}_1 + \dot{\alpha}\delta + \alpha\dot{\delta}\right)/\sin\phi + (\delta\varepsilon_2 + \gamma\beta)\cot\phi$$
$$- \varepsilon_2\left(\dot{\beta}/\sin\phi - \delta\cot\phi - \gamma\right) - \gamma\left(\dot{\delta}/\sin\phi + \beta\cot\phi\right)$$
$$- \varepsilon_1\left(\dot{\delta} - \alpha\right) + \text{O3rd}]$$

and from equation (2.55)

$$\varepsilon_1 = \frac{1}{a}(\dot{u} + w) \quad , \quad \varepsilon_2 = \frac{1}{a}\left(\dot{w} + u\cot\phi + \dot{v}/\sin\phi\right) \tag{2.67}$$

$$\alpha = \frac{1}{a}\dot{v} \quad , \quad \beta = \frac{1}{a}(u - \dot{w})$$

$$\gamma = \frac{1}{a}\left(\dot{u}/\sin\phi - v\cot\phi\right) \quad , \quad \delta = \frac{1}{a}\left(v - \dot{w}/\sin\phi\right)$$

The strain energy, U of the shell may now be expressed by equations (2.63) or (2.64) in terms of the displacements of the middle surface (u,v,w) with the aid of (2.66) and (2.67).

Within the limitations of the two Kirchhoff assumptions the expressions for the strain energy above contain all the terms in the displacements up to and including the fifth order terms in the membrane energy, and up to and including the cubic terms in the bending energy.

2.4 THE LOAD POTENTIAL ENERGY FOR SPHERICAL SHELLS

When the external loads are normal to the middle surface of the shell, the change in the load potential energy (J_L) due to the deformation of the shell is given by,

$$J_L = \iint P(\theta, \phi)\Delta V(\theta,\phi)\, d\theta d\phi \qquad (2.68)$$

Where $P(\theta, \phi)$ is the load density and $\Delta V(\theta, \phi)$ is the elemental change in volume, and integration is over the entire middle surface of the shell. The volume enclosed by a deformed spherical surface is given by,

$$V = \iint \frac{1}{2} R_\theta XY\, d\theta d\phi$$

R_θ, X, and Y are defined in Figure 2.5, and given below, in which u,v,w are the middle surface displacements nondimensionalised by the radius (a).

$$R_\theta = a(1 + w + u\cot\phi)\sin\phi$$
$$X = R_\theta d\phi + a\dot{u}\sin\phi\, d\phi - a\dot{w}\cos\phi\, d\phi$$
$$Y = R_\theta d\theta + a\dot{v}\, d\theta$$

For uniform external pressure loading, P is independent of θ and ϕ and the load potential energy may be written in the following form.

$$J_L = -P\iint \Delta V\, a^2 \sin\phi\, d\theta d\phi \qquad (2.69)$$

with ΔV given by

$$\begin{aligned}\Delta V = \frac{1}{2}[\ & 3(w + u\cot\phi) + \dot{v}/\sin\phi + \dot{u} - \dot{w}\cot\phi \\ & + 3(w + u\cot\phi)^2 + \left(\dot{u} - \dot{w}\cot\phi\right)\dot{v}/\sin\phi \\ & + 2(w + u\cot\phi)\left(\dot{v}/\sin\phi + \dot{u} - \dot{w}\cot\phi\right) \\ & + (w + u\cot\phi)^3 + (w + u\cot\phi)^2\left(\dot{v}/\sin\phi + \dot{u} - \dot{w}\cot\phi\right) \\ & + (w + u\cot\phi)\left(\dot{u} - \dot{w}\cot\phi\right)\dot{v}/\sin\phi\ ]\ a\sin^2\phi\end{aligned}$$

2.5 THE EQUILIBRIUM EQUATIONS FOR THIN SPHERICAL SHELLS

The total potential energy V of the system is given by the sum of strain energy U, equation (2.63), and the load potential energy J_L, equation (2.69).

$$V = U + J_L \tag{2.70}$$

By substitution from (2.55) and (2.56) we may write the total potential energy in the following form.

$$V = \int\int F\left(P, \theta, \phi, u, v, w, \dot{u}, \dot{v}, \dot{w}, \dot{u}, \dot{v}, \dot{w}, \ddot{w}, \dot{\ddot{w}}, \ddot{\ddot{w}}\right) d\theta d\phi \tag{2.71}$$

Where θ and ϕ are independent variables and u, v, and w are continuously differentiable functions of θ, ϕ to be determined, and the integration is carried out over the two dimensional middle surface of the shell.

For the system to be in a state of equilibrium it is necessary that the total potential energy be stationary with respect to any small kinematically admissible displacement function. For the equilibrium state to be stable the total potential energy must not only be stationary but must also be a complete relative minimum with respect to any small kinematically admissible displacements.

The calculus of variations, References [7] and [4], may be used to provide the necessary and sufficient conditions for equilibrium and stability.

By expanding the total potential energy as a Taylor series

$$V = \bar{V} + \Delta V \tag{2.72}$$

$$\Delta V = \varepsilon\delta V + \frac{1}{2!}\varepsilon^2\delta^2 V + \frac{1}{3!}\varepsilon^3\delta^3 V + \ldots$$

where δV, $\delta^2 V$, $\delta^3 V$ are the first, second and third variations of V, the necessary condition for equilibrium is given by,

$$\delta V = 0 \tag{2.73}$$

Stability of the equilibrium state requires that,

$$\Delta V = \frac{1}{2!}\varepsilon^2\delta^2 V + \frac{1}{3!}\varepsilon^3\delta^3 V + \ldots > 0 \tag{2.74}$$

and stability will depend upon the sign of the second variation, $\delta^2 V$. If the second variation is also zero then stability will depend on the third, fourth, etc. variations of the total potential energy.

For a uniform pressure loading P, independent of θ and ϕ, the necessary or stationary condition ($\delta V=0$) is given by the Euler equations as follows.

$$E_1 = 0; \quad \frac{d}{d\phi}\left(\frac{\partial F}{\partial \dot{u}}\right) + \frac{d}{d\theta}\left(\frac{\partial F}{\partial \ddot{u}}\right) - \frac{\partial F}{\partial u} = 0 \tag{2.75}$$

$$E_2 = 0; \quad \frac{d}{d\phi}\left(\frac{\partial F}{\partial \dot{v}}\right) + \frac{d}{d\theta}\left(\frac{\partial F}{\partial \ddot{v}}\right) - \frac{\partial F}{\partial v} = 0$$

$$E_3 = 0; \quad \frac{d^2}{d\phi^2}\left(\frac{\partial F}{\partial \ddot{w}}\right) + \frac{d^2}{d\theta d\phi}\left(\frac{\partial F}{\partial \dot{w}}\right) + \frac{d^2}{d\theta^2}\left(\frac{\partial F}{\partial \ddot{w}}\right) - \frac{d}{d\phi}\left(\frac{\partial F}{\partial \dot{w}}\right) - \frac{d}{d\theta}\left(\frac{\partial F}{\partial \ddot{w}}\right) + \frac{\partial F}{\partial w} = 0$$

The Euler equations (2.75) are used in Chapter Four to derive the equilibrium equations for both the fundamental and secondary paths.

2.6 BOUNDARY CONDITIONS FOR INCOMPLETE CLAMPED SPHERES

The presentation of the boundary conditions in this Chapter anticipates the exact circumferential modelling of the shell equations introduced by equation (4.40) in Chapter Four.

In general five boundary conditions (four for the axisymmetric case as $v = 0$) are required at that pole ($\phi=0,\pi$) and at the edges of the spherical cap in order that the displacements are uniquely defined and continuous over the shell, see Section 2.2.3. At the pole the shell is continuous and the five boundary conditions required may be derived by considering symmetry and anti-symmetry. While at the edge of the shell the fifth supplementary boundary condition has been derived by requiring that the first of the Euler equations $E_1 = 0$, equation (2.75), is satisfied at the boundary.

2.6.1 Boundary Conditions at the Pole

The equilibrium equations become singular at the poles ($\phi=0,\pi$) and have been replaced by the following sets of boundary conditions,

$i = 0$ $\quad u = \ddot{u} = 0$ (2.76)

$(v \equiv 0)$

$w = \dddot{w} = 0$

$i = 1, 3, 5, \ldots$ $\quad u = \dot{u} = 0$

$v = 0$

$w = \ddot{w} = 0$

$i = 2, 4, 6, \ldots$ $\quad u = \ddot{u} = 0$

$v = 0$

$w = \dot{w} = 0$

2.6.2 Boundary Conditions for Clamped Incomplete Spheres (Caps)

The 'natural' boundary conditions of the variational calculus, Reference [7], correspond to boundaries at which no work is done. Either the edge deflections and/or rotations are prescribed, or the corresponding edge forces and/or moments are zero.

If work is done at the boundary supports, the supports yield, then the appropriate energy components expressed in terms of boundary integrals must be included in the total potential energy of the system.

For a clamped edge at $\phi=\phi_b$ the following boundary conditions will be used for all circumferential modes (i=0, 1, 2...)

$$u = 0, \quad (u'\cot\phi_b + u'')(1+\alpha) + (w''\cot\phi_b + w''')\alpha = 0 \qquad (2.77)$$

$$v = 0$$

$$w = w' = 0$$

The second supplementary condition for u has been derived by requiring that the first Euler equation $E_1 = 0$, equation (2.75) is satisfied at the boundary.

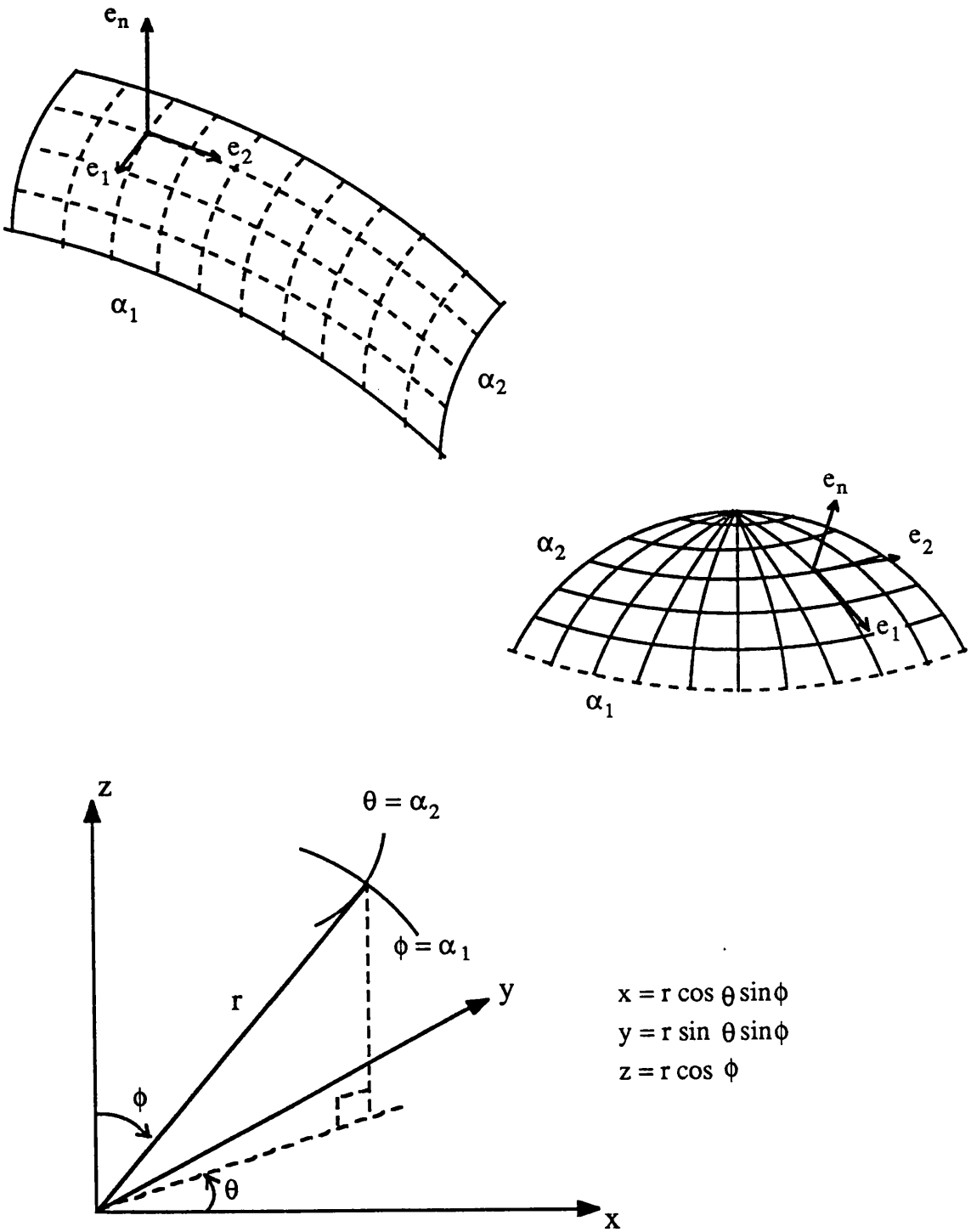


Figure 2.1 Shell Geometry

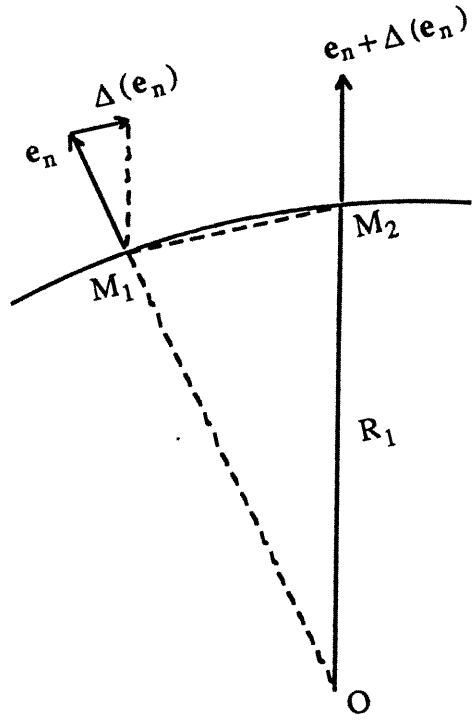


Figure 2.2 Differentiation of a unit vector.

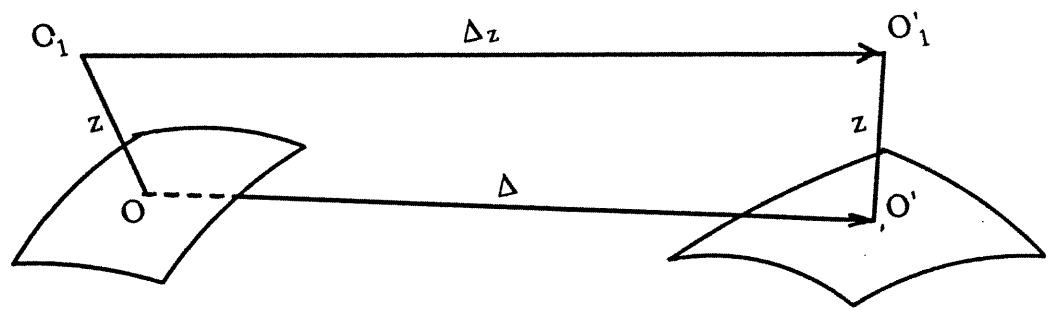


Figure 2.3 Deformations of the Shell.

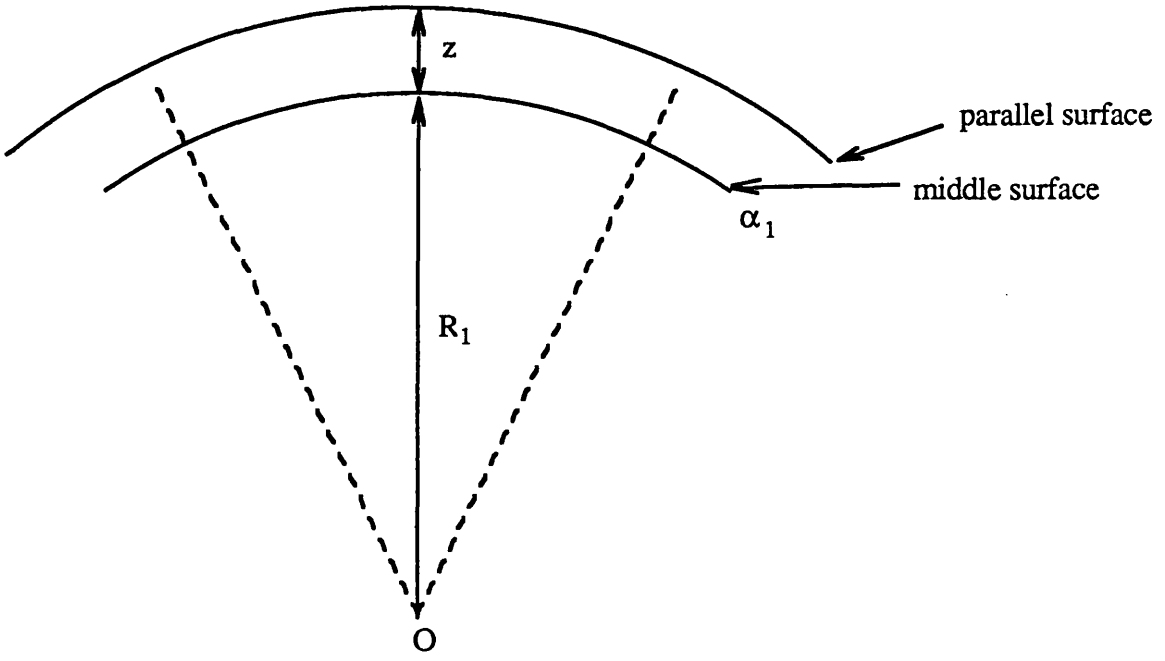
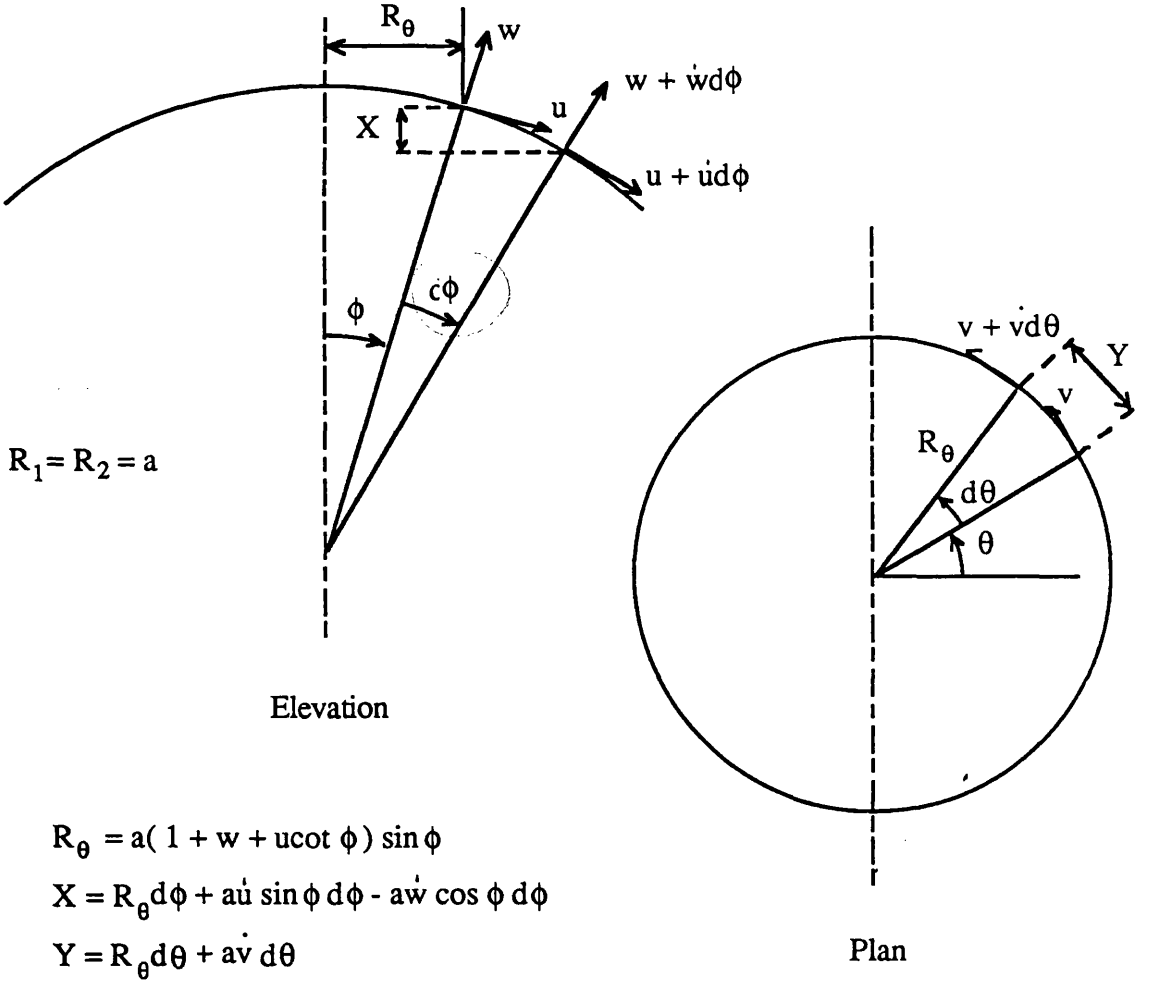


Figure 2.4 Parallel Surface.



$$R_\theta = a(1 + w + u\cot\phi)\sin\phi$$

$$X = R_\theta d\phi + a\dot{u}\sin\phi\, d\phi - a\dot{w}\cos\phi\, d\phi$$

$$Y = R_\theta d\theta + a\dot{v}\, d\theta$$

Figure 2.5 Volume of a Deformed Sphere

CHAPTER THREE

CLASSICAL AND AXISYMMETRIC POSTBUCKLING ANALYSIS OF COMPLETE SPHERICAL SHELLS

CHAPTER THREE

CONTENTS

CLASSICAL AND AXISYMMETRIC POSTBUCKLING ANALYSIS OF COMPLETE SPHERICAL SHELLS

3.1 INTRODUCTION

The first theoretical investigation of the elastic buckling of complete thin spherical shells under uniform pressure loading was undertaken by Zoelly in 1915. This method of analysis, and the lowest buckling or critical pressure that results, are frequently referred to as the classical analysis and the classical buckling or critical pressure of the perfect sphere. In the classical analysis of complete spherical shells buckling or bifurcation pressures are calculated from a linear eigenvalue problem, where the fundamental path consists of a pure radial contraction (membrane stresses) of the sphere, and the secondary or buckling mode is axisymmetric. This method of analysis has become a standard part of the analysis of complete spherical shells, see for example Timoshenko and Grere [8], Flugge [9] or Thompson and Hunt [3].

Although this method of analysis ignores buckling into nonaxisymmetric modes it has been claimed that, for the complete sphere, the eigenvalues associated with the nonaxisymmetric modes are coincident with the axisymmetric mode. This degeneracy being a consequence of the symmetry of the shell and loading. And that the nonaxisymmetric modes may be obtained by superposition of the axisymmetric modes with different orientations of the axes, while the number of possible modes increases with the radius to thickness ratio (r/t). A more detailed discussion of these coincident modes for the complete spherical shell is given by Hutchinson and Koiter [10], Thompson [11] and Silbiger [12].

Initial postbuckling analyses, for axisymmetric buckling of the complete spherical shell has been given by Koiter [13], Thompson [14], and Walker [15]. A multimode analysis which includes mode coupling between nonaxisymmetric and axisymmetric modes is also included in the work of Koiter [13]. In this work Koiter demonstrated that the range of validity of the axisymmetric postbuckling analysis tends to zero as the r/t ratio tends to infinity, as a result of mode coupling with the nonaxisymmetric periodic modes.

Although the classical analysis and the axisymmetric initial postbuckling analysis for complete spherical shells are limited by the considerations mentioned above, they are fundamental to the theoretical understanding of the elastic response of both complete and incomplete spherical shells, and are briefly reviewed below.

3.2 AXISYMMETRIC BUCKLING ANALYSIS OF COMPLETE SPHERICAL SHELLS

The notation used below for the axisymmetric buckling and the initial postbuckling analysis of complete spherical shells follows that used by Walker [15], the full derivation is contained in Appendix A, only the results of interest are presented below.

The axisymmetric strain displacement relations used in the classical analysis are given below, where the terms enclosed in the brackets, { }, are the terms derived in Chapter Two, that are required for consistency in the membrane and bending terms of the strain energy, these terms are neglected in the classical analysis.

$$\varepsilon_\phi = \dot{u} + w + \tfrac{1}{2}\dot{w}^2 + \{u^2 - 2u\dot{w} - \tfrac{1}{2}(\dot{u} + w)(\dot{u} - \dot{w})^2\} \tag{3.1}$$

$$\varepsilon_\theta = u\cot\phi + w$$

$$\chi_\phi = -\tfrac{1}{r}\ddot{w} + \{\tfrac{1}{r}\dot{u}\}$$

$$\chi_\theta = -\tfrac{1}{r}\dot{w}\cot\phi + \{\tfrac{1}{r}u\cot\phi\}$$

As a result of neglecting the terms in the brackets { } the expression for the strain energy, (A.2), is inconsistent in the quadratic terms of the bending energy, and the cubic and quartic terms of the membrane energy.

The load potential energy, (A.4) is given below, where similar linear, quadratic and cubic terms have been neglected from the complete expression given by equation (2.58).

$$J_L = -P2\pi r^3 \int_0^\pi w \sin\phi \, d\phi \tag{3.2}$$

As a result of using the above approximations, the fundamental path displacement is given by equation (A.6) as

$$w^f = \frac{Pr(1-\mu)}{2Et} \tag{3.3}$$

Where r is the radius of the shell, t is the thickness, E the modulus of elasticity, μ is Poisson's ratio, and P is the uniform pressure load, and w^f is the non-dimensional (with respect to r) radial displacement.

The classical critical pressure is then given by equation (A.15c) and (A.6) as

$$P_{cl} = \frac{2E}{\sqrt{3(1-\mu^2)}} \left(\frac{t^2}{r^2}\right) \tag{3.4}$$

The uniform radial contraction, W^f, at the classical critical pressure is given by,

$$\left.\frac{W^f}{t}\right|_{P = P_{cl}} = \frac{(1-\mu)}{\sqrt{3(1-\mu^2)}} \tag{3.5}$$

and for μ=0.3 the radial membrane contraction at the classical buckling pressure is 42.4% of the shell thickness.

The critical, buckling mode is given by the Legendre polynomial of order N, where N is given by (A.15b) as

$$N(N+1) \approx \sqrt{12(1-\mu^2)}\ \frac{r}{t} \tag{3.6}$$

and for large values of N, $N \approx \sqrt[4]{12(1-\mu^2)}\ \sqrt{\frac{r}{t}}$, and for $\mu = 0.3$ then $N \approx 1.8\sqrt{\frac{r}{t}}$.

The results of the classical buckling analysis for complete spherical shells described above are presented in the upper set of curves on Figure 3.1. The individual contributions to the quadratic term of the total potential energy, equation (A.9), that arise from the membrane and bending action of the shell, and the potential energy of the load are given by equation (A.36) of Appendix A. The axisymmetric nature of the buckling displacements in the classical analysis implies that these incremental displacements will develop circumferential bands of tension and compression. For the classical buckling of thin shells, equations (3.4) and (3.6), approximately 50% of the quadratic, buckling, strain energy of the shell is stored as circumferential membrane energy, with approximately the other 50% stored as meridional bending energy.

For buckling wavelengths that are longer than the classical critical wavelength, lower values of N, more than 50% of the incremental buckling strain energy of the shell is stored as circumferential membrane energy. The ability of the shell to develop circumferential bands of membrane compression will be sensitive to the presence of imperfections, therefore the effect of a reduction in the circumferential membrane stiffness on the buckling load will be of interest. The lower set of curves, on Figure 3.1, show the effect of performing the analysis described above with the circumferential membrane energy, $U_{2M\theta}$, set to zero. When the circumferential membrane energy is completely eliminated from the classical analysis the buckling wavelength increases and the buckling load reduces. For a shell with a r/t of 700 the buckling wavelength doubles by comparison to the classical wavelength, N reduces from 48 to 24, and the buckling load is reduced to approximately 20% of the classical value.

3.3 AXISYMMETRIC POSTBUCKLING ANALYSIS OF COMPLETE SPHERICAL SHELLS

The postbuckling path has been obtained by solving for the coefficients of a perturbation expansion about the critical, classical, bifurcation point, this is described in Appendix A. The perturbation series for the postbuckling path, equation (A.14), will contain cubic terms, as the expression for the total potential energy, equations (A.7) and (A.8), contained quartic terms

The postbuckling paths resulting from this analysis have been plotted in Figures 3.2 to 3.6 for values of the geometric slenderness parameter, λ, of between 12 and 70, where λ is defined as,

$$\lambda = \sqrt[4]{12(1-\mu^2)}\sqrt{\frac{r}{t}}\,\alpha \qquad (3.7)$$

in which α is the half angle of the spherical shell, for complete spheres $\alpha=\pi$.

For complete spheres the thin shell assumptions are not valid for values of λ less than about 26, corresponding to a r/t ratio of 20.7, however postbuckling paths have been calculated for λ values less than 26 as they may be representative of the behaviour exhibited by spherical caps, incomplete spherical shells, with equivalent λ values but larger radius to thickness ratios.

Figure 3.7 shows the effect of truncating the perturbation series after the linear, quadratic and cubic terms, for a spherical shell with a slenderness, λ, of 70.

Two points that arise from the classical buckling and postbuckling analysis of complete perfect spherical shells are of particular interest to the nonlinear elastic analysis of spherical caps. Firstly, the total prebuckling displacement, the fundamental path uniform radial contraction of the sphere, is approximately 42% of the shell thickness, equation (3.5), and the load carrying capacity of the shell on the postbuckling paths, Figure 3.2 to Figure 3.6, fall rapidly for small amplitudes of the incremental displacements. In most cases the load carrying capacity of the shell drops to zero for postbuckling displacements with an amplitude of about 20% of the thickness of the shell. Secondly, despite the relatively small amplitude of the displacements, it is at least necessary to model the postbuckling path using a cubic expression, that is the total potential energy must contain at least quartic terms.

The postbuckling behaviour of the complete perfect spherical shell will influence the response of imperfect complete spherical shells or incomplete spherical caps. Therefore, and as the results of the classical analyses described above indicate, it is expected that the nonlinear axisymmetric fundamental path analysis of incomplete spherical caps will require at least cubic terms to be included in the equilibrium equations, that is quartic terms in the expression for the total potential energy.

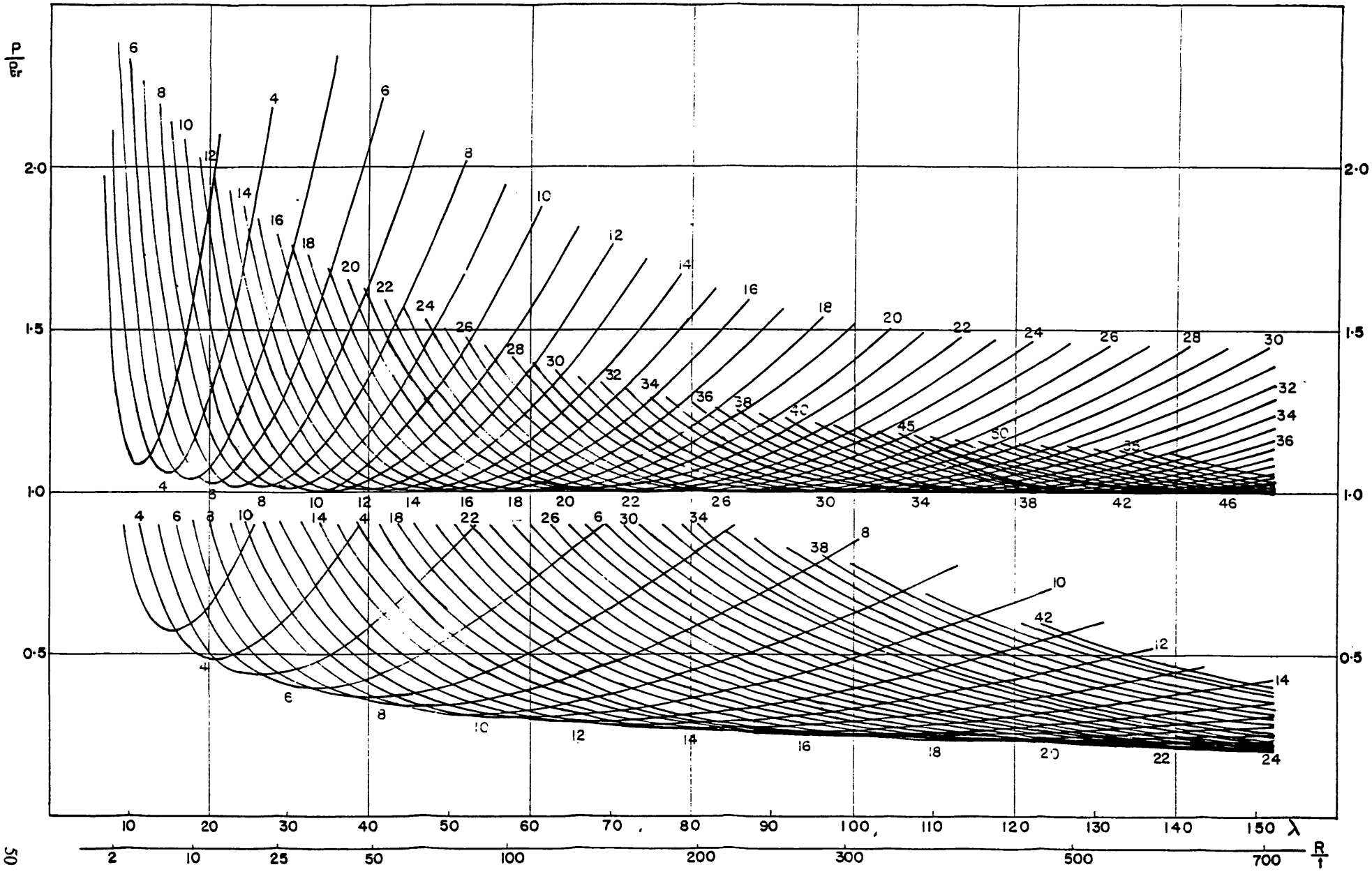


Figure 3.1 Classical Buckling Analysis of Perfect Complete Spherical Shells - upper set of curves, with $U_{2M\theta}=0$ - lower set of curves. The order of the Legendre polynomial is indicated.

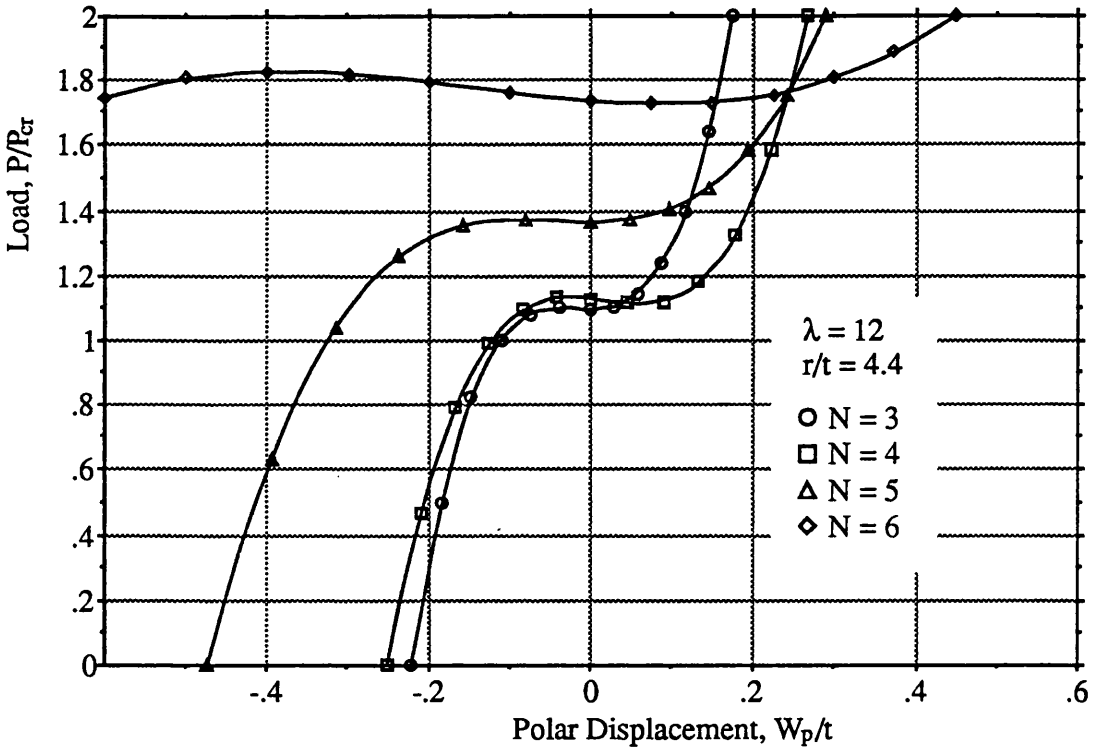


Figure 3.2 Post Buckling Paths for Complete Spherical Shells, $\lambda = 12$, N is the Order of the Legendre Polynomial of the Buckling mode.

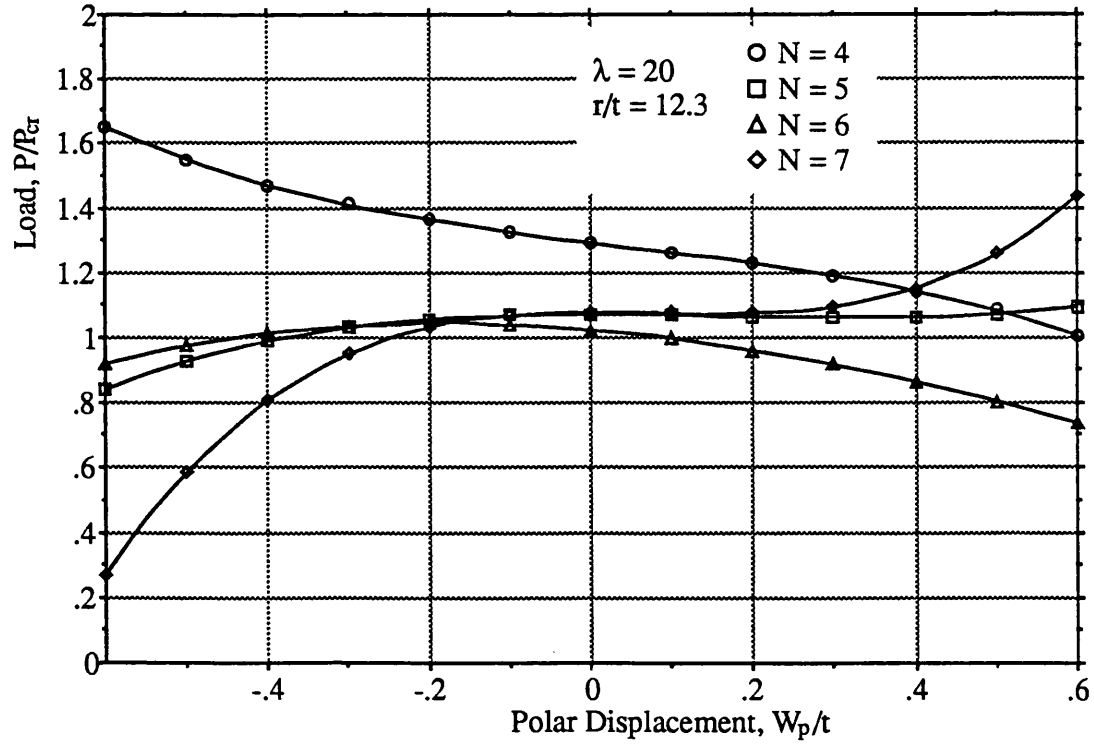


Figure 3.3 Post Buckling Paths for Complete Spherical Shells, $\lambda = 20$, N is the Order of the Legendre Polynomial of the Buckling mode.

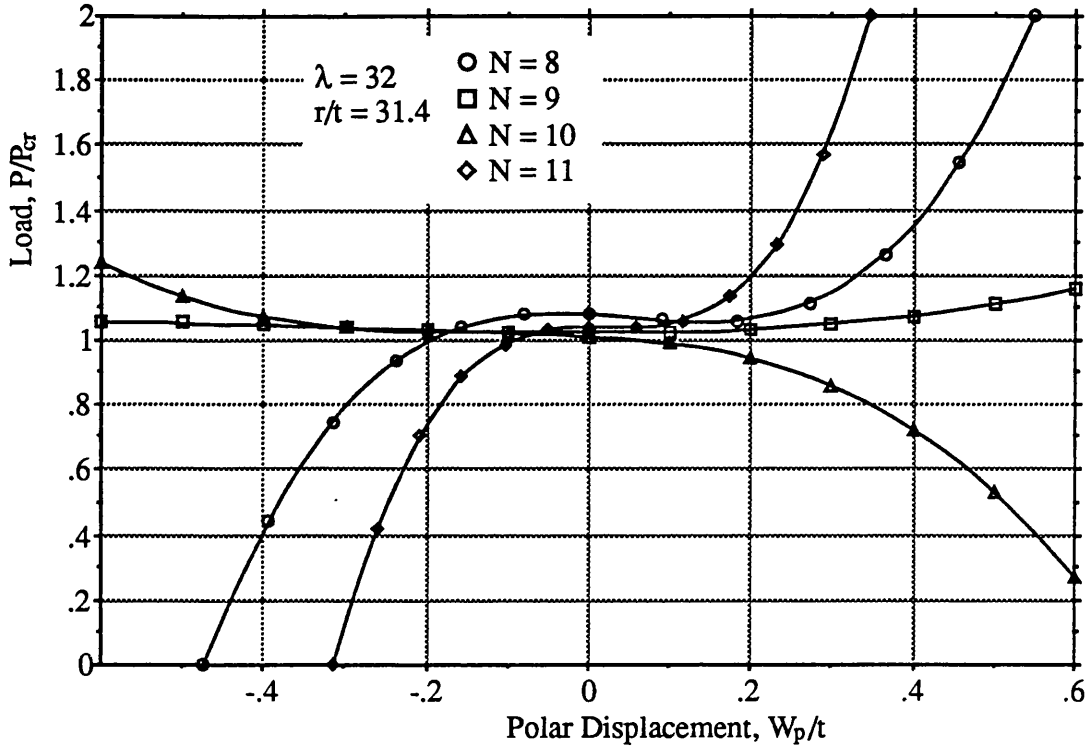


Figure 3.4 Post Buckling Paths for Complete Spherical Shells, $\lambda = 32$, N is the Order of the Legendre Polynomial of the Buckling mode.

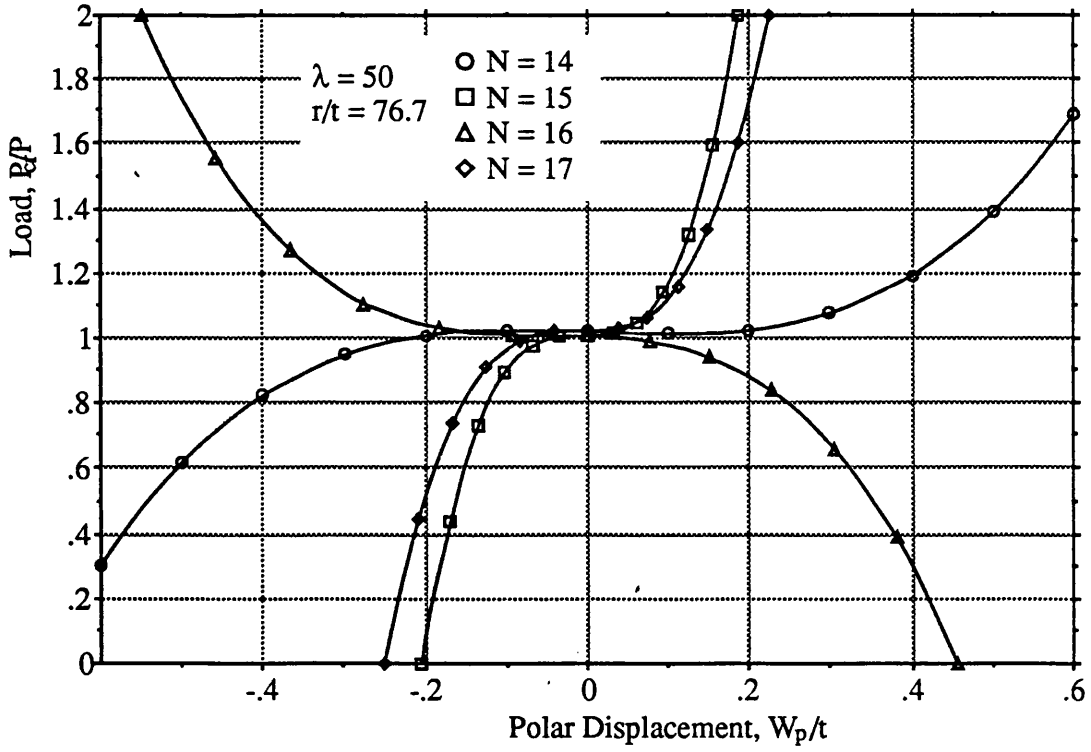


Figure 3.5 Post Buckling Paths for Complete Spherical Shells, $\lambda = 50$, N is the Order of the Legendre Polynomial of the Buckling mode.

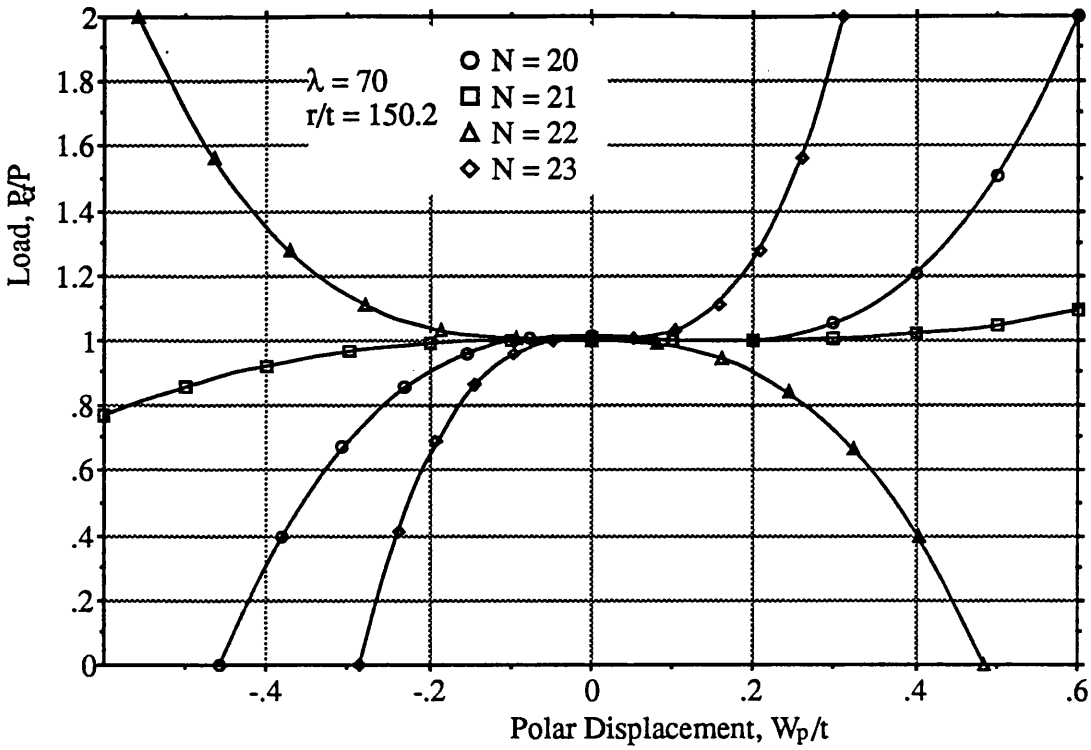


Figure 3.6 Post Buckling Paths for Complete Spherical Shells, $\lambda = 70$, N is the Order of the Legendre Polynomial of the Buckling mode.

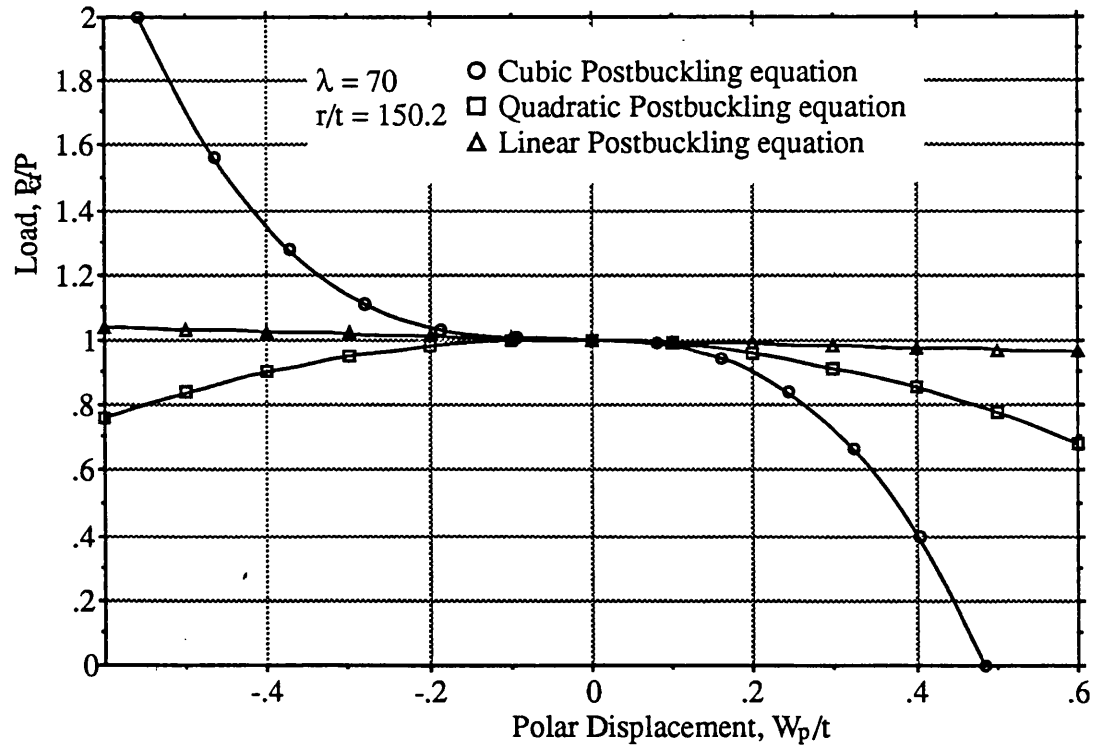


Figure 3.7 Comparisons of Cubic, Quadratic and Linear Post Buckling Paths for a Complete Spherical Shell, $\lambda = 70$.

CHAPTER FOUR

GENERAL SOLUTION METHODS FOR THE NONLINEAR FUNDAMENTAL PATH AND THE LINEARISED SECONDARY PATH

CHAPTER FOUR

CONTENTS

GENERAL SOLUTION METHODS FOR THE NONLINEAR FUNDAMENTAL PATH AND THE LINEARISED SECONDARY PATH

4.1 INTRODUCTION

General solution methods are presented in this Chapter for the nonlinear axisymmetric fundamental path, and the nonlinear eigenvalue problem that yields the location of the points of secondary path bifurcation and the initial mode shapes of the secondary paths at the points of bifurcation of the secondary paths from the nonlinear fundamental path. The nonlinear equilibrium equations of the shell will be formulated in such a way that they lend themselves to a solution procedure based on a finite difference discretization. Numerical solution algorithms, based on finite difference approximations of the partial derivatives that occur in the differential equations of this Chapter, are contained in Chapter Five.

Solution methods developed in this Chapter assume an axisymmetric fundamental path that may contain axisymmetric initial imperfections of a general nature. At zero pressure the initial displacement field and/or stress field will be axisymmetric or zero, and at low pressures the displacements of the shell will remain axisymmetric. As the pressure increases and deformations grow, so periodic components of the deformations may grow.

The fundamental equilibrium path (ρ_1) may be represented by the curve O'ABCD in Figure 4.1. At the maximum pressure, P_{bk}, the path would become unstable and the shell would snap buckle to some distant equilibrium state. A general power, or perturbation, series solution method capable of evaluating any nonlinear fundamental path is developed in Section 4.4 of this Chapter.

Secondary equilibrium paths (ρ_2), with displacements that are periodic in the circumferential direction, may intersect the fundamental path at any point along its length, before or after the buckling point B. Secondary path solutions will consist of solving the nonlinear eigenvalue problem that yields the critical pressure, P_{cr}, and the initial mode shape associated with the secondary path at the point of bifurcation, point A in Figure 4.1.

Figure 4.2 provides a key to the methods and equations used in developing the fundamental and secondary path solutions. The total potential energy functional, Section 4.2, provides the common starting point for both the fundamental and secondary path analysis. Application of the stationary conditions, Sections 4.3 and 4.5, yield the nonlinear differential equations that govern equilibrium of the fundamental and secondary paths. A perturbation method is developed in Section 4.4 for solving the nonlinear fundamental path problem.

The linear terms of the secondary path equation, that is the terms of the secondary path that are linear in the incremental displacements x, give rise to a nonlinear eigenvalue/vector problem, equation (4.39) of Section 4.5. This eigenvalue problem is nonlinear in terms of the

fundamental path displacements and load. An 'efficient' strategy for solving the nonlinear eigenvalue problem is presented in Section 5.4 of Chapter Five.

A perturbation technique similar to that used for the solution of the nonlinear fundamental path could be used to obtain the nonlinear post bifurcation, or secondary, path solution if desired. This nonlinear secondary path perturbation problem is indicated in Figure 4.2 by the dotted box, and has not been attempted in the present work.

Exact circumferential modelling of the middle surface displacements is introduced in Section 4.6, making the derivatives contained in the equations presented in this Chapter functions of the meridional variable only. The energy components of the nonlinear fundamental path (the components of V_0 and V_1) and the quadratic components of the secondary path (the components of V_2) are given in Section 4.7.

Table 4.1 defines the terms of the nonlinear axisymmetric equilibrium equation for thin spherical shells under uniform pressure loading, i.e. the nonlinear axisymmetric fundamental path equation, equation (4.31). And Table 4.2 defines the terms of the linearised secondary path equilibrium equation, i.e. the nonlinear eigenvalue problem, equation (4.39).

4.2 THE NON-LINEAR TOTAL POTENTIAL ENERGY FUNCTIONAL

4.2.1 Displacement Notation

In the following notation the total displacements at the general point, (ϕ, θ), on the middle surface of the spherical cap are written as,

$u^T \equiv u^T(\phi, \theta)$
$v^T \equiv v^T(\phi, \theta)$
$w^T \equiv w^T(\phi, \theta)$
and the uniform pressure, P, is independent of the coordinates ϕ and θ.

By introducing the standard sliding notation, extended to include the initial axisymmetric geometric imperfection, the total displacements (u^T, v^T, w^T) may be expressed as

$$u^T \equiv u^i + u^f + u \tag{4.1}$$

$$v^T \equiv v, \quad v^i \equiv v^f \equiv 0$$

$$w^T \equiv w^i + w^f + w$$

Where (u^i, w^i) are the axisymmetric initial geometric imperfections, and (u^f, w^f) are the axisymmetric fundamental path displacements, while (u, v, w) are the axisymmetric or periodic incremental displacements.

4.2.2 Non-linear strain displacement relations

Introducing the full nonlinear strain and the linear part of the curvature relations (2.66) and using x to denote the generalised displacements (u, v, w) with the superscript notation of (4.1) applying we have,

$$\varepsilon_\phi^T = \varepsilon_\phi(x^T) = \varepsilon_1^T + \frac{1}{2}\left\{\left(v^{T'}\right)^2 + \left(\beta^T\right)^2\right\} - \frac{1}{2}\varepsilon_1^T\left\{\left(v^{T'}\right)^2 + (\beta^T)^2\right\} \tag{4.2}$$
$$+\frac{1}{2}(\varepsilon_1^T)^2\left\{\left(v^{T'}\right)^2 + \left(\beta^T\right)^2\right\} - \frac{1}{8}\left\{\left(v^{T'}\right)^2 + \left(\beta^T\right)^2\right\}^2 + \ldots$$

$$\varepsilon_\theta^T = \varepsilon_\theta(x^T) = \varepsilon_2^T + \frac{1}{2}\left\{(\gamma^T)^2 + (\delta^T)^2\right\} - \frac{1}{2}(\varepsilon_2^T)\left\{(\gamma^T)^2 + (\delta^T)^2\right\}$$
$$+ \frac{1}{2}(\varepsilon_2^T)^2\left\{(\gamma^T)^2 + (\delta^T)^2\right\} - \frac{1}{8}\left\{(\gamma^T)^2 + (\delta^T)^2\right\}^2 + \ldots$$

$$\varepsilon_{\theta\phi}^T = \varepsilon_{\theta\phi}(x^T) = \frac{1}{2}[\left(\dot{v}^T + \gamma^T\right) + \left(\beta^T\delta^T - \varepsilon_1^T\dot{v}^T - \varepsilon_2^T\gamma\right)$$
$$- \frac{1}{2}\left(\dot{v}^T + \gamma^T\right)\left\{\left(\dot{v}^T\right)^2 + (\gamma^T)^2 + (\beta^T)^2 + (\delta^T)^2\right\}$$
$$+ \frac{1}{6}\left(\dot{v}^T + \gamma^T\right)^3 + \gamma^T(\varepsilon_2^T)^2 + \dot{v}^T(\varepsilon_1^T)^2$$
$$- \beta^T\gamma^T(\varepsilon_1^T + \varepsilon_2^T) - (\varepsilon_2^T)^3\gamma^T - (\varepsilon_1^T)^3\dot{v}^T$$
$$+ \left\{\left(\dot{v}^T\right)^2 + (\beta^T)^2\right\}\left(\frac{1}{2}\varepsilon_2^T\gamma^T + \frac{3}{2}\varepsilon_1^T\dot{v}^T + \varepsilon_1^T\gamma\right)$$
$$+ \left\{(\gamma^T)^2 + (\delta^T)^2\right\}\left(\frac{1}{2}\varepsilon_1^T\dot{v}^T + \frac{3}{2}\varepsilon_2^T\gamma^T + \varepsilon_2^T\gamma^T + \varepsilon_2^T\dot{v}^T\right)$$
$$+ \frac{1}{2}\left(\dot{v}^T + \gamma^T\right)^2\left(\beta^T\delta^T - \varepsilon_2^T\gamma^T - \varepsilon_1^T\dot{v}^T\right) + \ldots]$$

$$\chi_\phi^T = \chi_\phi(x^T) = \frac{1}{a}\dot{\beta}^T + \ldots$$

$$\chi_\theta^T = \chi_\theta(x^T) = \frac{1}{a}\left(\dot{\delta}^T/\sin\phi + \beta\cot\phi\right) + \ldots$$

$$\chi_{\theta\phi}^T = \chi_{\theta\phi}(x^T) = \frac{1}{2a}\left(\dot{\delta}^T + \dot{\beta}^T/\sin\phi - \delta^T\cot\phi\right) + \ldots$$

with the linear components given by

$$\varepsilon_1^T = \varepsilon_1(x^T) = \dot{u}^T + w^T \qquad (4.3)$$
$$\varepsilon_2^T = \varepsilon_2(x^T) = w^T + u^T\cot\phi + \dot{v}^T/\sin\phi$$
$$\beta^T = \beta(x^T) = u^T - \dot{w}^T$$
$$\gamma^T = \gamma(x^T) = \dot{u}^T/\sin\phi - v^T\cot\phi$$
$$\delta^T = \delta(x^T) = v^T - \dot{w}^T/\sin\phi$$

In equations (4.2) and (4.3) the sliding notation, introduced in (4.1), has been applied to the nonlinear strain and linear curvature expressions. As a consequence of the nonlinearity of the strain displacement relations, in general

$$\varepsilon_\phi^T \equiv \varepsilon_\phi(x^T = x^i + x^f + x) \neq \varepsilon_\phi(x^i) + \varepsilon_\phi(x^f) + \varepsilon_\phi(x) \equiv \varepsilon_\phi^i + \varepsilon_\phi^f + \varepsilon_\phi \qquad (4.4)$$

where ε^i_ϕ, ε^f_ϕ, and ε_ϕ are given by the strain displacement relations, equation (4.2), operating upon x^i, x^f, and x respectively.

At the initial unloaded point O' in Figure 4.1 the displacements, x^i, are axisymmetric and the strains and curvatures are given by

$$\varepsilon^i_\phi \equiv \varepsilon_\phi(x^i) = \varepsilon^i_1 + \frac{1}{2}(\beta^i)^2 - \frac{1}{2}\varepsilon^i_1(\beta^i)^2 + \frac{1}{2}(\varepsilon^i_1)^2(\beta^i)^2 - \frac{1}{8}(\beta^i)^4 + \ldots \tag{4.5}$$

$$\varepsilon^i_\theta \equiv \varepsilon_\theta(x^i) = \varepsilon^i_2$$

$$\varepsilon^i_{\theta\phi} \equiv 0$$

$$\chi^i_\phi = \frac{1}{a}\beta^{i\prime}$$

$$\chi^i_\theta = \frac{1}{a}\beta^i \cot\phi$$

$$\chi_{\theta\phi} \equiv 0$$

with the linear axisymmetric components given by

$$\varepsilon^i_1 \equiv \varepsilon_1(x^i) = u^{i\prime} + w^i \tag{4.6}$$

$$\varepsilon^i_2 \equiv \varepsilon_2(x^i) = w^i + u^i \cot\phi$$

$$\beta^i \equiv \beta(x^i) = u^i - w^{i\prime}$$

$$\gamma^i = \delta^i = v^i = 0$$

At any point along the axisymmetric fundamental path with displacements given by x^i+x^f, $(x \equiv 0)$, the strains are given by the axisymmetric expressions (4.5) and (4.6) operating upon (x^i+x^f). It should be noted that the change in meridional strain due to the fundamental path displacement x^f is given by $\varepsilon_\phi(x^i+x^f) - \varepsilon_\phi(x^i)$ and is not equal to $\varepsilon_\phi(x^f)$.

However, as the circumferential strains and both meridional and circumferential curvatures are linear for axisymmetric displacements of the shell, the following relations hold

$$\varepsilon_\theta(x^i+x^f) = \varepsilon_\theta(x^i)+\varepsilon_\theta(x^f) = \varepsilon_2(x^i)+\varepsilon_2(x^f) \tag{4.7}$$

$$\varepsilon_{\theta\phi}(x^i+x^f) = 0$$

$$\chi_\phi(x^i+x^f) = \chi_\phi(x^i)+\chi_\phi(x^f) = \frac{1}{a}\beta'(x^i)+\frac{1}{a}\beta'(x^f)$$

$$\chi_\theta(x^i+x^f) = \chi_\theta(x^i)+\chi_\phi(x^f) = \frac{1}{a}\beta(x^i)\cot\phi + \frac{1}{a}\beta(x^f)\cot\phi$$

$$\chi_{\theta\phi}(x^i+x^f) = 0$$

and in this case $\varepsilon_\theta{}^f$, $\chi_\phi{}^f$, $\chi_\theta{}^f$ may be interpreted as the change in strains and curvatures due to the fundamental path displacements (x^f).

Similarly the secondary path periodic displacements (x) are responsible for changes in the strains and curvatures of the shell. If the displacements at the point A in Figure 4.1 are given by (x^i+x^f), and the total displacements at a point on the secondary path ρ_2 are given by $x^T=x^i+x^f+x$, then the change in, say, the meridional strain from the initial unloaded state, O' in Figure 4.1, is given by $\varepsilon_\phi(x^T=x^i+x^f+x) - \varepsilon_\phi(x^i)$, which is not equal to $\varepsilon_\phi(x^f+x)$. Similar inequalities are also true for the other non-linear strains ε_θ and $\varepsilon_{\theta\phi}$.

Although the strain expressions (4.2) and (4.5) operating on the incremental periodic displacements (x) or the incremental fundamental displacement (x^f) respectively, are not in general equal to the corresponding incremental strains, they do provide a useful notation when expanding the total potential energy functional.

4.2.3 Stress Resultants and Strain Energy

The total membrane (N_ϕ^T, N_θ^T, $N_{\theta\phi}^T$) and bending (M_ϕ^T, M_θ^T, $M_{\theta\phi}^T$) stress resultants will be defined in terms of the corresponding strains by the use of equation (2.65), which is given below,

$$N_\phi^T \equiv N_\phi(x^T) = \frac{Et}{1-\mu^2}\left(\varepsilon_\phi^T + \mu\varepsilon_\theta^T\right) \tag{4.8}$$

$$N_\theta^T \equiv N_\theta(x^T) = \frac{Et}{1-\mu^2}\left(\varepsilon_\theta^T + \mu\varepsilon_\phi^T\right)$$

$$N_{\theta\phi}^T \equiv N_{\theta\phi}(x^T) = \frac{Et}{1+\mu}\varepsilon_{\theta\phi}^T$$

$$M_\phi^T \equiv M_\phi(x^T) = \frac{Et^3}{12(1-\mu^2)}\left(\chi_\phi^T + \mu\chi_\theta^T\right)$$

$$M_\theta^T \equiv M_\theta(x^T) = \frac{Et^3}{12(1-\mu^2)}\left(\chi_\theta^T + \mu\chi_\phi^T\right)$$

$$M_{\theta\phi}^T \equiv M_{\theta\phi}(x^T) = \frac{Et^3}{12(1+\mu)}\chi_{\theta\phi}^T$$

in which the strains and curvatures are defined in equations (4.2) and (4.3), and the superscript notation of (4.1) applies.

If at the initial unloaded state O' in Figure 4.1 the displacements (x^i) are accompanied by strains and curvatures $\varepsilon_\phi^i, \varepsilon_\theta^i, \chi_\phi^i, \chi_\theta^i$, giving rise to stress resultants N_ϕ^i, N_θ^i, M_ϕ^i, M_θ^i, then the total membrane and bending stress resultants at the general point $x^T = x^i + x^f + x$ are given by (4.8), while the corresponding change in the strains (from the initial point O' to the general point x^T) are given by $(\varepsilon_\phi^T - \varepsilon_\phi^i)$, $(\varepsilon_\theta^T - \varepsilon_\theta^i)$, etc.

However if the initial displacements (x^i) are unstressed (stress relieved), then due to the linearity between stresses and strains, (4.8), the stress resultants at the general point x^T are given by $N_\phi^T - N_\phi^i$, $N_\theta^T - N_\theta^i$ etc, where N_ϕ^i, N_θ^i, etc. are defined by (4.8) with the total strains ($\varepsilon_\phi^T, \varepsilon_\theta^T, \chi_\phi^T, \chi_\theta^T$) replaced by the corresponding initial strains ($\varepsilon_\phi^i, \varepsilon_\theta^i, \chi_\phi^i, \chi_\theta^i$).

It should be noted that the change in the strain, from the point O' to the general point x^T, remain the same, ($\varepsilon_\phi^T - \varepsilon_\phi^i$, etc.) for stressed or stress free initial imperfection displacements (x^i).

If we wish to include axisymmetric residual stress fields (N_ϕ^R, N_θ^R, M_ϕ^R, M_θ^R) that may be independent of the initial displacements (x^i), then these residual stress resultants are simply added to the stress resultants N_ϕ^T, or $N_\phi^T - N_\phi^i$, etc. discussed above.

The membrane U_M and bending U_B strain energies, (2.64), are given by

$$U_M = \frac{1}{2}\int\int\left[\bar{N}_\phi\left(\varepsilon_\phi^T - \varepsilon_\phi^i\right) + \bar{N}_\theta\left(\varepsilon_\theta^T - \varepsilon_\theta^i\right) + 2\bar{N}_{\theta\phi}\left(\varepsilon_{\theta\phi}^T - \varepsilon_{\theta\phi}^i\right)\right]a^2 \sin\phi d\theta d\phi \qquad (4.9)$$

$$U_B = \frac{1}{2}\int\int\left[\bar{M}_\phi\left(\chi_\phi^T - \chi_\phi^i\right) + \bar{M}_\theta\left(\chi_\theta^T - \chi_\theta^i\right) + 2\bar{M}_{\theta\phi}\left(\chi_{\theta\phi}^T - \chi_{\theta\phi}^i\right)\right]a^2 \sin\phi\, d\theta d\phi$$

where the integration is over the middle surface of the spherical cap.

In order that the strain energy is measured from a datum that corresponds to the unloaded initial state O' in Figure 4.1, the stress resultants are defined as

$$\begin{aligned}
\bar{N}_\phi &= N_\phi^T - \langle N_\phi^i \rangle + N_\phi^R \qquad (4.10)\\
\bar{N}_\theta &= N_\theta^T - \langle N_\theta^i \rangle + N_\theta^R\\
\bar{N}_{\theta\phi} &= N_{\theta\phi}^T\\
\bar{M}_\phi &= M_\phi^T - \langle M_\phi^i \rangle + M_\phi^R\\
\bar{M}_\theta &= M_\theta^T - \langle M_\theta^i \rangle + M_\theta^R\\
\bar{M}_{\theta\phi} &= M_{\theta\phi}^T
\end{aligned}$$

where the following interpretations must be placed on the brackets < >.

$$\begin{aligned}
\langle N_\phi^i \rangle &= 0\\
\langle N_\theta^i \rangle &= 0 \qquad \text{for stressed initial imperfections} \qquad (4.11)\\
\langle M_\phi^i \rangle &= 0\\
\langle M_\theta^i \rangle &= 0
\end{aligned}$$

$$\begin{aligned}
\langle N_\phi^i \rangle &= N_\phi^i\\
\langle N_\theta^i \rangle &= N_\theta^i \qquad \text{for stress - free initial imperfections}\\
\langle M_\phi^i \rangle &= M_\phi^i\\
\langle M_\theta^i \rangle &= M_\theta^i
\end{aligned}$$

The introduction of the brackets, < >, allows the derivation and solution of all axisymmetric imperfection types to be treated as a single case. It is only necessary to specify the initial imperfection displacements x^i, and whether they are stressed or stress free (the interpretation to be used for the brackets < >), along with the initial residual stress resultants N_ϕ^R etc., when solving the resulting fundamental and secondary path equations.

4.2.4 Load Potential Energy

The load potential energy, J_L, for a uniform pressure load is given by

$$J_L = -\frac{P}{2}\iint 2[\Delta V(x^T) - \Delta V(x^i)]\, a^2 \sin\phi \, d\theta d\phi \qquad (4.12)$$

Integration, over the middle surface of the shell, of the nonlinear change in volume $\Delta V(x^T)$ represents the difference in volume between the perfect spherical cap and the deformed shell with generalised displacements $x^T = x^i + x^f + x$, and $\Delta V(x^T)$ from equation (2.69), is given by

$$\begin{aligned}\Delta V^T \equiv \Delta V(x^T) = \frac{1}{2}\{&3(w^T + u^T\cot\phi) + \dot{v}^T/\sin\phi + \dot{u}^T - \dot{w}^T\cot\phi + 3(w^T + u^T\cot\phi)^2 \\ &+ 2(w^T + u^T\cot\phi)\left(\dot{v}^T/\sin\phi + \dot{u}^T - \dot{w}^T\cot\phi\right) \\ &+ \left(\dot{u}^T - \dot{w}^T\cot\phi\right)\dot{v}^T/\sin\phi + (w^T + u^T\cot\phi)^3 \\ &+ (w^T + u^T\cot\phi)^2\left(\dot{v}^T/\sin\phi + \dot{u}^T - \dot{w}^T\cot\phi\right) \\ &+ (w^T + u^T\cot\phi)\left(\dot{u}^T - \dot{w}^T\cot\phi\right)\dot{v}^T/\sin\phi\,\}\, a\sin^2\phi\end{aligned} \qquad (4.13)$$

4.2.5 The Total Potential Energy Functional

The total potential energy, V, may be written as the sum of the strain energy and the load potential energy as

$$V = U_M + U_B + J_L \qquad (4.14)$$

in which the membrane and bending strain energies (U_M, U_B) and the increase in the external load potential energy J_L for a uniformly loaded spherical shell are all measured from the initial unloaded state O' in Figure 4.1. Strain energies are given by equation (4.9) and may be written in terms of the generalised displacement by the use of equations (4.10), (4.8), (4.5), (4.6), (4.2), (4.3) and (4.1) with the notation of (4.11) defining the initial stress state, while the load potential energy is given by equation (4.12), and may be written in terms of the generalised displacements by using equation (4.13).

Expanding the total potential energy functional as a power series in the generalised incremental displacements (x), we have

$$V = V_0 + V_1 + V_2 + \ldots \qquad (4.15)$$

where V_0 is independent of the incremental displacement, (x), and V_1 and V_2 are linear and quadratic respectively in the incremental displacements (x).

Thus V_1 may be written as,

$$V_1 \equiv U_{1,M} + U_{1,B} + J_{1,L} \tag{4.16}$$

where the membrane ($U_{1,M}$), bending ($U_{1,B}$) and load potential ($J_{1,L}$) energies are given below

$$\begin{aligned}\frac{U_{1,M}}{Ka^2} = \iint [\ & \varepsilon_1\left\{(\varepsilon_\phi^s + \mu\varepsilon_\theta^s)\left(1 - \tfrac{1}{2}(\beta^s)^2 + \varepsilon_1^s(\beta^s)^2 + \ldots\right)\right\} \\ & + \beta\left\{(\varepsilon_\phi^s + \mu\varepsilon_\theta^s)\left(\beta^s - \varepsilon_1^s\beta^s + (\varepsilon_1^s)^2\beta^s - \tfrac{1}{2}(\beta^s)^3 + \ldots\right)\right\} \\ & + \varepsilon_2\{(\varepsilon_\theta^s + \mu\varepsilon_\phi^s)\}\ ]\ \sin\phi\, d\theta d\phi \\ -\tfrac{1}{2}\iint [\ & \varepsilon_1\left\{Q_\phi^i\left(1 - \tfrac{1}{2}(\beta^s)^2 + \varepsilon_1^s(\beta^s)^2 + \ldots\right)\right\} \\ & + \beta\left\{Q_\phi^i\left(\beta^s - \varepsilon_1^s\beta^s + (\varepsilon_1^s)^2\beta^s - \tfrac{1}{2}(\beta^s)^3 + \ldots\right)\right\} \\ & + \varepsilon_2\left\{Q_\theta^i\right\}\ ]\ \sin\phi\, d\theta d\phi\end{aligned} \tag{4.17}$$

$$\begin{aligned}\frac{U_{1,B}}{Ka^2} = \alpha\iint\Big[& \dot{\beta}\left\{\dot{\beta}^s + \mu\beta^s\cot\phi\right\} + \beta\left\{\beta^s\cot\phi + \mu\dot{\beta}^s\right\}\cot\phi\Big]\sin\phi d\theta d\phi \\ -\frac{\alpha}{2}\iint [\ & \dot{\beta}\left\{\dot{\beta}^i + \mu\beta^i\cot\phi\right\} + \beta\left\{\beta^i\cot\phi + \mu\dot{\beta}^i\right\}\cot\phi \\ & + < \dot{\beta}\left\{\dot{\beta}^i + \mu\beta^i\cot\phi\right\} + \beta\left\{\beta^i\cot\phi + \mu\dot{\beta}^i\right\}\cot\phi\ > \\ & + \dot{\beta}\left\{\frac{M_\phi^R}{D}\right\} + \beta\left\{\frac{M_\theta^R}{D}\right\}\ ]\ \sin\phi d\theta d\phi\end{aligned} \tag{4.18}$$

$$\begin{aligned}\frac{J_{1,L}}{Ka^2} = -\hat{p}\iint [\ & (w + u\cot\phi)\left\{3 + 6\varepsilon_2^s + 3(\varepsilon_2^s)^2 + 2\varepsilon_3^s + 2\varepsilon_2^s\,\varepsilon_3^s\right\} \\ & + \varepsilon_3\left\{1 + 2\varepsilon_2^s + (\varepsilon_2^s)^2\right\} + \dot{v}/\sin\phi\left\{1 + 2\varepsilon_2^s + \varepsilon_3^s + (\varepsilon_2^s)^2 + \varepsilon_2^s\varepsilon_3^s\right\}\ ]\ \sin^3\phi d\theta d\phi\end{aligned} \tag{4.19}$$

In equations (4.17), (4.18), and (4.19) the following notations and definitions have been introduced

$$K = \frac{Et}{1-\mu^2}, \qquad D = \frac{Et^3}{12(1-\mu^2)}, \qquad \alpha = \frac{D}{Ka^2} = \frac{t^2}{12a^2} \tag{4.20}$$

$$\hat{p} = \frac{Pa}{2K} = \frac{Pa(1-\mu^2)}{2Et} \tag{4.21}$$

$$\varepsilon_3 = \dot{u} - w\cot\phi \tag{4.22}$$

$$Q_\phi^i = \varepsilon_\phi^i + \mu\varepsilon_\theta^i + < \varepsilon_\phi^i + \mu\varepsilon_\theta^i > -\frac{N_\phi^R}{K} \qquad (4.23)$$

$$Q_\theta^i = \varepsilon_\theta^i + \mu\varepsilon_\phi^i + < \varepsilon_\theta^i + \mu\varepsilon_\phi^i > -\frac{N_\theta^R}{K}$$

and the superscript 's' has been introduced to denote displacements, strains and curvatures of the axisymmetric fundamental path, i.e. $x^s=x^i+x^f$, $e_\phi{}^s=\varepsilon_\phi(x^s)$ etc.

The terms of the total potential energy functional that are quadratic in the incremental displacements (x), may be treated in a similar manner to the linear terms as follows.

$$V_2 = U_{2,M} + U_{2,B} + J_{2,L} \qquad (4.24)$$

Where the membrane ($U_{2,M}$), bending ($U_{2,B}$) and load potential ($J_{2,L}$) energies are given below.

$$\frac{U_{2,M}}{Ka^2} = \frac{1}{2}\int\int\left[\varepsilon_1^2+\varepsilon_2^2+2\mu\varepsilon_1\varepsilon_2+\frac{1}{2}(1-\mu)\left(\dot{v}+\gamma\right)^2\right]\sin\phi d\theta d\phi \qquad (4.25)$$

$$+\frac{1}{2}\int\int 2[\ \frac{1}{2}\left(\dot{v}^2+\beta^2\right)\left\{\left(1-\varepsilon_1^s+(\varepsilon_1^s)^2-\frac{1}{2}(\beta^s)^2\right)\left(\varepsilon_1^s+\frac{1}{2}(\beta^s)^2-\frac{1}{2}(\beta^s)^2\varepsilon_1^s+\ldots+\mu\varepsilon_2^s\right)\right\}$$

$$+\frac{1}{2}\left(\gamma^2+\delta^2\right)\left\{\left(1-\varepsilon_2^s+(\varepsilon^s)^2\right)\left(\varepsilon_2^s+\mu\left(\varepsilon_1^s+\frac{1}{2}(\beta^s)^2-\frac{1}{2}(\beta^s)^2\varepsilon_1^s+\ldots\right)\right)\right\}$$

$$+\left[\varepsilon_1\beta\{-\beta^s+2\beta^s\varepsilon_1^s\}+\varepsilon_1^2\left\{\frac{1}{2}(\beta^s)^2\right\}+\beta^2\left\{-\frac{1}{2}(\beta^s)^2\right\}\right]\left[\mu\varepsilon_2^s+\varepsilon_1^s+\frac{1}{2}(\beta^s)^2\right]$$

$$+\varepsilon_1\beta\{\beta^s-\varepsilon_1^s\beta^s+(\varepsilon_1^s)^2\beta^s-(\beta^s)^3+\ldots\}$$

$$+\varepsilon_1^2\left\{-\frac{1}{2}(\beta^s)^2+\varepsilon_1^s(\beta^s)^2+\ldots\right\}+\beta^2\left\{\frac{1}{2}(\beta^s)^2-\varepsilon_1^s(\beta^s)^2\right\}$$

$$+\mu\varepsilon_2\beta\left\{(\beta^s)^2-\varepsilon_1^s\beta^s+(\varepsilon_1^s)^2\beta^s-\frac{1}{2}(\beta^s)^3\right\}$$

$$+\mu\varepsilon_1\varepsilon_2\left\{-\frac{1}{2}(\beta^s)^2+\varepsilon_1^s(\beta^s)^2\right\}$$

$$+(1-\mu)[\ \dot{v}^2\left\{-\varepsilon_1^s-\frac{1}{2}(\beta^s)^2+\frac{3}{2}(\varepsilon_1^s)^2-2(\varepsilon_1^s)^3+2\varepsilon_1^s(\beta^s)^2\right\}$$

$$+\gamma^2\left\{-\varepsilon_2^s-\frac{1}{2}(\beta^s)^2+\frac{3}{2}(\beta^s)^2-2(\varepsilon_2^s)^3+2\varepsilon_2^s(\beta^s)^2+\varepsilon_1^s(\beta^s)^2\right\}$$

$$+\delta^2\left\{\frac{1}{2}(\beta^s)^2-(\beta^s)^2(\varepsilon_1^s+\varepsilon_2^s)\right\}$$

$$+\dot{v}\gamma\{-(\varepsilon_1^s+\varepsilon_2^s)-(\beta^s)^2+(\varepsilon_1^s)^2+(\varepsilon_2^s)^2+\varepsilon_1^s\varepsilon_2^s-(\varepsilon_2^s)^3$$
$$-(\varepsilon_1^s)^3+\varepsilon_2^s(\beta^s)^2+3\varepsilon_1^s(\beta^s)^2-(\varepsilon_1^s)^2\varepsilon_2^s-\varepsilon_1^s(\varepsilon_2^s)^2\ \}$$

$$+\dot{v}\delta\left\{\beta^s-\beta^s(2\varepsilon_1^s+\varepsilon_2^s)-\frac{1}{2}(\beta^s)^3+2(\varepsilon_1^s)^2\beta^s+\varepsilon_1^s\varepsilon_2^s\beta^s\right\}$$

$$+\gamma\delta\left\{\beta^s-\beta^s(\varepsilon_1^s+2\varepsilon_2^s)-\frac{1}{2}(\beta^s)^3+2(\varepsilon_2^s)^2\beta^s+\varepsilon_1^s\varepsilon_2^s\beta^s\right\}\ ]\ ]\sin\phi d\theta d\phi$$

$$-\frac{1}{2}\int\int[\ \dot{v}^2\left\{\frac{1}{2}Q_\phi^i\left(1-\varepsilon_1^s+(\varepsilon_1^s)^2-\frac{1}{2}(\beta^s)^2+\ldots\right)\right\}$$

$$+\beta^2\left\{\frac{1}{2}Q_\phi^i\left(1-\varepsilon_1^s+(\varepsilon_1^s)^2-\frac{3}{2}(\beta^s)^2+\ldots\right)\right\}$$

$$+\gamma^2\left\{\frac{1}{2}Q_\theta^i(1-\varepsilon_2^s+(\varepsilon_2^s)^2)\right\}+\delta^2\left\{\frac{1}{2}Q_\theta^i(1-\varepsilon_2^s+(\varepsilon_2^s)^2)\right\}$$

$$+\varepsilon_1^2\left\{\frac{1}{2}Q_\phi^i(\beta^s)^2\right\}+\varepsilon_1\beta\left\{Q_\phi^i(-\beta^s+2\beta^s\varepsilon_1^s)\right\}\ ]\sin\phi d\theta d\phi$$

$$\frac{U_{2,B}}{Ka^2} = \frac{\alpha}{2}\int\int [\ \dot{\beta}^2 + \left(\dot{\delta}/\sin\phi + \beta\cot\phi\right)^2 + 2\mu\dot{\beta}\left(\dot{\delta}/\sin\phi + \beta\cot\phi\right) + \frac{1}{2}(1-\mu)\left(\dot{\delta} + \dot{\beta}/\sin\phi - \delta\cot\phi\right)^2\ ]\ \sin\phi\, d\theta d\phi \tag{4.26}$$

$$\frac{J_{2,L}}{Ka^2} = -\hat{p}\int\int [\ (w + u\cot\phi)^2\{3 + 3\varepsilon_2^s + \varepsilon_3^s\} + \varepsilon_3 \dot{v}/\sin\phi\{1 + \varepsilon_2^s\} + 2\left(\dot{w} + u\cot\phi\right)\left(\dot{v}/\sin\phi + \varepsilon_3\right)\{1 + \varepsilon_2^s\} \tag{4.27}$$

4.3 THE FUNDAMENTAL PATH EQUILIBRIUM EQUATION

A stationary value of the total potential energy with respect to the generalised coordinates (x) is necessary and sufficient for the equilibrium of the system. Hence for equilibrium of the system

$$\delta V = \delta V_1 + \delta V_2 + \ldots = 0 \tag{4.28}$$

On the fundamental path the incremental displacements, x, are identically zero, therefore

$$\delta V_2 = \delta V_3 = \ldots = 0 \tag{4.29}$$

and the fundamental path solution is given by the stationarity of V_1 with respect to any general incremental displacements (x). That is, $\delta V_1=0$ yields the fundamental path solution. The stationary condition (Chapter 2) is satisfied by the Euler equations, which for V_1, given by (4.16), (4.17), (4.18) and (4.19), reduce to,

$$E_1 \text{ or } E_u : \frac{d}{d\phi}\left(\frac{\partial F}{\partial u'}\right) - \frac{\partial F}{\partial u} = 0 \tag{4.30}$$

$$E_3 \text{ or } E_w : \frac{d^2}{d\phi^2}\left(\frac{\partial F}{\partial w''}\right) - \frac{d}{d\phi}\left(\frac{\partial F}{\partial w'}\right) + \frac{\partial F}{\partial w} = 0$$

where F is the complete integrand of V_1.

When the Euler equations (4.30) are applied to the linear (in x) potential energy functional (V_1) a pair of fourth order nonlinear (in x^f) differential equations of the following form result:

$$\sum_{j=0}^{4} A_{ij}(x^s)^j + \hat{p}\sum_{j=0}^{3} B_{ij}(x^s)^j = 0 \qquad \text{for } i = 1,3 \tag{4.31}$$

in which $x^s \equiv x^i + x^f$.

The terms of equation (4.31) have been calculated and are tabulated in Table 4.1, in which the $A_{i,0}$ and $A_{i,1}(x^s)$ terms are slightly different from those of expression (4.31). In anticipation of the solution method the constant and linear (in x^s) terms have been altered and are tabulated as follows:

$A_{i,0}$, of Table 4.1 = $A_{i,0}+A_{i,1}(x^i)$, of equation (4.31)

$A_{i,1}(x^f)$, of Table 4.1 = $A_{i,1}(x^s \equiv x^i+x^f)-A_{i,1}(x^i)$, of equation (4.31)

Fourth order expressions have been used for the strains and linear expressions for the curvatures in equation (4.2), allowing the fundamental path equilibrium equation, equation (4.31), to include consistent fourth order strain terms and linear curvature terms.

Originally the derivation of the equilibrium equations for the nonlinear fundamental path was based on the quartic strain displacement and the quadratic curvature displacement expressions given by equation (2.66). The eigenvalue problem that results from the linearised secondary path was based on strain displacement relations that contained all terms up to and including the cubic terms and only the 'largest' of the terms that arise from the quartic terms, while quadratic curvature displacement relations were used. Subsequently, these equations were solved, yielding the nonlinear axisymmetric fundamental path, and the initial mode shape and location (on the nonlinear axisymmetric fundamental path) of the secondary paths that bifurcate from the fundamental path. By neglecting higher order terms in the strain displacement and curvature displacement expressions and by comparing the order of magnitude of displacements, their derivatives, the linear strain components and the contributions to the total potential energy for both the fundamental and secondary paths, the order of accuracy required for the strain and curvature expressions was established, and this is presented in Section 6.2.3 of Chapter Six. It was found that cubic strain and linear curvature expressions are required for consistent and accurate modelling of the nonlinear elastic response of thin spherical caps, accurate but inconsistent modelling may be achieved with quadratic strain displacement and linear curvature displacement expression. As the terms that arise from the use of quadratic curvature expressions, equation (2.66), require the introduction of new, and unnecessary, notation they have been neglected in equation (4.2), and no quadratic curvature terms are given in Table 4.1.

4.4 THE FUNDAMENTAL PATH PERTURBATION METHOD

In order to solve the nonlinear differential equation (4.31) a perturbation technique will be adopted. If some general point on the fundamental path is given by $(\hat{p}_o, x^s{}_o \equiv x^i{+}x^f{}_o)$ which therefore satisfies the equilibrium equation (4.31), then we may construct the series

$$x^s = x^i + x^f = x^i + x^f_0 + x_1 e + x_2 e^2 + \ldots = x^s_0 + x_1 e + x_2 e^2 + \ldots \qquad (4.32)$$
$$\hat{p} = \hat{p}_0 + \hat{p}_1 e + \hat{p}_2 e^2 + \ldots$$

where e is some suitably defined perturbation parameter, and the fundamental path superscript 'f' has been omitted from the individual linear, x_1, quadratic, x_2, etc. terms of the nonlinear fundamental path displacement x^f. We would like to solve for x_1, x_2, x_3... and $\hat{p}_1$, $\hat{p}_2$, $\hat{p}_3$... such that x^s and $\hat{p}$ also satisfy equation (4.31).

Using the form of equation (4.31) defined in Table 4.1 and substituting for x^s and $\hat{p}$ from equation (4.32), and then collecting like powers of e together, we have an expression of the form

$$F_0 + F_1 e + F_2 e^2 + F_3 e^3 + \ldots = 0 \qquad (4.33)$$

in which the coefficients F_0, F_1, F_2,... are functions of x^i, $\hat{p}_o$, x_j, and $\hat{p}_j$ only. As equation (4.33) must be satisfied for all values of e it follows that the individual coefficients F_0, F_1, ... must be zero, and this condition yields the sequence of "perturbation equations" given below.

$$F_0 = 0 \Rightarrow A_{i,1}(x^s_0) + A_{i,2}\{(x^s_0)^2\} + A_{i,3}\{(x^s_0)^3\} + A_{i,4}\{(x^s_0)^4\} + A_{i,0} = \qquad (4.34)$$
$$= -\hat{p}_0[B_{i,1}\{x^s_0\} + B_{i,2}\{(x^s_o)^2\} + B_{i,0}]$$

$$F_1 = 0 \Rightarrow \bar{A}_i(x_1) = -\hat{p}_1\left[\bar{B}_{i,0}\right]$$

$$F_2 = 0 \Rightarrow \bar{A}_i(x_2) = -\hat{p}_2\left[\bar{B}_{i,0}\right] - [A_{i,2}\{(x_1)^2\} + A_{i,3}\{3(x^s_0)(x_1)^2\}$$
$$+ A_{i,4}\{6(x^s_0)^2(x_1)^2\} + \hat{p}_1 B_{i,1}\{x_1\} + \hat{p}_1 B_{i,2}\{2(x^s_0)x_1\}]$$

$$F_3 = 0 \Rightarrow \bar{A}_i(x_3) = -\hat{p}_3[\bar{B}_{i,0}] - [A_{i,2}\{2x_1x'_2\} + A_{i,3}\{(x_1)^3 + 6(x^s_0)x_1x_2\}$$
$$+ A_{i,4}\left\{4(x^s_0)x_1^3\right\} + A_{i,4}\{12(x^s_0)x_1x_2\} + \hat{p}_1 B_{i,1}\{x_2\}$$
$$+ \hat{p}_2 B_{i,1}\{x_1\} + \hat{p}_1 B_{i,2}\{(x_1)^2\} + \hat{p}_1 B_{i,2}\{(x_1)^2\}$$
$$+ \hat{p}_1 B_{i,2}\{2(x^s_0)x_2\} + \hat{p}_2 B_{i,2}\{2(x^s_0)x_1\} + \hat{p}_0 B_{i,2}\{2x_1x_2\}]$$

where

$$\bar{A}_i(x_j) = A_{i,1}(x_j) + A_{i,2}\{2(x_0^s)x_j\} + A_{i,3}\{3(x_0^s)^2 x_j\} + A_{i,4}\{4(x_0^s)^3 x_j\} + \hat{p}_0[B_{i,1}x_j + B_2\{2(x_0^s)x_j\}]$$

and

$$\bar{B}_{i,0} = B_{i,0} + B_{i,1}\{x_0^s\} + B_{i,2}\{(x_0^s)^2\}$$

In the above expressions i=1 and 3 and the terms $A_{i,j}(x)^j$ and $B_{i,j}(x)^j$ are defined in Table 4.1.

In order to solve sequentially equation (4.34) for $(\hat{p}_1, x_1)$, $(\hat{p}_2, x_2)$,... only an expression relating $\hat{p}_j$ and x_j is required, and by considering the choice of perturbation parameter we can find such a condition. For convenience it is desirable to choose the perturbation parameter so that it increases continuously along the fundamental equilibrium path. This will be achieved by identifying the perturbation parameter (e) with the arc length (s) of the fundamental path, where s is defined as

$$s = \int_0^e \left\{\left(\frac{d\hat{p}}{de}\right)^2 + \left(\frac{dx^f}{de}\right)^2\right\}^{\frac{1}{2}} de \qquad (4.35)$$

Other, physically more obvious, definitions of perturbation parameter were tried, including the load parameter ($\hat{p}$), the change in volume of the shell, and the radial displacement at the pole. None of these definitions proved adequate as the nonlinear fundamental state was not always uniquely defined by the perturbation parameter. That is, these definitions of the perturbation parameter do not yield a continuously increasing (or decreasing) value of the perturbation parameter along the fundamental path.

In order to ensure the unique definition of the fundamental state for any value of the perturbation parameter the fundamental path arc length definition has been adopted. Substitution of (4.32) into (4.35) yields

$$s = \int_0^e \left\{(\hat{p}_1 + 2\hat{p}_2 e + 3\hat{p}_3 e^2 + \ldots)^2 + (x_1 + 2x_2 e + 3x_3 e^2 + \ldots)^2\right\}^{\frac{1}{2}} de$$

hence

$$\frac{ds}{de} = \left\{[(\hat{p}_1)^2 + (x_1)^2] + 4[\hat{p}_1\hat{p}_2 + x_1x_2]e + [6\hat{p}_1\hat{p}_3 + 4(\hat{p}_2)^2 + 6x_1x_3 + 4(x_2)^2]e^2 + \ldots\right\}^{\frac{1}{2}}$$

Now if we identify the perturbation parameter with the arc length, by putting $s = a\,e$, where a is some arbitrary non-zero constant, we may write

$$\frac{ds}{de} = a = \left\{[(\hat{p}_1)^2 + (x_1)^2] + 4[\hat{p}_1\hat{p}_2 + x_1x_2]\,e + [6\hat{p}_1\hat{p}_3 + 4(\hat{p}_2)^2 + 6x_1x_3 + 4(x_2)^2]\,e^2 + \ldots\right\}^{\frac{1}{2}}$$

which yields the following sequence of equations:

$$(\hat{p}_1)^2 + (x_1)^2 = a^2 \qquad (4.36)$$

$$4\hat{p}_1\hat{p}_2 + 4x_1x_2 = 0$$

$$6\hat{p}_1\hat{p}_3 + 4(\hat{p}_2)^2 + 6x_1x_3 + 4(x_2)^2 = 0$$

$$8\hat{p}_1\hat{p}_4 + 12\hat{p}_2\hat{p}_3 + 8x_1x_4 + 12x_2x_3 = 0$$

etc.

Equations (4.34) and (4.36) may now be solved to yield the coefficients x_1, x_2, x_3 ... and $\hat{p}_1$, $\hat{p}_2$, $\hat{p}_3$ ... of the fundamental path series given by equation (4.32).

4.5 THE SECONDARY PATH EQUATION

The stationarity of V_1 with respect to the generalised displacement (x) has been used to determine the fundamental path solution. Hence the equilibrium of the system, equation (4.28), for general non-zero periodic or secondary path, displacements is given by:

$$\delta V_2 + \delta V_3 + \ldots = 0 \qquad (4.37)$$

The linearised secondary path solution consists of determining the critical pressure, P_{cr} and the initial mode shape of the secondary path at the point of bifurcation. This may be achieved by considering the solution of (4.37) which is linear in the generalised coordinate (x), that is $\delta V_2=0$.

If it were desired a perturbation method analogous to that used for the fundamental path solution could be set up for the secondary path. In which case the periodic displacement (x) and the uniform pressure load along the secondary path could be expressed in terms of a power series in the secondary path perturbation parameter. The solution of equation (4.37) would then be accomplished by setting the coefficients of like powers of the secondary path perturbation parameter to zero in a manner similar to that used in deriving equation (4.34). The present solution, given by putting $\delta V_2=0$, is identical to solving for the first term in the secondary path perturbation series.

The stationary condition, $\delta V_2=0$, is satisfied by the Euler equations

$$E_1 \text{ or } E_u ; \quad \frac{d}{d\phi}\left(\frac{\partial F}{\partial \dot{u}}\right) + \frac{d}{d\theta}\left(\frac{\partial F}{\partial \ddot{u}}\right) - \frac{\partial F}{\partial u} = 0 \qquad (4.38)$$

$$E_2 \text{ or } E_v ; \quad \frac{d}{d\phi}\left(\frac{\partial F}{\partial \dot{v}}\right) + \frac{d}{d\theta}\left(\frac{\partial F}{\partial \ddot{v}}\right) - \frac{\partial F}{\partial v} = 0$$

$$E_3 \text{ or } E_w ; \quad \frac{d^2}{d\phi^2}\left(\frac{\partial F}{\partial \ddot{w}}\right) + \frac{d^2}{d\theta^2}\left(\frac{\partial F}{\partial \ddot{w}}\right) + \frac{d^2}{d\phi d\theta}\left(\frac{\partial F}{\partial \dot{w}}\right) - \frac{d}{d\phi}\left(\frac{\partial F}{\partial \dot{w}}\right) - \frac{d}{d\theta}\left(\frac{\partial F}{\partial \dot{w}}\right) + \frac{\partial F}{\partial w} = 0$$

in which the quadratic (in the incremental displacements) components of the total potential energy V_2 are given by equations (4.24), (4.25), (4.26) and (4.27), and F is the complete integrand of V_2.

When the Euler equations (4.38) are applied to V_2 a set of fourth order differential equations, linear in the incremental displacements (x) and their derivatives, but non-linear in the axisymmetric, fundamental path displacement (x^s) and their derivatives results, and is of the following form:

$$\sum_{j=0}^{3}\left[A_{i,j}(x^s)^j\right](x) + \hat{p}\sum_{j=0}^{1}\left[B_{i,j}(x^s)^j\right](x) = 0 \qquad \text{for } i = 1,2,3 \qquad (4.39)$$

The terms of equation (4.39) have been calculated and are presented in Table 4.2.

On substitution for x^s and $\hat{p}$ from equation (4.32) into expression (4.39) a non-linear eigenvalue problem in terms of the fundamental path perturbation parameter (the eigenvalue) and the incremental displacement (x) (the eigenvector) results. It is the solution of this problem that yields the critical pressures and initial mode shapes of the secondary path displacement vector (x).

Fourth order expressions were used for the strain displacement and quadratic expressions for the curvatures displacement relations, equation (2.66), it is therefore possible to include consistent fourth order strain terms and quadratic curvature terms in equation (4.39). As for the fundamental path equilibrium equation, it is sufficient for consistent and accurate modelling of the elastic response of thin spherical caps to base the nonlinear eigenvalue problem, equation (4.39), on cubic strain and linear curvature expressions, and accurate but inconsistent modelling may be achieved with quadratic strain displacement and linear curvature displacement expression, see Chapter Six. For this reason the terms arising from the use of quadratic curvature expressions, equation (2.66), have been neglected in equation (4.2), and no quadratic curvature terms are given in Table 4.2. Also only the 'large' quartic terms (i.e. $[A_{i,3}(x^s)^3](x)$) arising from the use of quartic strain displacement relations, equation (4.2), are retained in Table 4.2.

4.6 THE CIRCUMFERENTIAL MODELLING OF DISPLACEMENTS.

Due to the axisymmetric properties of the spherical caps considered it is possible to express the displacements exactly as,

$$u = u(\phi,\theta) = \sum_{i=0}^{\infty} u_i(\phi) \cos i\theta \qquad (4.40)$$

$$v = v(\phi,\theta) = \sum_{i=0}^{\infty} v_i(\phi) \sin i\theta$$

$$w = w(\phi,\theta) = \sum_{i=0}^{\infty} w_i(\phi) \cos i\theta$$

The functions $u_i(\phi)$, $v_i(\phi)$ and $w_i(\phi)$ are dependent on ϕ only, and no confusion will arise if we use the shorter notation of u_i, v_i and w_i for these functions of ϕ.

For the axisymmetric fundamental path, i=0, the fundamental path displacements are represented by

$$u^f = u_0 \qquad (4.41)$$
$$v^f = v_0 = 0$$
$$w^f = w_0$$

Hence the fundamental path equations (4.32), (4.34) and (4.36) may be used exactly as they are, the displacements being functions of ϕ only.

For secondary path displacements i may take any zero or positive integer value. However the quadratic nature (quadratic in x) of the total potential energy functional (4.24) defining the secondary path, and the well known properties of the following integrals:

$$\int_0^{2\pi} \sin i\theta \sin j\theta \, d\theta = \int_0^{2\pi} \cos i\theta \cos j\theta \, d\theta = 0 \quad \text{for } i \neq j \qquad (4.42)$$
$$= \pi \quad \text{for } i = j$$

and

$$\int_0^{2\pi} \sin i\theta \cos j\theta \, d\theta = 0 \qquad \text{for all i and j}$$

allow equations (4.25), (4.26) and (4.27) to be used as they are, with the understanding that the circumferential mode number, i, is implied by u, v and w in the sense that

$$u = u_i \cos i\theta$$
$$v = v_i \sin i\theta \qquad (4.43)$$
$$w = w_i \cos i\theta$$

Equations (4.39) are linear in the incremental displacements u, v and w and their derivatives, and it may be seen, from Table 4.2, that a constant factor of $\cos(i\theta)$ can be cancelled from the E_1 and E_3 equations, and a factor of $\sin(i\theta)$ from the E_2 equation.

Finally equation (4.39) and Table 4.2 may be used, provided the following interpretation is placed on the incremental displacements and their differentials:

$$
\begin{array}{lll}
u = u_i & v = v_i & w = w_i \\
\dot{u} = -iu_i & \dot{v} = +iv_i & \dot{w} = -iw_i \\
\ddot{u} = -i^2u_i & \ddot{v} = -i^2v_i & \ddot{w} = -i^2w_i \\
 & \dddot{v} = -i^3v_i & \dddot{w} = +i^3w_i \\
 & & \ddddot{w} = +i^4w_i
\end{array}
\qquad (4.44)
$$

By specifying the circumferential mode number, i, the incremental displacements of the secondary path, equation (4.39), are dependent upon ϕ only.

4.7 THE ENERGY COMPONENTS OF THE FUNDAMENTAL AND SECONDARY PATHS

The total potential energy functional defined as

$$V = U_M + U_B + J_L$$

and equations (4.9), (4.12) and (4.13) are the common starting point for the derivation of both the fundamental path equations ($\delta V_1=0$) and the secondary path eigenvalue problem ($\delta V_2=0$).

The solution (x^f, p^f) and (x, p) of the nonlinear fundamental path problem and the nonlinear eigenvalue problem may be substituted into the total potential energy functional

$$V = V_0 + V_1 + V_2 + \ldots$$

This will give us an independent check on the derivation of the equilibrium equations, and the accuracy of the solution methods used. In effect we are able to check that the solution process has solved the original problem formulated by the total potential energy functional. The back substitution into the total potential energy functional also provides us with useful information as to the way in which the energy absorbed by the shell under loading is distributed through the shell. The subscript notation V_0, V_1, V_2 etc. will be used to denote the terms of the total potential energy that are independent, linear and quadratically dependent on the generalised incremental displacements.

4.7.1 Energy Components of the Fundamental Path (V_0)

The total potential energy at any point on the fundamental path (x=0), is given by,

$$V_0 = U_{0,M} + U_{0,B} + J_{0,L}$$

and for the axisymmetric fundamental path we have

$$\begin{aligned} U_{0,M} &= U_{0,M_\phi} + U_{0,M_\theta} \\ U_{0,B} &= U_{0,B_\phi} + U_{0,B_\theta} \\ U_{0,M_{\phi\theta}} &= U_{0,B_{\phi\theta}} = 0 \end{aligned} \qquad (4.45)$$

The meridional and circumferential membrane ($U_{0,M\phi}$ $U_{0,M\theta}$) and bending ($U_{0,B\phi}$ $U_{0,B\theta}$) energies are

$$\frac{U_{0,M_\phi}}{Ka^2} = \frac{1}{2}\int\int [\varepsilon_\phi^s(\varepsilon_\phi^s + \mu\varepsilon_\theta^s)] \sin\phi d\theta d\phi \qquad (4.46)$$
$$+ \frac{1}{2}\int\int\left[-\varepsilon_\phi^s\left\{\varepsilon_\phi^i + <\varepsilon_\phi^i + \mu\varepsilon_\theta^i> -\frac{N_\phi^R}{K}\right\} - \mu\varepsilon_\theta^s\varepsilon_\phi^i\right] \sin\phi d\theta d\phi$$
$$+ \frac{1}{2}\int\int\left[\varepsilon_\phi^i\left\{<\varepsilon_\phi^i + \mu\varepsilon_\theta^i> -\frac{N_\phi^R}{K}\right\}\right] \sin\phi d\theta d\phi$$

$$\frac{U_{0,M_\theta}}{Ka^2} = \frac{1}{2}\int\int [\varepsilon_\theta^s(\varepsilon_\theta^s + \mu\varepsilon_\phi^s)] \sin\phi d\theta d\phi$$
$$+ \frac{1}{2}\int\int\left[-\varepsilon_\theta^s\left\{\varepsilon_\theta^i + <\varepsilon_\theta^i + \mu\varepsilon_\phi^i> -\frac{N_\theta^R}{K}\right\} - \mu\varepsilon_\phi^s\varepsilon_\theta^i\right] \sin\phi d\theta d\phi$$
$$+ \frac{1}{2}\int\int\left[\varepsilon_\theta^i\left\{<\varepsilon_\theta^i + \mu\varepsilon_\phi^i> -\frac{N_\theta^R}{K}\right\}\right] \sin\phi d\theta d\phi$$

$$\frac{U_{0,B_\phi}}{Ka^2} = \frac{\alpha}{2}\int\int [\chi_\phi^s(\chi_\phi^s + \mu\chi_\theta^s)] \sin\phi d\theta d\phi$$
$$+ \frac{\alpha}{2}\int\int\left[-\chi_\phi^s\left\{\chi_\phi^i + <\chi_\phi^i + \mu\chi_\theta^i> -\frac{M_\phi^R}{D}\right\} - \mu\chi_\theta^s\chi_\phi^i\right] \sin\phi d\theta d\phi$$
$$+ \frac{\alpha}{2}\int\int\left[\chi_\phi^i\left\{<\chi_\phi^i + \mu\chi_\theta^i> -\frac{M_\phi^R}{D}\right\}\right] \sin\phi d\theta d\phi$$

$$\frac{U_{0,B_\theta}}{Ka^2} = \frac{\alpha}{2}\int\int [\chi_\theta^s(\chi_\theta^s + \mu\chi_\phi^s)] \sin\phi d\theta d\phi$$
$$+ \frac{\alpha}{2}\int\int\left[-\chi_\theta^s\left\{\chi_\theta^i + <\chi_\theta^i + \mu\chi_\phi^i> -\frac{M_\theta^R}{D}\right\} - \mu\chi_\phi^s\chi_\theta^i\right] \sin\phi d\theta d\phi$$
$$+ \frac{\alpha}{2}\int\int\left[\chi_\theta^i\left\{<\chi_\theta^i + \mu\chi_\phi^i> -\frac{M_\theta^R}{D}\right\}\right] \sin\phi d\theta d\phi$$

while the load potential energy, $J_{0,L}$ is

$$\frac{J_{0,L}}{Ka^2} = -\hat{p}\int\int [\{3\varepsilon_2^s + (\varepsilon_1^s - \varepsilon_2^s + \beta^s\cot\phi) + 3(\varepsilon_2^s)^2 + 2\varepsilon_2^s(\varepsilon_1^s - \varepsilon_2^s + \beta^s\cot\phi) + (\varepsilon_2^s)^3 \qquad (4.47)$$
$$+ (\varepsilon_2^s)^2(\varepsilon_1^s - \varepsilon_2^s + \beta^s\cot\phi)\} - \{3\varepsilon_2^i + (\varepsilon_1^i - \varepsilon_2^i + \beta^i\cot\phi) + 3(\varepsilon_2^i)^2$$
$$+ 2\varepsilon_2^i(\varepsilon_1^i - \varepsilon_2^i + \beta^i\cot\phi) + (\varepsilon_2^i)^3 + (\varepsilon_2^i)^2(\varepsilon_1^i - \varepsilon_2^i + \beta^i\cot\phi)\}] \sin\phi d\theta d\phi$$

4.7.2 Incremental Linear Energy Components (V_1)

At any point on the fundamental path (x=0), the energy components associated with the linear terms of the total potential energy (V_1) represent the rates of change of the energy components associated with the incremental displacements along the fundamental path.

These components are

(4.48)

$$\frac{U_{1,M_\phi}}{Ka^2} = \frac{1}{2}\int\int\Big[\varepsilon_1(2\varepsilon_\phi^s+\mu\varepsilon_\theta^s)\{1-\tfrac{1}{2}(\beta^s)^2\}+\beta(2\varepsilon_\phi^s+\mu\varepsilon_\theta^s)(\beta^s-\varepsilon_1^s\beta^s)+\mu\varepsilon_2\varepsilon_\phi^s\Big]\sin\phi d\theta d\phi$$
$$-\frac{1}{2}\int\int\Big[\Big\{\varepsilon_1\{1-\tfrac{1}{2}(\beta^s)^2\}+\beta(\beta^s-\varepsilon_1^s\beta^s)\Big\}\Big\{\varepsilon_\phi^i+<\varepsilon_\phi^i+\mu\varepsilon_\theta^i>-\frac{N_\phi^R}{K}\Big\}-\mu\varepsilon_2\varepsilon_\phi^i\Big]\sin\phi d\theta d\phi$$

$$\frac{U_{1,M_\theta}}{Ka^2} = \frac{1}{2}\int\int\Big[\varepsilon_2(2\varepsilon_\theta^s+\mu\varepsilon_\phi^s)+\varepsilon_1\Big\{1-\frac{1}{2}(\beta^s)^2\Big\}\mu\varepsilon_2^s+\beta(\beta^s-\varepsilon_1^s\beta^s)\mu\varepsilon_2^s\Big]\sin\phi d\theta d\phi$$
$$-\frac{1}{2}\int\int\Big[\varepsilon_2\Big(\varepsilon_\theta^i+<\varepsilon_\theta^i+\mu\varepsilon_\phi^i>-\frac{N_\theta^R}{K}\Big)+\varepsilon_1\Big\{1-\frac{1}{2}(\beta^s)^2\Big\}\mu\varepsilon_\theta^i+\beta(\beta^s-\varepsilon_1^s\beta^s)\mu\varepsilon_\theta^i\Big]\sin\phi d\theta d\phi$$

$$\frac{U_{1,B_\phi}}{Ka^2} = \frac{\alpha}{2}\int\int\Big[\beta(2\dot\beta^s+\mu\dot\beta^s\cot\phi)+\beta(\mu\dot\beta^s\cot\phi\Big]\sin\phi d\theta d\phi$$
$$-\frac{\alpha}{2}\int\int\Big[\beta(\dot\beta^i+<\dot\beta^i+\mu\dot\beta^i\cot\phi>-\frac{M_\phi^R}{D})+\beta(\mu\dot\beta^i\cot\phi)\Big]\sin\phi d\theta d\phi$$

$$\frac{U_{1,B_\theta}}{Ka^2} = \frac{\alpha}{2}\int\int\Big[\beta\cot\phi\,(2\dot\beta^s\cot\phi+\mu\dot\beta^s)+\beta(\mu\dot\beta^s\cot\phi)\Big]\sin\phi d\theta d\phi$$
$$-\frac{\alpha}{2}\int\int\Big[\beta\cot\phi\,(\dot\beta^i\cot\phi+<\dot\beta^i\cot\phi+\mu\dot\beta^i>-\frac{M_\theta^R}{D})+\beta(\mu\dot\beta^i\cot\phi)\Big]\sin\phi d\theta d\phi$$

while the rate of change of the load potential energy $J_{1,L}$ is

$$\frac{J_{1,L}}{Ka^2} = -\hat p\int\int[\,(w+u\cot\phi)\{3+6\varepsilon_2^s+3(\varepsilon_2^s)^2+2\varepsilon_3^s+2\varepsilon_2^s\varepsilon_3^s\} \qquad (4.49)$$
$$+(u-w\cot\phi)\{1+2\varepsilon_2^s+(\varepsilon_2^s)^2\}]\sin^3\phi\,d\theta d\phi$$

4.7.3 Incremental Quadratic Energy Components (V_2)

Due to the non-linear strain displacement relations the incremental energy components associated with the initial mode shape of the secondary path, at the point of bifurcation from the nonlinear fundamental path, may be subdivided as follows

$$U_{2,M_\phi} = \frac{1}{2}\int\int\left[\overset{\prime}{N}_\phi\overset{\prime}{\varepsilon}_\phi + \overset{f,\prime}{N}_\phi\overset{f,\prime}{\varepsilon}_\phi + \overset{f}{N}_\phi\overset{f,''}{\varepsilon}_\phi + \overset{f,''}{N}_\phi\overset{f}{\varepsilon}_\phi\right]\sin\phi\, d\theta d\phi \tag{4.50}$$

$$U_{2,M_\theta} = \frac{1}{2}\int\int\left[\overset{\prime}{N}_\theta\overset{\prime}{\varepsilon}_\theta + \overset{f,\prime}{N}_\theta\overset{f,\prime}{\varepsilon}_\theta + \overset{f}{N}_\theta\overset{f,''}{\varepsilon}_\theta + \overset{f,''}{N}_\theta\overset{f}{\varepsilon}_\theta\right]\sin\phi\, d\theta d\phi$$

$$U_{2,M_{\theta\phi}} = \frac{1}{2}\int\int 2\left[\overset{\prime}{N}_{\theta\phi}\overset{\prime}{\varepsilon}_{\theta\phi} + \overset{f,\prime}{N}_{\theta\phi}\overset{f,\prime}{\varepsilon}_{\theta\phi}\right]\sin\phi\, d\theta d\phi$$

$$U_{2,B_\phi} = \frac{1}{2}\int\int \overset{\prime}{M}_\phi\overset{\prime}{\chi}_\phi\, a^2\sin\phi\, d\theta d\phi$$

$$U_{2,B_\theta} = \frac{1}{2}\int\int \overset{\prime}{M}_\theta\overset{\prime}{\chi}_\theta\, a^2\sin\phi\, d\theta d\phi$$

$$U_{2,B_{\theta\phi}} = \frac{1}{2}\int\int 2\overset{\prime}{M}_{\theta\phi}\overset{\prime}{\chi}_{\theta\phi}\, a^2\sin\phi\, d\theta d\phi$$

$$J_{2,L} = \frac{-P}{2}\int\int 2\left[\overset{''}{\Delta v} + \overset{f,''}{\Delta v}\right]a^2\sin\phi\, d\theta d\phi$$

In equation (4.50) the superscript notation has been introduced in order to indicate the dependence of the energy components upon the fundamental and secondary path displacements. In this notation a prime (') is used to indicate a strain or stress resultant that is linear in the periodic incremental displacements, a double prime (") indicates quadratic components, and the (f) indicates linear or nonlinear dependence on the fundamental path displacements. Using the same notation we may write the individual energy components as

$$\frac{\overset{\prime}{N}_\phi\overset{\prime}{\varepsilon}_\phi}{Ka^2} = \frac{1}{2}\int\int \varepsilon_1(\varepsilon_1 + \mu\varepsilon_2)\sin\phi\, d\theta d\phi \tag{4.51}$$

$$\frac{\overset{\prime}{N}_\theta\overset{\prime}{\varepsilon}_\theta}{Ka^2} = \frac{1}{2}\int\int \varepsilon_2(\varepsilon_2 + \mu\varepsilon_1)\sin\phi\, d\theta d\phi$$

$$\frac{\overset{\prime}{N}_{\theta\phi}\overset{\prime}{\varepsilon}_{\theta\phi}}{Ka^2} = \frac{1}{2}\int\int \frac{1}{2}(1-\mu)(\overset{\prime}{v}+\gamma)^2\sin\phi\, d\theta d\phi$$

$$\frac{\overset{f,\prime}{N}_\phi\overset{f,\prime}{\varepsilon}_\phi}{Ka^2} = \frac{1}{2}\int\int\Bigg[\,2\varepsilon_1\beta\left\{\beta^s - \varepsilon_1^s\beta^s + (\varepsilon_1^s)^2\beta^s - \frac{1}{2}(\beta^s)^3\right\} + ((\varepsilon_1)^2 - (\beta)^2)\left\{-\frac{1}{2}(\beta^s)^2 + (\beta^s)^2\varepsilon_1^s\right\}$$
$$+\mu\left\{\varepsilon_2\beta\{\beta^s - \varepsilon_1^s\beta^s + (\varepsilon_1^s)^2\beta^s + \frac{1}{2}(\beta^s)^3\} + \varepsilon_1\varepsilon_2\{-\frac{1}{2}(\beta^s)^2 + (\beta^s)^2\varepsilon_1^s\}\right\}\Bigg]\sin\phi\, d\theta d\phi$$

$$\frac{\overset{f,\prime}{N}_\theta\overset{f,\prime}{\varepsilon}_\theta}{Ka^2} = \frac{1}{2}\int\int\mu\left[\varepsilon_2\beta\left\{\beta^s - \varepsilon_1^s\beta^s + (\varepsilon_1^s)^2\beta^s + \frac{1}{2}(\beta^s)^3\right\} + \varepsilon_1\varepsilon_2\left\{-\frac{1}{2}(\beta^s)^2 + (\beta^s)^2\varepsilon_1^s\right\}\right]\sin\phi\, d\theta d\phi$$

(4.51 contd.)

$$\frac{\overset{f,'}{N}_{\theta\phi}\overset{f,'}{\varepsilon}_{\theta\phi}}{Ka^2} = \frac{1}{2}\int\int(1-\mu)\,[\,\dot{v}^2\left\{-\varepsilon_1^s - \frac{1}{2}(\beta^s)^2 + \frac{3}{2}(\varepsilon_1^s)^2 - 2(\varepsilon_1^s)^3 + 2\varepsilon_1^s(\beta^s)^2\right\}$$

$$+\gamma^2\left\{-\varepsilon_2^s - \frac{1}{2}(\beta^s)^2 + \frac{3}{2}(\varepsilon_2^s)^2 - 2(\varepsilon_2^s)^3 + \varepsilon_2^s(\beta^s)^2 + \varepsilon_1^s(\beta^s)^2\right\}$$

$$+\delta^2\left\{\frac{1}{2}(\beta^s)^2 - \varepsilon_1^s(\beta^s)^2 - \varepsilon_2{}^s(\beta^s)^2\right\}$$

$$+\dot{v}\gamma\,\{-\varepsilon_1^s - \varepsilon_2^s - (\beta^s)^2 + (\varepsilon_1^s)^2 + (\varepsilon_2^s)^2 + \varepsilon_1^s\varepsilon_2^s - (\varepsilon_2^s)^3$$

$$-(\varepsilon_1^s)^3 + \varepsilon_2^s(\beta^s)^2 + 3\varepsilon_1^s(\beta^s)^2 - \varepsilon_2^s(\varepsilon_1^s)^2 - \varepsilon_1^s(\varepsilon_2^s)^2\,\}$$

$$+\dot{v}\delta\left\{\beta^s(1-2\varepsilon_1^s+\varepsilon_2^s) - \frac{1}{2}(\beta^s)^3 + 2(\varepsilon_1^s)^2\beta^s + \varepsilon_1^s\varepsilon_2^s\beta^s\right\}$$

$$+\gamma\delta\left\{\beta^s(1-2\varepsilon_2^s+\varepsilon_1^s) - \frac{1}{2}(\beta^s)^3 + 2(\varepsilon_2^s)^2\beta^s + \varepsilon_1^s\varepsilon_2^s\beta^s\right\}\,]\,\sin\phi\,d\theta d\phi$$

$$\frac{\overset{f}{N}_{\phi}\overset{f,''}{\varepsilon}_{\phi}}{Ka^2} = \frac{1}{2}\int\int[\,\frac{1}{2}(\dot{v}^2+\beta^2)\left\{\varepsilon_1^s - (\varepsilon_1^s)^2 + \frac{1}{2}(\beta^s)^2 - \frac{3}{2}\varepsilon_1^s(\beta^s)^2 + (\varepsilon_1^s)^3\right\}$$

$$+\varepsilon_1\beta\left\{-\varepsilon_1^s\beta^s - 2(\varepsilon_1^s)^2\beta^s - \frac{1}{2}(\beta^s)^3\right\} + (\varepsilon_1^2-\beta^2)\left\{\frac{1}{2}(\beta^s)^2\varepsilon_1^s\right\}$$

$$+\mu\{\,\frac{1}{2}(\dot{v}^2+\beta^2)\left\{\varepsilon_2^s - \varepsilon_2^s\varepsilon_1^s + \varepsilon_2^s(\varepsilon_1^s)^2 - \frac{1}{2}(\beta^s)^2\varepsilon_2^s\right\}$$

$$+\varepsilon_1\beta(-\varepsilon_2^s\beta^s - 2\varepsilon_1^s\varepsilon_2^s\beta^s) + (\varepsilon_1^2-\beta^2)\{\frac{1}{2}(\beta^s)^2\varepsilon_2^s\}\}\,]\,\sin\phi\,d\theta d\phi$$

$$\frac{\overset{f,''}{N}_{\phi}\overset{f}{\varepsilon}_{\phi}}{Ka^2} = \frac{1}{2}\int\int[\,\frac{1}{2}(\dot{v}^2+\beta^2)\left\{\varepsilon_1^s - (\varepsilon_1^s)^2 + \frac{1}{2}(\beta^s)^2 - \frac{3}{2}\varepsilon_1^s(\beta^s)^2 + (\varepsilon_1^s)^3\right\}$$

$$+\varepsilon_1\beta\{-\varepsilon_2^s\beta^s - 2\varepsilon_1^s\varepsilon_2^s\beta^s\} + (\varepsilon_1^2-\beta^2)\left\{\frac{1}{2}(\beta^s)^2\varepsilon_1^s\right\}$$

$$+\mu\left\{\frac{1}{2}(\gamma^2+\delta^2)\left\{\varepsilon_1^s + \frac{1}{2}(\beta^s)^2 - \varepsilon_1^s\varepsilon_2^s + \varepsilon_1^s(\varepsilon_2^s)^2 - \frac{1}{2}\varepsilon_2^s(\beta^s)^2 - \frac{1}{2}\varepsilon_1^s(\beta^s)^2\right\}\right\}]\,\sin\phi\,d\theta d\phi$$

$$\frac{\overset{f}{N}_{\theta}\overset{f,''}{\varepsilon}_{\theta}}{Ka^2} = \frac{1}{2}\int\int[\,\frac{1}{2}(\gamma^2+\delta^2)\{\varepsilon_2^s - (\varepsilon_2^s)^2 + (\varepsilon_2^s)^3\}$$

$$+\frac{1}{2}\mu(\gamma^2+\delta^2)\left\{\varepsilon_1^s - \varepsilon_1^s\varepsilon_2^s + \frac{1}{2}(\beta^s)^2 - \frac{1}{2}\varepsilon_2^s(\beta^s)^2 - \frac{1}{2}\varepsilon_1^s(\beta^s)^2 + \varepsilon_1^s(\varepsilon_2^s)^2\right\}\,]\,\sin\phi\,d\theta d\phi$$

$$\frac{\overset{f,"}{N}_\theta \overset{f}{\varepsilon}_\theta}{Ka^2} = \frac{1}{2}\int\int [\ \frac{1}{2}(\gamma^2+\delta^2)\{\varepsilon_2^s - (\varepsilon_2^s)^2 + (\varepsilon_2^s)^3\}$$

$$+\mu\{\ (\dot{v}^2+\beta^2)\left\{\varepsilon_2^s - \varepsilon_1^s\varepsilon_2^s + (\varepsilon_1^s)^2\varepsilon_2^s - \frac{1}{2}(\beta^s)^2\varepsilon_2^s\right\}$$

$$+\varepsilon_1\beta(-\varepsilon_2^s\beta^s + 2\varepsilon_1^s\varepsilon_2^s\beta^s) + (\varepsilon_1^2 - \beta^2)\{\frac{1}{2}(\beta^s)^2\varepsilon_2^s\}\}\ ]\ \sin\phi\ d\theta d\phi$$

(4.51 contd.)

$$\frac{U_{2,B_\phi}}{Ka^2} = \frac{\alpha}{2}\int\int\dot{\beta}\left[\dot{\beta} + \mu(\delta/\sin\phi + \beta\cot\phi)\right]\sin\phi\ d\theta d\phi$$

$$\frac{U_{2,B_\theta}}{Ka^2} = \frac{\alpha}{2}\int\int(\dot{\delta}/\sin\phi + \beta\cot\phi)(\dot{\delta}/\sin\phi + \beta\cot\phi + \mu\dot{\beta})\sin\phi\ d\theta d\phi$$

$$\frac{U_{2,B_{\theta\phi}}}{Ka^2} = \frac{\alpha}{2}\int\int\frac{1}{2}(1-\mu)(\dot{\delta} + \dot{\beta}/\sin\phi - \delta\cot\phi)^2\sin\phi\ d\theta d\phi$$

$$\frac{\overset{"}{P}(\Delta V)^*}{Ka^2} = \hat{p}\int\int[\ 3(w+u\cot\phi)^2 + (\dot{u} - \dot{w}\cot\phi)\dot{v}/\sin\phi$$

$$+2(w+u\cot\phi)(\dot{u} - \dot{w}\cot\phi + \dot{v}/\sin\phi)\ ]\ \sin^3\phi\ d\theta d\phi$$

$$\frac{\overset{f,"}{P}(\Delta V)}{Ka^2} = \hat{p}\int\int[\ (w+u\cot\phi)^2(3\varepsilon_2^s + \varepsilon_3^s) + (\dot{u} - \dot{w}\cot\phi)\dot{v}/\sin\phi(\varepsilon_2^s)$$

$$+2(w+u\cot\phi)(\dot{u} - \dot{w}\cot\phi + \dot{v}/\sin\phi)(\varepsilon_2^s)$$

$$+(w+u\cot\phi)\dot{v}/\sin\phi(\varepsilon_3^s)\ ]\ \sin^3\phi\ d\theta d\phi$$

TABLE 4.1; Definition of terms in equation (4.31), page 1 of 2.

$A_{1,1}(x^f)$ =	$u^f\left[-(1+\alpha)(\cot^2\phi+\mu)\right]+\dot{u}^f[(1+\alpha)\cot\phi]+\ddot{u}^f[1+\alpha]$ $+\dot{w}^f\left[1+\mu+\alpha(\cot^2\phi+\mu)\right]+\ddot{w}^f[-\alpha\cot\phi]+\dddot{w}^f[-\alpha]+u^f[\frac{1}{2}\dot{Q}^i_\phi]+\dot{w}^f[-\frac{1}{2}Q^i_\phi]$
$A_{3,1}(x^f)$ =	$u^f\left[(1+\mu)\cot\phi-\alpha(1-\mu+1/\sin^2\phi)\cot\phi\right]+\dot{u}^f\left[(1+\mu)+\alpha(\mu+1/\sin^2\phi)\right]$ $+\ddot{u}^f[-2\alpha\cot\phi]+\dddot{u}^f[-\alpha]+w^f[2(1+\mu)]+\dot{w}^f[\alpha(1-\mu+1/\sin^2\phi)\cot\phi]$ $+\ddot{w}^f[-\alpha(\mu+1/\sin^2\phi)]+\dddot{w}^f[2\alpha\cot\phi]+\ddddot{w}^f[\alpha]$ $+u^f[-\frac{1}{2}\dot{Q}^i_\phi-\frac{1}{2}Q^i_\phi\cot\phi]+\dot{u}^f[-\frac{1}{2}Q^i_\phi]+\dot{w}^f[\frac{1}{2}\dot{Q}^i_\phi+\frac{1}{2}Q^i_\phi\cot\phi]+\ddot{w}^f[\frac{1}{2}Q^i_\phi]$
$A_{1,2}(x^s)^2$ =	$(\beta^s)^2[\frac{1}{2}(1-\mu)\cot\phi]+\beta^s\dot{\beta}^s-\beta^s\varepsilon^s_1[-1]+\beta^s\varepsilon^s_2[-\mu]$ $+(\beta^s)^2[\frac{1}{4}\dot{Q}^i_\phi+\frac{1}{4}Q^i_\phi\cot\phi]+\beta^s\dot{\beta}^s[\frac{1}{2}Q^i_\phi]+\beta^s\varepsilon^s_1[-\frac{1}{2}Q^i_\phi]$
$A_{3,2}(x^s)^2$ =	$(\beta^s)^2[\frac{1}{2}(1+\mu)]+\beta^s\varepsilon^s_1[\cot\phi]+\beta^s\varepsilon^s_2[\mu\cot\phi]+\dot{\beta}^s\varepsilon^s_1+\dot{\beta}^s\varepsilon^s_2[\mu]+\beta^s\dot{\varepsilon}^s_1$ $+\beta^s\dot{\varepsilon}^s_2[\mu]+(\beta^s)^2[\frac{1}{4}\dot{Q}^i_\phi]+\beta^s\varepsilon^s_1[\frac{1}{2}\dot{Q}^i_\phi\cot\phi+\frac{1}{2}Q^i_\phi]+\beta^s\varepsilon^s_1[\frac{1}{2}Q^i_\phi]+\varepsilon^s_1\dot{\beta}^s[\frac{1}{2}Q^i_\phi]$
$A_{1,3}(x^s)^3$ =	$(\beta^s)^3[-\frac{1}{2}]+(\beta^s)^2\varepsilon^s_1[-(1-\frac{1}{2}\mu)\cot\phi]+(\beta^s)^2\varepsilon^s_2[-\frac{1}{2}\mu\cot\phi]+\beta^s\dot{\beta}^s\varepsilon^s_1[-2]$ $+\beta^s(\varepsilon^s_1)^2+\beta^s\varepsilon^s_1\varepsilon^s_2[\mu]+(\beta^s)^2\dot{\varepsilon}^s_2[-\frac{1}{2}\mu]-(\beta^s)^2\dot{\varepsilon}^s_1+\beta^s\dot{\beta}^s\varepsilon^s_2[-\mu]$
$A_{3,3}(x^s)^3$ =	$(\beta^s)^3[\frac{1}{2}\cot\phi]+(\beta^s)^2\beta^s[\frac{3}{2}]+(\beta^s)^2\varepsilon^s_1[-1-\frac{1}{2}\mu]+(\beta^s)^2\varepsilon^s_2[-\frac{1}{2}\mu]$ $+\beta^s(\varepsilon^s_1)^2[-\cot\phi]+\beta^s\varepsilon^s_1\varepsilon^s_2[-\mu\cot\phi]-\beta^s(\varepsilon^s_1)^2+\beta^s\varepsilon^s_1\varepsilon^s_2[-\mu]$ $+\beta^s\varepsilon^s_1\dot{\varepsilon}^s_1[-2]+\beta^s\varepsilon^s_1\varepsilon^s_2[-\mu]+\beta^s\dot{\varepsilon}^s_1\varepsilon^s_2[-\mu]$
$A_{1,4}(x^s)^4$ =	$(\beta^s)^4[\frac{1}{8}\cot\phi(\mu-3)]+(\beta^s)^3\beta^s[-\frac{3}{2}]+(\beta^s)^3\varepsilon^s_1[\frac{3}{2}]+(\beta^s)^3\varepsilon^s_2[\frac{1}{2}\mu]$ $+(\beta^s)^2(\varepsilon^s_1)^2[\frac{1}{2}\cot\phi(3-\mu)]+(\beta^s)^2\varepsilon^s_1\varepsilon^s_2[\mu\cot\phi]+(\beta^s)^2\varepsilon^s_1\dot{\varepsilon}^s_1[3]+(\beta^s)^2\dot{\varepsilon}^s_1\varepsilon^s_2[\mu]$ $+(\beta^s)^2\dot{\varepsilon}^s_1\varepsilon^s_2[\mu]+\beta^s\dot{\beta}^s(\varepsilon^s_1)^2[3]+\beta^s\dot{\beta}^s\varepsilon^s_1\varepsilon^s_2[2\mu]-\beta^s(\varepsilon^s_1)^3+\beta^s(\varepsilon^s_1)^2\varepsilon^s_2[-\mu]$
$A_{3,4}(x^s)^4$ =	$(\beta^s)^4[-\frac{1}{8}(3+\mu)]+(\beta^s)^3\varepsilon^s_1[-\frac{3}{2}\cot\phi]+(\beta^s)^3\varepsilon^s_2[-\frac{1}{2}\mu\cot\phi]+(\beta^s)^3\dot{\varepsilon}^s_1[-\frac{3}{2}]$ $+(\beta^s)^3\dot{\varepsilon}^s_2[-\frac{1}{2}\mu]+(\beta^s)^2\dot{\beta}^s\varepsilon^s_1[-\frac{9}{2}]+(\beta^s)^2\dot{\beta}^s\varepsilon^s_2[-\frac{3}{2}\mu]+(\beta^s)^2(\varepsilon^s_1)^2[\frac{1}{2}(3+\mu)]$ $+(\beta^s)^2\varepsilon^s_1\varepsilon^s_2[\mu]+\beta^s(\varepsilon^s_1)^3[\cot\phi]+\beta^s(\varepsilon^s_1)^2\varepsilon^s_2[\mu\cot\phi]+\beta^s(\varepsilon^s_1)^2\dot{\varepsilon}^s_2[\mu]$ $+\beta^s(\varepsilon^s_1)^2\dot{\varepsilon}^s_2[\mu]+\beta^s\varepsilon^s_1\varepsilon^s_2\dot{\varepsilon}^s_1[2\mu]+\beta^s(\varepsilon^s_1)^3+\beta^s(\varepsilon^s_1)^2\dot{\varepsilon}^s_1[3]$

TABLE 4.1; Definition of terms in equation (4.31), page 2 of 2.

$A_{1,0}$	=	$\beta^i[\frac{1}{2}\beta^i(1-\mu)\cot\phi+\dot{\beta}^i-\dot{\varepsilon}_1^i-\mu\dot{\varepsilon}_2^i]+\beta^i\varepsilon_1^i[-2\dot{\beta}^i+\dot{\varepsilon}_1^i+\mu\dot{\varepsilon}_2^i]$ $+(\beta^i)^2[-\frac{1}{2}\beta^i-(1-\frac{1}{2}\mu)\varepsilon_1^i\cot\phi-\frac{1}{2}\mu\varepsilon_2^i\cot\phi-\frac{1}{2}\mu\dot{\varepsilon}_2^i-\dot{\varepsilon}_1^i]+\beta^i\dot{\beta}^i\varepsilon_2^i[-\mu]$ $+(\beta^i)^2[\frac{1}{4}\dot{Q}_\phi^i+\frac{1}{4}Q_\phi^i\cot\phi]+\varepsilon_1^i\beta^i[-\frac{1}{2}Q_\phi^i]+\dot{\beta}^i\beta^i[\frac{1}{2}Q_\phi^i]$
$A_{3,0}$	=	$\beta^i[\frac{1}{2}(1+\mu)\beta^i+\varepsilon_1^i\cot\phi+\mu\varepsilon_2^i\cot\phi+\dot{\varepsilon}_1^i+\mu\dot{\varepsilon}_2^i]+\dot{\beta}^i[\varepsilon_1^i+\mu\varepsilon_2^i]$ $+(\beta^i)^2[\frac{1}{2}\beta^i\cot\phi+\frac{3}{2}\beta^i-(1+\frac{1}{2}\mu)\varepsilon_1^i-\frac{1}{2}\mu\varepsilon_2^i]+\beta^i\varepsilon_1^i[-\dot{\varepsilon}_1^i-\mu\dot{\varepsilon}_2^i]$ $+\beta^i\varepsilon_1^i[-\varepsilon_1^i\cot\phi-\mu\varepsilon_2^i\cot\phi-2\dot{\varepsilon}_1^i-\mu\dot{\varepsilon}_2^i]+\beta^i\dot{\varepsilon}_1^i\varepsilon_2^i[-\mu]+(\beta^i)^2[\frac{1}{4}Q_\phi^i]$ $+\beta^i\varepsilon_1^i[\frac{1}{2}Q_\phi^i\cot\phi+\frac{1}{2}\dot{Q}_\phi^i]+\beta^i\dot{\varepsilon}_1^i[\frac{1}{2}Q_\phi^i]+\dot{\varepsilon}_1^i\beta^i[\frac{1}{2}Q_\phi^i]$
$B_{1,1}(x^s)$	=	$2\beta^s$
$B_{3,1}(x^s)$	=	$-2(\varepsilon_1^s+\varepsilon_2^s)$
$B_{1,2}(x^s)^2$	=	$2\beta^s\varepsilon_2^s$
$B_{3,2}(x^s)^2$	=	$-2\varepsilon_1^s\varepsilon_2^s$
$B_{1,0}$	=	0
$B_{3,0}$	=	-2

TABLE 4.2; Definition of terms in equation (4.39), page 1 of 6.

$[A_{1,0}]x$	=	$u[-(1+\alpha)(\mu+\cot^2\phi)] + \dot{u}[(1+\alpha)\cot\phi] + \ddot{u}[(1+\alpha)]$ $+ \ddot{u}\left[\frac{1}{2}(1+\alpha)(1-\mu)/\sin^2\phi\right] + w[(1+\mu)+\alpha(\mu+\cot^2\phi)] + \dot{w}[-\alpha\cot\phi]$ $+ \ddot{w}[-\alpha] + \dot{w}[2\alpha\cot\phi/\sin^2\phi] + \dot{w}[-\alpha/\sin^2\phi] + v[-\frac{1}{2}(1+\alpha)(3-\mu\cot\phi/\sin\phi]$ $+ \dot{v}[\frac{1}{2}(1+\alpha)(1+\mu)] + u[\frac{1}{2}Q^i_\phi] + w[-\frac{1}{2}Q^i_\phi] + \ddot{u}[-\frac{1}{2}Q^i_\theta/\sin^2\phi] + \dot{v}[\frac{1}{2}Q^i_\theta\cot\phi/\sin\phi]$
$[A_{2,0}]x$	=	$\dot{u}[\frac{1}{2}(1+\alpha)(3-\mu)\cot\phi/\sin\phi] + \dot{u}[\frac{1}{2}(1+\alpha)(1+\mu)/\sin\phi]$ $+ v[\frac{1}{2}(1+\alpha)(1-\mu)(1-\cot^2\phi)] + \dot{v}[\frac{1}{2}(1+\alpha)(1-\mu)\cot\phi] + \ddot{v}[\frac{1}{2}(1+\alpha)(1-\mu)]$ $+ \dot{v}[(1+\alpha)/\sin^2\phi] + \dot{w}\left[-\alpha(1-\mu)/\sin\phi + (1+\mu)/\sin\phi\right] + \dot{w}[-\alpha\cot\phi/\sin\phi]$ $+ \ddot{w}[-\alpha/\sin\phi] + \dot{w}[-\alpha/\sin^3\phi] + \dot{u}[-\frac{1}{2}Q^i_\theta\cot\phi/\sin\phi] + v[\frac{1}{2}Q^i_\theta/\sin^2\phi]$ $+ \dot{w}[-\frac{1}{2}Q^i_\theta/\sin\phi] + \dot{v}[-\frac{1}{2}\dot{Q}^i_\phi - \frac{1}{2}Q^i_\phi\cot\phi] + \ddot{v}[-\frac{1}{2}Q^i_\phi]$
$[A_{3,0}]x$	=	$u[-\alpha(\cot\phi/\sin^2\phi + (1-\mu)\cot\phi) + (1+\mu)\cot\phi] + \dot{u}[(1+\alpha)(1+\mu) + \alpha\cot^2\phi]$ $+ \ddot{u}[-2\alpha\cot\phi] + \dddot{u}[-\alpha] + \ddot{u}[-\alpha\cot\phi/\sin^2\phi] + \dot{u}[-\alpha/\sin^2\phi]$ $+ \dot{v}[-\alpha(1/\sin^3\phi + (1-\mu)/\sin\phi) + (1+\mu)/\sin\phi] + \dot{v}[\alpha\cot\phi/\sin\phi] + \ddot{v}[-\alpha/\sin\phi]$ $+ \dddot{v}[-\alpha/\sin^3\phi] + w[2(1+\mu)] + \dot{w}[\alpha(\cot\phi/\sin^2\phi + (1-\mu)\cot\phi]$ $+ \ddot{w}[-\alpha(1+\mu+\cot^2\phi)] + \dddot{w}[2\alpha\cot\phi] + \ddddot{w}[\alpha]$ $+ \ddot{w}[4\cot^2\phi/\sin^2\phi + (3-\mu)/\sin^2\phi] + \ddot{w}[-2\alpha\cot\phi/\sin^2\phi] + \ddot{w}[2\alpha/\sin^2\phi]$ $+ \ddddot{w}[\alpha/\sin^4\phi] + u[-\frac{1}{2}\dot{Q}^i_\phi - \frac{1}{2}Q^i_\phi\cot\phi] + \dot{v}[-\frac{1}{2}Q^i_\theta/\sin\phi] + \dot{u}[-\frac{1}{2}Q^i_\phi]$ $+ \dot{w}[\frac{1}{2}\dot{Q}^i_\phi + \frac{1}{2}Q^i_\phi\cot\phi] + \ddot{w}[\frac{1}{2}Q^i_\phi] + \ddot{w}[\frac{1}{2}Q^i_\theta\sin^2\phi]$

TABLE 4.2; Definition of terms in equation (4.39), page 2 of 6.

$[A_{1,1}(x^s)]x$ =	$u[-\varepsilon_1^s - \mu\varepsilon_2^s + \dot{\beta}^s + (1-2\mu)\beta^s\cot\phi] + w[-(1+\mu)\dot{\beta}^s]$ $+ \dot{w}[\varepsilon_1^s + \mu\varepsilon_2^s - \dot{\beta}^s - (1-\mu)\beta^s\cot\phi] + \ddot{w}[-\dot{\beta}^s] + \ddot{u}[\mu(\varepsilon_1^s + \varepsilon_2^s)/\sin^2\phi]$ $+ \ddot{w}[-\frac{1}{2}(1-\mu)\beta^s/\sin^2\phi] + \dot{v}[-\mu(\varepsilon_1^s + \varepsilon_2^s)\cot\phi/\sin\phi + \frac{1}{2}(1-3\mu)\dot{\beta}^s/\sin\phi]$ $+ \dot{v}[-\frac{1}{2}(1-\mu)(\varepsilon_1^s + \varepsilon_2^s)/\sin\phi] + u[\frac{1}{2}Q_\phi^i\beta^s - \frac{1}{2}Q_\phi^i(\beta^s + \beta^s\cot\phi - \varepsilon_1^s)] + w[-\frac{1}{2}Q_\phi^i\dot{\beta}^s]$ $+ \dot{w}[-\frac{1}{2}Q_\phi^i\beta^s + \frac{1}{2}Q_\phi^i(\varepsilon_1^s - \dot{\beta}^s - \beta^s\cot\phi)] + \ddot{u}[\frac{1}{2}Q_\theta^i\varepsilon_2^s/\sin^2\phi] + \dot{v}[-\frac{1}{2}Q_\theta^i\varepsilon_2^s\cot\phi/\sin\phi]$ $+ \ddot{w}[-\frac{1}{2}Q_\phi^i\dot{\beta}^s]$
$[A_{2,1}(x^s)]x$ =	$\dot{u}[\mu(\varepsilon_1^s + \varepsilon_2^s)\cot\phi/\sin\phi - \frac{1}{2}(1-\mu)(\dot{\varepsilon}_1^s + \dot{\varepsilon}_2^s)/\sin\phi - \frac{1}{2}(1-3\mu)\dot{\beta}^s/\sin\phi]$ $+ v[(1 - \frac{1}{2}(1+\mu) - \mu\cot^2\phi)\dot{\varepsilon}_1^s + (-\frac{1}{2}(3-\mu) - \mu\cot^2\phi)\dot{\varepsilon}_2^s + \frac{1}{2}(1-\mu)(\dot{\varepsilon}_1^s + \dot{\varepsilon}_2^s)\cot\phi$ $+ \frac{3}{2}(1-\mu)\dot{\beta}^s\cot\phi + \frac{1}{2}(1-\mu)\dot{\beta}^s]$ $+ \dot{w}[\mu\dot{\varepsilon}_1^s/\sin\phi + \dot{\varepsilon}_2^s/\sin\phi - \frac{1}{2}(1-\mu)\dot{\beta}^s/\sin\phi - \frac{1}{2}(1-\mu)\dot{\beta}^s\cot\phi/\sin\phi]$ $+ \dot{u}[-\frac{1}{2}(1-\mu)(\dot{\varepsilon}_1^s + \dot{\varepsilon}_2^s)/\sin\phi] + \dot{v}[\mu(\dot{\varepsilon}_1^s + \dot{\varepsilon}_2^s) + \mu(\dot{\varepsilon}_1^s + \dot{\varepsilon}_2^s)\cot\phi]$ $+ \dot{w}[-\frac{1}{2}(1+\mu)\dot{\beta}^s/\sin\phi] + \ddot{v}[\mu(\dot{\varepsilon}_1^s + \dot{\varepsilon}_2^s)] + \dot{u}[\frac{1}{2}Q_\theta^i\dot{\varepsilon}_2^s\cot\phi/\sin\phi]$ $+ v[-\frac{1}{2}Q_\theta^i\dot{\varepsilon}_2^s/\sin^2\phi] + \dot{w}[\frac{1}{2}Q_\theta^i\dot{\varepsilon}_2^s/\sin\phi] + \dot{v}[\frac{1}{2}Q_\phi^i\dot{\varepsilon}_1^s + \frac{1}{2}Q_\phi^i(\dot{\varepsilon}_1^s + \dot{\varepsilon}_1^s\cot\phi)]$ $+ \ddot{v}[\frac{1}{2}Q_\phi^i\dot{\varepsilon}_1^s]$
$[A_{3,1}(x^s)]x$ =	$u[(\varepsilon_1^s + \mu\varepsilon_2^s)\cot\phi + \dot{\varepsilon}_1^s + \mu\dot{\varepsilon}_2^s + \dot{\beta}^s + \mu\dot{\beta}^s\cot\phi]$ $+ \dot{v}[(\mu\dot{\varepsilon}_1^s + \dot{\varepsilon}_2^s)/\sin\phi + \mu\dot{\beta}^s/\sin\phi - \frac{1}{2}(1-\mu)\dot{\beta}^s\cot\phi/\sin\phi]$ $+ w[(1+\mu)(\dot{\beta}^s + \dot{\beta}^s\cot\phi)] + \dot{u}[\dot{\varepsilon}_1^s + \mu\dot{\varepsilon}_2^s + \dot{\beta}^s + (1+\mu)\dot{\beta}^s\cot\phi]$ $+ \dot{v}[\frac{1}{2}(1+\mu)\dot{\beta}^s/\sin\phi] + \dot{w}[-\dot{\varepsilon}_1^s - \mu\dot{\varepsilon}_2^s - (\dot{\varepsilon}_1^s + \mu\dot{\varepsilon}_2^s)\cot\phi] + \ddot{u}[\dot{\beta}^s] + \ddot{w}[-\dot{\varepsilon}_1^s + \mu\dot{\varepsilon}_2^s]$ $+ \ddot{w}[-(\dot{\varepsilon}_2^s + \mu\dot{\varepsilon}_1^s)/\sin^2\phi] + \ddot{u}[\frac{1}{2}(1-\mu)\dot{\beta}^s/\sin^2\phi]$ $+ u[\frac{1}{2}Q_\phi^i\dot{\varepsilon}_1^s + \frac{1}{2}Q_\phi^i(\dot{\varepsilon}_1^s + \dot{\varepsilon}_1^s\cot\phi + \dot{\beta}^s)] + \dot{v}[\frac{1}{2}Q_\theta^i\dot{\varepsilon}_2^s/\sin\phi]$ $+ w[\frac{1}{2}Q_\phi^i\dot{\beta}^s + \frac{1}{2}Q_\phi^i(\dot{\beta}^s + \beta^s\cot\phi)] + \dot{u}[\frac{1}{2}Q_\phi^i\dot{\beta}^s + \frac{1}{2}Q_\phi^i(\dot{\varepsilon}_1^s + \dot{\beta}^s + \beta^s\cot\phi)]$ $+ \dot{w}[-\frac{1}{2}Q_\phi^i\dot{\varepsilon}_1^s - \frac{1}{2}Q_\phi^i(\dot{\varepsilon}_1^s + \dot{\varepsilon}_1^s\cot\phi)] + \ddot{u}[\frac{1}{2}Q_\phi^i\dot{\beta}^s] + \ddot{w}[-\frac{1}{2}Q_\phi^i\dot{\varepsilon}_1^s] + \ddot{w}[-\frac{1}{2}Q_\theta^i\dot{\varepsilon}_2^s/\sin^2\phi]$

TABLE 4.2; Definition of terms in equation (4.39), page 3 of 6.

$\left[A_{1,2}(x^s)^2\right]x =$	$u\,[-\frac{1}{2}(3-\mu)(\beta^s)^2+(\varepsilon_1^s)^2+\mu\varepsilon_1^s\varepsilon_2^s-\beta^s(2\dot{\varepsilon}_1^s+\mu\dot{\varepsilon}_2^s)-\dot{\beta}^s(2\varepsilon_1^s+\mu\varepsilon_2^s)$ $-\beta^s(2\varepsilon_1^s+\mu\varepsilon_2^s)\cot\phi+2\mu\varepsilon_1^s\beta^s\cot\phi-\mu\beta^s\dot{\beta}^s\cot\phi\,]$ $+\dot{u}[-2\beta^s\dot{\beta}^s-(\beta^s)^2\cot\phi]+\ddot{u}[-(\beta^s)^2]$ $+\ddot{u}[-\frac{1}{2}(1-2\mu)(\beta^s)^2+\frac{1}{2}(1-3\mu)(\varepsilon_2^s)^2-\mu\varepsilon_1^s\varepsilon_2^s)/\sin^2\phi]$ $+w[-(2+\mu)\beta^s\dot{\beta}^s+(2+\mu)\beta^s\varepsilon_1^s+\beta^s\varepsilon_2^s-(\beta^s)^2\cot\phi]$ $+\dot{w}\,[\frac{1}{2}(1-\mu)(\beta^s)^2-\varepsilon_1^s(\varepsilon_1^s+\mu\varepsilon_2^s)+\beta^s(2\dot{\varepsilon}_1^s+\mu\dot{\varepsilon}_2^s)+\dot{\beta}^s(2\varepsilon_1^s+\mu\varepsilon_2^s)$ $+\beta^s(2\varepsilon_1^s+\mu\varepsilon_2^s)\cot\phi-\mu\beta^s\varepsilon_1^s\cot\phi\,]$ $+\ddot{w}[\beta^s(2\varepsilon_1^s+\mu\varepsilon_2^s)]+\ddot{w}[\frac{1}{2}(1-\mu)\beta^s(\varepsilon_1^s+2\varepsilon_2^s)/\sin^2\phi]$ $+\dot{v}\,[\,\{\frac{1}{2}(1-2\mu)(\beta^s)^2-\frac{1}{2}(1-3\mu)(\varepsilon_2^s)^2+\mu\varepsilon_1^s\varepsilon_2^s\}\cot\phi/\sin\phi$ $+\{\mu\varepsilon_1^s\beta^s-\mu\beta^s\dot{\beta}^s-\frac{1}{2}(1-\mu)\beta^s(\varepsilon_1^s+2\varepsilon_2^s)\}/\sin\phi]$ $+\dot{v}[-\frac{1}{2}\mu(\beta^s)^2/\sin\phi+\frac{1}{2}(1-\mu)\left((\varepsilon_1^s)^2+(\varepsilon_2^s)^2-(\beta^s)^2+\varepsilon_1^s\varepsilon_2^s\right)/\sin\phi]$ $+u[\frac{1}{2}Q_\phi^i((\varepsilon_1^s)^2-\frac{3}{2}(\beta^s)^2-2\beta^s\varepsilon_1^s-2\dot{\beta}^s\varepsilon_1^s-2\beta^s\dot{\varepsilon}_1^s\cot\phi)-\dot{Q}_\phi^i\varepsilon_1^s\beta^s]$ $+\dot{u}[-\frac{1}{2}\dot{Q}_\phi^i(\beta^s)^2-\frac{1}{2}Q_\phi^i((\beta^s)^2\cot\phi+2\beta^s\dot{\beta}^s)]$ $+w[-\frac{1}{2}\dot{Q}_\phi^i(\beta^s)^2-\frac{1}{2}Q_\phi^i((\beta^s)^2\cot\phi+2\beta^s\dot{\beta}^s+2\dot{\beta}^s\varepsilon_1^s)]$ $+\dot{w}[-\frac{1}{2}\dot{Q}_\phi^i((\varepsilon_1^s)^2-\frac{1}{2}(\beta^s)^2-2\beta^s\dot{\varepsilon}_1^s-2\dot{\beta}^s\varepsilon_1^s-2\beta^s\varepsilon_1^s\cot\phi)-\dot{Q}_\phi^i\beta^s\varepsilon_1^s]$ $+\ddot{u}[-\frac{1}{2}Q_\theta^i(\varepsilon_2^s)^2/\sin^2\phi]+\dot{v}[\frac{1}{2}\dot{Q}_\theta^i\varepsilon_2^s\cot\phi/\sin\phi]+\ddot{u}[-\frac{1}{2}\dot{Q}_\phi^i(\beta^s)^2]+\ddot{w}[\dot{Q}_\phi^i\beta^s\varepsilon_1^s]$

TABLE 4.2; Definition of terms in equation (4.39), page 4 of 6.

$\left[A_{2,2}(x^s)^2\right]x =$	$\dot{u}\,[\,\{-\frac{1}{2}(1-2\mu)(\beta^s)^2+\frac{1}{2}(1-3\mu)(\varepsilon_2^s)^2-\mu\varepsilon_1^s\varepsilon_2^s\}\cot\phi/\sin\phi-\mu\varepsilon_1^s\beta^s/\sin\phi$
	$+(1-\mu)\{-\beta^s\dot{\beta}^s+\varepsilon_1^s\,(\dot{\varepsilon}_1^s+\tfrac{1}{2}\dot{\varepsilon}_2^s)+\varepsilon_2^s\,(\dot{\varepsilon}_2^s+\tfrac{1}{2}\dot{\varepsilon}_1^s)+\tfrac{1}{2}\beta^s(\dot{\varepsilon}_1^s+2\dot{\varepsilon}_2^s)\}/\sin\phi]$
	$+v[\,\{\tfrac{1}{2}(1-2\mu)(\beta^s)^2-\tfrac{1}{2}(1-3\mu)(\varepsilon_2^s)^2+\mu\varepsilon_1^s\varepsilon_2^s\}\cot^2\phi-\tfrac{1}{2}(\beta^s)^2+\varepsilon_2^s(\varepsilon_2^s+\mu\varepsilon_1^s)$
	$+(1-\mu)\{\beta^s\dot{\beta}^s-\varepsilon_1^s\,(\dot{\varepsilon}_1^s+\tfrac{1}{2}\dot{\varepsilon}_2^s)-\varepsilon_2^s\,(\dot{\varepsilon}_2^s+\tfrac{1}{2}\dot{\varepsilon}_1^s)\}\cot\phi$
	$+(1-\mu)\{-\tfrac{1}{2}(\beta^s)^2+\frac{1}{2}(\varepsilon_1^s)^2+\tfrac{1}{2}(\varepsilon_2^s)^2+\tfrac{1}{2}\varepsilon_1^s\varepsilon_2^s\}$
	$-(1-\mu)\{\tfrac{1}{2}\beta^s(2\dot{\varepsilon}_1^s+\dot{\varepsilon}_2^s)+\tfrac{1}{2}\dot{\beta}^s(2\varepsilon_1^s+\varepsilon_2^s)\}$
	$-(1-\mu)\{\tfrac{1}{2}\beta^s(2\dot{\varepsilon}_1^s+\dot{\varepsilon}_2^s)+\beta^s(\dot{\varepsilon}_1^s+2\dot{\varepsilon}_2^s)\}\cot\phi]$
	$+\dot{w}\,[\,\{\tfrac{1}{2}(1-\mu)(\beta^s)^2-\varepsilon_2^s(\mu\varepsilon_1^s+\varepsilon_2^s)\}/\sin\phi$
	$+(1-\mu)\{\tfrac{1}{2}\beta^s(2\dot{\varepsilon}_1^s+\dot{\varepsilon}_2^s)+\tfrac{1}{2}\dot{\beta}^s(2\varepsilon_1^s+\varepsilon_2^s)\}/\sin\phi$
	$+\tfrac{1}{2}(1-\mu)\beta^s(\dot{\varepsilon}_1^s+2\dot{\varepsilon}_2^s)\cot\phi/\sin\phi\,]$
	$+\dot{u}[-\tfrac{1}{2}(\beta^s)^2/\sin\phi+\tfrac{1}{2}(1-\mu)\left\{(\varepsilon_1^s)^2+(\varepsilon_2^s)^2+\varepsilon_1^s\varepsilon_2^s\right\}/\sin\phi]$
	$+\dot{w}[\beta^s\varepsilon_1^s/\sin\phi+\tfrac{1}{2}(1-\mu)\beta^s\varepsilon_2^s/\sin\phi]$
	$+\dot{v}[\mu\beta^s\dot{\beta}^s+(1-3\mu)\varepsilon_1^s\dot{\varepsilon}_1^s-\mu(\dot{\varepsilon}_1^s\varepsilon_2^s+\varepsilon_1^s\dot{\varepsilon}_2^s)$
	$+\{\tfrac{1}{2}\mu(\beta^s)^2+\tfrac{1}{2}(1-3\mu)(\varepsilon_1^s)^2-\mu\varepsilon_1^s\varepsilon_2^s\}\cot\phi]$
	$+\ddot{v}[\tfrac{1}{2}\mu(\beta^s)^2+\tfrac{1}{2}(1-3\mu)(\varepsilon_1^s)^2-\mu\varepsilon_1^s\varepsilon_2^s]+\dot{u}[-\tfrac{1}{2}Q_\theta^i(\varepsilon_2^s)^2\cot\phi/\sin\phi]$
	$+v[\tfrac{1}{2}Q_\theta^i(\varepsilon_2^s)^2\cot^2\phi+\tfrac{1}{2}Q_\theta^i(\varepsilon_2^s)^2]+\dot{w}[-\tfrac{1}{2}Q_\theta^i(\varepsilon_2^s)^2/\sin\phi]$
	$+\dot{v}[-\tfrac{1}{2}\dot{Q}_\phi^i\{(\varepsilon_1^s)^2-\frac{1}{2}(\beta^s)^2\}-\tfrac{1}{2}Q_\phi^i\{2\dot{\varepsilon}_1^s\varepsilon_1^s-\beta^s\dot{\beta}^s+(\varepsilon_1^s)^2\cot\phi-\tfrac{1}{2}(\beta^s)^2\cot\phi\}]$
	$+\ddot{v}[-\tfrac{1}{2}Q_\phi^i\{(\varepsilon_1^s)^2-\frac{1}{2}(\beta^s)^2\}]$

TABLE 4.2; Definition of terms in equation (4.39), page 5 of 6.

$\left[A_{3,2}(x^s)^2x\right] =$	$u[\,3\beta^s\beta^S - 2\varepsilon_1^s\varepsilon_1^S - \mu(\beta^S\varepsilon_2^s + \varepsilon_1^s\varepsilon_2^S) + \beta^s(-2\varepsilon_1^s - \mu\varepsilon_2^s)$ $+ \{\frac{3}{2}(\beta^s)^2 - (\varepsilon_1^s)^2 - \mu\varepsilon_1^s\varepsilon_2^s - \mu\beta^s\varepsilon_1^S - \mu\beta^S\varepsilon_1^s - \frac{1}{2}\mu(\beta^s)^2\}\cot\phi]$ $+ \dot{v}[\,\{\frac{1}{2}(1-\mu)(\beta^s)^2 - (\varepsilon_2^s)^2 - \mu\varepsilon_1^s\varepsilon_2^s - \mu\beta^s\varepsilon_1^S - \mu\beta^S\varepsilon_1^s\}/\sin\phi$ $+ \frac{1}{2}(1-\mu)\beta^s(2\varepsilon_2^s + \varepsilon_1^s)\cot\phi/\sin\phi]$ $+ w[\,-(1+\mu)(\beta^s)^2 - \beta^s(2\varepsilon_1^S + \mu\varepsilon_2^S) - \beta^S(2\varepsilon_1^s + \mu\varepsilon_2^s) - \mu(\varepsilon_1^S\beta^s + \varepsilon_1^s\beta^S)$ $- \beta^s(2\varepsilon_1^s + \mu\varepsilon_2^s)\cot\phi - \mu\varepsilon_1^s\beta^s\cot\phi]$ $+ \dot{u}[\,-\frac{1}{2}(1+\mu)(\beta^s)^2 - \varepsilon_1^s(\varepsilon_1^s + \mu\varepsilon_2^s) - \beta^s(2\varepsilon_1^S + \mu\varepsilon_2^S) - \beta^S(2\varepsilon_1^s + \mu\varepsilon_2^s)$ $- \beta^s(2\varepsilon_1^s + \mu\varepsilon_2^s)\cot\phi - \mu\varepsilon_1^s\beta^s\cot\phi]$ $+ \dot{w}[-3\beta^s\beta^S + 2\varepsilon_1^s\varepsilon_1^S + \mu(\varepsilon_1^S\varepsilon_2^s + \varepsilon_1^s\varepsilon_2^S) + \{-\frac{3}{2}(\beta^s)^2 + (\varepsilon_1^s)^2 + \mu\varepsilon_1^s\varepsilon_2^s\}\cot\phi]$ $+ \dot{v}[-\mu\varepsilon_1^s\beta^s/\sin\phi - \frac{1}{2}(1-\mu)\beta^s(2\varepsilon_1^s + \varepsilon_2^s)/\sin\phi] + \ddot{u}[-\beta^s(2\varepsilon_1^s + \mu\varepsilon_2^s)]$ $+ \ddot{w}[-\frac{3}{2}(\beta^s)^2 + (\varepsilon_1^s)^2 + \mu\varepsilon_1^s\varepsilon_2^s] + \ddot{w}[\{-\frac{1}{2}(\beta^s)^2 + (\varepsilon_2^s)^2 + \mu\varepsilon_1^s\varepsilon_2^s\}/\sin^2\phi]$ $+ \ddot{u}[-\frac{1}{2}(1-\mu)\beta^s(\varepsilon_1^s + 2\varepsilon_2^s)/\sin^2\phi]$ $+ u[\,-\frac{1}{2}Q_\phi^i\{2\varepsilon_1^s\varepsilon_1^S - 3\beta^s\beta^S + (\varepsilon_1^s)^2\cot\phi - \frac{3}{2}(\beta^s)^2\cot\phi + 2\beta^s\varepsilon_1^s\}$ $- \frac{1}{2}Q_\phi^i\{(\varepsilon_1^s)^2 - \frac{3}{2}(\beta^s)^2\}]$ $+ \dot{v}[-\frac{1}{2}Q_\theta^i(\varepsilon_2^s)^2/\sin\phi]$ $+ w[-Q_\phi^i\beta^s\varepsilon_1^s - \frac{1}{2}Q_\phi^i\{2\beta^s\varepsilon_1^S + 2\beta^S\varepsilon_1^s + 2\beta^s\varepsilon_1^s\cot\phi + (\beta^s)^2\}]$ $+ \dot{u}[-Q_\phi^i\beta^s\varepsilon_1^s - \frac{1}{2}Q_\phi^i\{2\beta^s\varepsilon_1^S + 2\beta^S\varepsilon_1^s + 2\beta^s\varepsilon_1^s\cot\phi + (\varepsilon_1^s)^2 - \frac{1}{2}(\beta^s)^2\}]$ $+ \dot{w}[\frac{1}{2}Q_\phi^i\{(\varepsilon_1^s)^2 - \frac{3}{2}(\beta^s)^2\} + \frac{1}{2}Q_\phi^i\{2\varepsilon_1^s\varepsilon_1^S - 3\beta^s\beta^S + (\varepsilon_1^s)^2\cot\phi - \frac{3}{2}(\beta^s)^2\cot\phi\}]$ $+ \ddot{u}[-Q_\phi^i\beta^s\varepsilon_1^s] + \ddot{w}[\frac{1}{2}Q_\phi^i\{(\varepsilon_1^s)^2 - \frac{3}{2}(\beta^s)^2\}] + \ddot{w}[\frac{1}{2}Q_\theta^i(\varepsilon_2^s)^2/\sin^2\phi]$
$\left[A_{1,3}(x^s)^3\right]x =$	$u[-\frac{9}{2}\beta^s(\beta^s)^2] + \dot{u}[2\beta^s\beta^s(3\varepsilon_1^s + \mu\varepsilon_2^s) + (\beta^s)^2(3\varepsilon_1^s + \mu\varepsilon_2^s)\cot\phi]$ $+ w[2\beta^s\beta^s(3\varepsilon_1^s + \mu\varepsilon_1^s + \mu\varepsilon_2^s) + (\beta^s)^2(3\varepsilon_1^s + \mu\varepsilon_2^s)\cot\phi] + \dot{w}[\frac{9}{2}\beta^s(\beta^s)^2$ $+ \frac{1}{2}(3-\mu)(\beta^s)^3\cot\phi] + \ddot{u}[\frac{1}{2}(2-3\mu)(\beta^s)^2(\varepsilon_1^s + \varepsilon_2^s)/\sin^2\phi]$ $+ \ddot{w}[\frac{1}{4}(1-\mu)(\beta^s)^3/\sin^2\phi] + \dot{v}[2\mu\varepsilon_1^s\beta^s\beta^s/\sin\phi] + \ddot{u}[(\beta^s)^2(3\varepsilon_1^s + \mu\varepsilon_2^s)]$ $+ \ddot{w}[\frac{3}{2}(\beta^s)^3] + \dot{v}[\mu\varepsilon_1^s(\beta^s)^2/\sin\phi + \frac{1}{2}(1-\mu)(\beta^s)^2(3\varepsilon_1^s + \varepsilon_2^s)/\sin\phi]$

TABLE 4.2; Definition of terms in equation (4.39), page 6 of 6.

$\left[A_{2,3}(x^s)^3\right]x =$	$\dot{u}[\frac{1}{2}(2-3\mu)(\beta^s)^2(\varepsilon_1^s+\varepsilon_2^s)\cot\phi/\sin\phi+(1-\mu)\beta^s\dot{\beta}^s(3\varepsilon_1^s+\varepsilon_2^s)/\sin\phi]$ $+v[-\frac{1}{2}(2-3\mu)(\beta^s)^2(\varepsilon_1^s+\varepsilon_2^s)\cot^2\phi-\frac{3}{4}(1-\mu)(\beta^s)^2\dot{\beta}^s$ $-(1-\mu)\beta^s\dot{\beta}^s(3\varepsilon_1^s+\varepsilon_2^s)\cot\phi]$ $+\dot{w}[\frac{3}{4}(1-\mu)(\beta^s)^2\dot{\beta}^s/\sin\phi]+\dot{u}[\mu\varepsilon_1^s(\beta^s)^2/\sin\phi+\frac{1}{2}(1-\mu)(\beta^s)^2(3\varepsilon_1^s+\varepsilon_2^s)/\sin\phi]$ $+\dot{v}[\beta^s\dot{\beta}^s(-\mu\varepsilon_2^s+(1-4\mu)\varepsilon_1^s)]$
$\left[A_{3,3}(x^s)^3\right]x =$	$u[\dot{\beta}^s\beta^s(-9\varepsilon_1^s-3\mu\varepsilon_2^s)-\frac{3}{2}(\beta^s)^3-\frac{3}{2}\mu(\beta^s)^2\dot{\beta}^s\cot\phi]+\dot{v}[-\frac{3}{2}\mu(\beta^s)^2\dot{\beta}^s/\sin\phi]$ $+w[-\frac{9}{2}(\beta^s)^2\dot{\beta}^s-\frac{3}{2}(\beta^s)^3\cot\phi-\frac{3}{2}\mu(\beta^s)^2\dot{\beta}^s-\frac{1}{2}\mu(\beta^s)^3\cot\phi]$ $+\dot{u}[-\frac{9}{2}(\beta^s)^2\dot{\beta}^s-\frac{1}{2}(3+\mu)(\beta^s)^3\cot\phi]$ $+\dot{w}[\beta^s\dot{\beta}^s(9\varepsilon_1^s+3\mu\varepsilon_2^s)+\frac{1}{2}(\beta^s)^2(9\varepsilon_1^s+3\mu\varepsilon_2^s)\cot\phi]+\ddot{u}[-\frac{3}{2}(\beta^s)^3]$ $+\ddot{w}[\frac{1}{2}(\beta^s)^2(9\varepsilon_1^s+3\mu\varepsilon_2^s)]+\ddot{w}[\frac{1}{2}(2-\mu)(\beta^s)^2(\varepsilon_1^s+\varepsilon_2^s)/\sin^2\phi]$ $+\ddot{u}[-\frac{1}{4}(1-\mu)(\beta^s)^3/\sin^2\phi]$
$[B_{1,0}]x =$	$u[-2]+\dot{w}[2]+\dot{v}[\sin\phi]$
$[B_{2,0}]x =$	$\dot{u}[2\cos\phi]+\dot{u}[\sin\phi]+\dot{w}[2\sin\phi]+\dot{w}[-\cos\phi]$
$[B_{3,0}]x =$	$u[2\cot\phi]+w[4]+\dot{u}[2]+\dot{v}[1/\sin\phi]+\dot{v}[\cos\phi]$
$[B_{1,1}(x^s)]x =$	$\dot{u}[-2\varepsilon_2^s-2\beta^s\cot\phi]+w[-2\beta^s]+\dot{v}[-\beta^s/\sin\phi]+\dot{w}[2\varepsilon_2^s]+\dot{v}[\varepsilon_2^s\sin\phi]$
$[B_{2,1}(x^s)]x =$	$\dot{u}[(\varepsilon_1^s+\varepsilon_2^s+\beta^s\cot\phi)\cos\phi]+\dot{u}[\varepsilon_2^s\sin\phi]+\dot{w}[(\varepsilon_1^s+\varepsilon_2^s+\beta^s\cot\phi)\sin\phi]$ $+\dot{w}[-\varepsilon_2^s\cos\phi]$
$[B_{3,1}(x^s)]x =$	$u[2\varepsilon_1^s\cot\phi]+w[2\varepsilon_1^s+2\varepsilon_2^s]+\dot{u}[2\varepsilon_2^s]+\dot{v}[\varepsilon_1^s/\sin\phi]+\dot{v}[\varepsilon_2^s\cos\phi]$

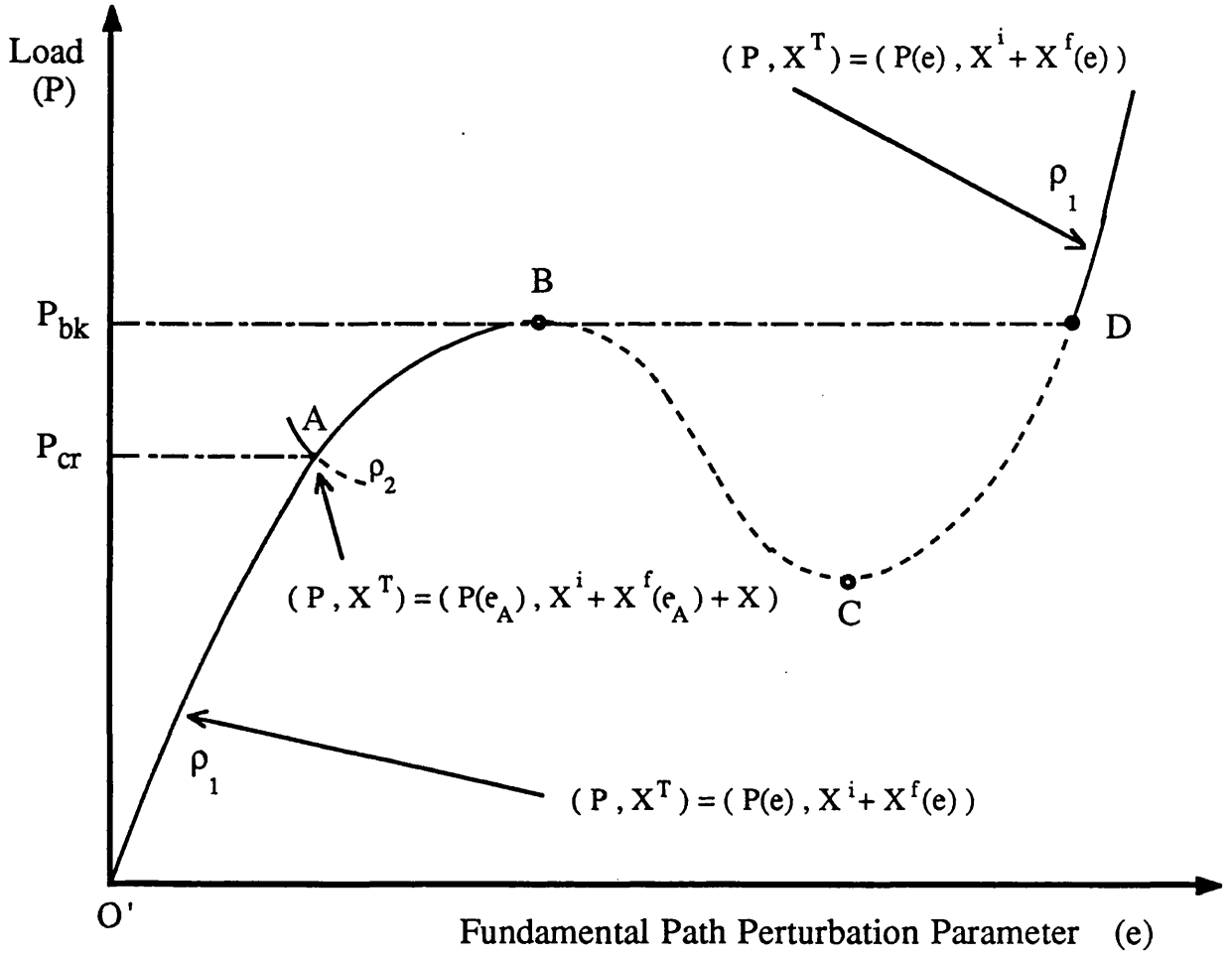


Figure 4.1 Fundamental and Secondary Equilibrium paths, Load -vs- Perturbation Parameter.

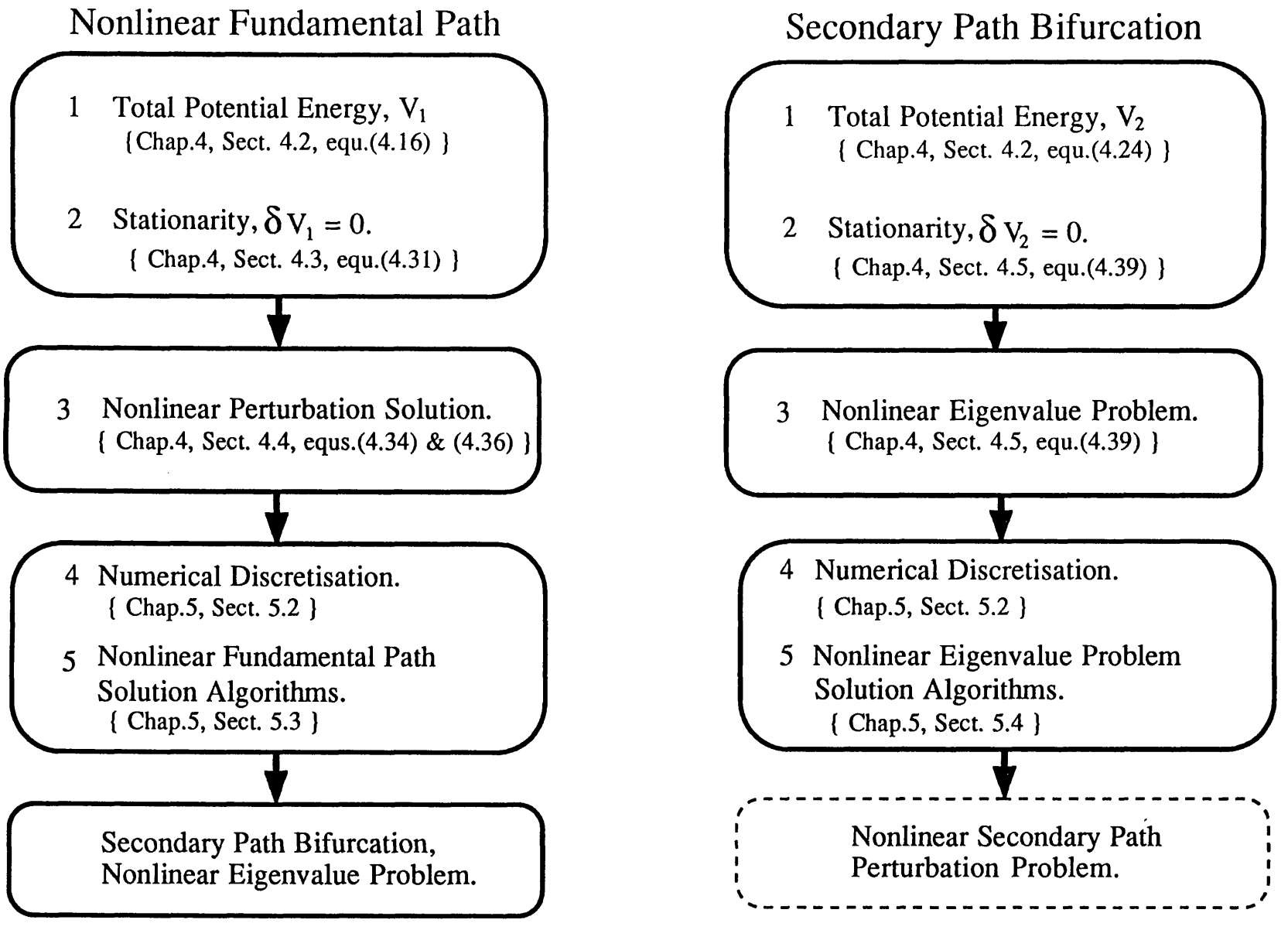


Figure 4.2 Key to Solution Methods for the Nonlinear Fundamental Path and Secondary Path Bifurcation

CHAPTER FIVE

NUMERICAL SOLUTION ALGORITHMS BASED ON FINITE DIFFERENCE SOLUTIONS OF THE KIRCHHOFF EQUATIONS

CHAPTER FIVE

CONTENTS

NUMERICAL SOLUTION ALGORITHMS BASED ON FINITE DIFFERENCE SOLUTIONS OF THE KIRCHHOFF EQUATIONS

5.1 INTRODUCTION

The partial differential equations describing the fundamental path and the linearised secondary path equilibrium equations have been presented in Chapter Four. Exact circumferential modelling (the θ coordinate direction) was introduced in Chapter Four, leaving the generalised displacements x=(u, v, w) as functions of only the meridional coordinate, ϕ. In this chapter numerical solution methods, based on finite difference approximations, are presented.

The accuracy with which finite difference expressions approximate the exact functions over the range of the function is (non-linearly) dependent upon the order of the finite difference expressions and the mesh or node spacing adopted. Storage requirements for banded matrices and vectors are linearly dependent on the nodal spacing, the order of the matrix or vector, while the order of the finite difference expressions have a weaker effect on the bandwidth of the matrices, and hence on storage requirements. For banded matrices the time required for computation varies for different parts of the solution algorithms, it is typically linearly dependent upon the order of the matrix, and either quadratically (Gaussian elimination) or linearly (back-substitution) dependent on the bandwidth. Due to these dependencies simple second order central finite difference expressions will not be the optimum expressions for use with the solution algorithms.

The computer program was written in such a way that sets of finite difference expressions, and the portion of the shell surface over which they are to be applied, form part of the input data to the program. The expressions given in Section 5.2 are those that were adopted after conducting trials using various expressions. The effect of the order of finite difference expressions and nodal spacing on convergence is considered in more detail in Chapter Six.

The fundamental and secondary path equations that are the subject of the solution algorithms are both equilibrium equations, derived from the total potential energy functional. At various points during the solution method a check on the accuracy of the present solution is required, and these checks are performed within the context of the equilibrium equations.

The fundamental path solution algorithm is presented in Section 5.3 and the secondary path solution algorithm in Section 5.4. Once the fundamental and secondary path solutions have been determined, it is possible to calculate the various contributions to the strain and load terms of the total potential energy (Section 4.7). For a conservative system the sum of these various incremental energy contributions must be zero, therefore the error in the sum of these energy contributions may be used to provide a more independent check on the solution method and the internal accuracy required for the solution of the equilibrium equations.

5.2 THE FINITE DIFFERENCE DISCRETIZATION

Finite difference methods are based on approximating the dependent variable, $x(\phi)$, by a localised polynomial function $\bar{x}$, over a subregion of ϕ. If $x(\phi)$ is defined over the domain, $0 \leq \phi \leq \ell$, and the domain is divided into n equal intervals, $h=\ell/n$, then over the sub region r_j in the vicinity of ϕ_j the function x is approximated by $\bar{x}_j$

$$x \cong \bar{x}_j = \sum_{k=0}^{t} a_k z^k \tag{5.1}$$

where z is the local coordinate, $\phi = \phi_j + z$, as shown in Figure 5.1.

The derivatives in the sub region r_j are therefore approximated by

$$\frac{dx}{d\phi} \cong \frac{d\bar{x}_j}{dz} = \sum_{k=1}^{t} k a_k z^{k-1} \tag{5.2}$$

$$\frac{d^2x}{d\phi^2} \cong \frac{d^2\bar{x}_j}{dz^2} = \sum_{k=2}^{t} k(k-1)\, a_k\, z^{k-2}$$

$$\frac{d^3x}{d\phi^3} \cong \frac{d^3\bar{x}_j}{dz^3} = \sum_{k=3}^{t} k(k-1)(k-2)\, a_k\, z^{k-3}$$

$$\frac{d^4x}{d\phi^4} \cong \frac{d^4\bar{x}_j}{dz^4} = \sum_{k=4}^{t} k(k-1)(k-2)(k-3)\, a_k z^{k-4}$$

The coefficients a_k of the polynomial are defined by matching the values of $\bar{x}_j$ with those of x at the t+1 nodes (..., ϕ_{j-3}, ϕ_{j-2}, ϕ_{j-1}, ϕ_j, ϕ_{j+1}, ϕ_{j+2},..), which may be chosen symmetrically or not about ϕ_j. Choosing to match $\bar{x}_j$ with x symmetrically about ϕ_j yields central finite differences, while varying degrees of forward or backward finite difference expressions result from appropriate non-symmetrical matching.

$$\bar{x}_j\big|_{z=ih} = x\big|_{\phi_{j+i}} = x_{j+i} \quad ; \quad i = \ldots -3,\ -2,\ -1,\ 0,\ 1,\ 2,\ \ldots \tag{5.3}$$

Equation (5.1) subject to the conditions (5.3) results in a set of t+1 simultaneous equations which may be written as,

$$[H]\,\mathbf{a} = \mathbf{x} \tag{5.4}$$

and the solution is given by,

$$\mathbf{a} = [H]^{-1}\mathbf{x} \tag{5.5}$$

where **a** is the coefficient vector $(a_0, a_1, a_2, \ldots a_t)^T$ and **x** is the nodal value vector $(\ldots, \bar{x}_{j-3}, \bar{x}_{j-2}, \bar{x}_{j-1}, \bar{x}_j, \bar{x}_{j+1}, \bar{x}_{j+2}, \ldots)^T$, and the elements of [H] are dependent on the node spacing h.

The derivatives of the local polynomial function $\bar{x}$, evaluated at ϕ_j, (z=0), are given by equation (5.2) as,

$$\left.\frac{d^s \bar{x}_j}{d\phi^s}\right|_{z=0} = s! a_s \qquad (5.6)$$

and these are the finite difference approximations used in the discretization of the continuous differential equations presented in Chapter Four.

The accuracy of these expressions depends on the order of the polynomial, t, the order of the derivative, s, and the nodal spacing, h. By considering the largest of the neglected terms due to truncating the expression (5.2) at k=t, the error over the region r_j; ($|z| \leq h/2$) may be evaluated and is of the order h^{t-s}.

The optimum choice of finite difference expressions for the discretization and subsequent computer solution for a particular problem is dependent on three main interrelated factors. The storage required by the program, the accuracy required, and the time taken to solve the problem; these all depend on the finite difference expressions selected. These factors influenced the choice of the finite difference expressions that were used, and will be discussed in Chapter Six with reference to the solution algorithms used by the computer program.

A computer program capable of generating finite difference expressions of any type and order of accuracy by the method outlined above was used. The finite difference expressions that were used are listed in Table 5.1, where they are written in the form,

$$\left.\frac{d^s x}{d\phi^s}\right|_{\phi_j} \cong \frac{d^s \bar{x}_j}{d\phi^s} = f_s \sum_{r=-5}^{+5} \alpha_{s,r}\, x_{j-r} \qquad (5.7)$$

The coefficients, $\alpha_{s,r}$, and f_s, along with the order of accuracy of the expressions are all presented in Table 5.1.

The finite difference expressions are presented in seven groups for use with the shell equations, and two groups (one forward, one backward) for use with the boundary conditions. Figure 5.2 shows schematically at which of the equally spaced nodes on the spherical shell the various groups of finite difference expressions are used.

The fictitious nodes (0 and n+1) in Figure 5.2 are required for the application of the shell equations near the boundaries, and are defined by the boundary conditions at $\phi = 0$ and $\phi = \phi_b$.

Forward and backward finite difference expressions are used near the boundaries in order to avoid a need for more fictitious off shell nodes.

5.3 THE FUNDAMENTAL PATH SOLUTION ALGORITHMS

The solution algorithm for the nonlinear fundamental path is presented below in a general matrix/vector form.

The application of the finite difference expressions of Table 5.1 to equations (4.32), (4.34) and (4.36) allows them to be written in matrix form as follows.

From equation (4.32) we have,

$$\bar{x}^s = \bar{x}^i + \bar{x}^f = \bar{x}^i + \bar{x}_0^f + \bar{x}_1 e + \bar{x}_2 e^2 + \ldots = \bar{x}_0^s + \bar{x}_1 e + \bar{x}_2 e^2 + \ldots \tag{5.8}$$
$$\hat{p} = \hat{p}_0 + \hat{p}_1 e + \hat{p}_2 e^2 + \ldots$$

where $(\bar{x})^T = (u_1, w_1, u_2, w_2, \ldots u_n, w_n)$

Equation (4.34) yields a set of simultaneous equations which may be written in the form

$$[\dot{A}]\,(\bar{x}_r) = -\hat{p}_r(\bar{b}) - (\bar{d}_{r-1}) \tag{5.9}$$

Where the matrix $[\dot{A}]$ is given by

$$\left[\dot{A}\right] = \left[A_1 + A_2\{2\bar{x}_0^s\} + A_3\{3(\bar{x}_0^s)^2\} + A_4\{4(\bar{x}_0^s)^3\} + \hat{p}_0\{B_1 + B_2 2\bar{x}_0^s\}\right] \tag{5.10}$$

and the vectors $\bar{b}$ and $\bar{d}_{r-1}$ are defined by

$$\bar{b} = B_0 + B_1\{\bar{x}_0^s\} + B_2\{(\bar{x}_0^s)^2\} \tag{5.11}$$

$$\bar{d}_0 \equiv 0$$

$$\bar{d}_1 = -A_2\{\bar{x}_1^2\} - A_3\{3(\bar{x}_0^s)\bar{x}_1^2\} - A_4\{6(\bar{x}_0^s)^2\bar{x}_1^2\} - \hat{p}_1 B_1\{\bar{x}_1\} - \hat{p}_1 B_2\{2(\bar{x}_0^s)\bar{x}_1\}$$

$$\begin{aligned}\bar{d}_2 = {} & -A_2\{2\bar{x}_1\bar{x}_2\} - A_3\{3(\bar{x}_1^3) + 6(\bar{x}_0^s)\bar{x}_1\bar{x}_2\} - A_4\{4(\bar{x}_0^s)\bar{x}_1^3\} \\ & - A_4\{12(\bar{x}_0^s)\bar{x}_1\bar{x}_2\} - \hat{p}_1 B_1\{\bar{x}_2\} - \hat{p}_2 B_1\{\bar{x}_1\} - \hat{p}_1 B_2\{\bar{x}_1^2\} \\ & - \hat{p}_1 B_2\{2(\bar{x}_0^s)\bar{x}_2\} - \hat{p}_2 B_2\{2(\bar{x}_0^s)\bar{x}_1\} - \hat{p}_0 B_2\{2\bar{x}_1\bar{x}_2\}\end{aligned}$$

etc.

The matrix [A] is a 2n x 2n banded matrix, and the vectors (x_r), (b) and (d_{r-1}) are all of length 2n. In arriving at equations (5.9), (5.10) and (5.11) the equilibrium equations, E_i i=1 and 3 in equation (4.34), are taken in pairs at each of the on shell nodes of the finite difference grid, nodes 1 to n in Figure 5.2. And the boundary conditions, equations (2.76) and (2.77), are used to eliminate the non zero elements in the first and last four columns of the matrix, corresponding to the boundary and off shell fictitious nodes.

Finally equation (4.36) on application of the finite difference expression yields vector equations of the following form

$$(\hat{p}_1)^2 + (\bar{x}_1)^T\bar{x}_1 = a = \text{constant} \qquad (5.12)$$

$$4\hat{p}_1\hat{p}_2 + 4(\bar{x}_1)^T\bar{x}_2 = 0$$

$$6\hat{p}_1\hat{p}_3 + 4(\hat{p}_2)^2 + 6(\bar{x}_1)^T\bar{x}_3 + 4(\bar{x}_2)^T\bar{x}_2 = 0$$

$$8\hat{p}_1\hat{p}_4 + 12\hat{p}_2\hat{p}_3 + 8(\bar{x}_1)^T\bar{x}_4 + 12(\bar{x}_2)^T\bar{x}_3 = 0$$

etc.

To solve the sequence of equations (5.9) for r=1, 2, 3,... we may choose $\hat{p}_1$ arbitrarily and so solve for $\bar{x}_1$; in general we will need to make use of the condition (5.12) to solve for $\hat{p}_r$ and $\bar{x}_r$, when $r \neq 1$.

Equation (5.9) may be manipulated to yield

$$\left[\bar{A}\right](\bar{x}_r) = \frac{\hat{p}_r}{\hat{p}_1}\left[\bar{A}\right](\bar{x}_1) + \left[\bar{A}\right](\bar{x}_r^*) \qquad (5.13)$$

in which

$$\left[\bar{A}\right](\bar{x}_1) = -\hat{p}_1(\bar{b})$$

$$\left[\bar{A}\right](\bar{x}_r^*) = -(\bar{d}_{r-1})$$

and

$$\bar{x}_r = \bar{x}_r^* + \frac{\hat{p}_r}{\hat{p}_1}\bar{x}_1$$

Now $(\bar{x}_r^*)$ is dependent on the previously known terms in $(\bar{d}_{r-1})$, and substituting for $\bar{x}_r$ from (5.13) into (5.12) we arrive at,

$$(\hat{p}_1)^2 + (\bar{x}_1)^T\bar{x}_1 = a = \text{constant} \qquad (5.14)$$

$$\hat{p}_2 = \frac{-4(\bar{x}_1)^T\bar{x}_2^*}{4(\hat{p}_1 + \frac{1}{\hat{p}_1}(\bar{x}_1)^T\bar{x}_1}$$

$$\hat{p}_3 = \frac{-\left[4(\hat{p}_2)^2 + 4(\bar{x}_2)^T \bar{x}_2 + 6(\bar{x}_1)^T \bar{x}_3^*\right]}{6\left[\hat{p}_1 + \frac{1}{\hat{p}_1}(\bar{x}_1)^T \bar{x}_1\right]} \qquad \text{(5.14 contd.)}$$

$$\hat{p}_4 = \frac{-\left[12\hat{p}_2\hat{p}_3 + 12(\bar{x}_2)^T \bar{x}_3 + 8(\bar{x}_1)^T \bar{x}_4^*\right]}{8\left[\hat{p}_1 + \frac{1}{\hat{p}_1}(\bar{x}_1)^T \bar{x}_1\right]}$$

etc.

And $\hat{p}_r$ is now defined by the previously known terms $\hat{p}_{r-1}$, $\hat{p}_{r-2}$,... and $\bar{x}_{r-1}$, $\bar{x}_{r-2}$,...., allowing both $\hat{p}_r$ and $\bar{x}_r$ to be evaluated. The solution algorithm is given schematically in Figure 5.3, where it is the portion of the total algorithm within the dotted line.

In setting up the $[\bar{A}]$ matrix and the $(\bar{b})$ vector, see Figure 5.3, equations (5.10) and (5.11) require $\hat{p}_0$, $\bar{x}_0$ and $\bar{x}^i$,($\bar{x}_0 = \bar{x}^i + \bar{x}_0$) to be specified. In general none of these are necessarily zero, but for the first entry (t=1) into the algorithm $\hat{p}_0$ and $\bar{x}_0$ will be zero, while $\bar{x}^i$ will take the value specified by the initial imperfection profile.

In using the perturbation method outlined here it was found that the fundamental path could not adequately be represented by a single polynomial series, equation (5.8). That is, the error in $\bar{x}^f$ and $\hat{p}$ increased, with 'e', reaching unacceptable levels within the range of 'e' that is of interest. It should be noted that this error is not entirely dependent on the number of terms in the series or on the choice of perturbation parameter, i.e. equation (5.14). That is equations (5.13) and $(\bar{d}_{r-1})$ defined by equation (5.11) are the result of specifying a perturbation series of the type (polynomial) given by equation (5.8). Choosing to identify the perturbation parameter with some physically or theoretically measurable path parameter, yields a condition, of the type given by equation (5.14), enabling $\hat{p}_r$ to be specified in (5.13). Hence the magnitude of the vector $\bar{x}_1$ that is added to $\bar{x}_r^*$ to give $\bar{x}_r$, equation (5.13), is determined. The convergence of the series, given by equation (5.8), at large values of 'e' will be limited by the inclusion of the $\bar{x}_1$ components in $\bar{x}_r$. And the $\bar{x}_r$ vector may become divergent, increasing in magnitude and oscillating in sign as r increases.

The repeated use of the initial rate of change of the displacement profile, $\bar{x}_1$, to force $\bar{x}_r$ to satisfy the perturbation condition may be overcome by a repeated use of the perturbation method, so that it is the 'local initial' displacement vector $\bar{x}_{t,1}$ that is used to satisfy the perturbation condition.

We may write the general perturbation series as

$$\bar{x}_t^f = \bar{x}_{t,0} + \bar{x}_{t,1}\, e_\ell + \bar{x}_{t,2}\, e_\ell^2 + \ldots \tag{5.15}$$

$$\hat{p}_t = \hat{p}_{t,0} + \hat{p}_{t,1} e_\ell + \hat{p}_{t,2} e_\ell^2 + \ldots$$

where $\quad e_\ell = e_g - a_{t-1} \quad$ for $\quad a_{t-1} \le e_g \le a_t$

and e_ℓ is the local perturbation parameter, e_g the global perturbation parameter, and a_{t-1} and a_t are the lower and upper limits for e_g of the particular local perturbation series under consideration. The terms $\bar{x}_{t,0}$ and $\hat{p}_{t,0}$ are defined by

$$\bar{x}_{t,0} = \bar{x}_{t-1}^f \Big|_{e_g = a_{t-1}} \tag{5.16}$$

$$\hat{p}_{t,0} = \hat{p}_{t-1} \Big|_{e_g = a_{t-1}}$$

The solution algorithms described previously may be used to solve for the coefficients of the series given by equation (5.15). While the choice of $\hat{p}_{t,1}$ remains arbitrary, it will be helpful at a later stage to choose $\hat{p}_{t,1}$ such that the slope of the 'load-vs-e' graph is continuous, that is.

$$\hat{p}_{t,1} = \frac{d\hat{p}_{t-1}}{de_\ell} \Big|_{e_g = a_{t-1}} \tag{5.17}$$

After determining the coefficients of equation (5.15) an upper limit on e_g, say a_t, must be established, and this may be done by evaluating the shell equations (4.31) and checking the relative error between terms. The finite difference expressions, when applied to (4.31) yield the following vector equations

$$\bar{v}_1 = \bar{v}_2 - \bar{v}_3 \tag{5.18}$$

where

$$\bar{v}_1 = [A_{i,1}] \left(\bar{x}^f\right) \qquad \text{for } i = 1, 3$$

$$\bar{v}_2 = -\left\{ \sum_{j=2}^{4} [A_{i,j}] \left(\bar{x}^s\right)^j + \hat{p} \sum_{j=0}^{2} [B_{i,j}] \left(\bar{x}^s\right)^j \right\}$$

$$\bar{v}_3 = A_{i,0}$$

The maximum relative error in equation (5.18) can be represented by

$$\text{ERROR} = \max \left(\frac{\bar{v}_1 - \bar{v}_2 + \bar{v}_3}{|\bar{v}_1| + |\bar{v}_2| + |\bar{v}_3|} \right) \tag{5.19}$$

where equation (5.19) is evaluated by comparing the individual elements of the vectors $\bar{v}_1$, $\bar{v}_2$, and $\bar{v}_3$. The value of a_t is then set so that the error given by equation (5.19) is just less than or equal to some acceptable level when $e_g=a_t$.

Before reentering the solution routine for t+1 and r=1, it is necessary to improve the accuracy of $\bar{x}_{t+1,0}$ and $\hat{p}_{t+1,0}$ given by equation (5.16) to avoid a_{t+1} eventually being returned with the same value as a_t. This improvement of $(\bar{x}_{t+1,0}, \hat{p}_{t+1,0})$ may be accomplished by assuming that the total error is in the displacement vector. Then a Newton-Raphson iteration technique may be used to improve the vector, $\bar{x}$, until the error in the shell equation, given by equation (5.19), is reduced to an acceptable level.

The iteration scheme is given by

$$\bar{x}_{N+1} = \bar{x}_N - \bar{x}_{C,N} \tag{5.20}$$

where

$$f'(\bar{x}_N)\,\bar{x}_{C,N} = f(\bar{x}_N)$$

and $\bar{x}_{C,N}$ is the correction vector to $\bar{x}_N$. Using the shell equation as given in equation (5.18), the iteration scheme, equation (5.20), may be written as

$$\left[\dot{A}\right](\bar{x}_{C,N}) = -\bar{v}_1 + \bar{v}_2 - \bar{v}_3$$

where

$$\left[\dot{A}\right](\bar{x}_{C,N}) = \sum_{j=1}^{4} [A_{i,j}]\left(j(\bar{x}_N)^{j-1}\right)(\bar{x}_{C,N}) + \hat{p}_{t+1,0} \sum_{j=1}^{2} [B_{i,j}]\left(j(\bar{x}_N)^{j-1}\right)(\bar{x}_{C,N}) \tag{5.21}$$

$$\bar{v}_1 = [A_{i,1}]\,(\bar{x}_N) \qquad \text{for } i = 1,3$$

$$\bar{v}_2 = -\left\{\sum_{j=2}^{4} [A_{i,j}](\bar{x}_N)^j + \hat{p}_{t+1,0} \sum_{j=0}^{2} [B_{i,j}](\bar{x}_N)^j\right\}$$

$$\bar{v}_3 = A_{i,0}$$

In setting up the matrix $[\bar{A}]$, equation (5.21), the same subroutines may be used that set up $[\bar{A}]$ for equation (5.10), similarly the vectors $\bar{v}_1$ and $\bar{v}_2$ may be evaluated by the subroutines used for equation (5.18).

The complete solution algorithm for the fundamental path is outlined in Figure 5.3. Using this algorithm the $[\bar{A}]$ matrix is only set up, and decomposed using a Gaussian elimination method with pivoting in the outer, t, loop which results in a routine that is fairly fast.

The resulting fundamental path may be represented by the dotted line in Figure 5.4. The error, or departure of the approximate path from the exact path, is controlled by specifying the allowable value of the error in (5.19) when determining a_t.

5.4 THE SECONDARY PATH SOLUTION ALGORITHM

The nonlinear eigenvalue problem that results from the introduction of finite difference expressions into equation (4.39) is solved by the use of iterative methods. First an inverse iteration method will be used to achieve an initial approximation to the eigenvalue and vector, then several shifts of origin followed by inverse iteration are used to achieve the desired accuracy on both the eigenvalue and vector. An orthogonalization of the eigenvector is also introduced into the iterative procedures, allowing the subdominant eigenvalues and vectors to be obtained.

Due to the somewhat cumbersome nature of the shorthand notation used for the equations of the nonlinear eigenvalue problem the method will first be described with reference to the linear eigenvalue problem.

5.4.1 The Linear Eigenvalue Problem

The inverse iteration schemes discussed here will lead to convergence on the eigenvalues of lowest absolute value, as we are interested in the lowest buckling pressures. It is usual, in the literature, Reference [16], to pose the eigenvalue problem so that convergence is on the largest eigenvalue first, and from this usage the inverse iterative process takes its name.

The linear eigenvalue problem

$$[A](\bar{x}) + \lambda[B](\bar{x}) = 0 \tag{5.22}$$

may be solved by using inverse iteration without a shift of origin, the solution algorithm is given by

$$[A]\,\bar{v}_{r+1} = -[B]\,\bar{u}_r$$

$$\bar{u}_{r+1} = \frac{\bar{v}_{r+1}}{\max(\bar{v}_{r+1})} \tag{5.23}$$

and

$$\lambda_{r+1} = \frac{1}{\max(\bar{v}_{r+1})}$$

where the notation max($\bar{v}$) is used to denote the element of maximum modulus of the vector $\bar{v}$. In the limit as $r \to \infty$, $\bar{u}_r \to \bar{x}_1$ and $\lambda_r \to \lambda_1$.

The rate of convergence may be established by considering the underlying reason for the method working. The property of eigenvectors, $\bar{x}_1, \bar{x}_2, \ldots \bar{x}_N$ of spanning the N-dimensional vector space, allows any general N-dimensional vector, $\bar{z}$, to be written as

$$\bar{z} = \sum_{i=1}^{N} \alpha_i \bar{x}_i \tag{5.24}$$

where the α_i's are constants depending on the vector $\bar{z}$. Without loss of generality we may assume that the eigenvalue corresponding to the eigenvector $\bar{x}_i$ is λ_i, for i=1,2,3,...N and that

$$|\lambda_1| \leq |\lambda_2| \leq |\lambda_3| \leq \ldots \leq |\lambda_N| \tag{5.25}$$

Consider the effect on the general vector $\bar{z}$ of a single step of the iterative process, we have

$$[A]\left(\bar{z}_{i+1}\right) = -[B]\left(\bar{z}_i\right) \tag{5.26}$$

or

$$[A]\left(\frac{\alpha_1 \bar{x}_1}{\lambda_1} + \frac{\alpha_2 \bar{x}_2}{\lambda_2} + \ldots \frac{\alpha_N \bar{x}_N}{\lambda_N}\right) = -[B]\left(\alpha_1 \bar{x}_1 + \alpha_2 \bar{x}_2 + \ldots \alpha_N \bar{x}_N\right)$$

It may be seen that the component of $\bar{z}_{i+1}$ in the direction of $\bar{x}_1$ has increased by a factor of at least λ_2/λ_1 with respect to the other components. The rate of convergence is dependent on the ratio λ_2/λ_1 and the method will converge on the smallest (dominant) eigenvalue and vector $(\lambda_1, \bar{x}_1)$.

If the eigenvalue problem contains coincident roots, say $\lambda_1 = \lambda_2 = \ldots = \lambda_r$ and $|\lambda_1| < |\lambda_{r+1}| \leq |\lambda_{r+2}| \ldots \leq |\lambda_N|$, then the process will still converge on the dominant, coincident, eigenvalue. However the vector will not be uniquely defined, but will be dependent on the initial vector $\bar{u}_o$, and will lie in the subspace spanned by $\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_r$.

If the ratio λ_2/λ_1 is close to unity then convergence is slow, and it may be desirable to use a shift of origin with the inverse iteration technique to improve the rate of convergence. If we have an approximation to λ_1, say λ_e, then we may write the eigenvalues as

$$\lambda_i = \lambda_e + \delta_i \qquad \text{and} \qquad |\delta_1| << |\delta_2| \leq |\delta_3| \ldots \leq |\delta_N| \tag{5.27}$$

Equation (5.22) may be written in the following form

$$[A+\lambda_e B](\bar{x}) = -\delta[B](\bar{x}) \tag{5.28}$$

or

$$[\bar{A}](\bar{x}) = -\delta[B](\bar{x})$$

where

$$[\bar{A}] = [A+\lambda_e B]$$

The solution algorithm is now given by

$$[\bar{A}]\,\bar{v}_{r+1} = -[B]\bar{u}_r \tag{5.29}$$

$$\bar{u}_{r+1} = \frac{\bar{v}_{r+1}}{\max(\bar{v}_{r+1})}$$

and

$$\delta_{r+1} = \frac{1}{\max(\bar{v}_{r+1})}$$

The rate of convergence is now dependent on the ratio δ_2/δ_1, and may be improved by further shifts of the origin to improved estimates of λ_e if this is required. An orthogonalization scheme may be introduced into the iterative cycle allowing subdominant eigenvalues and vectors to be determined. If we have found the first k eigenvalues and vectors then the (k+1)'th eigenvalue and vector may be found using the following algorithm

$$[A]\,\bar{v}_{r+1} = -[B]\,\bar{u}_r \tag{5.30}$$

$$\bar{w}_{r+1} = \bar{v}_{r+1} - \sum_{i=1}^{k} \frac{(\bar{v}_{r+1})^T \bar{x}_i}{\|\bar{x}_i\|}\,\bar{x}_i$$

$$\bar{u}_{r+1} = \frac{\bar{w}_{r+1}}{\max(\bar{w}_{r+1})}$$

$$\lambda_{r+1} = \frac{1}{\max(\bar{w}_{r+1})}$$

where

$$\|\bar{x}_i\| = (\bar{x}_i)^T \bar{x}_i$$

The vector $\bar{w}_{r+1}$ is orthogonal to $\bar{x}_1$, $\bar{x}_2$,...$\bar{x}_k$ hence $\alpha_1,\alpha_2,..\alpha_k$ in the expression for $\bar{z}$ in equation (5.24) are zero and hence convergence on λ_1, λ_2,...λ_k will be suppressed. It is not

necessary to orthogonalize at each step of the iteration. When a shift of the origin is used with a reasonable estimate to λ_{k+1} then the orthogonalization may be dropped completely from the inverse iteration as λ_{k+1} is dominant in any case.

The evaluation of subdominant eigenvalues enables the possibility of coincident eigenvalues to be investigated, and the verification that the lowest eigenvalue (buckling pressure) has been found. It is possible for the numerical procedures based on these algorithms to converge on subdominant (λ_i, $i \neq 1$) eigenvalues first, if the initial vector $\bar{u}_0$ has a large component in the direction of a subdominant eigenvector.

4.4.2 The Nonlinear Eigenvalue Problem

The application of finite difference expressions to the secondary path equation (4.39) results in a nonlinear eigenvalue problem which may be written in the following form

$$\sum_{j=0}^{3}\left[A_j\left(\bar{x}^s\right)^j\right](\bar{x}) + \hat{p}\sum_{j=0}^{1}\left[B_j\left(\bar{x}^s\right)^j\right](\bar{x}) = 0 \qquad (5.31)$$

where

$$\bar{x}^s = \bar{x}^i + \bar{x}^f$$

and

$$\bar{x}^f_t = \bar{x}_{t,0} + \bar{x}_{t,1}\, e_\ell + \bar{x}_{t,2}\, e_\ell^2 + \ldots \qquad (5.32)$$

$$\hat{p} = \hat{p}_{t,0} + \hat{p}_{t,1} e_\ell + \hat{p}_{t,2}\, e_\ell^2 + \ldots$$

for

$$a_{t-1} \leq e_g \leq a_t \qquad \text{and} \qquad e_\ell = e_g - a_{t-1}$$

The generalised displacement vectors are given by

$$\left(\bar{x}^s\right)^T = (u_1^s, w_1^s, u_2^s, w_2^s, \ldots, u_n^s, w_n^s) \qquad (5.33)$$

and

$$(\bar{x})^T = (u_1, v_1, w_1, u_2, v_2, w_2, \ldots, u_n, v_n, w_n)$$

Equation (5.31) is a nonlinear eigenvalue problem in e_g and x; by substituting for $\bar{x}^s$ and $\hat{p}$ from (5.32) it is possible to recast (5.31) in the standard form

$$[D_0](\bar{x}) + e_g[D_1](\bar{x}) + e_g^2[D_2](\bar{x}) + \ldots e_g^m\,[D_m](\bar{x}) = 0 \qquad (5.34)$$

Fortunately it is not necessary for the iterative solution method that the nonlinear eigenvalue problem be posed in the standard form given by equation (5.34). The iterative method based on the naturally occurring form of the eigenvalue problem given by equations (5.31) and (5.32) has the advantage that the nonlinearity is all contained in equation (5.32).

The iterative procedure for the solution of the nonlinear eigenvalue problem, without a shift of the origin, is given by analogy with equation (5.23) as follows

$$e_r = \lambda_r - a_{t-1} \qquad \text{where} \qquad a_{t-1} \le \lambda_r \le a_t$$

$$\bar{x}^s_r = \bar{x}^i + \bar{x}_{t,0} + \bar{x}_{t,1}\, e_r + \bar{x}_{t,2}\, e_r^2 + \ldots$$

$$\hat{p}_r = \hat{p}_{t,0} + \hat{p}_{t,1}\, e_r + \hat{p}_{t,2}\, e_r^2 + \ldots$$

$$[A_o]\,\bar{v}_{r+1} = -\frac{1}{\lambda_r}\left\{\sum_{j=1}^{3}\left[A_j\left(\bar{x}^s_r\right)^j\right]\left(\bar{u}_r\right) + \hat{p}_r\sum_{j=0}^{1}\left[B_j\left(\bar{x}^s_r\right)^j\right]\left(\bar{u}_r\right)\right\} \tag{5.35}$$

$$\bar{w}_{r+1} = \bar{v}_{r+1} - \sum_{i=1}^{K}\frac{\left(\bar{v}_{r+1}\right)^T \bar{x}_i}{\left\|\bar{x}_i\right\|}\,\bar{x}_i$$

$$\bar{u}_{r+1} = \frac{\bar{w}_{r+1}}{\max\left(\bar{w}_{r+1}\right)}$$

$$\lambda_{r+1} = \frac{1}{\max\left(\bar{w}_{r+1}\right)}$$

In equation (5.35) the eigenvalue λ is the global perturbation parameter, $\lambda \equiv e_g$ and e_r is the r'th iterative estimate to the local perturbation parameter e_ℓ.

Similarly by analogy with equations (5.28) and (5.30) the iterative procedure for use with a shift of origin is given by equations (5.36) and (5.37). The shift of origin to λ_e implies an estimate to the fundamental path displacements and load, given by $\bar{x}^s_{\lambda e}$ and $\hat{p}_{\lambda e}$ and hence to the $[\,\bar{A}_0\,]$ matrix.

$$e_e = \lambda_e - a_{t-1} \qquad \text{for} \qquad a_{t-1} \le \lambda_e \le a_t \tag{5.36}$$

with

$$\bar{x}^s_{\lambda_e} = \bar{x}^i + \bar{x}_{t,0} + \bar{x}_{t,1}\, e_e + \bar{x}_{t,2}\, e_e^2 + \ldots$$

$$\hat{p}_{\lambda_e} = \hat{p}_{t,0} + \hat{p}_{t,1}\, e_e + \hat{p}_{t,2}\, e_e^2 + \ldots$$

and

$$\left[\bar{A}_0\right] = \sum_{j=0}^{3}\left[A_j\left(\bar{x}^s_{\lambda_e}\right)^j\right] + \hat{p}_{\lambda_e}\sum_{j=0}^{1}\left[B_j\left(\bar{x}^s_{\lambda_e}\right)^j\right]$$

The iterative scheme with a shift of origin is now given by:

$$\lambda_r = \lambda_e + \delta_r \qquad \text{for} \qquad a_{t-1} \le \lambda_r \le a_t \tag{5.37}$$

$$e_r = \lambda_r - a_{t-1}$$

$$\bar{x}_r^s = \bar{x}^i + \bar{x}_{t,0} + \bar{x}_{t,1}\, e_r + \bar{x}_{t,2}\, e_r^2 + \ldots$$

$$\hat{p}_r = \hat{p}_{t,0} + \hat{p}_{t,1}\, e_r + \hat{p}_{t,2}\, e_r^2 + \ldots$$

and

$$\left[\bar{A}_0\right]\bar{v}_{r+1} = \frac{-1}{\delta_r}\left\{\sum_{j=1}^{3}\left[A_j\left(\bar{x}_r^s\right)^j\right]\bar{u}_r + \hat{p}_r\sum_{j=0}^{1}\left[B_j\left(\bar{x}_r^s\right)^j\right]\bar{u}_r - \sum_{j=1}^{3}\left[A_j\left(\bar{x}_{\lambda_e}^s\right)^j\right]\bar{u}_r - \hat{p}_{\lambda_e}\sum_{j=0}^{1}\left[B_j\left(\bar{x}_{\lambda_e}^s\right)^j\right]\bar{u}_r\right\}$$

$$\bar{w}_{r+1} = \bar{v}_{r+1} - \sum_{i=1}^{K}\frac{\left(\bar{v}_{r+1}\right)^T\bar{x}_i}{\left\|\bar{x}_i\right\|}\bar{x}_i$$

$$\bar{u}_{r+1} = \frac{\bar{w}_{r+1}}{\max\left(\bar{w}_{r+1}\right)}$$

$$\delta_{r+1} = \frac{1}{\max\left(\bar{w}_{r+1}\right)}$$

The convergence of these iterative procedures, equations (5.35) and (5.37), apart from numerical stability, is again dependent on the n eigenvectors of equation (5.31) spanning the n-dimensional vector space. As equation (5.31) may be written in the standard form given by equation (5.34) it is sufficient to show that the eigenvectors of equation (5.34) span the n-dimensional vector space. The nonlinear eigenvalue problem, equation (5.34) may be written in the form of a standard linear eigenvalue problem, Reference [16], as follows.

$$\begin{bmatrix} 0 & I & 0 & 0 & \ldots & 0 \\ 0 & 0 & I & 0 & \ldots & 0 \\ 0 & 0 & 0 & I & \ldots. & 0 \\ . & & & & & \\ . & & & & & \\ . & & & & & \\ . & & & & & \\ 0 & 0 & 0 & 0 & \ldots & I \\ D_m^{-1}D_0 & D_m^{-1}D_1 & D_m^{-1}D_2 & D_m^{-1}D_3 & \ldots. & D_m^{-1}D_{m-1} \end{bmatrix} \begin{bmatrix} X \\ Y_1 \\ Y_2 \\ . \\ . \\ . \\ . \\ Y_{m-2} \\ Y_{m-1} \end{bmatrix} = e_g \begin{bmatrix} X \\ Y_1 \\ Y_2 \\ . \\ . \\ . \\ . \\ Y_{m-2} \\ Y_{m-1} \end{bmatrix} \tag{5.38}$$

Where the partitioned matrix is mn by mn and the X, Y_i vectors are of length n, therefore the eigenvectors of equation (5.34) are independent.

The iterative routines given by equations (5.35), (5.36) and (5.37) are combined to form a solution algorithm, as shown in Figure 5.5. The convergence test used for iteration without a shift of origin is on the eigenvalue only, and is given by.

$$\text{ERROR} = \left| \frac{\lambda_{r+1} - \lambda_r}{\lambda_r} \right| \tag{5.39}$$

For iteration with a shift of origin convergence is checked on both the eigenvalue and vector, the error in the eigenvector is given by,

$$\text{ERROR} = \max\left(\frac{\bar{u}_{r+1} - \bar{u}_r}{\bar{u}_r} \right) \tag{5.40}$$

in which the error is the maximum value of the relative error as corresponding elements of the vectors $\bar{u}_{r+1}$ and $\bar{u}_r$ are compared. These convergence checks indicate that the iterative procedures are yielding stable eigenvalues and vectors, not that the eigenvalue and vector satisfy equation (5.31) or the original total potential energy functional equation (4.24).

LOCATION (Figure 5.2)	REGION I EXPRESSIONS				REGION II & VI EXPRESSIONS			
DERIVATIVE	$\frac{dx}{d\phi}$, s = 1	$\frac{d^2x}{d\phi^2}$, s = 2	$\frac{d^3x}{d\phi^3}$, s = 3	$\frac{d^4x}{d\phi^4}$, s = 4	$\frac{dx}{d\phi}$, s = 1	$\frac{d^2x}{d\phi^2}$, s = 2	$\frac{d^3x}{d\phi^3}$, s = 3	$\frac{d^4x}{d\phi^4}$, s = 4
ACCURACY	h^6	h^6	h^4	h^4	h^6	h^6	h^4	h^4
f_s / r	$\frac{1}{60h}$	$\frac{1}{180h^2}$	$\frac{1}{8h^3}$	$\frac{1}{6h^4}$	$\frac{1}{60h}$	$\frac{1}{180h^2}$	$\frac{1}{8h^3}$	$\frac{1}{6h^4}$
-5	0	0	0	0	0	0	0	0
-4	0	0	0	0	0	0	0	0
-3	0	0	0	0	-1	2	1	-1
-2	2	-11	-1	4	9	-27	-8	12
-1	-24	214	-8	-11	-45	270	13	-39
0	-35	-378	35	0	0	-490	0	56
1	80	130	-48	31	45	270	-13	-39
2	-30	85	29	-44	-9	-27	8	12
3	8	-54	-8	27	1	2	-1	-1
4	-1	16	1	-8	0	0	0	0
5	0	-2	0	1	0	0	0	0

LOCATION (Figure 5.2)	REGION III & V EXPRESSIONS				REGION IV EXPRESSIONS			
DERIVATIVE	$\frac{dx}{d\phi}$, s = 1	$\frac{d^2x}{d\phi^2}$, s = 2	$\frac{d^3x}{d\phi^3}$, s = 3	$\frac{d^4x}{d\phi^4}$, s = 4	$\frac{dx}{d\phi}$, s = 1	$\frac{d^2x}{d\phi^2}$, s = 2	$\frac{d^3x}{d\phi^3}$, s = 3	$\frac{d^4x}{d\phi^4}$, s = 4
ACCURACY	h^8	h^8	h^6	h^6	h^{10}	h^{10}	h^8	h^8
f_s / r	$\frac{1}{840h}$	$\frac{1}{5040h^2}$	$\frac{1}{240h^3}$	$\frac{1}{240h^4}$	$\frac{1}{2520h}$	$\frac{1}{25200h^2}$	$\frac{1}{30240h^3}$	$\frac{1}{15120h^4}$
-5	0	0	0	0	-2	8	205	-82
-4	3	-9	-7	7	25	-125	-2522	1261
-3	-32	128	72	-96	-150	1000	14607	-9738
-2	168	-1008	-338	676	600	-6000	-52428	52428
-1	-672	8064	448	-1952	-2100	42000	70098	-140196
0	0	-14350	0	2730	0	-73766	0	192654
1	672	8064	-448	-1952	2100	42000	-70098	-140196
2	-168	-1008	338	676	-600	-6000	52428	52428
3	32	128	-72	-96	150	1000	-14607	-9738
4	-3	-9	7	7	-25	-125	2522	1261
5	0	0	0	0	2	8	-205	-82

TABLE 5.1 Coeffecients $\alpha_{s,r}$ and f_s of the Finite Difference Expressions;

$$\left.\frac{d^s x}{d\phi^s}\right|_{\phi_j} \cong \frac{d^s \tilde{x}_j}{d\phi^s} = f_s \sum_{r=-5}^{+5} \alpha_{s,r}\, x_{j-r} \text{ , equation (5.7).}$$

LOCATION (Figure 5.2)	REGION VII EXPRESSIONS			
DERIVATIVE	$\frac{dx}{d\phi}$, s = 1	$\frac{d^2x}{d\phi^2}$, s = 2	$\frac{d^3x}{d\phi^3}$, s = 3	$\frac{d^4x}{d\phi^4}$, s = 4
ACCURACY	h^6	h^6	h^4	h^4
f_s / r	$\frac{1}{60h}$	$\frac{1}{180h^2}$	$\frac{1}{8h^3}$	$\frac{1}{6h^4}$
-5	0	-2	0	1
-4	1	16	-1	-8
-3	-8	-54	8	27
-2	30	85	-29	-44
-1	-80	130	48	31
0	35	-378	-35	0
1	24	214	8	-11
2	-2	-11	1	4
3	0	0	0	0
4	0	0	0	0
5	0	0	0	0

LOCATION (Figure 5.2)	POLE BOUNDARY EXPRESSIONS		
DERIVATIVE	$\frac{dx}{d\phi}$, s = 1	$\frac{d^2x}{d\phi^2}$, s = 2	$\frac{d^3x}{d\phi^3}$, s = 3
ACCURACY	h^6	h^6	h^6
f_s / r	$\frac{1}{60h}$	$\frac{1}{180h^2}$	$\frac{1}{240h^3}$
-1	-10	126	-469
0	-77	-70	1818
1	150	-486	2924
2	-100	855	2690
3	50	-670	-1710
4	-15	324	814
5	2	-90	-268
6	0	11	54
7	0	0	-5

LOCATION (Figure 5.2)	EDGE BOUNDARY EXPRESSIONS		
DERIVATIVE	$\frac{dx}{d\phi}$, s = 1	$\frac{d^2x}{d\phi^2}$, s = 2	$\frac{d^3x}{d\phi^3}$, s = 3
ACCURACY	h^6	h^6	h^6
f_s / r	$\frac{1}{60h}$	$\frac{1}{180h^2}$	$\frac{1}{240h^3}$
-6	0	11	0
-5	-2	-90	1
-4	15	324	-8
-3	-50	-670	29
-2	100	855	-64
-1	-150	-486	83
0	77	-70	-56
1	10	126	15

TABLE 5.1 Coeffecients $\alpha_{s,r}$ and f_s of the Finite Difference Expressions;

$$\left.\frac{d^s x}{d\phi^s}\right|_{\phi_j} \cong \frac{d^s \tilde{x}_j}{d\phi^s} = f_s \sum_{r=-6}^{+7} \alpha_{s,r}\, x_{j-r}\,, \text{ equation (5.7).}$$

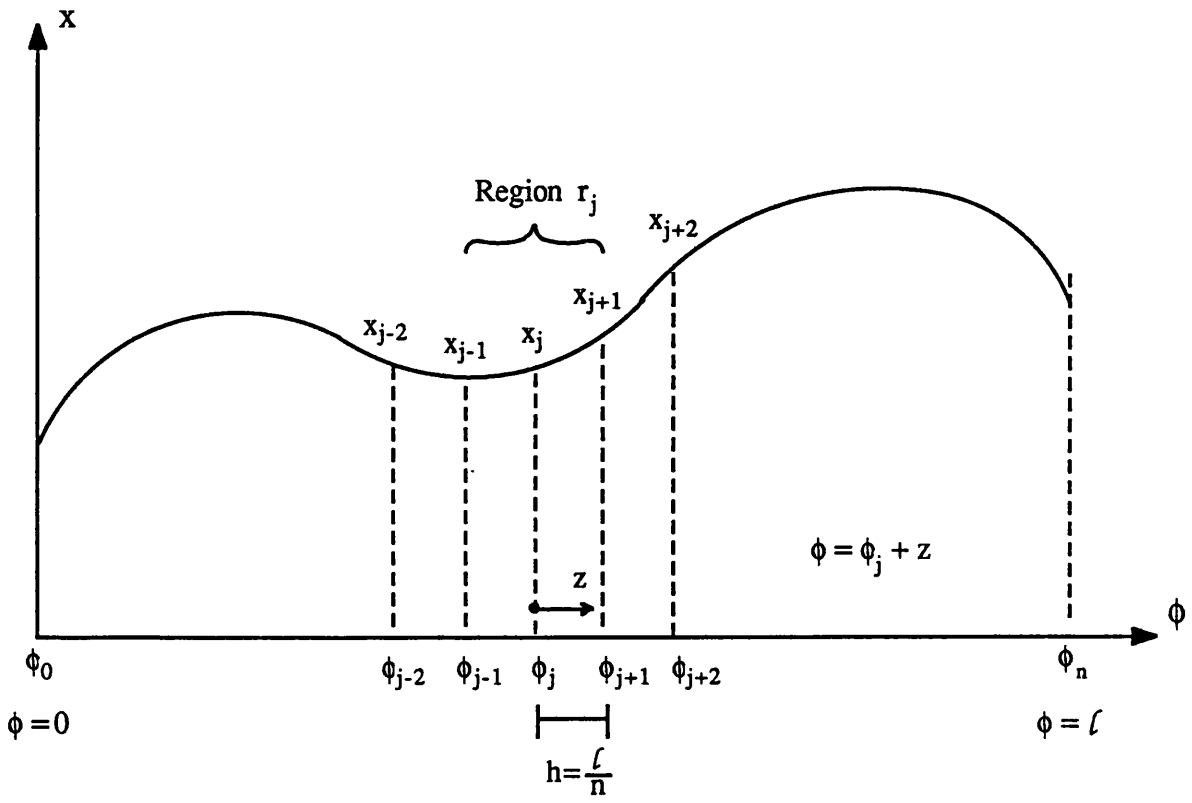


Figure 5.1 Finite Difference Collocation.

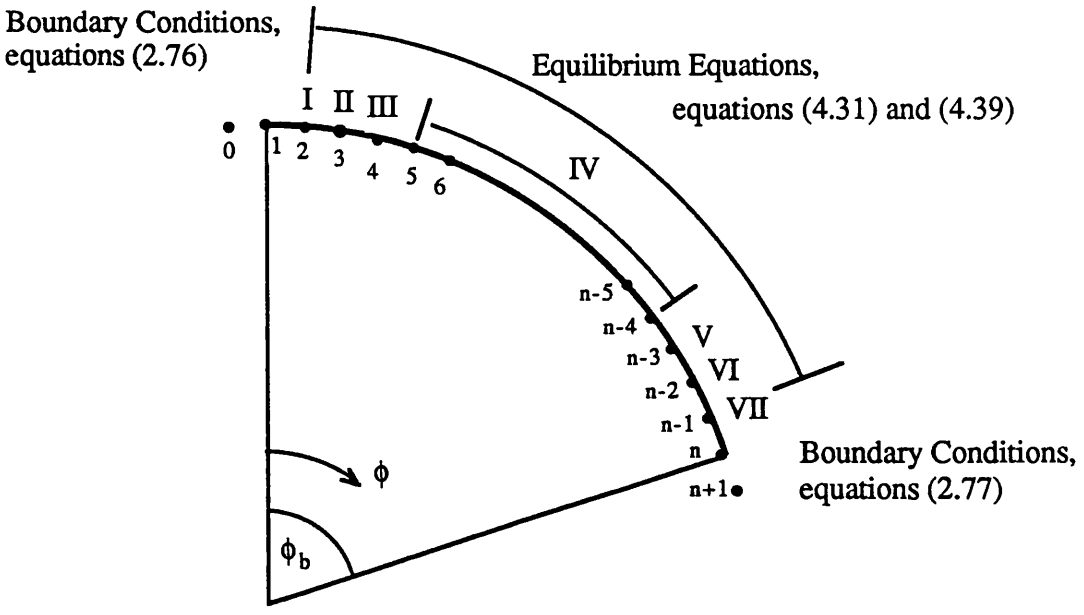


Figure 5.2 Finite Difference Mesh and Shell Regions (Table 5.1).

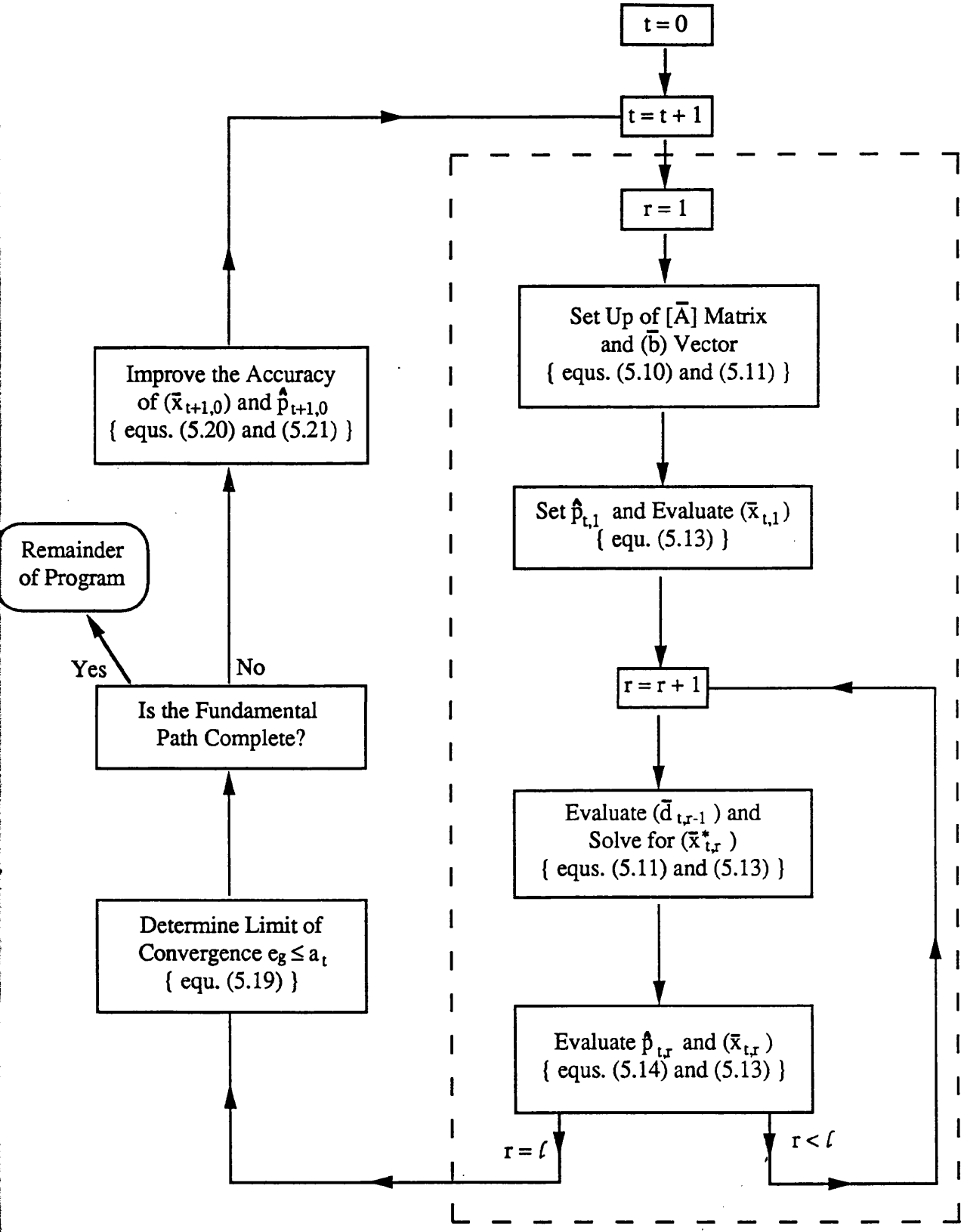


Figure 5.3 Solution Algorithm For The Nonlinear Fundamental Path

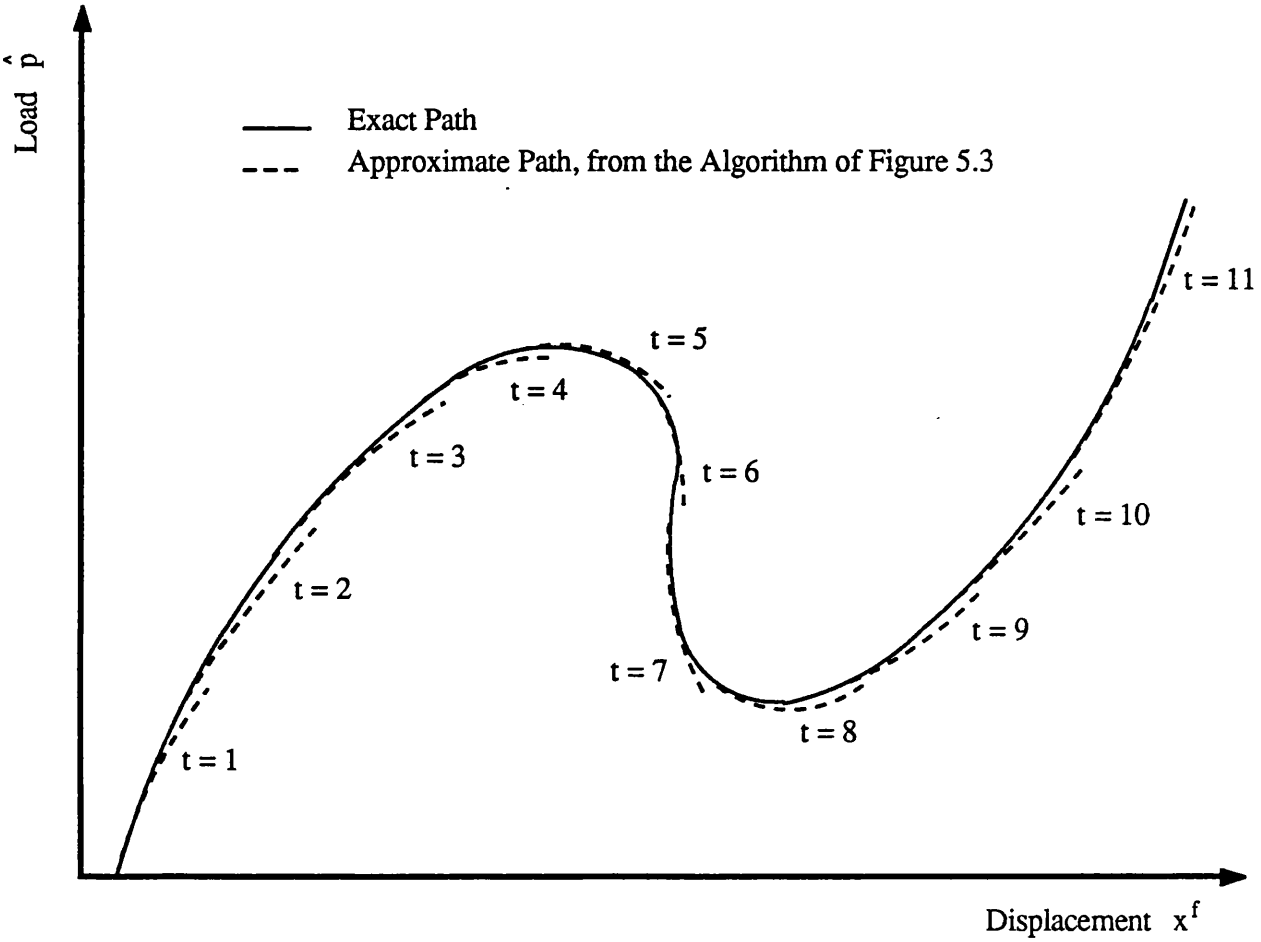


Figure 5.4 Piece-wise Continuous Modelling of the Nonlinear Fundamental Path

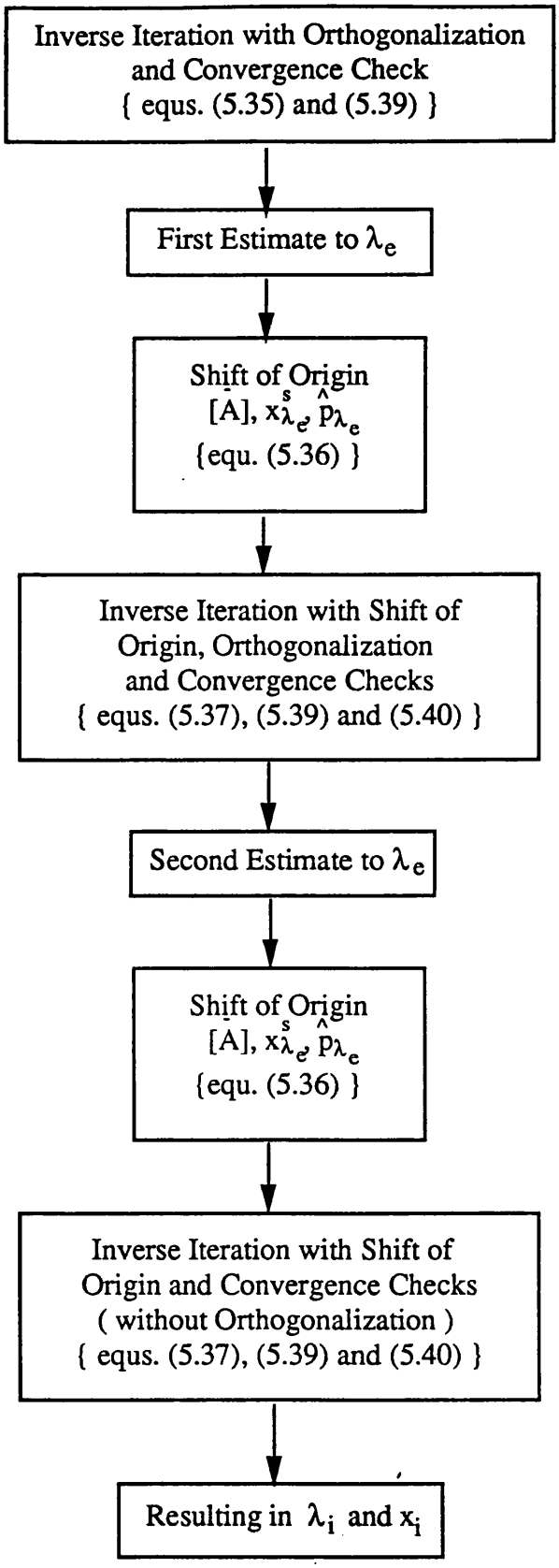


Figure 5.5 Solution Alogrithms for the Nonlinear Eigenvalue Problem

CHAPTER SIX

VALIDATION OF THE COMPUTER PROGRAM

CHAPTER SIX

CONTENTS

VALIDATION OF THE COMPUTER PROGRAM

6.1 INTRODUCTION

Three computer programs were written in the course of the present work. Two of these programs were written to perform the classical axisymmetric bifurcation analysis, and the post bifurcation analysis for complete spherical shells. The mathematical basis of these programs has been described in Chapter Three and Appendix A. The validation of these two programs was performed by simple comparisons with results reported in the literature, and will not be reported here.

The third program, NLSPHERE, which implements the solution methods described in Chapters Four and Five, and in particular the validation of this program, are the concern of this Chapter. It is not intended to present a complete validation and description of the program NLSPHERE, but to draw attention to those features of the program and its operation which influence the accuracy and efficiency. The source listing (Fortran) and a typical output file are included in Appendix B.

The task of program validation has been separated into two parts. First, attention is focused on internal error limits and accuracy in Section 6.2. Then, comparisons between the present numerical results, and results obtained independently by other workers using nonlinear shallow shell theory are presented in Section 6.3.

6.2 INTERNAL CONVERGENCE AND ERROR LIMITS

6.2.1 Finite Difference Expression and Convergence

Replacing the continuous partial differentials of the 'exact' differential equations of Chapter Four with discrete finite difference approximations results in the set of simultaneous equations described in Chapter Five. The matrices that result will be 'banded', that is non-zero elements will be clustered about the principal diagonal of the matrix, all the solution techniques that have been used allow for the banded nature of the matrices to reduce both storage and the number of arithmetic operations required to perform a given computation. The accuracy with which the set of simultaneous equations models the 'exact' differential equations is nonlinearly dependent on both the accuracy of the finite difference expressions used in the transformation and the mesh spacing, h. The error may be reduced by using more accurate finite difference expressions, resulting in an increase in the number of non-zero elements in each row of the matrices, increasing the bandwidth K, or by reducing the mesh spacing h, resulting in a greater number of simultaneous equations to be solved, which increases both the order of the matrices and the length of the vectors, N. The use of central finite difference expressions with a high order of accuracy at or near the boundaries of the shell will increase the number of off shell 'fictitious nodes', while the use of highly accurate partial forward or backward expressions at or near the boundary may introduce errors due to the extra 'weight' attached to the near boundary nodes.

The computer program NLSPHERE requires the finite difference expression to be supplied as part of the input data file, and allows as many as seven sets of finite difference expressions to be used over the surface of the shell, the 'shell nodes'. The finite difference expressions used at the boundaries, the 'boundary nodes', are specified separately. This has been described in Chapter Five, in particular Figure 5.2 shows the node numbering and Table 5.1 presents the finite difference expressions that were eventually used.

The optimum (to minimise memory and CPU time required) choice of finite difference expressions and number of nodes will depend on the geometry of the shell under consideration. The degree of accuracy (the node spacing and the order of the finite difference expressions) required by the finite difference modelling of the elastic behaviour of the spherical shell will depend on the shortest wavelength present in the midsurface displacements. For spherical shells the wavelength of the buckling mode (both the axisymmetric buckling and periodic bifurcation modes) decreases as the geometric Bartdoff parameter, λ, increases. From the classical analysis of complete spherical shells $n \approx 1.81\sqrt{r/t}$, and for $\lambda \approx 56$ or $r/t=100$, then $n \approx 18$ where n is the order of the Legendre polynomial describing the buckling mode. In the numerical solution it is only necessary to model half of the spherical cap, therefore n is equal to the number of half waves modelled by the finite difference representation. The need for a

highly accurate finite difference discretization results from the presence of these short wavelength components of the midsurface displacements.

In the overall solution algorithm the boundary conditions are used to eliminate the boundary and off shell fictitious nodes from the matrices and vectors before the solution techniques described in Chapter Five are used. If the boundary and off shell fictitious nodes were not eliminated from the matrices of the nonlinear eigenvalue problem they would give rise to spurious eigen-values and vectors.

Storage requirements for banded matrices and vectors are linearly dependent on the nodal spacing, i.e. the order of the matrix or vector. The accuracy, or order, of the finite difference expressions will have a much weaker effect on the bandwidth of the matrices and will not affect the length of the vectors at all, hence the accuracy of the finite difference expressions used will have a weak effect on the overall storage requirements of the program. The product N(K-1) where N is the order of the matrices, [N=3x(number of 'shell nodes')] and K is the bandwidth of the matrices [K=3x(number of nodes used by the finite difference expressions)], is representative of the number of arithmetic operations, and therefore the CPU time, required to evaluate the derivative of a vector or perform back substitution of a decomposed matrix. Also the CPU time taken to perform Gaussian elimination with partial pivoting on banded matrices is proportional to $N(K-1)^2$. As a result of these nonlinear dependencies simple second order central finite difference expressions will not be the optimum expressions for use with the solution algorithms.

Convergence was studied in two parts, due to the large number of combinations of finite difference expressions and node spacings that were considered and tested. First a linear fundamental path problem ($w^f = Pr(1-\mu)/2Et$) with bifurcation into axisymmetric and periodic modes was used to select the optimum set of finite difference expressions. A typical comparison between finite difference expressions is shown in Figure 6.1, for a spherical cap with $\lambda \approx 32$. Figure 6.1 compares standard central finite difference expressions with a set of finite difference expressions in which all error terms are of order h^4 and with the set of finite difference expressions that were eventually used, i.e. the finite difference expressions in Table 5.1. After selecting the optimum set of finite difference expressions from the linear fundamental path convergence study, a second convergence study using the full nonlinear capabilities of the program was undertaken. Results from the second convergence study are shown in Figure 6.2, where the axisymmetric buckling pressure, $\hat{p}_{bk}$ (i=0), and the minimum critical bifurcation pressure, $\hat{p}_{cr}$ (i=2 and i=26), for two shells (λ=6 and λ=38) are compared, as the number of nodes is increased.

As a result of these convergence studies the finite difference expressions given in Table 5.1 with 51 shell and boundary nodes and two off shell fictitious nodes were used throughout. It is estimated that the relative errors introduced into the final numerical results due to the finite

difference discretization will for all the shells considered in this work, $3.5 \leq \lambda \leq 60$, be less than 1%, and usually less than 0.1%.

6.2.2 The Perturbation Series and Relative Error Limits used in the Solution of the Equilibrium Equations

The overall accuracy of the program depends on the accuracy with which the nonlinear fundamental path is modelled. The tendency, in the nonlinear modelling of the fundamental path, for errors to increase with increasing values of the perturbation parameter has been discussed in Chapter Five, (Section 5.3). As a consequence the local perturbation series was introduced, equation (5.15), and a measure of the relative error based on the equilibrium equations was defined, equation (5.19). Subsequently a Newton-Raphson iterative scheme for the improvement of the displacement vector prior to its use as the initial vector in the next local perturbation series was given by equation (5.20).

The computer program NLSPHERE requires the user to specify the number of terms used in a local perturbation series and the number of these local perturbation series that are used to model the complete fundamental path. Trials were undertaken in which the order of the local perturbation series was varied, resulting in the adoption of sixteen local perturbation series of ten terms each. As a result of tests, the upper bound relative error limit, equation (5.19), used in the evaluation of a_t was set at 0.005, with the Newton-Raphson iteration reducing this error to 10^{-6}, this 'improved' displacement vector is then used as the initial displacement vector for the next local perturbation series. Due to the adoption of the error limits discussed above, the relative errors introduced into the final numerical results by the nonlinear fundamental path solution method will in all cases be less than 0.5%.

The iterative solution method used for determining the eigen-values and vectors have been described in Chapter Five, (Section 5.2). The measure of the relative errors in the eigenvalues and eigenvectors have been defined by equations (5.39) and (5.40). Various error limits are used within the solution algorithms to determine when shifts of origin are to be performed. Each shift of origin requires the recalculation and decomposition (Gaussian elimination) of the banded [A] matrix, equation (5.28), this improves the speed with which the iterative solution method converges, equation (5.29). Trials performed using various combinations of error limits and origin shifts resulted in the following rules being incorporated in the solution routines for the nonlinear eigenvalue problem.

First - Inverse iteration and orthogonalization (if $j \geq 2$) are used until the relative change in the eigenvalue, equation (5.39), is less than 10^{-3}

Second - Inverse iteration with a shift of origin to λ_e and orthogonalization (if $j \geq 2$) are used until the relative change in the eigen-value and vector, equations (5.39) and (5.40), are both less than 10^{-5}.

Third - Inverse iteration with a shift of origin to λ_e without orthogonalization is used until the relative error in the eigen-value and vector, equations (5.39) and (5.40), are both less than 10^{-8}.

In practice the above rules were found to be a reasonable compromise between the increased rate of convergence that results from a shift of origin, and the CPU time used in setting up the $[\bar{A}]$ matrix and performing the Gaussian elimination with partial pivoting required by each shift of origin. It was found that the second and third error criteria, 10^{-5} and 10^{-8} above, were usually satisfied in less than ten iterations, when the estimate to the eigenvalue used for the shift of origin, λ_e, was given by the final value of the previous iterative step. The error limit of 10^{-8} on the final calculated value of the eigenvectors is important when subdominant eigenvalues are required, as the error introduced by the orthogonalization of the subdominant vectors during iteration depends on the errors present in the previously calculated 'dominant' eigenvectors. Also, as a result of using the final relative error limit of 10^{-8} on the eigenvalue and vector, the numerical solution algorithm used for the nonlinear eigenvalue problem, will be responsible for errors that are entirely negligible in comparison with those introduced by the finite difference discretization (0.1 to 1%) and the nonlinear fundamental path solution algorithm (<0.5%).

6.2.3 Quadratic, Cubic and Quartic Terms of the Strain Displacement Relations and the Equilibrium Equation

The fundamental path equilibrium equation, equation (4.31) and Table 4.1, contains the terms $A_1(x^s)$ and $\hat{p}B_o$, $A_2(x^s)^2$ and $\hat{p}B_1(x^s)$, $A_3(x^s)^3$ and $\hat{p}B_2(x^s)^2$, and $A_4(x^s)^4$; these are the linear, quadratic, cubic and quartic terms of the equilibrium equation. As mentioned in Chapter Four the derivation of the equilibrium equations for both the nonlinear fundamental path and the linearised secondary path were based on quartic strain and quadratic curvature expressions, equation (2.66). However for spherical shells with radius to thickness ratios greater than 100 ($\alpha = t^2/12r^2 < 8.3 \times 10^{-6}$) the quadratic curvature terms were found to have a negligible influence on the numerical results. As the inclusion of the quadratic curvature terms requires the notation used for the bending energy to be extended, and as for all cases considered these terms will for all practical purposes be zero, only the linear curvature terms are presented and discussed in the present work.

Figure 6.3 shows the effect on the nonlinear fundamental path for two, $\lambda=6$ and $\lambda=26$, perfect spherical caps. Pressure is plotted against perturbation parameter, when the equilibrium equations, equation (4.31), are used and all consistent terms up to and including the quadratic, the cubic, and the quartic are maintained. The necessity of maintaining terms in the equilibrium

equation up to and including the cubic terms is clearly demonstrated. The difference between the numerical results obtained from the consistent cubic and quartic expressions at most affected only the fifth significant figure.

Cubic terms in the equilibrium equation arise from quartic terms in the total potential energy functional. Which in turn arise from the product of cubic terms of the change in volume and the pressure load, equation (2.69), from the product of the quadratic terms of the strain displacement relations, and from the product of the cubic and linear terms of the strain displacement relations. Therefore consistent cubic terms in the equilibrium equation may only be achieved when the strain displacement relations and the change in volume are also consistent in their cubic terms. The 'largest' of the cubic terms in the equilibrium equation arise from the quadratic terms of the strain displacement relations, the classical axisymmetric analysis reported in Chapter Three is based on only the largest of the quadratic terms of the strain displacement relations, equation (3.1). The nonlinear, up to and including all cubic terms, strain displacement relations are given by

$$\varepsilon_\phi = \varepsilon_\phi(x) = \varepsilon_1 + \frac{1}{2}\left\{\left(\dot{v}\right)^2 + (\beta)^2\right\} - \frac{1}{2}\varepsilon_1\left\{\left(\dot{v}\right)^2 + (\beta)^2\right\} + \ldots \qquad (6.1)$$

$$\varepsilon_\theta = \varepsilon_\theta(x) = \varepsilon_2 + \frac{1}{2}\left\{(\gamma)^2 + (\delta)^2\right\} - \frac{1}{2}(\varepsilon_2)\left\{(\gamma)^2 + (\delta)^2\right\} + \ldots$$

$$\varepsilon_{\theta\phi} = \varepsilon_{\theta\phi}(x) = \frac{1}{2}\left[\left(\dot{v}+\gamma\right) + \left(\beta\delta - \varepsilon_1\dot{v} - \varepsilon_2\gamma\right) - \frac{1}{2}\left(\dot{v}+\gamma\right)\left\{\left(\dot{v}\right)^2 + (\gamma)^2 + (\beta)^2 + (\delta)^2\right\}\right.$$
$$\left. + \frac{1}{6}\left(\dot{v}+\gamma\right)^3 + \gamma(\varepsilon_2)^2 + \dot{v}(\varepsilon_1)^2 - \beta\gamma(\varepsilon_1+\varepsilon_2) + \ldots\right]$$

where

$\varepsilon_1 = \dot{u} + w$

$\varepsilon_2 = w + u\cot\phi + \dot{v}/\sin\phi$

$\beta = \dot{u} - w$

$\gamma = \dot{u}/\sin\phi - v\cot\phi$

$\delta = v - \dot{w}/\sin\phi$

Profiles of the fundamental path displacements and their derivatives as well as the linear strain components, and the quartic contributions to the total membrane energy, are all presented in Figure 6.5 and Figure 6.6 at the axisymmetric buckling point, the fundamental paths for these shells, λ=6 and λ=26, are shown in Figure 6.3 and Figure 6.4. Profiles of the fundamental path displacements (normalised by the shell thickness) and their derivatives are shown in Figure 6.5 and Figure 6.6, (a) and (b), where the inplane displacement, u, and its derivative, u', have been amplified by a factor of fifty (x 50) to allow presentation on the same graph as the radial displacement, w, and its derivative, w'. The profiles of the axisymmetric linear

strain components, ε_1, ε_2, and β, are depicted in Figure 6.5 (c) and Figure 6.6 (c), where ε_1 and ε_2 have been amplified by a factor of ten (x 10). Finally, in Figure 6.5 (d) and Figure 6.6 (d) the contributions to the total membrane energy, $U_{1,M}$, arising from the quadratic, $\beta^4/4$, and cubic, $(\varepsilon_1^2+\mu\varepsilon_1\varepsilon_2)\beta^2$, terms of the strain displacement relations are shown separately, the contribution from the cubic terms of strain displacement relations have been amplified by a factor of fifty (x 50).

Figure 6.5 (d) and Figure 6.6 (d), show that the quartic contribution to the fundamental path membrane energy, $U_{1,M}$, from the quadratic terms, $\beta^4/4$, of the strain displacement relations decreases from approximately 10% for λ=6 to 5.5% for λ=26, at the maximum point on the profile. And the corresponding contribution from the cubic terms, $(\varepsilon_1^2+\mu\varepsilon_1\varepsilon_2)\beta^2$, of the strain displacement relations is approximately 0.1% for λ=6 and 0.05% for λ=26.

The linear terms of the secondary path equilibrium equation, the nonlinear eigenvalue problem given by equations (4.39) or (5.31) and Table 4.2, contain terms of orders $A_0(x)$, $[A_1(x^s)](x)$ and $\hat{p}B_0(x)$, $[A_2(x^s)^2](x)$ and $\hat{p}[B_1(x^s)](x)$, and $[A_3(x^s)^3](x)$, these are the terms that are linear, quadratic, cubic, and quartic in the generalised coordinates $(\hat{p},x^T)$, $x^T=x^i+x^f+x$.

In the derivation of the quartic terms of the equilibrium equation, $[A_3(x^s)^3](x)$, only the 'largest' terms (those containing the radial displacement w and its derivatives) were maintained and included in the computer coding of the eigenvalue solution routines. Comparisons of eigenvalues and vectors resulting from the maintenance of all the cubic strain displacement terms, consistent cubic equilibrium equations, in both the fundamental path solution algorithm and the solution algorithm for the eigenvalue problem with the corresponding results when the equilibrium equations for the fundamental path contained consistent quartics, and the equilibrium equation for the secondary path contained consistent cubics and 'large' quartics, were performed, and resulted in differences in the fifth significant figure.

Bifurcation from the axisymmetric fundamental path into periodic modes occurs before the axisymmetric buckling pressure, P_{bk}, is reached for the two shells discussed above the critical periodic modes and buckling pressures are i=2 and P_{cr}/P_{cl}=0.7618 for λ=6 and i=18 and P_{cr}/P_{cl}=0.7924 for λ=26. Figure 6.7 and Figure 6.8 present profiles of the initial secondary path incremental displacements (the eigenvector), their derivatives, and the linear strain components; as these are all based on the eigenvector the vertical scale is arbitrary. Figure 6.7 and Figure 6.8, (a) and (b), use the same arbitrary vertical scale, and the inplane displacements, u and v, and their derivatives, $\dot{u}$, $\acute{u}$, $\dot{v}$, and $\acute{v}$, have all been amplified by a factor of fifty (x 50) compared to the radial displacement, w, and its derivatives, $\dot{w}$ and $\acute{w}$. In Figure 6.7 (c) and Figure 6.8 (c) the profiles of the initial secondary path linear strain components have been normalised by the largest value, and the components ε_1, ε_2, v, and γ have been amplified by a factor of fifty (x 50) in comparison to the components β and δ.

By considering the contributions to the quartic terms of the strain energy that result from the inclusion of cubic terms of the strain displacement relations and by inspection of Figure 6.5 through Figure 6.8, it may be concluded that the inclusion of cubic terms in the strain displacement relations will in general make a difference to the strain energy of spherical shells of approximately 0.1%. The total potential energy and the equilibrium equations that result from the use of quadratic strain displacement relations will be inconsistent in their quartic and cubic terms respectively; however the errors introduced into the analysis by neglecting the cubic terms of the strain displacement relations should also be of the order of 0.1%.

For mathematical consistency and for practical reasons, as the extra terms that result from the use of cubic strain displacement expressions were already included in the coding of the computer program, and as the effect of including these extra terms on the CPU time and storage required for execution of the code are entirely negligible, the equilibrium equations for both the fundamental and the secondary paths used in this study are based on the cubic strain displacement expressions given by equation (6.1).

6.2.4 Remarks on Efficiency and Accuracy.

The solution algorithms presented in Chapter Five have been used with the finite difference discretization, fundamental path modelling, eigenvalue problem solution strategy, and error and accuracy limits discussed above, in the computer program NLSPHERE. It is of interest to note that the algorithm used for the solution of the nonlinear eigenvalue problem, equations (5.35) and (5.37), results in the CPU time required for the solution being very weakly dependent on the number of terms used in the local perturbation series, and completely independent of the total number of terms in the fundamental path. This program was compiled (Minnesota Fortran, MNF, compiler) and executed on the University of London Computer Centre CDC 7600 computer.

The program NLSPHERE evaluates a number of path parameters, mode shapes, and energy components associated with the fundamental and secondary paths, a sample output for a perfect spherical cap, λ=12 (r/t=100, $0 \leq i \leq 2$), is contained in Appendix B. The energy contributions to V_o, V_1, and V_2 of the total potential energy, equations (4.46) to (4.51), are calculated at four points along the fundamental path (contribution to V_o), at the point of axisymmetric buckling (contributions to V_0 and V_1) and at the point of 'bifurcation' into axisymmetric or periodic modes (contribution to V_0, V_1 and V_2). The individual components of the membrane, bending and load potential energy are evaluated in the meridional direction by numerical integration using Simpson's rule.

The total potential energy functional, equations (4.14) and (4.15), is the starting point for the derivation of the equilibrium equations, therefore a final check on the accuracy of both the fundamental and secondary path solutions may be based on back substituting the displacement

vector and load into the total potential energy functional. This check is significant as it is independent of the solution methods used. The computer program evaluates the error in the terms of the total potential energy functional at the axisymmetric buckling point (V_1) and at all points of periodic bifurcation (V_1 and V_2). The relative error has been defined as twice the sum of all the individually integrated energy components divided by the sum of the absolute values of the same energy components, where the energy components are defined by equations (4.46) to (4.51). The relative error in the total potential energy based on this definition is typically of the order of 0.1%, this final and important check demonstrates that the fundamental path displacements, the bifurcation modes and the loads satisfy the total potential energy functional, which is the fundamental equation on which the analysis is based.

6.3 COMPARISONS WITH OTHER NUMERICAL RESULTS

The internal limits on errors and checks by back substitution, described in the first part of this chapter provide only a partial confirmation of the validity of the numerical results obtained from the program NLSPHERE. Independent confirmation of the present results by comparison with the published results of Huang, Reference [17], and Uchiyama and Yamada, Reference [18], are presented below.

Both Huang and Uchiyama and Yamada use shallow shell theory; the spherical geometry of the shell is replaced by a paraboloid and with the introduction of a stress function into the analysis they arrive at Marguerre's nonlinear differential equation, in polar coordinates for shallow spherical caps under uniform pressure loading. The derivation of Marguerre's nonlinear differential equation incorporates the Kirchhoff assumptions, discussed in Chapter Two, as well as the following assumptions.

- The ratio of rise height, $h = r(1-\cos\alpha)$, to the base diameter or chord, $D = 2r\sin\alpha$, of the spherical cap is less than say 1/8, or equivalently half the open angle of the cap, α, is less than 28 degrees. This is the shallow shell assumption, and the spherical geometry is represented by a paraboloid.

- The inplane meridional and circumferential displacements u and v are small in comparison with the radial displacement w. That is, the only nonlinear terms used in the expression for the middle surface strains are quadratic terms involving the radial displacement w.

The introduction of these assumptions in the work of Huang and Uchiyama and Yamada results in a different method of analysis and different solution techniques to those used in the present work. For this reason comparisons between the present results and those of Huang and Uchiyama and Yamada are comparisons between the results of different shell theories.

6.3.1 Comparison of Axisymmetric and Periodic Buckling Pressures for Perfect Spherical Caps with Clamped Edges.

The present results (μ=0.3) are compared with those of Huang (μ=1/3) and Uchiyama and Yamada (μ=0.3) in Figure 6.9 where the axisymmetric (i=0) and periodic (i=1,2,..) buckling pressures have been plotted against the slenderness or Bartdoff parameter λ for perfect clamped shallow spherical caps.

The present results in Figure 6.9 are seen to be in good agreement with those of Huang, Reference [17], as well as Uchiyama and Yamada, Reference [18], especially when the different values of Poisson's ratio, μ, used by Huang and the differences between the shallow

shell derivations and the present analysis are considered. In general the difference between the present results and those based on shallow shell theory are of the same order as the differences between the two results obtained using shallow shell theory with slightly different values of Poisson's ratio. The greatest variation in results occurs for the antisymmetric (i=1) buckling mode, even so the differences in buckling pressures are less than 5%. The locus of the antisymmetric buckling pressures given by the present results have a similar shape to that of Uchiyama and Yamada and have an opposite curvature to the results of Huang. When displacement profiles of the fundamental path or periodic bifurcation modes were visually compared, as Uchiyama and Yamada give their results graphically, no differences between the profiles could be detected.

6.3.2 Comparisons of the Axisymmetric Nonlinear Fundamental Path Solution with Initial Imperfections.

Various comparisons between the results from the computer program NLSPHERE and those reported by Uchiyama and Yamada, Reference [18], for spherical caps with initial axisymmetric imperfections have been undertaken. Agreement between the results of the two methods are remarkably good, with relative errors of less than 5% in all cases. A single typical comparison will be presented.

Uchiyama and Yamada present graphs of pressure (Q) against displacement at the pole (δ) for a shallow spherical cap, $\lambda \approx 5.75$ or H=5 for different amplitudes of initial imperfection (A_2^o, W_2^o), in figure 10.2 of Reference [18].

The following notation is used by Uchiyama and Yamada,

$$Q = \frac{8H^2}{\sqrt{3(1-\mu^2)}} \left(\frac{P}{P_{cl}}\right) \qquad (6.2)$$

$$H = \frac{h}{t}$$

$$\lambda = 2\sqrt[4]{3(1-\mu^2)H^2}$$

$$W_2^o = (1-\xi^2)^2$$

$$\xi = \frac{\tan\phi}{\tan\alpha}$$

$$\mu = 0.3$$

A_2^o = imperfection amplitude

in which h is the rise height and t the shell thickness. The parameter ξ, ($0 \le \xi \le 1$), used by Uchiyama and Yamada is the nondimensional distance measured along the chord of the cap from beneath the pole or apex, and is given above in terms of the meridional coordinate ϕ and the half open angle of the cap, α.

The same shape of initial imperfection (W_2°), was used with the program NLSPHERE, resulting in the excellent agreement shown in Figure 6.10. The values of the imperfection amplitude shown in Figure 6.10 have been nondimensionalised with respect to the shell thickness and positive values indicate an initial imperfection with an outward displacement at the pole of the cap. Comparisons of other cases incorporating initial imperfections reported by Uchiyama and Yamada resulted in equally good agreement with the predictions from the program NLSPHERE.

As a result of the validation exercises reported in this Chapter, and other tests undertaken during the course of the work, it is believed that no significant 'bugs' or errors remain in the program source code.

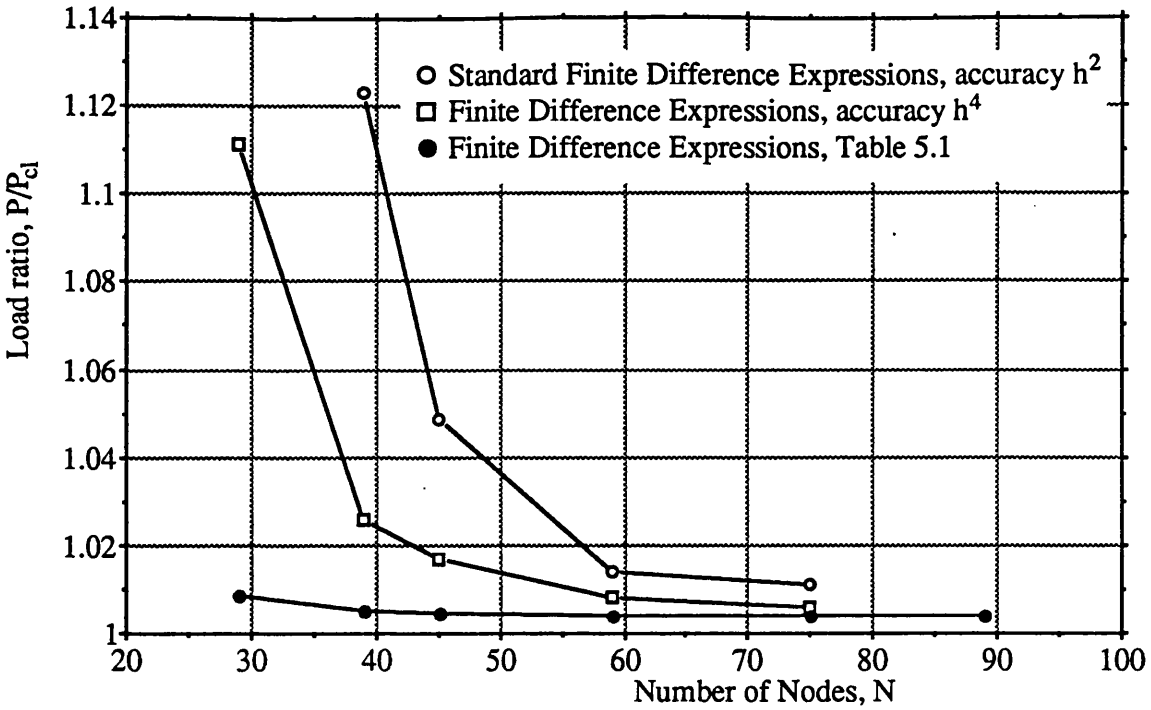


Figure 6.1 Comparison of Finite Difference Expressions, Lowest Buckling Load of Clamped Spherical Cap, Membrane Fundamental Path, $\lambda = 32$.

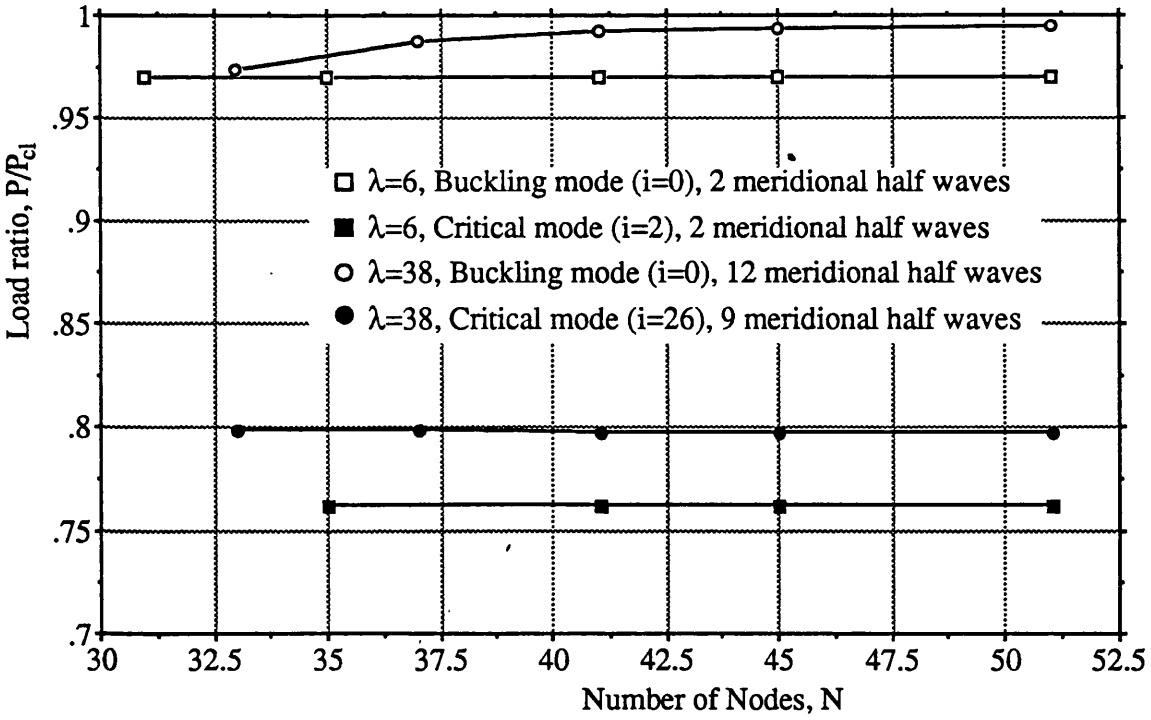


Figure 6.2 Convergence Study, Finite Difference Expressions in Table 5.1, Buckling and Critical Modes of Clamped Spherical Cap, Nonlinear Fundamental Path, $\lambda = 6$ and $\lambda = 38$

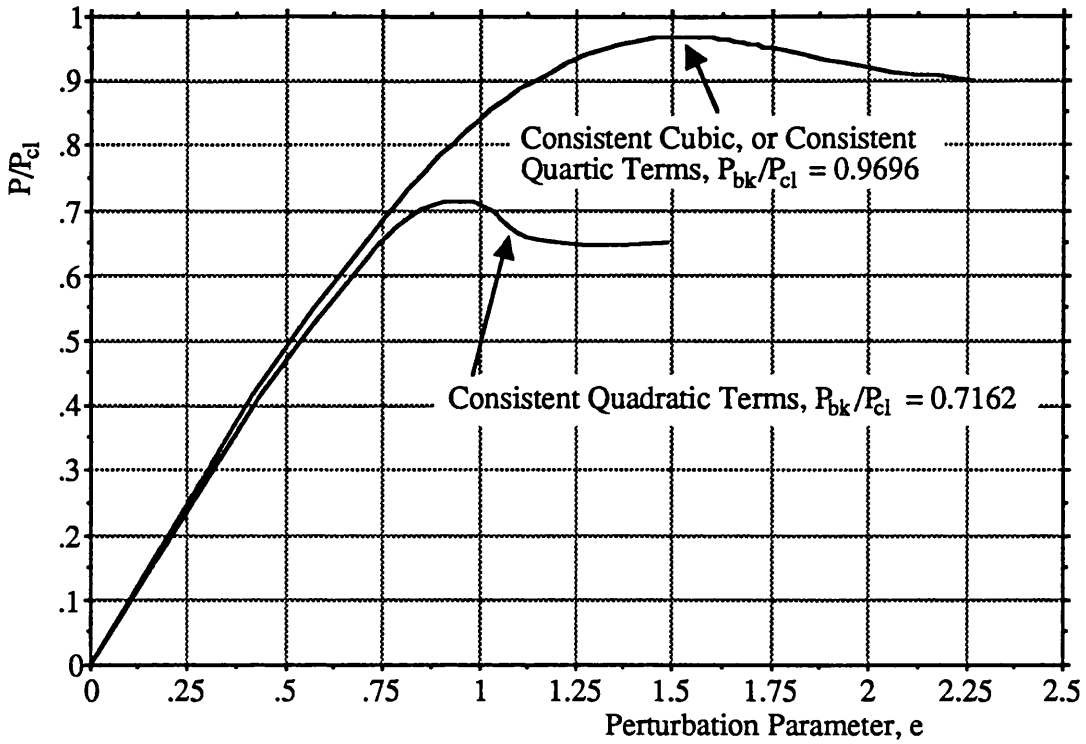


Figure 6.3 Nonlinear Fundamental Paths, based on Quadratic, Cubic, and "Quartic" Equilbrium Equations, for a Spherical Cap with $\lambda = 6$.

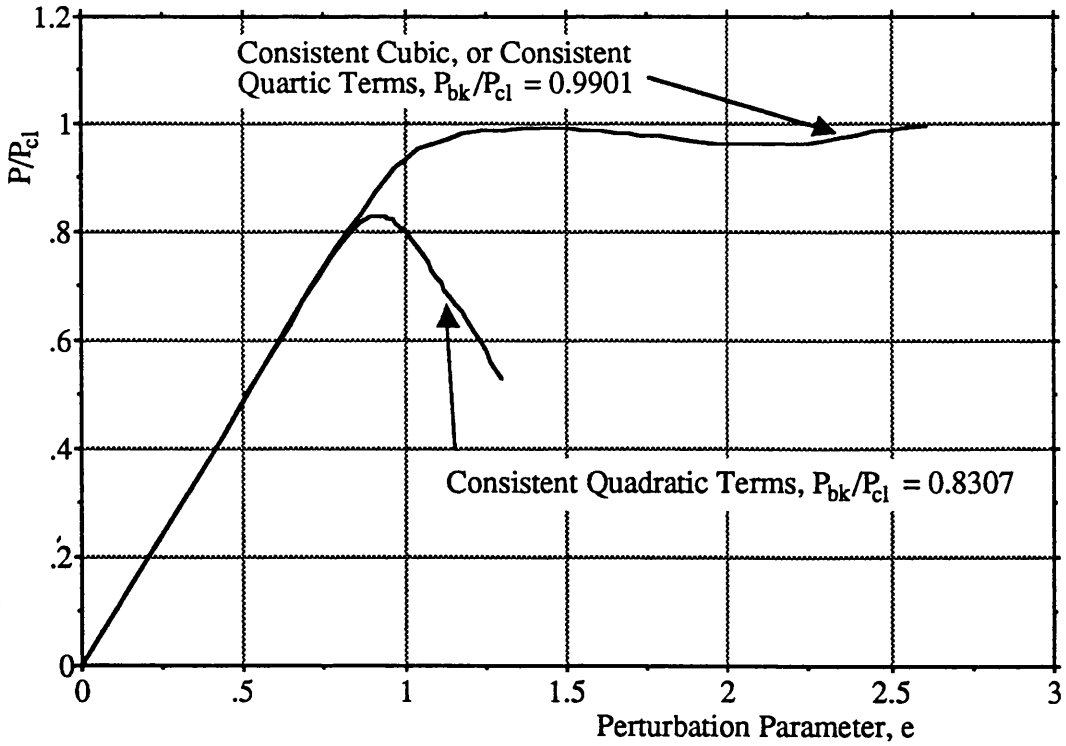
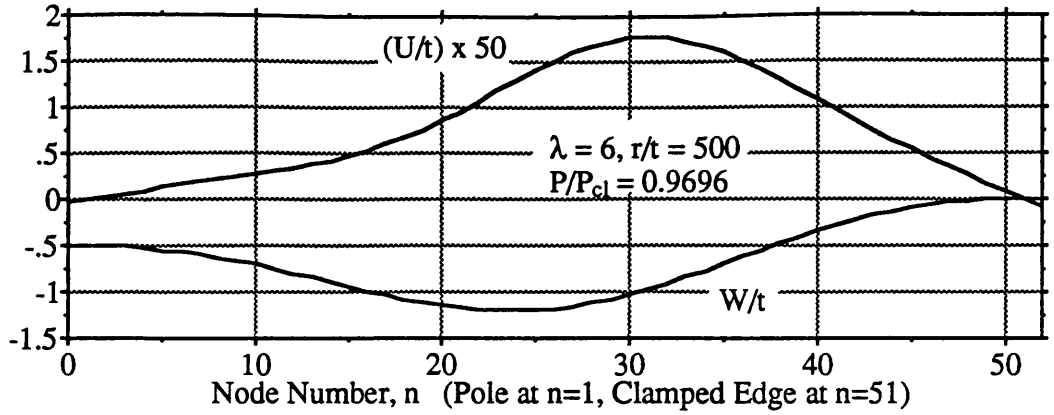
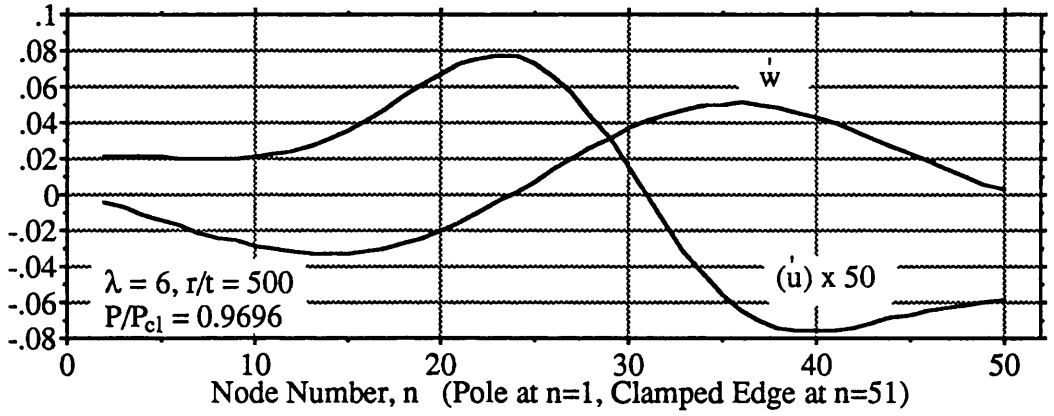


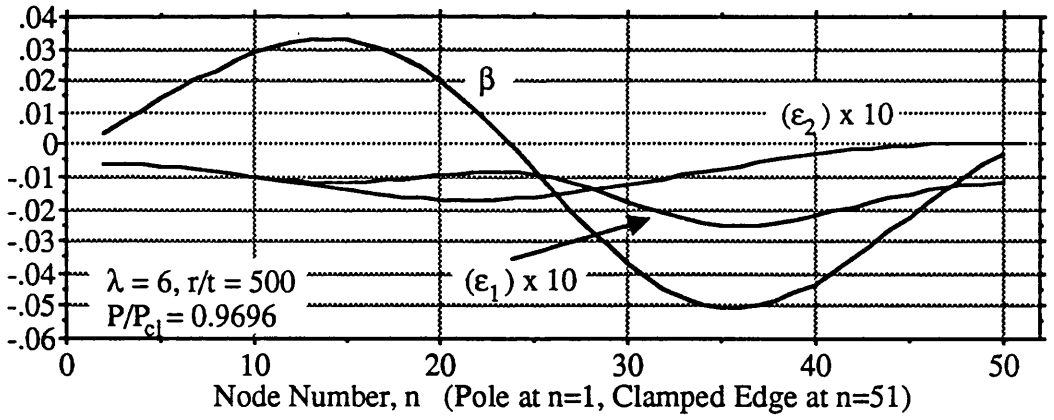
Figure 6.4 Nonlinear Fundamental Paths, based on Quadratic, Cubic, and "Quartic" Equilbrium Equations, for a Spherical Cap with $\lambda = 26$.



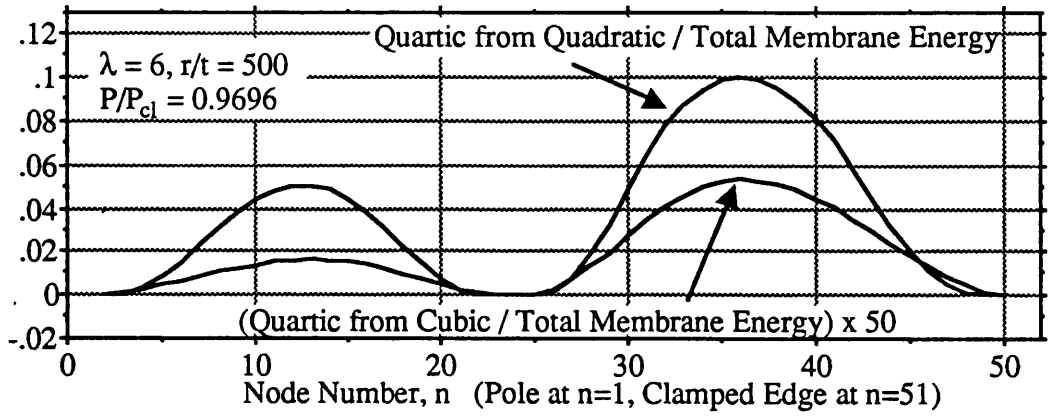
(a) Displacements



(b) Derivatives

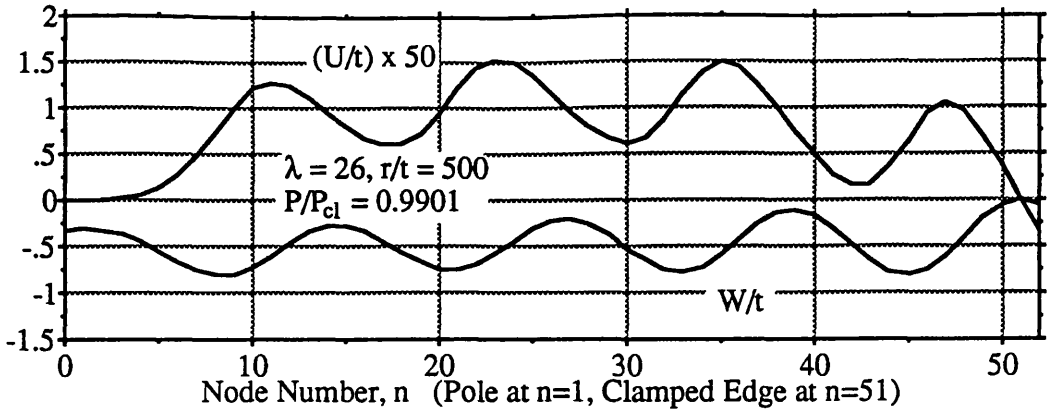


(c) Strain Components

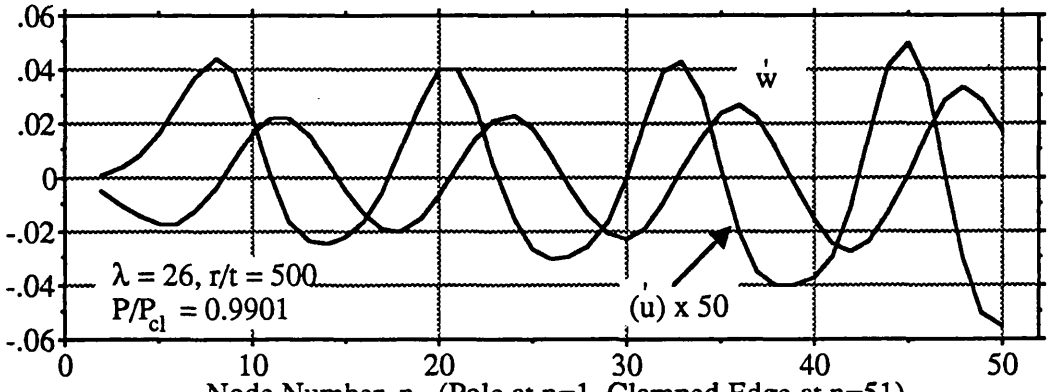


(d) Contributions to the Quartic Terms of the Membrane Strain Energy

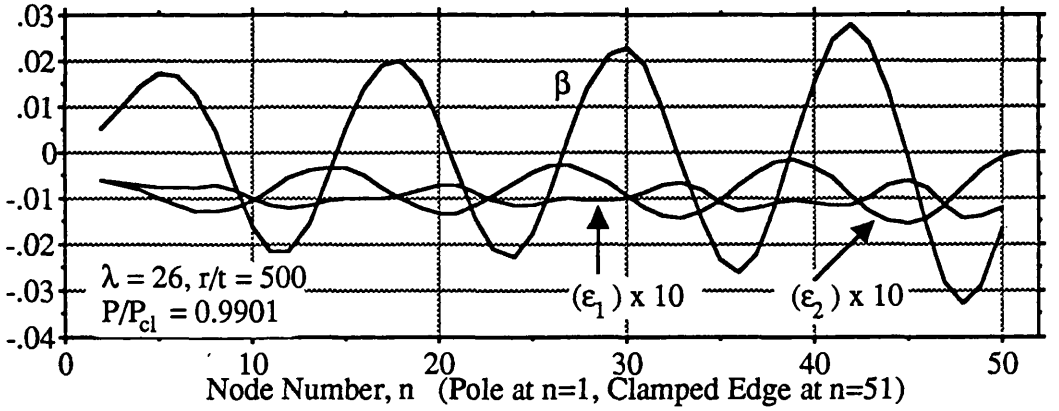
Figure 6.5 Fundamental Path Displacement, Strain, and Membrane Energy Profiles at the Axisymmetric Buckling Point, for $\lambda = 6$.



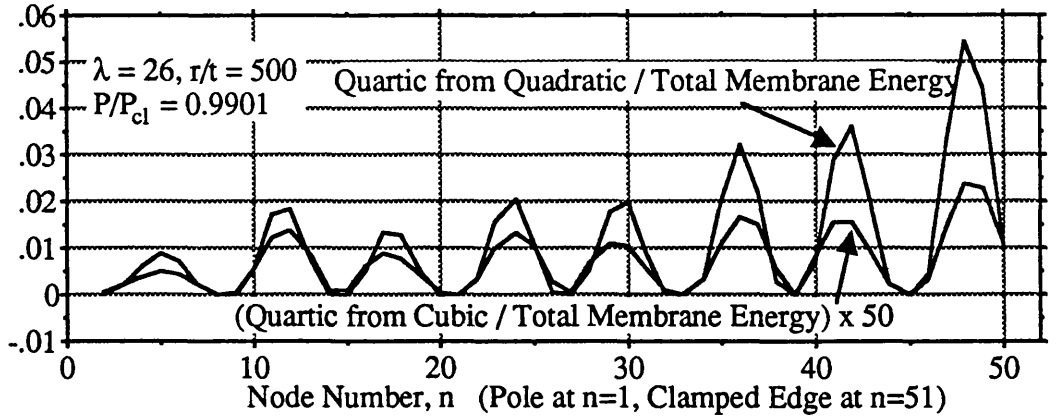
(a) Displacements



(b) Derivatives

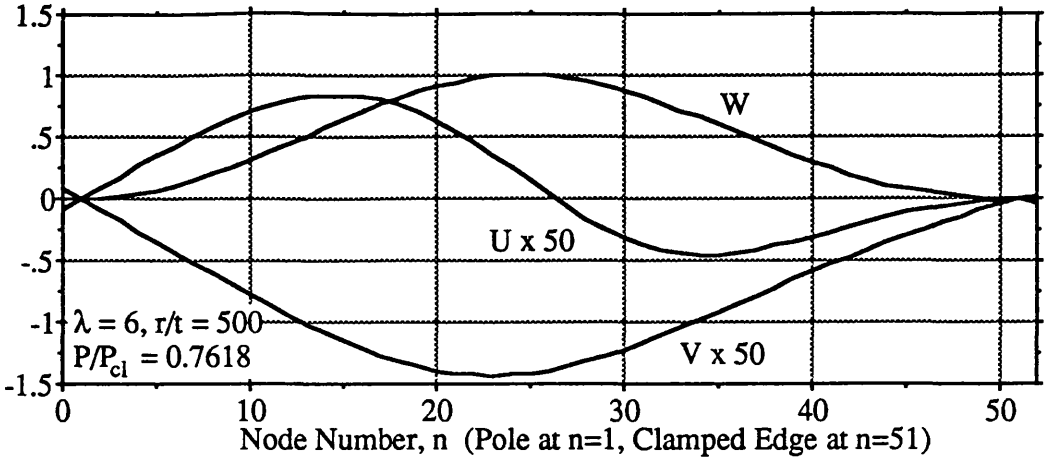


(c) Strain Components

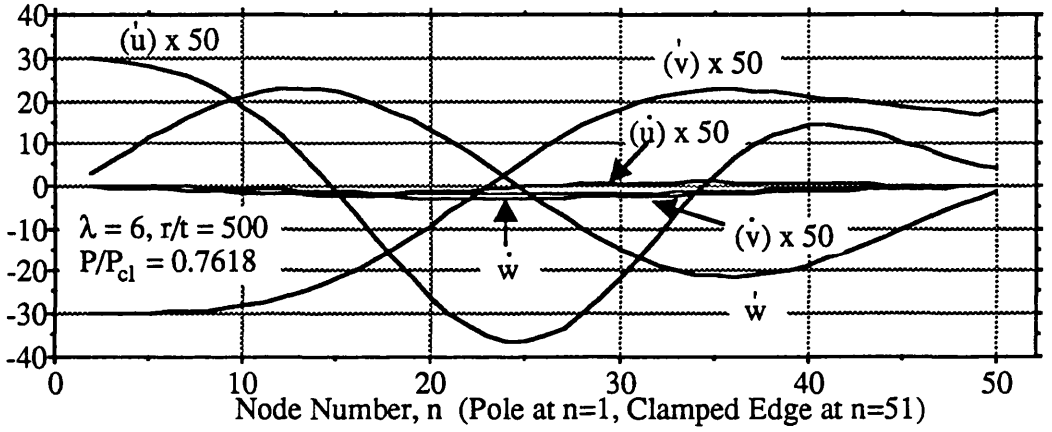


(d) Contributions to the Quartic Terms of the Membrane Strain Energy

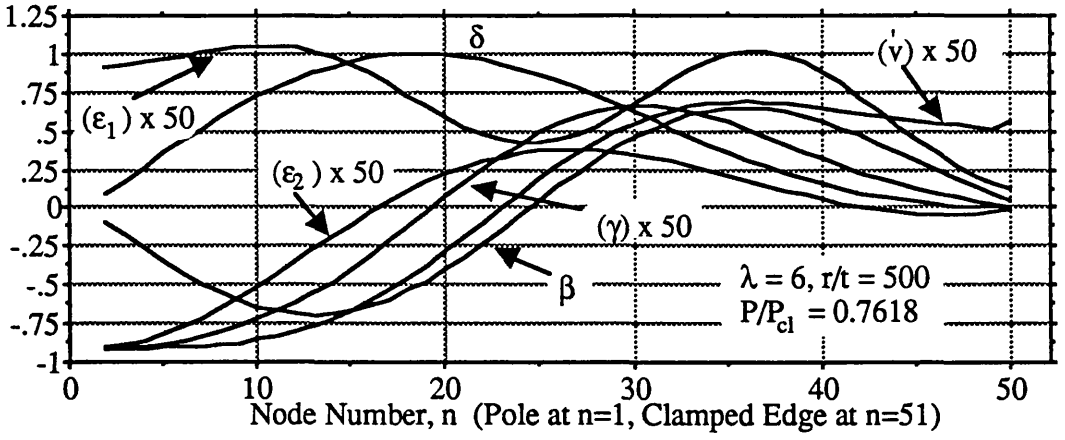
Figure 6.6 Fundamental Path Displacement, Strain, and Membrane Energy Profiles at the Axisymmetric Buckling Point, for $\lambda = 26$.



(a) Normalised Displacements

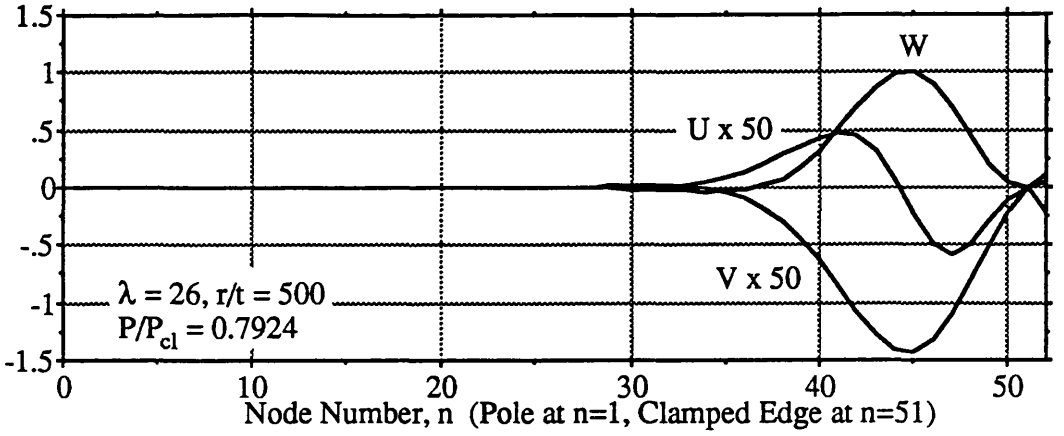


(b) Normalised Derivatives

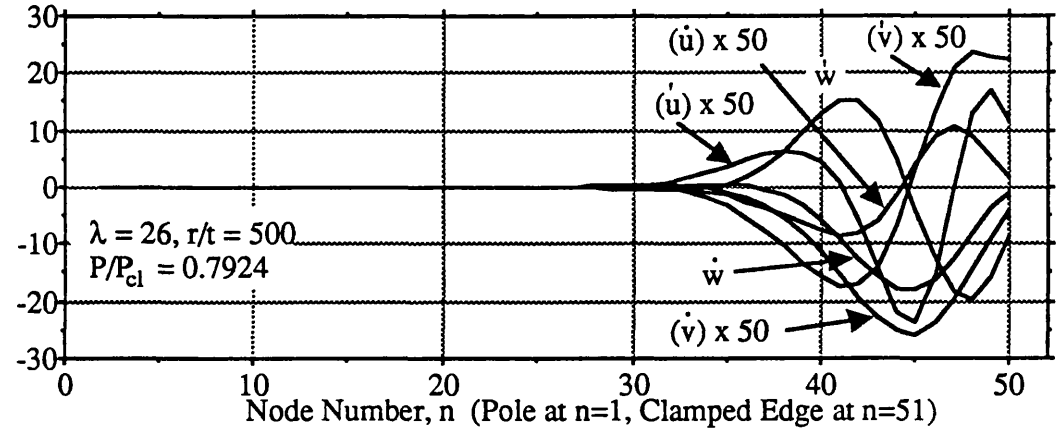


(c) Normalised Strain Components

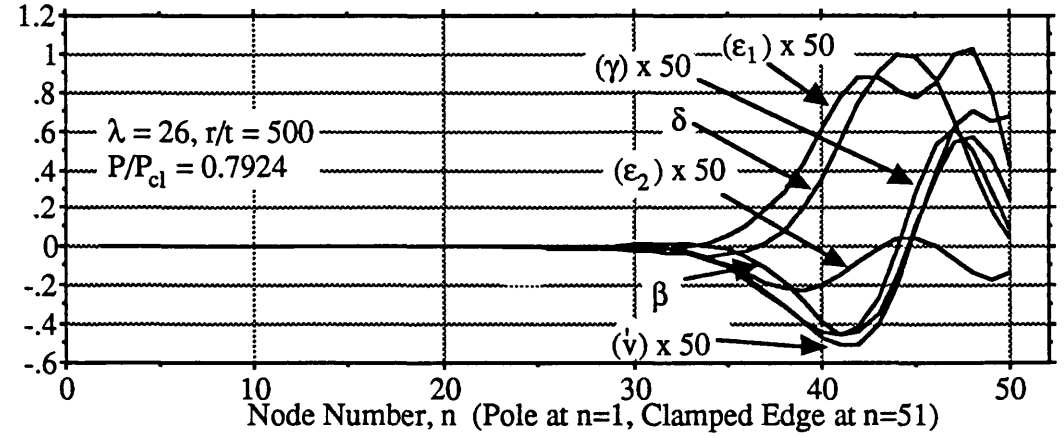
Figure 6.7 Secondary Path Initial Displacements, Derivatives, and Strain Components at the Point of Bifurcation into the Critical Periodic Mode, i = 2, for $\lambda = 6$.



(a) Normalised Displacements



(b) Normalised Derivatives



(c) Normalised Strain Components

Figure 6.8 Secondary Path Initial Displacements, Derivatives, and Strain Components at the Point of Bifurcation into the Critical Periodic Mode, i = 18, for $\lambda = 26$.

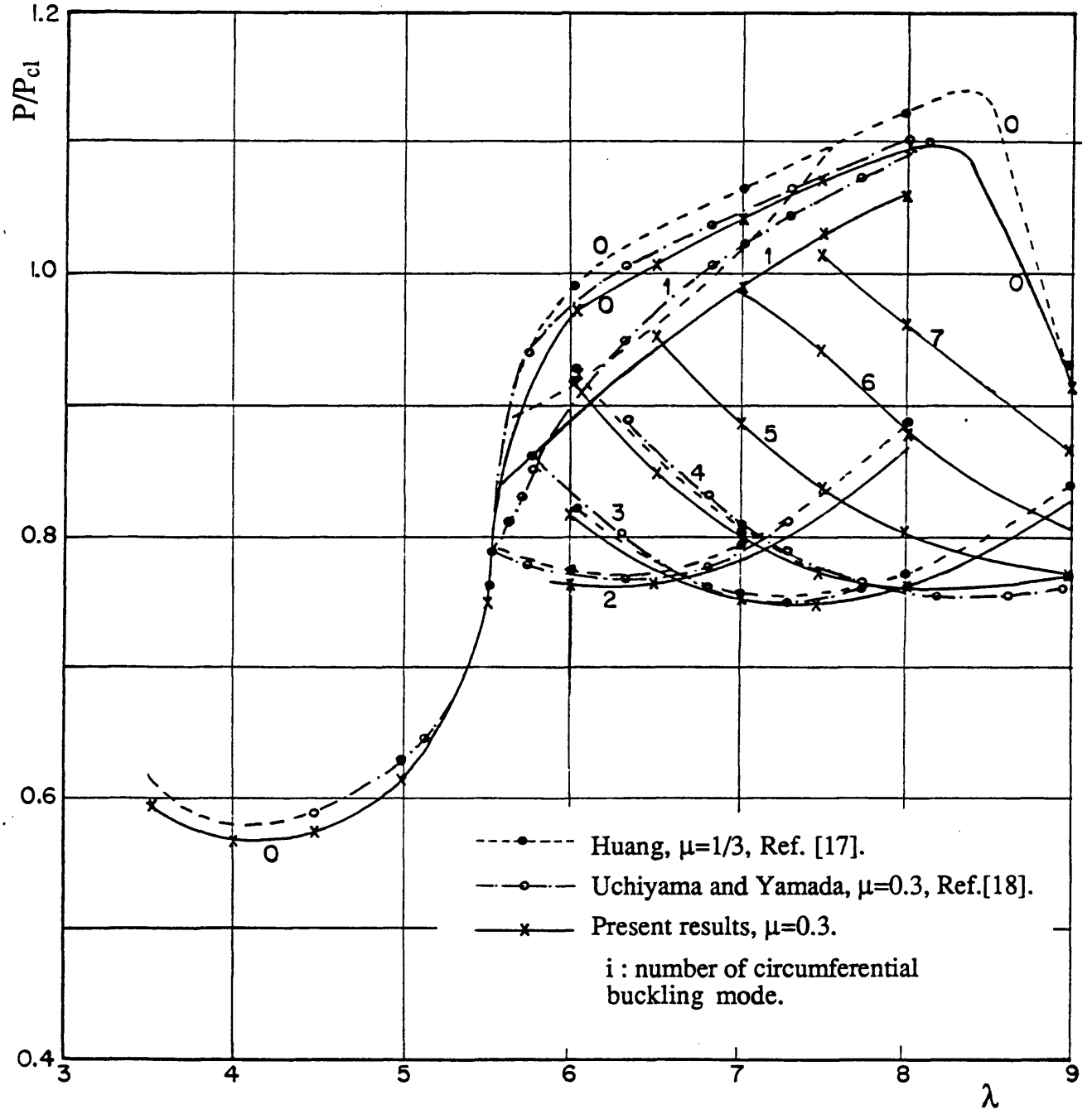


Figure 6.9 Comparisons of Axisymmetric and Periodic Buckling Pressures for Clamped Perfect Spherical Caps with Shallow Shell Theory Results, References [17] and [18]

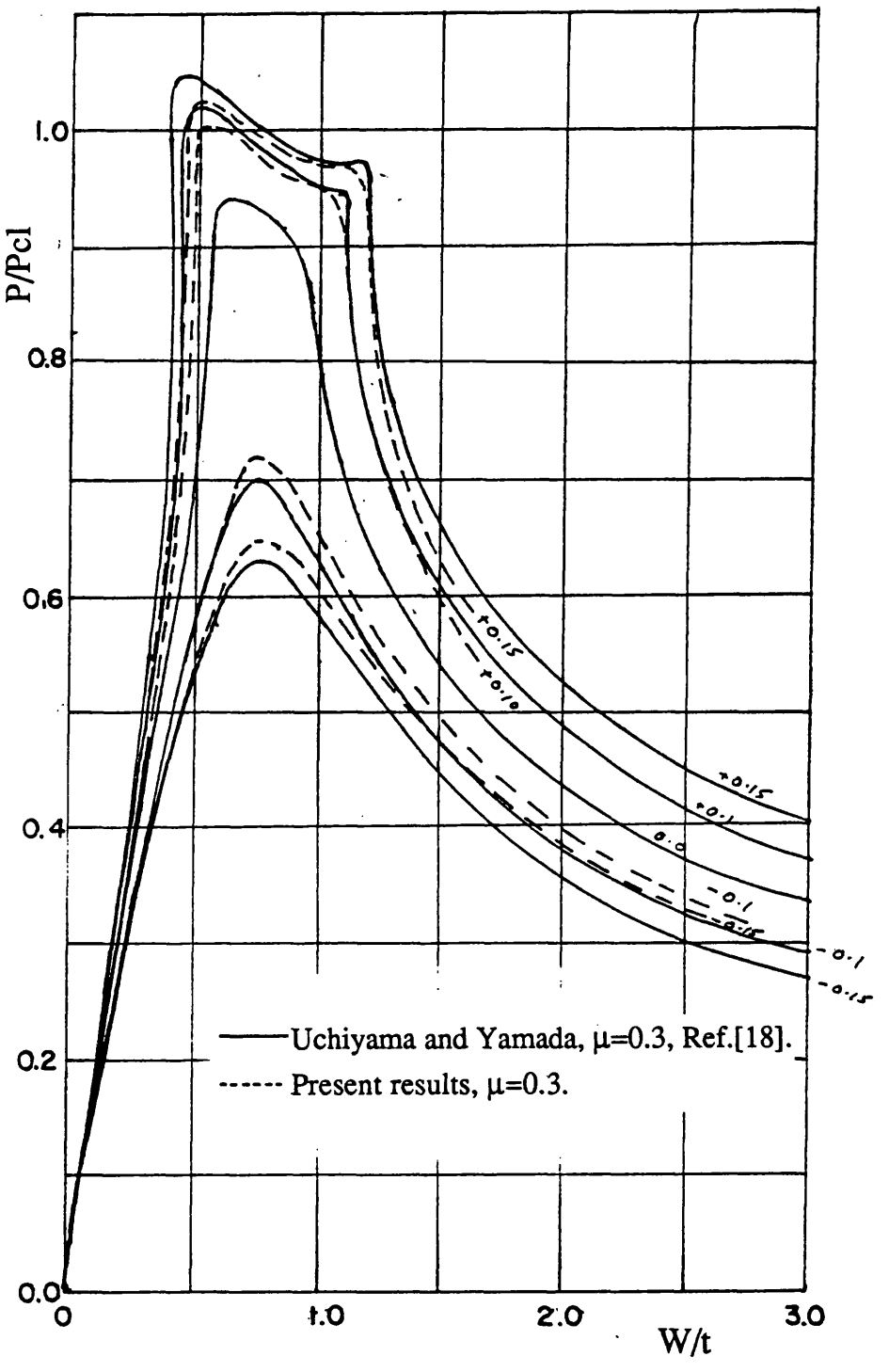


Figure 6.10 Comparisons of Nonlinear Fundamental Path for Clamped Imperfect Spherical Cap, λ=5.75, with Results from Shallow Shell Theory, Reference [18], Imperfection Amplitude is Indicated, see equation (6.2).

CHAPTER SEVEN

PRESENTATION OF RESULTS

CHAPTER SEVEN

CONTENTS

PRESENTATION OF RESULTS

7.1 INTRODUCTION

In this Chapter results from the computer program NLSPHERE, described in Chapters Five and validated in Chapter Six, are presented for perfect and imperfect spherical caps clamped at the boundary. Before the main results are presented in Section 7.3 and Section 7.4, results from different shells with the same value of the geometric slenderness or Bartdoff parameter, λ, are compared in Section 7.2. As a result of the comparisons presented in Section 7.2 the geometry of the spherical caps are described by the single slenderness or Bartdoff parameter λ in the remainder of the Chapter.

In Section 7.3 detailed results are presented for six perfect spherical caps with slenderness values of 3.5, 4, 6, 9, 12, and 30. And general results are given for perfect spherical caps with slenderness values between 3.5 and 60.

Section 7.4 presents detailed results for five spherical caps with initial axisymmetric imperfections and slenderness values of 4, 6, 9, 12, and 30. The initial axisymmetric imperfection mode that is used in the results presented in this Chapter is the buckling mode that results from a linear membrane fundamental path problem with boundary constraint imposed at the axisymmetric or periodic buckling point. This results in a linear eigenvalue problem, the eigenvector associated with the smallest eigenvalue is in all cases axisymmetric, and the radial component of the eigenvector associated with the smallest eigenvalue is used as the initial stress-free imperfection mode shape.

7.2 THE GEOMETRIC SLENDERNESS OR BARTDOFF PARAMETER λ

The geometric Bartdoff parameter λ for isotropic spherical shells is given by,

$$\lambda = \sqrt[4]{12(1-\mu^2)}\,\sqrt{\frac{r}{t}}\,\alpha \qquad (7.1)$$

where

μ = poissons ratio

r = radius of the shell

t = the thickness of the shell

α = the open angle of half the shell, for a complete sphere $\alpha = \pi$

The geometric parameter λ is proportional to the ratio of the arc length, ℓ ($\ell = \alpha r$) to the characteristic shell length ($\sqrt{rt}$), and by analogy with cylindrical shells is often referred to as the slenderness parameter. For complete spherical shells $\alpha = \pi$ and the slenderness parameter λ is proportional to $\sqrt{r/t}$. Also the order of the Legendre polynomial, N of the critical axisymmetric buckling mode is given by

$$N(N+1) \approx \left(\frac{\lambda}{\pi}\right)^2 = \sqrt{12(1-\mu^2)}\,\frac{r}{t} \qquad (7.2)$$

The wavelength of the critical buckling mode, the critical wavelength, is inversely proportional to N for slender shells, large values of λ and N. The ratio of the characteristic shell length, $\sqrt{r/t}$, to the arc length, ℓ, may be expressed as

$$\frac{C}{\lambda} = \frac{\sqrt[4]{12(1-\mu^2)}}{\lambda} = \frac{\sqrt{rt}}{\ell} \qquad (7.3)$$

and this nondimensional parameter is indicated on many of the figures presented in this Chapter.

When shallow shell assumptions are used in the analysis of spherical caps the equilibrium equations governing the elastic behaviour of the shell may be written in a nondimensional form, which incorporates the shell geometry in terms of the single parameter λ. In the present work no attempt has been made in the derivation of the deep shell equations contained in Chapters Four and Five to express the shell geometry in terms of the single parameter λ. In the computer program NLSPHERE the geometry of the shell is specified by two parameters, the radius to thickness ratio r/t, and the open angle of the half shell α. For isotropic spherical caps, clamped at the boundary, the results of the nonlinear fundamental path and bifurcation analyses, when suitably nondimensionalised, were found to be dependent on λ alone.

For example Figure 7.1 (3 pages) compares results for two different shells with the same slenderness value of λ=12. These shells were specified by r/t=100 and α=0.660 which does not satisfy the shallow shell conditions, and r/t=500 and α=0.295 which does satisfy the shallow shell conditions.

In Figure 7.1 (a), (b) and (c) the nondimensional load, P/P_{cl}, is plotted against the perturbation parameter, e, the relative change in volume, $\Delta V/V$, and the displacement at the pole, δ, respectively. In the solution of the perturbation problem, Section 5.3 of Chapter Five, the value of $p_{1,1}$ is arbitrary. In the computer program NLSPHERE the first term of the perturbation series for the load, $p_{1,1}$, was set equal to the classical buckling pressure for the complete sphere, consequently the linear fundamental path solution, the solution for the complete sphere, would in all cases pass through the origin and the point with coordinates (1,1) in Figure 7.1 (a). Figure 7.1 (b) again shows the fundamental path, the load is plotted against the change in volume of the spherical cap, where the change in volume, ΔV, is normalised by the initial volume, V, of the spherical cap, and this is not the 'correct' nondimensionalisation. It may be shown that the 'correct' nondimensionalisation will result when the change in volume of the spherical cap is normalised by the change in volume of the complete sphere at the buckling load. The fundamental path is shown once again in Figure 7.1 (c), in this case the nondimensional load has been plotted against the displacement at the pole, δ, normalised by the shell thickness. For the complete spherical shell the radial contraction is proportional to the load and the contraction at the point of buckling is given by equation (3.5) of Chapter Three, for Poisson's ratio of 0.3 the radial contraction at buckling is equal to 42.4% of the shell thickness. The linear fundamental path solution, the solution for the complete sphere, would in all cases pass through the origin and the point with coordinates (0.424,1) in Figure 7.1 (c).

The difference in the axisymmetric buckling pressures ($P_{bk}/P_{cl}\approx$0.960) for the two shells is 0.11% which is within the limits of accuracy of the program NLSPHERE discussed in Chapter Six.

The radial, W, and inplane, U, displacements normalised by the shell thickness at the point of buckling are shown in Figure 7.1 (d) and (e) respectively for the two spherical caps. The radial displacements, Figure 7.1 (d), agree within the accuracy of the program, however there is a factor of $\sqrt{5}$ between the inplane displacements, Figure 7.1 (e). The 'correct' length to use in normalising the inplane, U, displacements is the arc length, $\ell=r\alpha$, of the spherical cap.

The difference in the critical periodic buckling pressures ($P_{cr}/P_{cl}\approx$0.772, i_{cr}=7) for the two shells is approximately 0.13%, which is within the limits of accuracy of the program. Figure 7.1 (f) and (g) show the meridional, σ_ϕ, and circumferential, σ_θ, stress profiles at the critical load, $P_{cr}/P_{cl}\approx$0.772, and at the axisymmetric buckling load, $P_{bk}/P_{cl}\approx$0.960, and these stresses are normalised by the classical buckling stress, σ_{cl}, for the complete spherical shell. It is of

interest to note that the critical bifurcation load, P_{cr}/P_{cl}≈0.772, occurs when the circumferential stress, σ_θ, first reaches the classical value, σ_{cl}. Comparisons of the critical meridional mode shape, i_{cr}=7, that result from the nonlinear eigenvalue problem have not been shown as the amplitude of the displacements is necessarily arbitrary.

When an axisymmetric initial imperfection with an amplitude of one tenth of the shell thickness is introduced into the analysis no periodic bifurcation from the nonlinear axisymmetric fundamental path occurs. This magnitude of initial imperfection reduces the axisymmetric buckling pressure for the two shells (λ=12, r/t=100 and r/t=500) to 0.596 of the classical pressure (P_{bk}/P_{cl}≈ 0.596), and the difference between the results obtained for the two shells is less than 0.3%.

The geometry of the spherical caps will be described by the single slenderness or Bartdoff parameter λ in this Chapter. As a point of interest a radius to thickness ratio of 500 has been used in all the cases presented below.

7.3 PRESSURE LOADED PERFECT SPHERICAL CAPS CLAMPED AT THE BOUNDARY

Periodic buckling occurs at a lower pressure than axisymmetric buckling for spherical caps with slenderness values, λ, of more than about 5.5, see Figure 6.9. In the detailed discussion and presentation of the results for perfect spherical caps below only the lowest bifurcation or critical load is mentioned. However as the slenderness increases, λ increases, the number of periodic bifurcation modes increase and the proximity of these periodic bifurcation modes also increases. The position of the points of bifurcation on the fundamental path of the dominant and subdominant periodic modes with circumferential mode numbers, i, between 16 and 33 are shown in Figure 7.2 for a perfect spherical cap with a slenderness value of 30.

In Figures 7.3 to 7.8, the nonlinear fundamental path, the fundamental path displacement profiles, the meridional and circumferential stress profiles, and the lowest, or critical, periodic buckling mode, eigenvector, are shown for caps with λ values of 3.5, 4, 6, 9, 12, and 30. In the graphs of the fundamental path radial displacement and stress profiles contained in Figures 7.3. to 7.8, the parameter 's' is the normalised arc length measured from the pole (s=0) to the clamped edge (s=1) of the spherical cap. To avoid confusion the value of the perturbation parameter, e, is used to indicate the location on the fundamental path to which the fundamental path displacement or stress profiles refer. The amplitude of the eigenvector resulting from the linearised eigenvalue problem is indeterminate, and the amplitude of the incremental mode shapes shown in Figures 7.5 to 7.8 have been normalised such that the maximum displacement is one.

Figure 7.3 λ=3.5

Figure 7.3 (a) shows the fundamental path, load is plotted against perturbation parameter, for a spherical cap with a λ value of 3.5. Axisymmetric buckling, a local maximum, occurs at e=1.35, when the load P_{bk}/P_{cl}=0.597, and a local minimum occurs at e=2.54, when the load P/P_{cl}=0.559. Figure 7.3 (b) and (c) show the radial, W, and inplane, U, displacement profiles at various positions along the fundamental path. For this spherical cap the rise height h=1.85t, and from inspection of Figure 7.3 (b) it may be seen that buckling, a local maximum, occurs at e=1.35 when the radial displacement at the pole $\delta \approx -1.1t$, and the local minimum occurs at e=2.54 when the radial displacement at the pole $\delta \approx -2.0t$, that is the shell turns 'inside out'. The meridional and circumferential stress profiles are shown in Figure 7.3 (d) and (e), the circumferential stress, σ_θ, at buckling, e=1.35, has a maximum value of approximately $1.1\sigma_{cl}$, Figure 7.3 (e). Both the meridional and circumferential stresses become tensile, negative in Figure 7.3 (d) and (e), near the pole for large values of the perturbation parameter, e.g. e=4.89. No periodic bifurcations from the fundamental path were found.

Figure 7.4, λ=4

The fundamental path for a spherical cap with a λ value of 4 is shown in Figure 7.4 (a) and (b), in (a) the load is plotted against perturbation parameter, and in (b) the load is plotted against the relative change in volume of the spherical cap. Axisymmetric buckling, a local maximum, occurs at e=1.24, when the load P_{bk}/P_{cl}=0.563, and a local minimum occurs at e=3.84, when the load P/P_{cl}=0.381. Figure 7.4 (c) and (d) show the radial, W, and inplane, U, displacement profiles at various positions along the fundamental path. For this spherical cap the rise height h=2.42t, and from inspection of Figure 7.4 (c) it may be seen that buckling occurs, a local maximum, at e=1.24 when the radial displacement at the pole $\delta\approx$-1.0t, and the local minimum occurs at e=3.84, when the radial displacement at the pole $\delta\approx$-3.1t. The behaviour of this spherical cap, λ=4, is similar to that described above for λ=3.5, the shell turns 'inside out' between the buckling load and the minimum load. The meridional and circumferential stress profiles are shown in Figure 7.4 (e) and (f), the circumferential stress, σ_θ, at buckling, e=1.24, has a maximum value of approximately $1.0\sigma_{cl}$, Figure 7.4 (f). Both the meridional and circumferential stresses become tensile near the pole for large values of the perturbation parameter. Once again no periodic bifurcations from the fundamental path occur.

Figure 7.5, λ=6

In Figure 7.5 (a), (b) and (c) the fundamental path for a spherical cap with a λ value of 6 is shown, in (a) the load is plotted against perturbation parameter, in (b) the load is plotted against the relative change in volume of the spherical cap, and in (c) the load is plotted against the radial displacement at the pole. Axisymmetric buckling occurs at e=1.58, when the load P_{bk}/P_{cl}=0.970. The radial, W, and inplane, U, displacement profiles at various positions along the fundamental path are shown in Figure 7.5 (d) and (e). From inspection of Figure 7.5 (c) and (d) it may be seen that axisymmetric buckling occurs, at e=1.58, when the radial displacement at the pole $\delta\approx$-0.5t, however at this load the maximum radial displacement is approximately -1.2t and occurs at $s\approx$0.46, Figure 7.5 (d).

For values of the perturbation parameter, e, of less than about 0.83 the behaviour of the spherical cap is fairly linear, while for values of the perturbation parameter greater than 0.83 the behaviour becomes highly nonlinear, Figure 7.5 (a), (b) and (c). Prior to buckling, 0.83<e<1.58, the radial displacement at $s\approx$0.46 grows rapidly with only small changes in the radial displacement at the pole. The radial displacement profile, Figure 7.5 (d), at buckling contains two points of inflection, j=2, and the radial displacement at the pole is a relative minimum. On the postbuckling part of the axisymmetric fundamental path, 1.58<e<3.33, the displacement mode shape changes, the radial displacements at the pole increase rapidly, Figure 7.5 (c) and (d), the radial displacement at the pole becomes the maximum j=1, while the radial

displacement at s≈0.46 decreases rapidly. This change in mode shape results in a decrease in the change in volume of the deformed shell, Figure 7.5 (b).

The meridional and circumferential stress profiles are shown in Figure 7.5 (f) and (g), the circumferential stress, σ_θ, at the axisymmetric buckling load, e=1.58, has a maximum value of approximately $1.82\sigma_{cl}$, Figure 7.5 (g).

Bifurcation from the axisymmetric fundamental path into periodic modes occurs before the axisymmetric buckling point is reached. The minimum or critical bifurcation is in the circumferential mode $i_{cr}=2$ at e=0.90 when the load $P_{cr}/P_{cl}=0.762$, and this occurs when the maximum value of the circumferential stress, σ_θ, is approximately equal to the classical value, σ_{cl}, Figure 7.5 (g). The shape of the critical mode is shown in Figure 7.5 (h).

Figure 7.6, $\lambda=9$

The fundamental path for a spherical cap with a λ value of 9 is shown in Figure 7.6 (a), (b) and (c). Axisymmetric buckling occurs at e=1.14, when the load $P_{bk}/P_{cl}=0.915$. The radial, W, and inplane, U, displacement profiles at various positions along the fundamental path are shown in Figure 7.6 (d) and (e). From Figure 7.6 (c) and (d) it may be seen that axisymmetric buckling occurs when the radial displacement at the pole $\delta\approx-0.6t$, however at this load the maximum radial displacement is approximately -0.75t and occurs at s≈0.65, Figure 7.6 (d).

The radial displacement profile, Figure 7.6 (d), at buckling contains three points of inflection, j=3, and the radial displacement at the pole is a relative maximum. The behaviour of the spherical cap is fairly linear for values of the perturbation parameter, e, of less than about 1.1, while for values of the perturbation parameter greater than 1.1 the behaviour becomes highly nonlinear, Figure 7.6 (a), (b) and (c). On the postbuckling part of the axisymmetric fundamental path, $1.14<e<9.26$, the displacement mode shape changes rapidly, the radial displacements at the pole increase, Figure 7.6 (c) and (d), and the radial displacement mode shape changes from j=3 to j=1. This change in mode shape results in a decrease in the change in volume of the deformed shell, Figure 7.6 (b), as the inward radial displacements at s≈0.65 change to outward displacements.

The meridional and circumferential stress profiles are shown in Figure 7.6 (f) and (g), the circumferential stress, σ_θ, at the axisymmetric buckling load, e=1.14, has a maximum value of approximately $1.75\sigma_{cl}$, Figure 7.6 (g).

Before the axisymmetric buckling point is reached bifurcation from the axisymmetric fundamental path into periodic modes occurs. The minimum or critical bifurcation is in the circumferential mode $i_{cr}=4$ at e=0.80 when the load $P_{cr}/P_{cl}=0.767$, and this occurs when the

maximum value, at s≈0.61, of the circumferential stress is approximately equal to the classical value, Figure 7.6 (g). The shape of the critical mode is shown in Figure 7.6 (h).

Figure 7.7, λ=12

Figure 7.7 (a), (b) and (c) show the fundamental path for a spherical cap with a λ value of 12. Axisymmetric buckling occurs at e=1.28, when the load P_{bk}/P_{cl}=0.959. Radial, W, and inplane, U, displacement profiles at various positions along the fundamental path are shown in Figure 7.7 (d) and (e). It may be seen from inspection of Figure 7.7 (c) and (d) that axisymmetric buckling occurs when the radial displacement at the pole δ≈-0.55t is a local minimum, however at this load the radial displacement profile has two local maxima of approximately -0.75t at s≈0.2 and at s≈0.74, Figure 7.7 (d).

For values of the perturbation parameter, e, of less than about 0.90 the behaviour of the spherical cap is fairly linear, while for values of the perturbation parameter greater than 0.90 the behaviour becomes highly nonlinear, Figure 7.7 (a), (b) and (c). Prior to buckling, 0.97<e<1.28, the radial displacements at s≈0.2 and s≈0.74 grow rapidly with only small changes in the radial displacement at the pole. The radial displacement profile, Figure 7.7 (d), at buckling contains four points of inflection, j=4, and the radial displacement at the pole is a relative minimum. On the postbuckling part of the axisymmetric fundamental path the displacement mode shape changes, the radial displacements at the pole increase from approximately -0.55t to approximately -0.91t as e increases from 1.28 to 1.94, Figure 7.7 (c) and (d), and the radial displacement at the pole becomes a local maximum j=3, while the radial displacements at s≈0.2 and s≈0.74 decrease. This change in mode shape results in a decrease in the change in volume of the deformed shell, Figure 7.7 (b).

The meridional and circumferential stress profiles are shown in Figure 7.7 (f) and (g), the circumferential stress, σ_θ, at the axisymmetric buckling load, e=1.28, has a maximum value of approximately $1.5\sigma_{cl}$, Figure 7.5 (g).

Bifurcation from the axisymmetric fundamental path into periodic modes occurs before the axisymmetric buckling point is reached. The critical bifurcation is in the circumferential mode i_{cr}=7 at e=0.82 when the load P_{cr}/P_{cl}=0.772, and this occurs when the maximum value of the circumferential stress, σ_θ, is approximately equal to the classical value, σ_{cl}, Figure 7.7 (g). The shape of the critical mode is shown in Figure 7.7 (h).

Figure 7.8, λ=30

The fundamental path for a spherical cap with a λ value of 30 is shown in Figure 7.8 (a), (b) and (c). Axisymmetric buckling occurs at e=1.05, when the load P_{bk}/P_{cl}=0.952. From Figure 7.8 (c) and (d) it may be seen that axisymmetric buckling occurs when the radial displacement

at the pole $\delta \approx -0.52t$, is a local maximum. The radial displacement profile, at the buckling load, has four other local maxima the largest of which is approximately -0.68t at $s \approx 0.90$, Figure 7.8 (d).

At buckling the radial displacement profile, Figure 7.8 (d), contains nine points of inflection, j=9, and the radial displacement at the pole is a relative maximum. The behaviour of the spherical cap is fairly linear for values of the perturbation parameter, e, of less than about 0.9, while for values of the perturbation parameter greater than 0.9 the behaviour becomes highly nonlinear, Figure 7.8 (a), (b) and (c). On the postbuckling part of the axisymmetric fundamental path the displacement mode shape changes, the radial displacements at the pole increase from approximately -0.52t to approximately -1.16t as e increases from 1.05 to 1.55, Figure 7.8 (c) and (d). The radial displacement mode shape also changes from j=9 at e=1.05 to j=7 at e=1.55, the radial displacement at the pole becoming the maximum displacement at e=1.55. This change in mode shape results in a decrease in the change in volume of the deformed shell, and the unloading path in Figure 7.8 (b) is virtually indistinguishable from the loading path.

The meridional and circumferential stress profiles are shown in Figure 7.8 (f) and (g), the circumferential stress at the axisymmetric buckling load, e=1.05, has a maximum value of approximately $1.38\sigma_{cl}$, Figure 7.8 (g).

Before the axisymmetric buckling point is reached bifurcation from the axisymmetric fundamental path into periodic modes occurs. The minimum or critical bifurcation is in the circumferential mode $i_{cr}=21$ at e=0.80 when the load $P_{cr}/P_{cl}=0.795$, and this occurs when the maximum value, at $s \approx 0.89$, of the circumferential stress is approximately equal to the classical value, Figure 7.8 (g). The shape of the critical mode is shown in Figure 7.8 (h).

Summary

The axisymmetric buckling and periodic bifurcation or critical pressures for clamped spherical caps where the slenderness parameter λ is in the range $3.5 \leq \lambda \leq 60$ are presented in Table 7.1(a) and (b) and graphically in Figure 7.9. When two values of the normalised axisymmetric buckling pressure are given in Table 7.1 this indicates that the fundamental path solution algorithm found a second (minimum) turning point, and this value is given as the second figure in Table 7.1. For example, when $\lambda=3.5$, see Figure 7.3, maxima and minima occur on the fundamental path at values of P/P_{cl} equal to 0.597 and 0.559 respectively. Also two values of the normalised periodic bifurcation pressure appearing in Table 7.1 for the same circumferential mode number, i, indicate that subdominant eigenvalues and vectors have been found by the computer program NLSPHERE. When subdominant eigenvalues have been found the load corresponding to the first bifurcation (lowest value of perturbation parameter) is given first. In

the case λ=15 and i=5, where the first value given in Table 7.1(a) is larger than the second, the second bifurcation occurs after the axisymmetric buckling point.

For spherical caps with values of the geometric parameter λ greater than about 5.5 bifurcation into periodic modes occurs at a lower pressure than axisymmetric buckling. The maximum radial displacement of the periodic critical mode and the maximum radial displacement of the axisymmetric fundamental path displacements at the critical load occur at approximately the same position on the shell. As λ increases the periodic buckling displacements affect a smaller area of the shell adjacent to the clamped boundary and the critical circumferential mode number i_{cr} increases, as may be seen from Figures 7.5 to 7.8. The number of periodic bifurcation modes, and their proximity to the critical mode also increases as λ increases. For a cap with a λ value of 30 some of these subcritical modes are shown in Figure 7.2.

For all spherical caps with λ values greater than 4 axisymmetric buckling, $4 \leq \lambda \leq 5.5$, or the lowest periodic bifurcation, $\lambda > 5.5$, occurs when the circumferential stress, σ_θ, first reaches the classical buckling stress, σ_{cl}, for the complete sphere with the same radius to thickness ratio.

For axisymmetric displacements j has been used to denote the meridional mode shape, and corresponds to the number of points of inflection ($j \geq 1$) occurring on the half shell ($0 \leq s \leq 1$), while N is reserved for the order of the Legendre polynomial of the buckling mode of the complete sphere. The values of j and N for clamped spherical caps and complete spheres respectively are also given in Figure 7.9, and are approximately equal for shells of equal slenderness, λ. Inspection of Figures 7.3 to 7.8 reveals that for axisymmetric displacements odd values of j occur when the displacement at the pole is a local maximum and even values of j are associated with a local minimum of the radial displacement at the pole. The undulating nature of the locus of axisymmetric buckling points in Figure 7.9 reflects the transition between odd and even values of j. Axisymmetric buckling pressures being locally less for odd values of j, and the odd j modes extend over a larger range of λ than the modes corresponding to even values of j.

Although the periodic post bifurcation paths have not been calculated in the present work, it may be of interest to speculate on the behaviour that they would reveal. For slender shells periodic bifurcation occurs before axisymmetric buckling and the critical mode shape affects the portion of the cap close to the boundary, the remainder of the cap being less affected by the initial critical bifurcation mode. However, if we consider the 'unaffected' part of the cap as a separate cap with a smaller slenderness value, λ, and 'softer' boundary conditions then it is reasonable to suppose that this part of the cap will have a lower buckling or critical load than the entire cap, see Figure 7.9. In this way it may be imagined that elastic buckling behaviour of slender spherical caps will start with periodic bifurcation near the boundary and then propagate towards the pole.

The terms of the total potential energy, V, that are independent, V_0, linearly dependent, V_1, and quadratically dependent, V_2, on the incremental displacements are given in Chapter Four, by equations (4.46) to (4.51), and the program evaluates these terms at various points along the fundamental path. In evaluating the energy contributions to V_1, the rate of change of the fundamental path displacement vector is used as the incremental vector, as all contributions to V_1 arising from periodic displacements are necessarily zero. The linearised eigenvalue problem that results from the application of the stationary condition to V_2 has axisymmetric solutions corresponding to the stationary points of the fundamental path, and periodic solutions when $\lambda>5.5$ corresponding to the points of bifurcation to a periodic secondary path from the axisymmetric fundamental path. The contributions to the quadratic term of the total potential energy, V_2, are evaluated for all eigenvalues and vectors, that is at both the points of axisymmetric 'snap' buckling and at the points of periodic bifurcation. In the sample output in Appendix B the evaluation of these energy contributions is shown.

For a spherical cap with a λ value of 12 the contributions to the quadratic terms of the total potential energy V_2 are plotted against the circumferential mode number, i, in Figure 7.10. As the sum of the quadratic energy contributions is zero and the amplitude of the eigenvector is indeterminate the quadratic energy contributions shown in Figure 7.10 have been normalised such that the total positive (stabilising) or negative (destabilising) energy is equal to one or minus one.

The destabilising energy contributions are due to the interaction between the axisymmetric fundamental path displacements and the periodic secondary path displacements. The stabilising contributions to the quadratic energy in all cases come from the terms that are functions of the secondary path displacements only, that is the positive energy is independent of the fundamental path displacements at the point of bifurcation. The quadratic components of the load potential energy J_{2L} given by equation (4.50) are not shown in Figure 7.10 as they are small by comparison to the other terms. Typically J_{2L} is responsible for less than 0.1% of the total destabilising energy.

The critical circumferential buckling mode, $i_{cr}=7$, occurs where the stabilising secondary path membrane and bending energies are equal in magnitude, as far as this is possible for integer values of i.

The stabilising and destabilising energy contributions to the circumferential membrane energy exhibit opposite trends, as the periodic buckling mode number, i, varies. At the critical buckling mode number, $i_{cr}=7$, the stabilising energy resulting from the secondary path circumferential membrane energy, $\overset{\prime}{N}_\theta\overset{\prime}{\varepsilon}_\theta$, is small and decreases with increasing values of i. However the interaction between the secondary path and fundamental path circumferential stresses and strains, $\overset{f\prime}{N}_\theta\overset{f\prime}{\varepsilon}_\theta$, $\overset{f}{N}_\theta\overset{\prime\prime}{\varepsilon}_\theta$ and $\overset{\prime\prime}{N}_\theta\overset{f}{\varepsilon}_\theta$, are responsible for approximately 50% of the destabilising energy at the critical buckling mode, and the relative magnitude of these energy

components increases with increasing i. This is a result of the interaction between the fundamental and secondary path displacement fields shown in Figures 7.5 to 7.8. Periodic bifurcation results in a release of this fundamental path circumferential membrane energy, which is present in the form of compressive strain energy.

The total bending energy ($U_{2B}= U_{2,B\phi} + U_{2,B\theta} + U_{2,B\theta\phi}$) is stabilising and increases when the periodic buckling mode number increases, this is almost entirely due to the increase in circumferential bending energy $U_{2,B\theta}$ which increases as i increases. All other energy contributions to both the stabilising and destabilising energy are either fairly insensitive to changes in the circumferential buckling mode number i, $\overset{f4}{N}_{\theta\phi}\overset{f1}{\varepsilon}_{\theta\phi}$ and $U_{2,B\theta\phi}$, or decrease as i increases, $\overset{f1}{N}_{\phi}\overset{f1}{\varepsilon}_{\phi} + \overset{f}{N}_{\phi}\overset{II}{\varepsilon}_{\phi} + \overset{II}{N}_{\phi}\overset{f}{\varepsilon}_{\phi}$, $\overset{I}{N}_{\theta\phi}\overset{I}{\varepsilon}_{\theta\phi}$, $\overset{I}{N}_{\phi}\overset{I}{\varepsilon}_{\phi}$, $\overset{I}{N}_{\theta}\overset{I}{\varepsilon}_{\theta}$, $U_{2,B\phi}$.

7.4 PRESSURE LOADED SPHERICAL CAPS CLAMPED AT THE BOUNDARY WITH AXISYMMETRIC INITIAL IMPERFECTIONS

The mode shape of the axisymmetric initial imperfection, the nonlinear fundamental path, the imperfection sensitivity, and the fundamental path radial displacement profiles and the meridional and circumferential stress profiles for the perfect spherical cap and for a cap with an axisymmetric imperfection with an amplitude of one tenth of the shell thickness, are compared for caps with λ values of 4, 6, 9, 12, and 30 in Figures 7.11 to 7.15 (two pages each). To avoid confusion the value of the perturbation parameter, e, is used to indicate the location on the fundamental path to which the fundamental path displacement or stress profiles refer.

A number of different initial axisymmetric imperfection mode shapes were tested, these included the buckling mode from the linear membrane fundamental path problem, the damped Legendre solution for a complete sphere of equivalent λ, the buckling mode resulting from the nonlinear solution of the perfect cap, the axisymmetric mode with the same meridional mode shape as the critical mode for the perfect cap, and the mode shape that results from a uniform radial edge displacement of the cap. Comparisons of the results for the different types of initial imperfections showed that the imperfection sensitivity was greatest, and almost identical, for the first two types of imperfection mentioned above, the buckling mode from the linear membrane fundamental path problem, and the damped Legendre mode for a complete sphere of equivalent λ. Results will be presented below for the initial axisymmetric imperfections that have the same radial shape as the buckling mode of the linear membrane fundamental path problem. That is a linear membrane fundamental path solution, radial displacements only, was used with boundary constraint, fully clamped, imposed at the axisymmetric or periodic buckling point, resulting in a linear eigenvalue problem; the eigenvector associated with the smallest eigenvalue is in all cases axisymmetric, and the radial component of the eigenvector associated with the smallest eigenvalue is the imperfection mode shape used below. In the computer program NLSPHERE these initial imperfections were treated as stress-free imperfections as described in Chapter Four by equations (4.9) to (4.11).

In all cases when the amplitude of the initial imperfection was greater than one tenth of the thickness of the spherical cap all periodic bifurcation was eliminated and the behaviour of the cap was entirely axisymmetric. In the results presented below no mention is made of periodic bifurcations from the imperfect axisymmetric fundamental path, as relatively small axisymmetric imperfections in the mode used are sufficient to eliminate any periodic behaviour.

Figure 7.11, λ=4

The stress-free initial imperfection mode shape, W^*, is shown in Figure 7.11 (a), for the spherical cap with a λ value of 4. The fundamental path is shown in Figure 7.11 (b), (c) and (d), in (b) the load is plotted against perturbation parameter, in (c) the load is plotted against the

relative change in volume of the spherical cap, and in (d) the load is plotted against the radial displacement at the pole, δ. The nondimensional amplitude of the initial imperfection with respect to the thickness of the shell, W^*/t, is indicated on Figure 7.11 (b), a negative amplitude indicates an inward initial displacement. Imperfection sensitivity, the locus of the maximum values or buckling loads on the fundamental path is shown in Figure 7.11 (e). Radial displacement profiles, and meridional and circumferential stress profiles at both the local maximum and minimum of the fundamental path, are shown in Figures 7.11 (f), (g), and (h) respectively.

For this spherical cap, λ=4, the rise height h=2.42t, and for all imperfections shown in Figure 7.11 the cap turns 'inside out' between the local maximum and minimum on the fundamental path, and the local minimum occurs for positive values of load. As the amplitude of the inward imperfection increases, the difference between the local maximum and minimum decreases, and for an imperfection amplitude of W^*/t=-0.3 no turning points were detected on the fundamental path, the maximum and minimum were completely eliminated and the load increased monotonically. The imperfection sensitivity shown in Figure 7.11 (e) represents the load at which buckling occurs, although in this case the spherical cap will support larger loads in the 'inside out' configuration.

Figure 7.12, λ=6

Figure 7.12 (a) shows the stress-free initial imperfection mode shape, W^*, for the spherical cap with a λ value of 6, and Figure 7.12 (b), (c) and (d) show the nonlinear fundamental path. The nondimensional amplitude of the initial imperfection with respect to the thickness of the shell, W^*/t, is indicated on Figure 7.12 (b), a negative amplitude indicates an inward initial displacement. Imperfection sensitivity, the locus of the maximum values or buckling loads on the fundamental path is shown in Figure 7.12 (e). The nonlinear fundamental path solution routine, in the program NLSPHERE, found no minimum value within the one hundred and sixty terms allowed for the fundamental path, and Figure 7.12 (b) suggests that the minimum value will occur for negative values of load. Radial displacement profiles, and meridional and circumferential stress profiles at the local maximum, e=1.58 for the perfect cap W^*/t=0.0, and e=0.89 for the imperfect cap W^*/t=-0.1, of the fundamental path and at the postbuckling points given by e=2.50, P/P_{cl}=0.893 for W^*/t=0.0 and e=2.37, P/P_{cl}=0.329 for W^*/t=-0.1 are shown in Figures 7.12 (f), (g), and (h).

For this spherical cap, λ=6, the rise height h=5.44t, and for all fundamental paths shown in Figure 7.12 no minimum load has been found, and the cap has not yet turned 'inside out', i.e. δ<5.44t in Figure 7.12 (d). For the perfect cap periodic buckling occurs before axisymmetric buckling, when the circumferential stress first reaches the classical value for the complete spherical shell. For the imperfect spherical cap with an imperfection amplitude of -0.1t the maximum meridional stress at buckling is approximately $1.1\sigma_{cl}$ at s≈0.28, and the maximum

circumferential stress at buckling is approximately $0.9\sigma_{cl}$ at the pole, s=0, Figure 7.12 (g) and (h). The imperfection sensitivity is shown in Figure 7.12 (e), the axisymmetric buckling load is reduced by approximately 60% for an inwards imperfection with an amplitude of 30% of the shell thickness.

Figure 7.13, λ=9

For the spherical cap with a λ value of 9, Figure 7.13 (a) shows the stress-free initial imperfection mode shape, W*, and Figure 7.13 (b), (c), and (d) show the nonlinear fundamental path. The nondimensional amplitude of the initial imperfection is indicated on Figure 7.13 (b). Figure 7.13 (e) shows the imperfection sensitivity for this value of the slenderness parameter, λ=9. The nonlinear fundamental path solution routine, in the program NLSPHERE, found no minimum values of load although negative loads were obtained for large values of the perturbation parameter, e, in a number of cases. Radial displacement profiles, and meridional and circumferential stress profiles at the local maximum, e=1.14 for the perfect cap W*/t=0.0, and e=0.86 for the imperfect cap W*/t=-0.1, on the fundamental path and at the postbuckling points given by e=3.09, P/P_{cl}=0.549 for W*/t=0.0 and e=1.56, P/P_{cl}=0.441 for W*/t=-0.1 are shown in Figures 7.13 (f), (g), and (h).

For the imperfect spherical cap with an imperfection amplitude of -0.1t the maximum meridional stress at buckling is approximately $1.25\sigma_{cl}$ at s≈0.18 and the maximum circumferential stress at buckling is approximately $0.85\sigma_{cl}$ at s≈0.12, Figure 7.13 (g) and (h). The imperfection sensitivity is shown in Figure 7.13 (e), the axisymmetric buckling load is reduced by approximately 55% for an inwards imperfection with an amplitude of 30% of the shell thickness.

Figure 7.14, λ=12

Figure 7.14 (a) shows the stress-free initial imperfection mode shape, W*, and Figure 7.14 (b), (c), and (d) show the nonlinear fundamental path for the spherical cap with a λ value of 12, and the amplitude of the initial imperfection is indicated on Figure 7.14 (b). Figure 7.14 (e) shows the imperfection sensitivity. The nonlinear fundamental path solution routine found no minimum values of load although in a number of cases negative loads were obtained for large values of the perturbation parameter. Radial displacement profiles, and meridional and circumferential stress profiles at the local maximum, e=1.28 for the perfect cap W*/t=0.0, and e=0.81 for the imperfect cap W*/t=-0.1, on the fundamental path and at the postbuckling points given by e=1.94, P/P_{cl}=0.786 for W*/t=0.0 and e=2.37, P/P_{cl}=0.316 for W*/t=-0.1 are shown in Figures 7.14 (f), (g), and (h).

The imperfect spherical cap with an imperfection amplitude of -0.1t buckles axisymmetrically when the maximum meridional stress is approximately $1.20\sigma_{cl}$ at s≈0.13 and the maximum

circumferential stress is approximately $0.80\sigma_{cl}$ at s≈0.08, Figure 7.14 (g) and (h). The imperfection sensitivity is shown in Figure 7.14 (e). The axisymmetric buckling load is reduced by approximately 60% for an inwards imperfection with an amplitude of 30% of the shell thickness.

Figure 7.15, λ=30

The stress-free initial imperfection mode shape, W°, is shown in Figure 7.15 (a). Figure 7.15 (b), (c), and (d) show the nonlinear fundamental path, and the amplitude of the initial imperfection is indicated on Figure 7.15 (b). The imperfection sensitivity is shown in Figure 7.14 (e). Once again the nonlinear fundamental path solution routine found no minimum values of load although in a number of cases negative loads were obtained for large values of the perturbation parameter. Figures 7.15 (f), (g), and (h) show the radial displacement profiles, and the meridional and circumferential stress profiles at the local maximum, e=1.05 for the perfect cap W°/t=0.0, and e=0.74 for the imperfect cap W°/t=-0.1, on the fundamental path and at the postbuckling points given by e=1.55, P/P_{cl}=0.682 for W°/t=0.0 and e=2.16, P/P_{cl}=0.321 for W°/t=-0.1.

The imperfect spherical cap with an imperfection amplitude of -0.1t buckles axisymmetrically when the maximum meridional stress is approximately $1.20\sigma_{cl}$ at s≈0.06 and the maximum circumferential stress is approximately $0.80\sigma_{cl}$ at 0<s<0.04, Figure 7.15 (g) and (h). The imperfection sensitivity is shown in Figure 7.15 (e). The axisymmetric buckling load is reduced by approximately 60% for an inwards imperfection with an amplitude of 30% of the shell thickness.

Summary

As mentioned above when the initial axisymmetric imperfection is inwards and of the type used above, and when the amplitude of the imperfection is greater than one tenth of the thickness of the spherical cap all periodic behaviour of the cap is eliminated. The spherical cap buckles axisymmetrically with the radial displacement at the pole 'leading' the buckle. In some, but not all, cases a small outward initial imperfection of the type used resulted in a small increase in the axisymmetric buckling load. For spherical caps with slenderness values, λ, greater than 6 the results presented above indicate that the postbuckling path becomes negative before the cap turns 'inside out'.

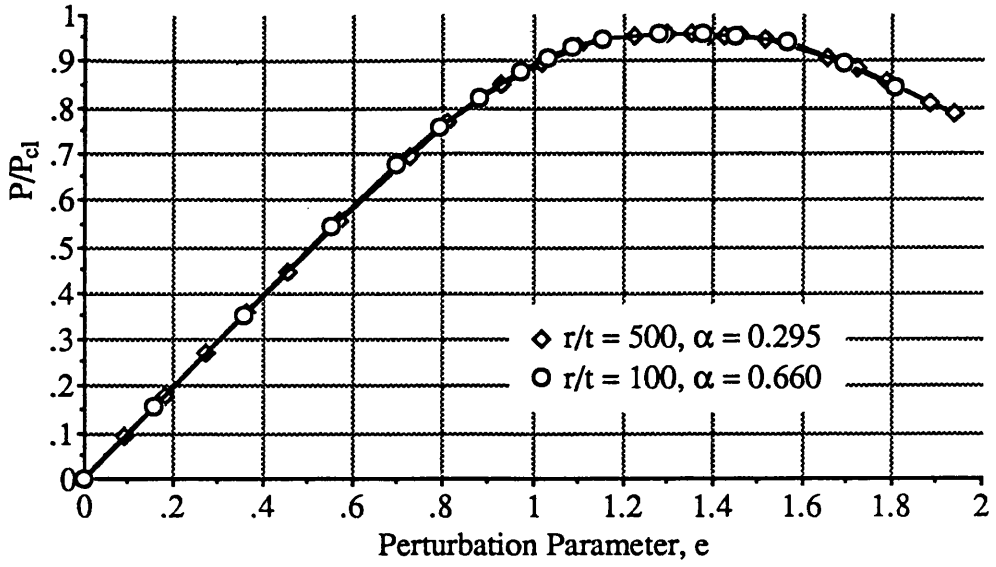
As a matter of interest the fundamental path solution algorithm implemented by the computer program NLSPHERE stops when the load reaches twice the classical buckling value, or when negative values of load are encountered, or when the one hundred and sixty terms allowed for fundamental path have all been utilised. In a large number of cases negative values of the load were encountered before the limit of one hundred and sixty terms was reached.

I \ λ	3·5	4	4·5	5	5·5	6	6·5	7	7·5	8	9	10	11	12	13	14	15	16	17	18
0	0·597 0·559	0·563 0·381	0·573	0·616	0·746	0·970	1·009	1·042	1·073	1·102	0·915	0·811	0·807	0·959	0·964 0·904	0·977 0·896	0·988	0·902	0·888	0·922
1						0·886	0·944 0·943	0·990 0·980	1·029 1·073	1·065 1·095										
2						0·762	0·760	0·780	0·819	0·873										
3						0·818	0·772	0·750	0·748	0·763	0·831									
4						0·920	0·853	0·804	0·773	0·758	0·767	0·803		0·884						
5							0·954	0·892	0·842	0·806	0·769	0·767	0·789	0·826	0·857	0·882 0·931	0·914 0·899			
6								0·988	0·954	0·880	0·809	0·772	0·767	0·786	0·812	0·838 0·968	0·862 0·934	0·881		
7									1·018	0·965	0·870	0·805	0·776	0·772	0·783	0·802	0·825	0·845	0·864	0·882 0·901
8										1·045			0·804	0·782	0·775	0·782	0·797	0·815	0·833	0·850 0·908
9														0·809	0·785	0·778	0·782	0·793	0·808	0·823
10														0·848	0·810	0·788	0·781	0·782	0·791	0·802
11														0·893	0·844	0·811	0·792	0·783	0·783	0·789
12														0·938	0·884	0·842	0·813	0·794	0·786	0·784
13															0·925	0·878	0·841	0·814	0·797	0·788
14															0·972	0·916	0·874	0·840	0·815	0·799
15															1·021	0·952	0·909	0·870	0·839	0·817
16															1·058	1·020	0·944		0·867	0·839
17																1·061				0·865
18																1·096				0·893

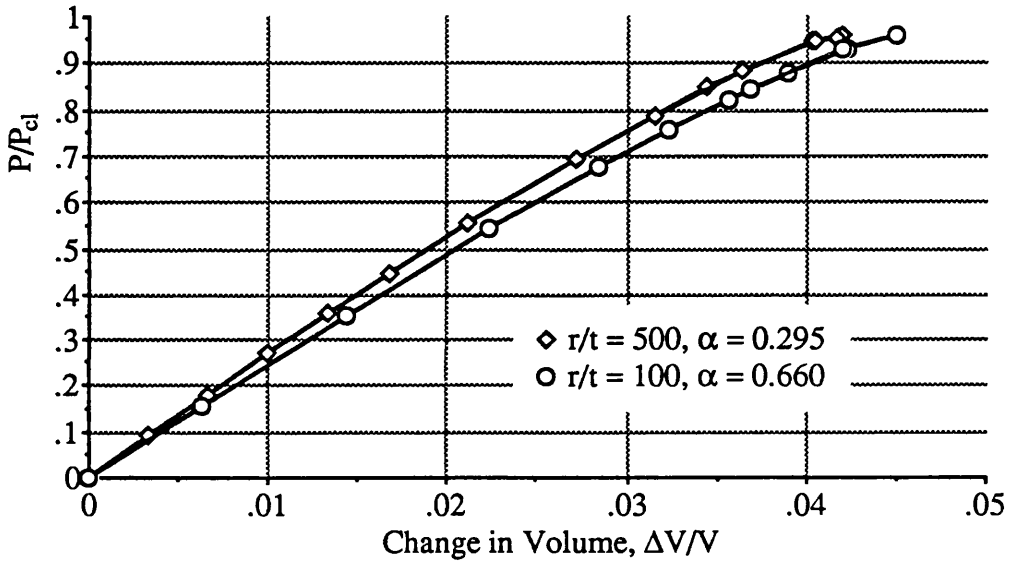
Table 7.1(a) Axisymmetric and Periodic Buckling Pressures for Clamped Perfect Spherical Caps, $3.5 \leq \lambda \leq 18$.

I \ λ	20	22	24	26	28	30	32	34	36	38	40	42·5	45	47·5	50	52·5	55	57·5	60
0	0·985 0·939	0·943	0·938	0·990 0·960	0·967	0·952	0·993 0·973	0·985	0·962	0·995 0·983	0·997	0·971	0·995	0·976	0·993 0·990	0·990	0·979	0·983	0·973
9	0·854 0·905	0·883 0·912																	
10	0·829 0·911	0·857 0·906																	
11	0·810 0·925	0·834 0·907	0·858 0·910																
12	0·796 0·943	0·816 0·914	0·838 0·907	0·859 0·915															
13	0·788 0·961	0·802 0·927	0·820 0·909	0·840 0·910															
14	0·787 0·976	0·793 0·942	0·807 0·916	0·824 0·909	0·842 0·913														
15	0·793 0·985	0·789	0·797 0·927	0·811 0·912	0·827 0·913														
16	0·804	0·791	0·792	0·801 0·919	0·814 0·910	0·828 0·912	0·843 0·918	0·857 0·950	0·870 0·952										
17	0·820	0·797	0·791	0·795 0·929	0·804 0·914	0·816 0·911	0·829 0·915	0·842 0·920	0·855 0·951	0·867 0·932									
18	0·840	0·808	0·794	0·792 0·941	0·798 0·921	0·807 0·912	0·818 0·912	0·830 0·916	0·842 0·951	0·853 0·952	0·863								
19	0·862	0·823	0·802	0·793 0·953	0·794 0·931	0·800 0·917	0·809 0·912	0·819 0·914	0·830 0·918	0·840 0·923	0·851 0·953								
20		0·841	0·812	0·798 0·965	0·794 0·941	0·796 0·924	0·802 0·914	0·810 0·913	0·820 0·915	0·830 0·919	0·839 0·923	0·850	0·860 0·956						
21		0·861	0·826	0·806 0·975	0·796 0·953	0·795 0·932	0·798 0·919	0·804 0·914	0·812 0·914	0·820 0·916	0·829 0·920	0·839 0·925	0·849 0·931						
22		0·883	0·843	0·817 0·983	0·802 0·963	0·796 0·942	0·796 0·926	0·799 0·917	0·805 0·914	0·812 0·915	0·820 0·917	0·829 0·922	0·839 0·954						
23		0·906	0·862	0·830 0·995	0·810	0·800	0·796 0·934	0·797 0·922	0·801 0·916	0·806 0·914	0·812 0·916	0·821 0·919	0·829 0·923	0·837 0·928					
24			0·882	0·845	0·821	0·806	0·799 0·943	0·797 0·929	0·798 0·919	0·802 0·915	0·806 0·915	0·814 0·917	0·821 0·920	0·829 0·924	0·836 0·928				
25			0·902	0·863	0·834	0·815	0·804 0·953	0·798 0·936	0·797 0·925	0·799 0·918	0·802 0·915	0·808 0·916	0·814 0·918	0·821 0·921	0·827 0·925		0·839 0·932		
26				0·881	0·849	0·826	0·811 0·962	0·802 0·945	0·798 0·931	0·798 0·922	0·800 0·917	0·804 0·916	0·809 0·917	0·815 0·919	0·820 0·922	0·826 0·925	0·831 0·928		
27					0·865	0·838	0·820 0·970	0·808 0·954	0·802 0·939	0·799 0·928	0·799 0·920	0·801 0·916	0·805 0·916	0·809 0·918	0·814 0·920	0·819 0·923	0·824 0·925	0·828 0·928	
28					0·882	0·852	0·831 0·978	0·816 0·962	0·806 0·947	0·801 0·934	0·799 0·925	0·799 0·919	0·802 0·916	0·805 0·917	0·809 0·918	0·814 0·921	0·818 0·923	0·821 0·925	0·824 0·927
29						0·867	0·843 0·984	0·825 0·970	0·813 0·955	0·806 0·942	0·801 0·931	0·800 0·922	0·800 0·918	0·803 0·917	0·806 0·917	0·809 0·919	0·812 0·921	0·816 0·923	0·818 0·924
30						0·883		0·836 0·977	0·821	0·811 0·949	0·805 0·937	0·801 0·927	0·800 0·920	0·801 0·918	0·803 0·917	0·806 0·918	0·809 0·919	0·811 0·921	0·814 0·922
31								0·848 0·983		0·819 0·957	0·810 0·944	0·805	0·802 0·924	0·801 0·920	0·802 0·918	0·803 0·918	0·806 0·918	0·808 0·919	0·810 0·920
32										0·827 0·964	0·817 0·952	0·809 0·938	0·804 0·929	0·802 0·923	0·802 0·920	0·802 0·918	0·804 0·918	0·805 0·918	0·807 0·919
33											0·825 0·959	0·814 0·945	0·808 0·934	0·804 0·925	0·803 0·922	0·802 0·919	0·803 0·918	0·804 0·918	0·805 0·918
34												0·821 0·952	0·813 0·940	0·808 0·931	0·805 0·925	0·804 0·922	0·803 0·920	0·804 0·919	0·805 0·918
35													0·819 0·947	0·812 0·937	0·808 0·929	0·806 0·925	0·805 0·922	0·805 0·920	0·805 0·919
36													0·826 0·953		0·813 0·934	0·809 0·928	0·807 0·924	0·806 0·922	0·806 0·920
37													0·835 0·959		0·818 0·940	0·814 0·933	0·811 0·928	0·809 0·924	0·809 0·922
38															0·825 0·945		0·815 0·932	0·813	0·812 0·925
39															0·832 0·951		0·821 0·937	0·818 0·932	0·816 0·928
40																	0·827 0·942		0·821 0·932
41																	0·834 0·947		0·827 0·936
42																			0·834 0·941
43																			0·841 0·946

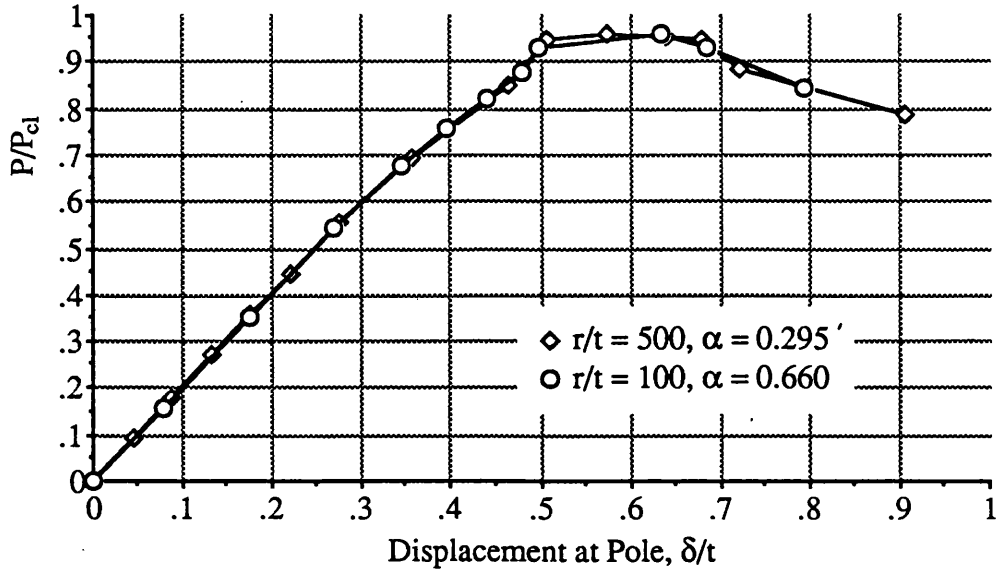
Table 7.1(b) Axisymmetric and Periodic Buckling Pressures for Clamped Perfect Spherical Caps, $20 \leq \lambda \leq 60$.



(a) Fundamental Path, P -vs- e

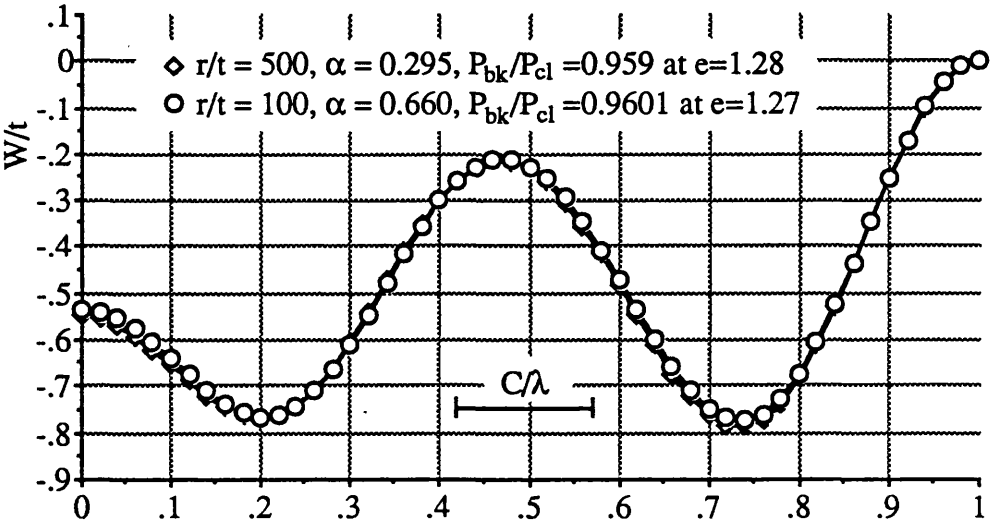


(b) Fundamental Path, P -vs- ΔV

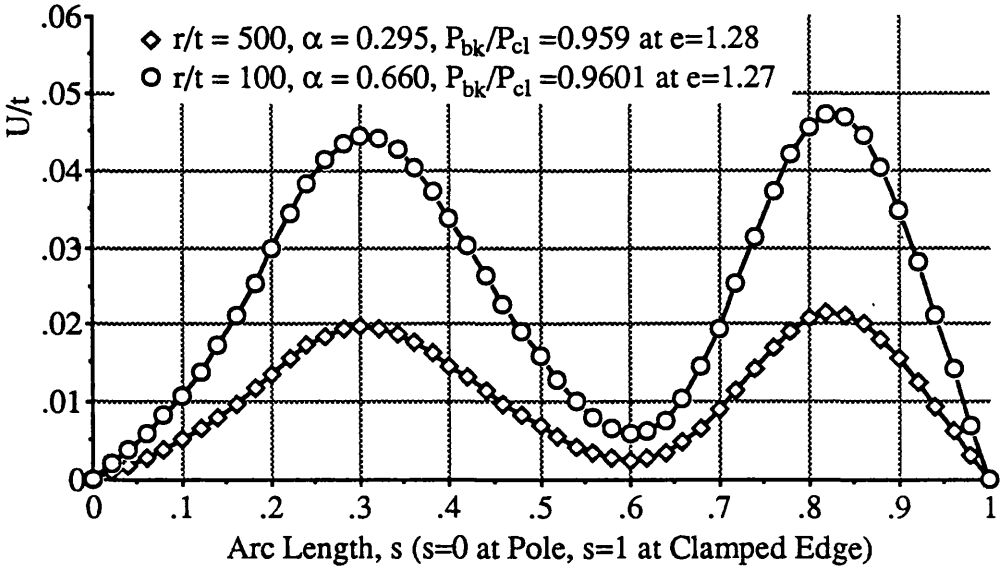


(c) Fundamental Path, P -vs- δ

Figure 7.1 Fundamental Path and Shell Profiles for Perfect Shells, λ=12, r/t=500 and r/t=100.

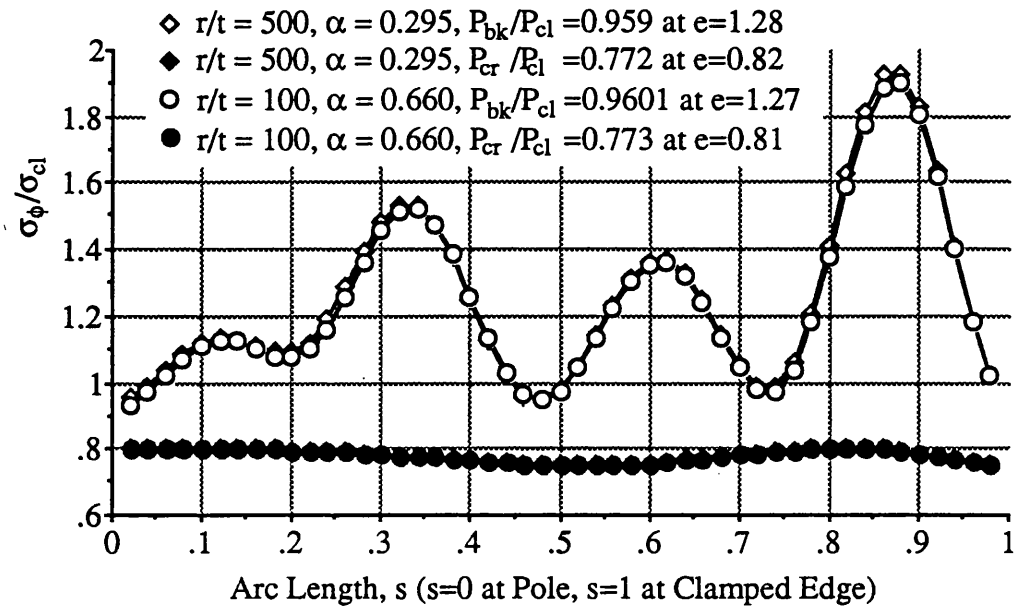


(d) Radial Displacement Profiles at Point of Buckling

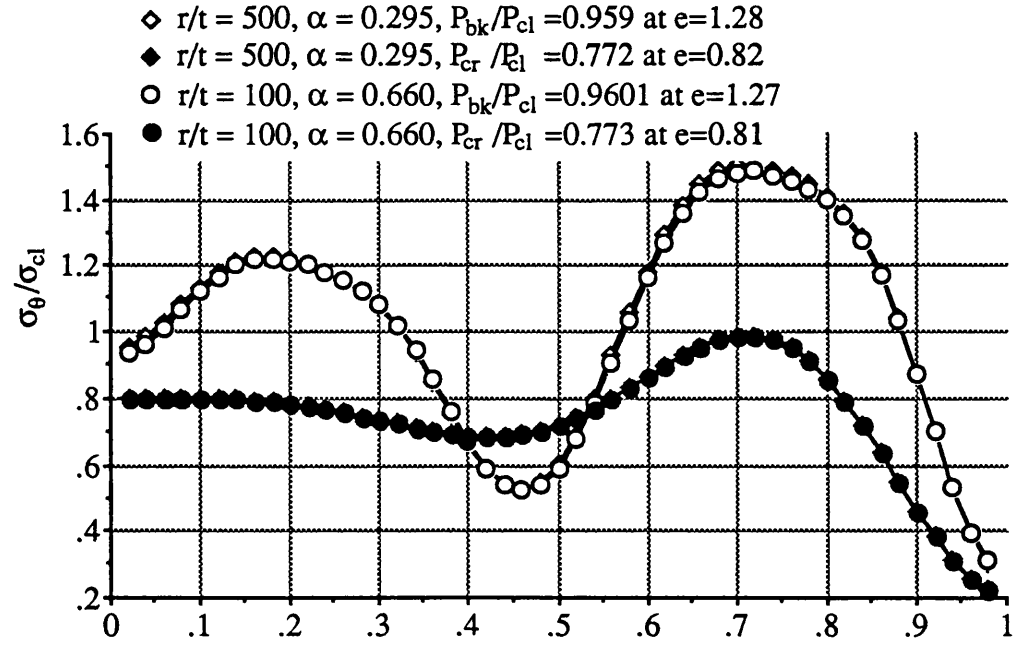


(e) Inplane Displacement Profiles at Point of Buckling

Figure 7.1 Fundamental Path and Shell Profiles for Perfect Shells, λ=12, r/t=500 and r/t=100.

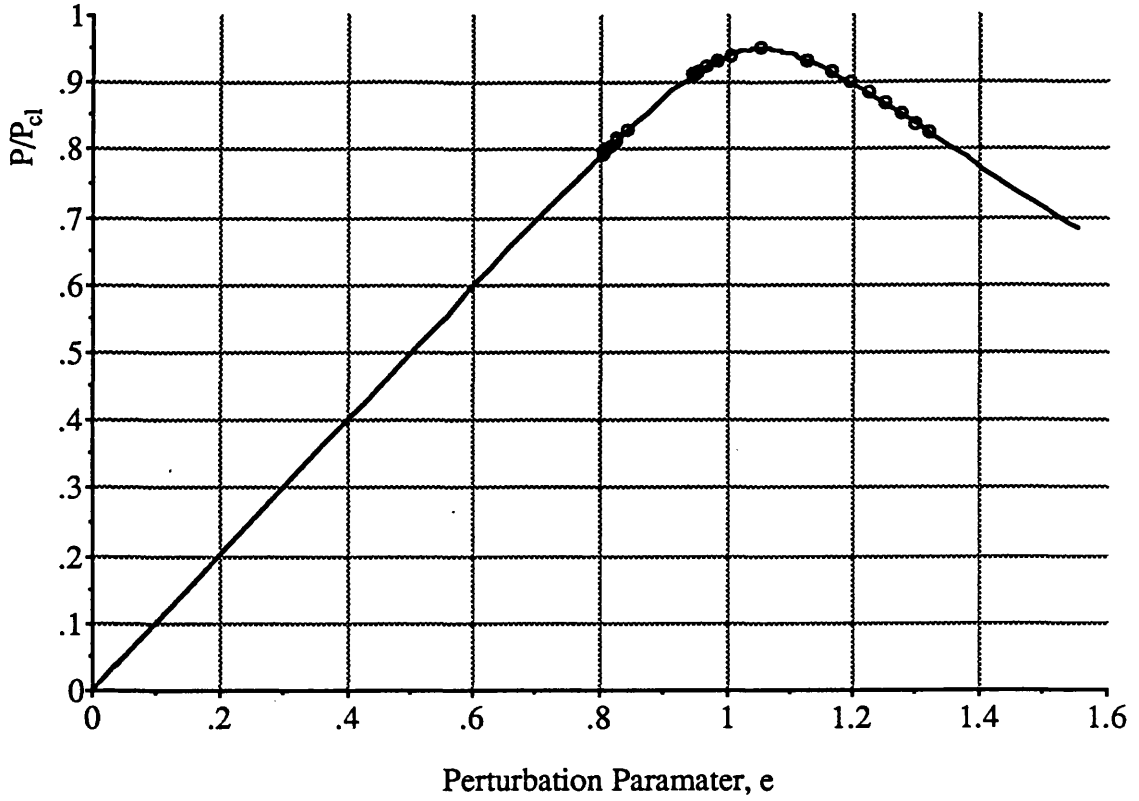


(f) Meridional Stress Profiles at Buckling Point and Critical Point

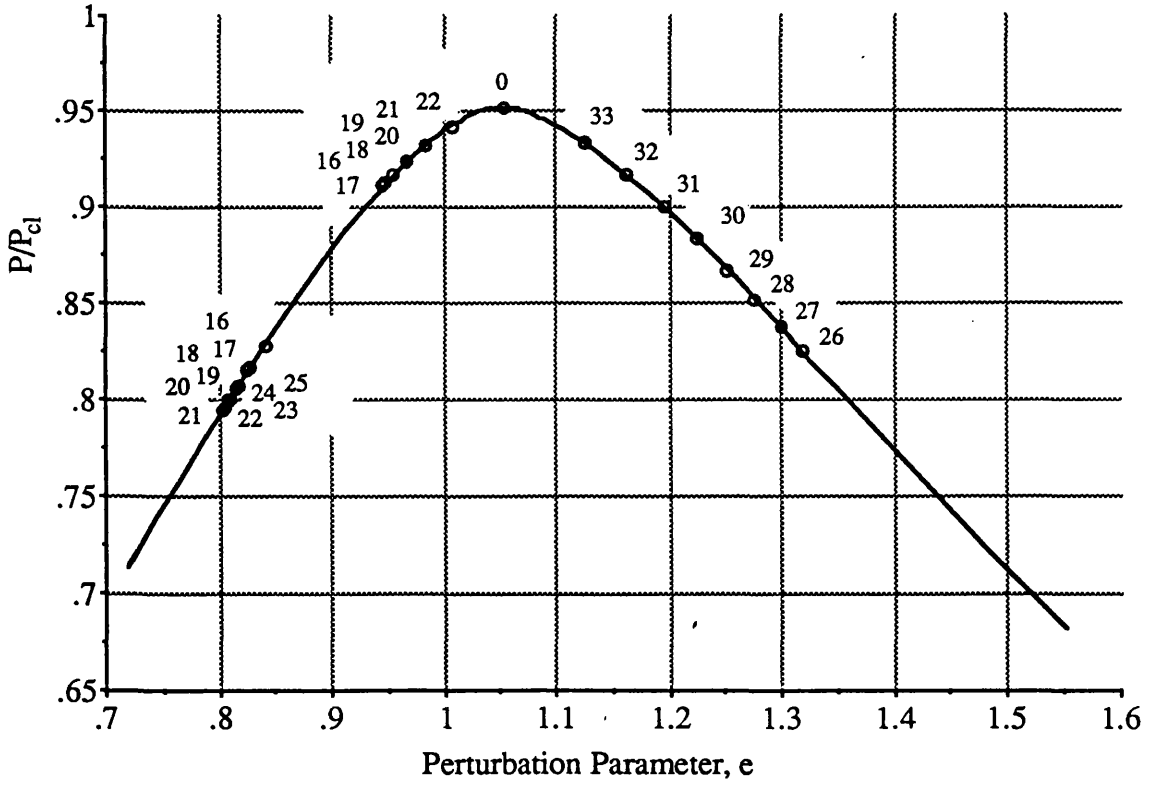


(g) Circumferential Stress Profiles at Buckling Point and Critical Point

Figure 7.1 Fundamental Path and Shell Profiles for Perfect Shells, λ=12, r/t=500 and r/t=100.

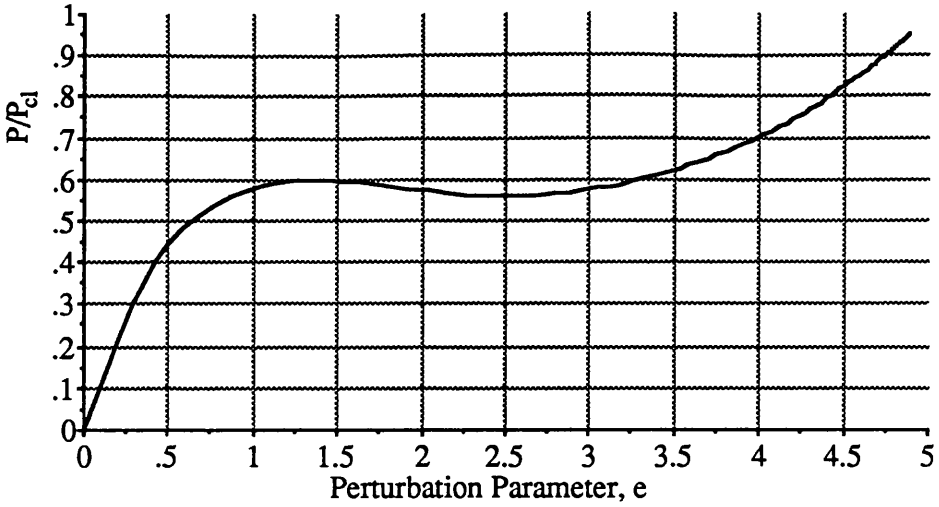


(a) Fundamental Path, P -vs- e, Showing Secondary Path Bifurcations

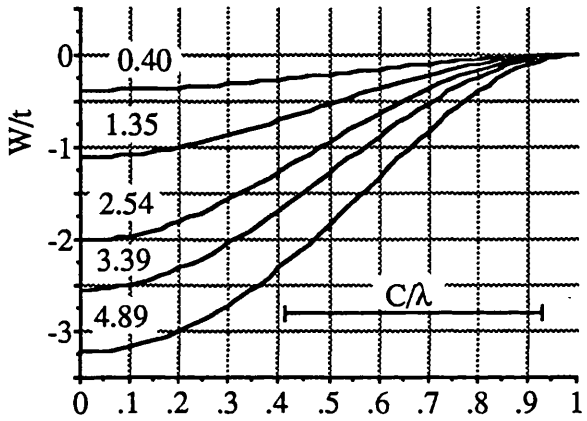


(b) Enlargement of Fundamental Path, P -vs- e, Showing Secondary Path Bifurcations

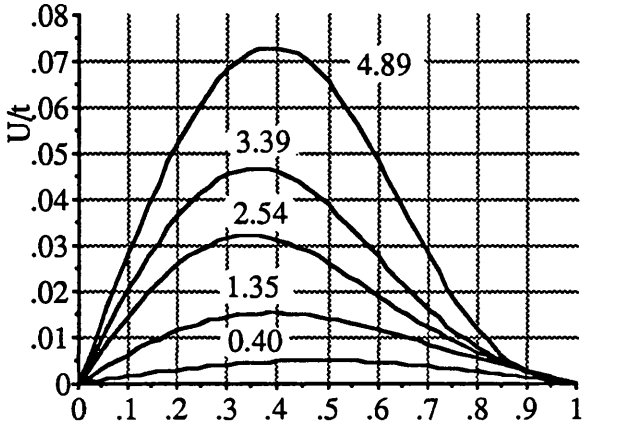
Figure 7.2 Fundamental Path and Secondary Path Bifurcations for Perfect Shell, λ=30, r/t=500. (on figure (b) the circumferential mode numbers, i, are indicated)



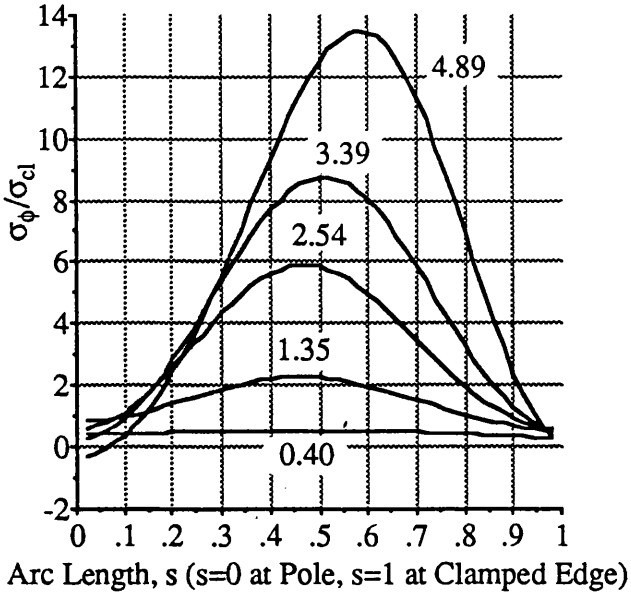
(a) Fundamental Path, P -vs- e



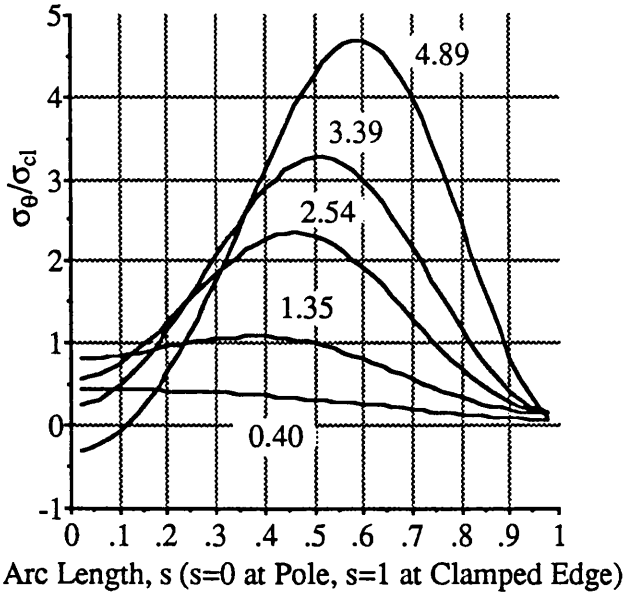
(b) Radial Displacement Profiles



(c) Inplane Displacement Profiles

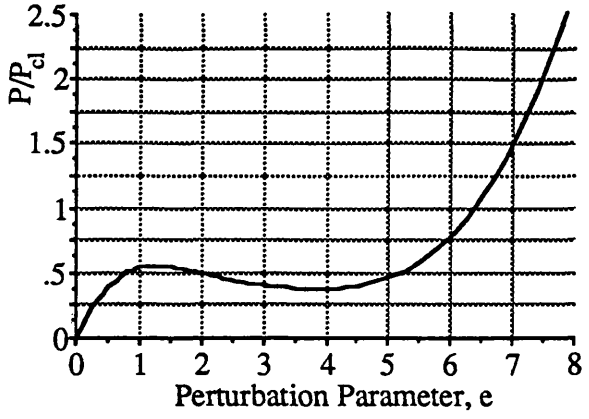


(d) Meridional Stress Profiles

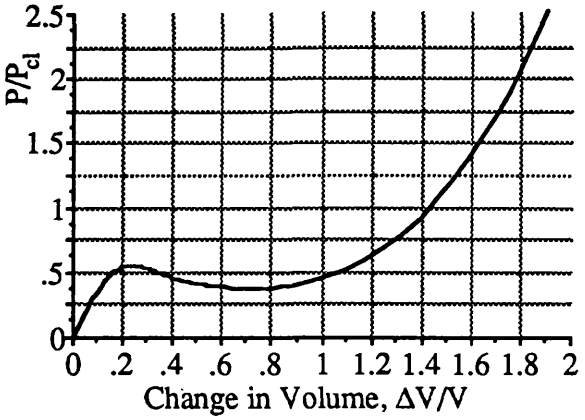


(e) Circumferential Stress Profiles

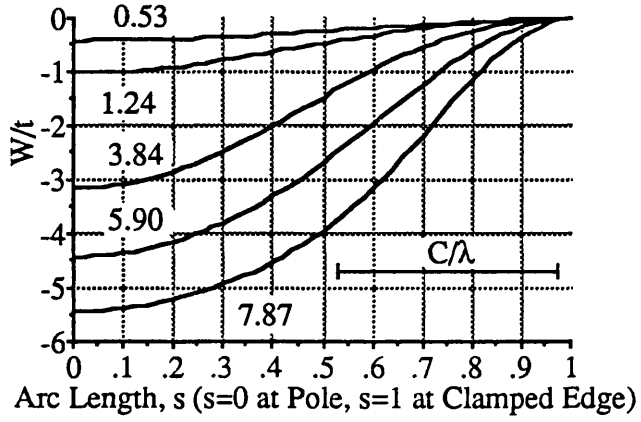
Figure 7.3 Fundamental Path and Shell Profiles for Perfect Shell, λ=3.5, r/t=500. P_{bk}/P_{cl} =0.597, at e=1.35



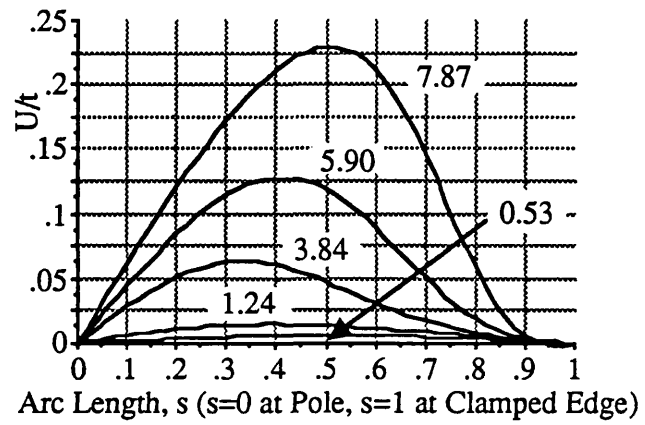
(a) Fundamental Path, P -vs- e



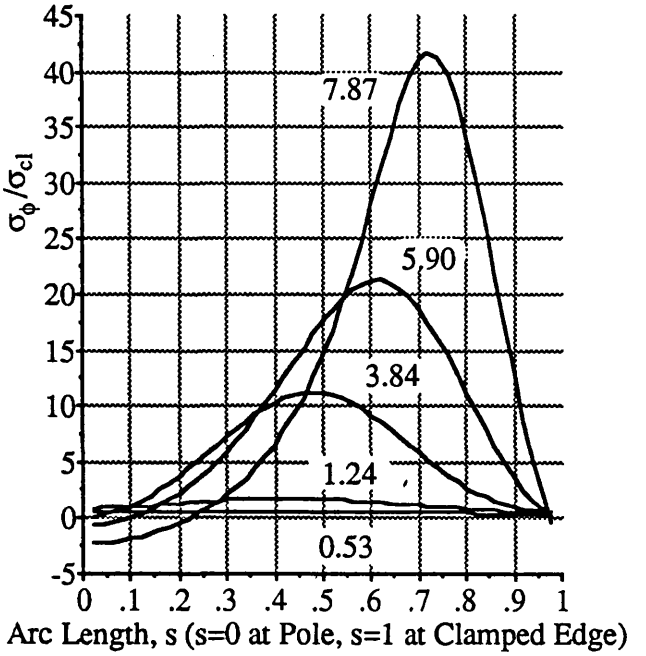
(b) Fundamental Path, P -vs- ΔV



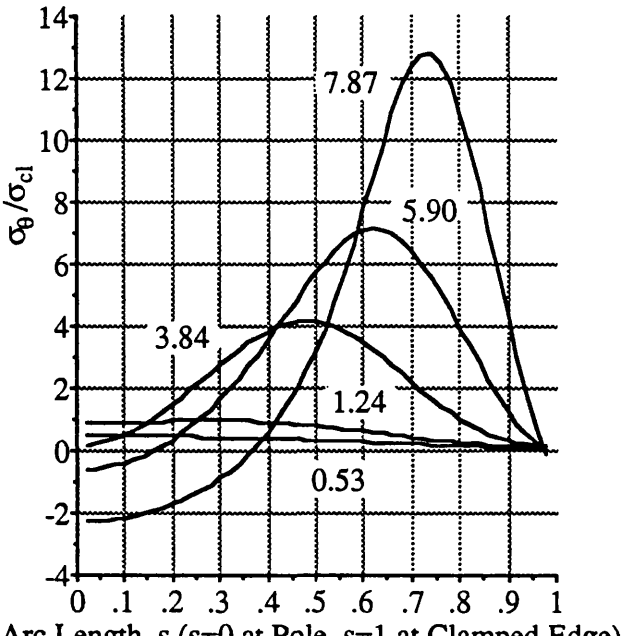
(c) Radial Displacement Profiles



(d) Inplane Displacement Profiles

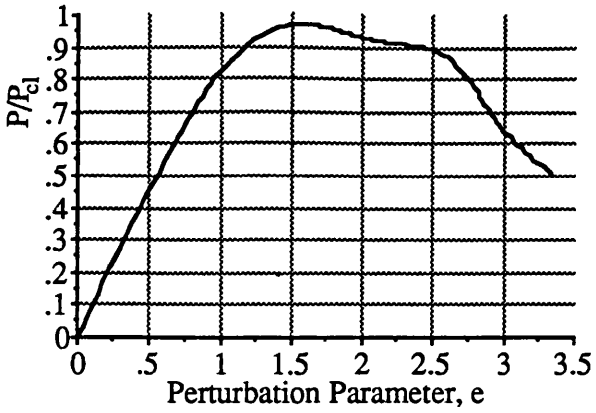


(e) Meridional Stress Profiles

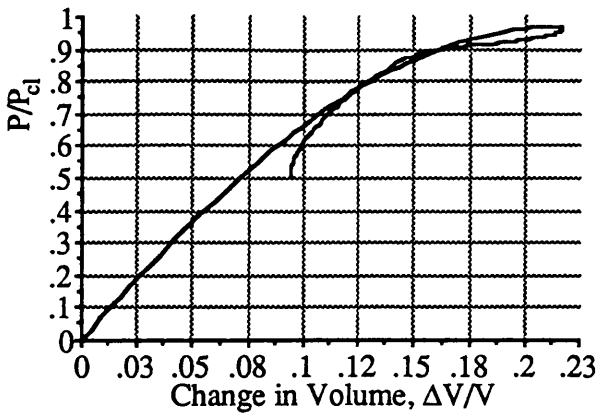


(f) Circumferential Stress Profiles

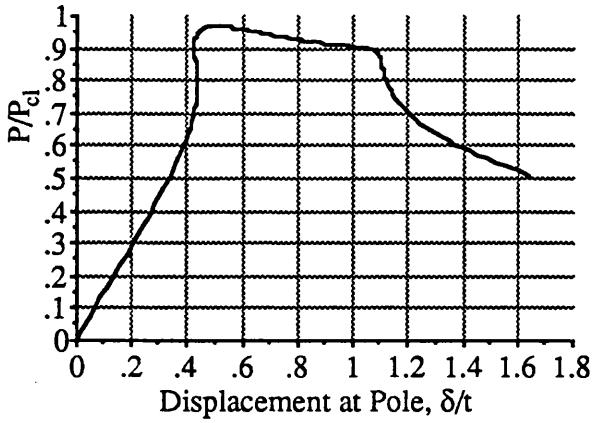
Figure 7.4 Fundamental Path and Shell Profiles for Perfect Shell, λ=4, r/t=500. P_{bk}/P_{cl} =0.563 at e=1.24.



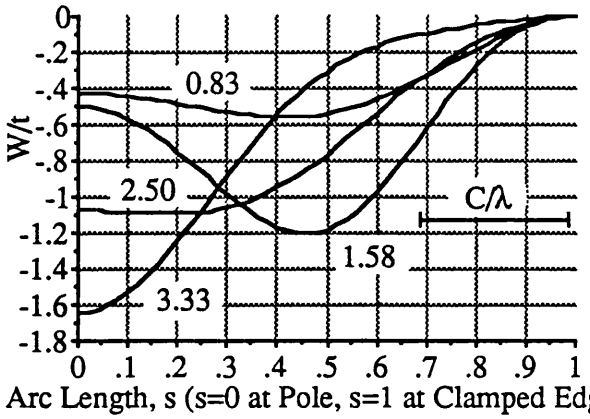
(a) Fundamental Path, P -vs- e



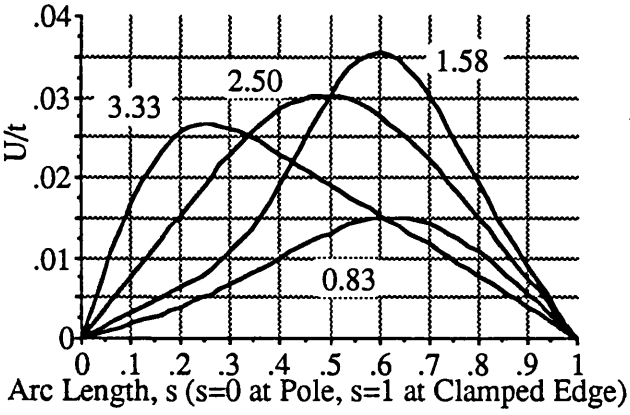
(b) Fundamental Path, P -vs- ΔV



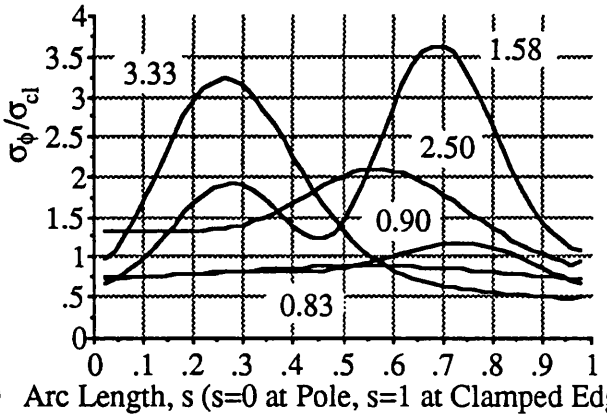
(c) Fundamental Path, P -vs- δ



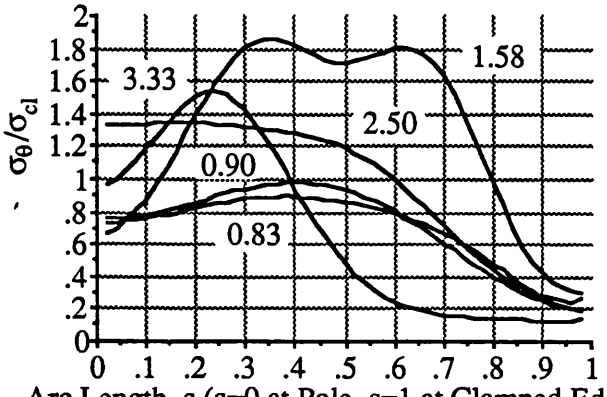
(d) Radial Displacement Profiles



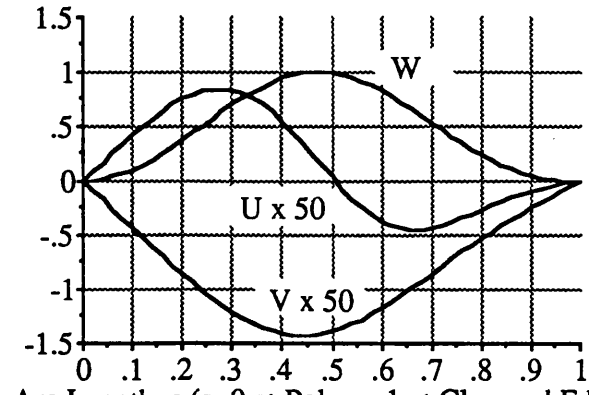
(e) Inplane Displacement Profiles



(f) Meridional Stress Profiles

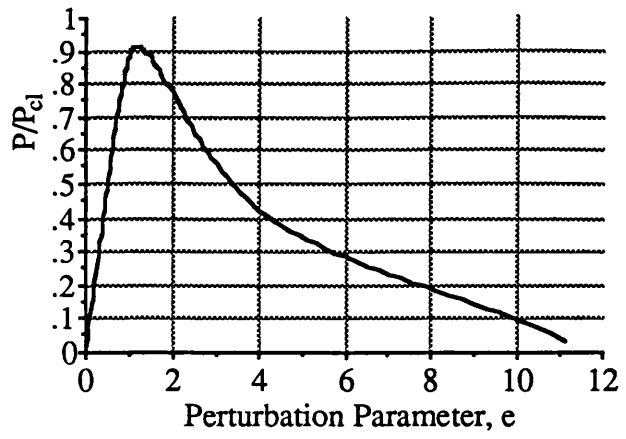


(g) Circumferential Stress Profiles

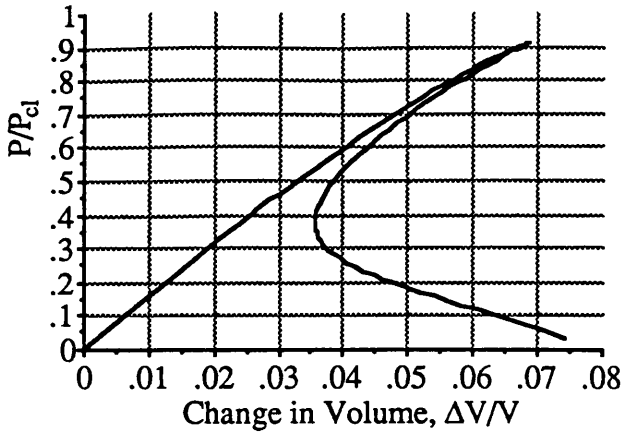


(h) Critical, i=2, Displacement Profiles

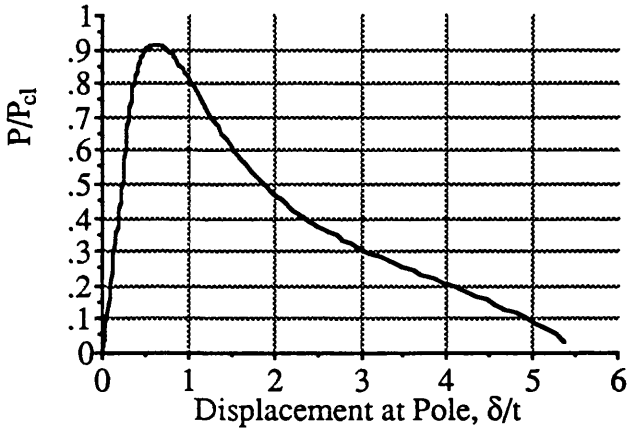
Figure 7.5 Fundamental Path and Shell Profiles for Perfect Shell, λ=6, r/t=500. P_{bk}/P_{cl} =0.970, at e=1.58, P_{cr}/P_{cl} =0.775, i_{cr}=2, at e=0.90.



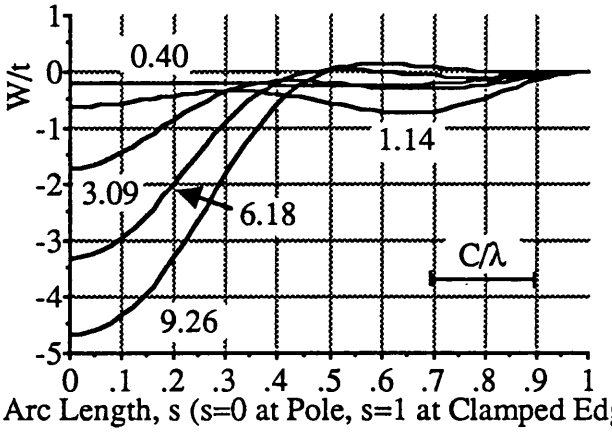
(a) Fundamental Path, P -vs- e



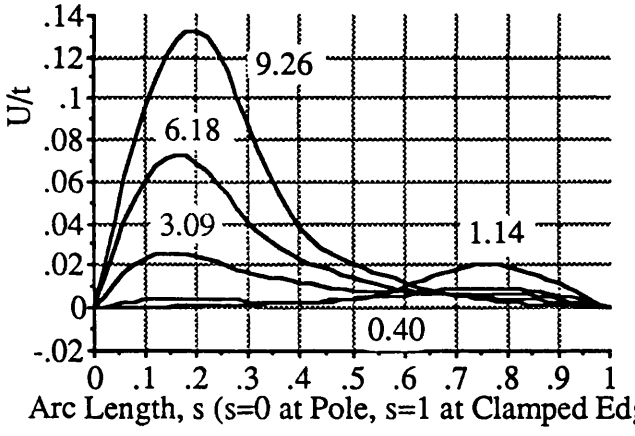
(b) Fundamental Path, P -vs- ΔV



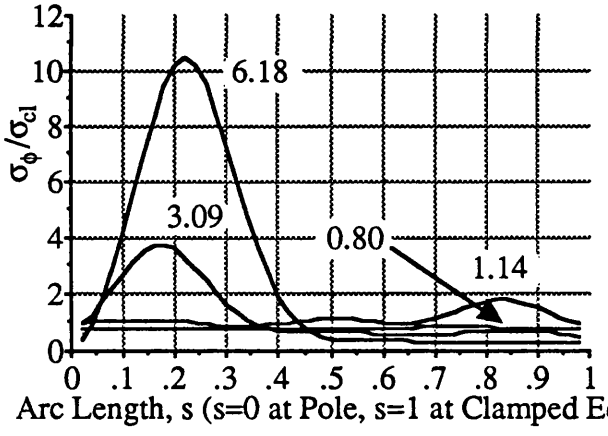
(c) Fundamental Path, P -vs- δ



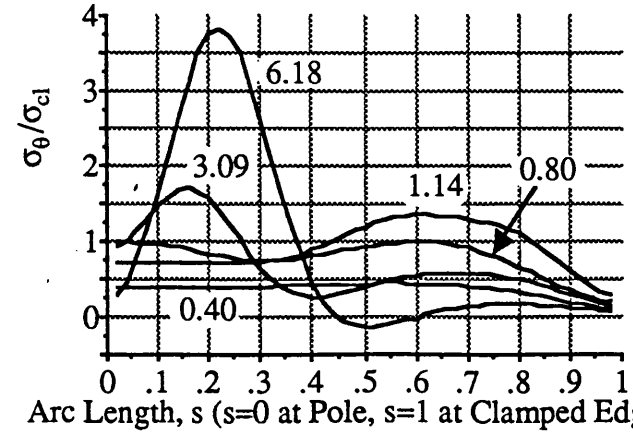
(d) Radial Displacement Profiles



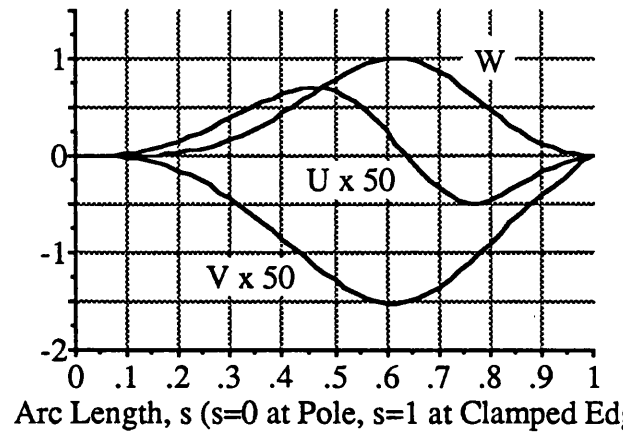
(e) Inplane Displacement Profiles



(f) Meridional Stress Profiles

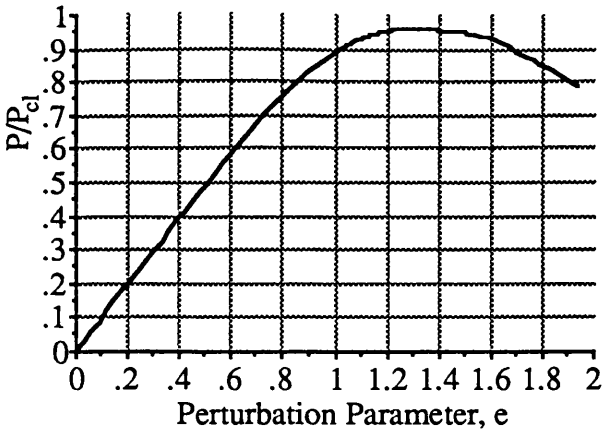


(g) Circumferential Stress Profiles

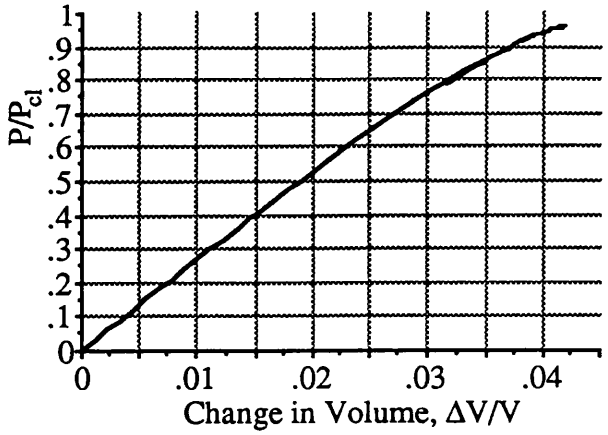


(h) Critical, i=4, Displacement Profiles

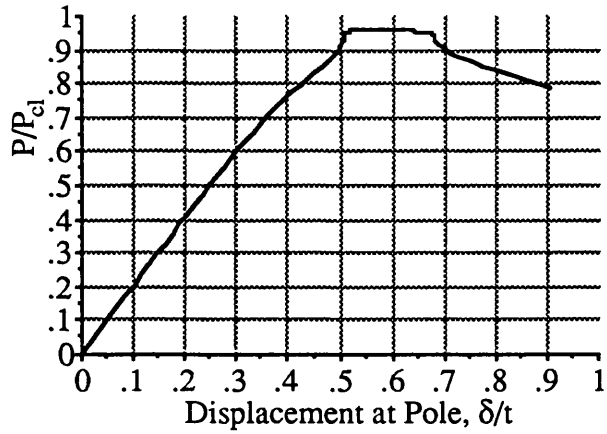
Figure 7.6 Fundamental Path and Shell Profiles for Perfect Shell, λ=9, r/t=500. P_{bk}/P_{cl}=0.915 at e=1.14, P_{cr}/P_{cl}=0.767, i_{cr}=4, at e=0.80.



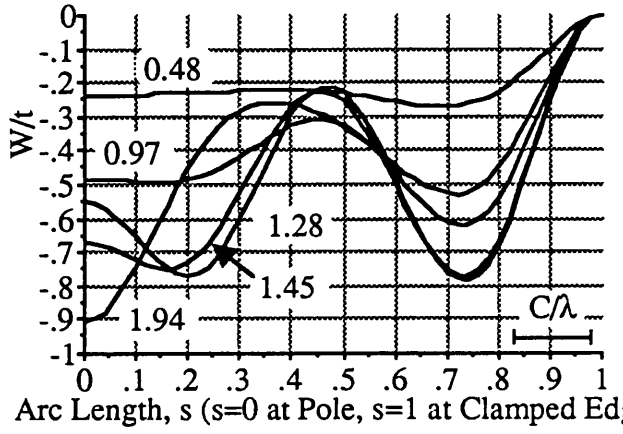
(a) Fundamental Path, P -vs- e



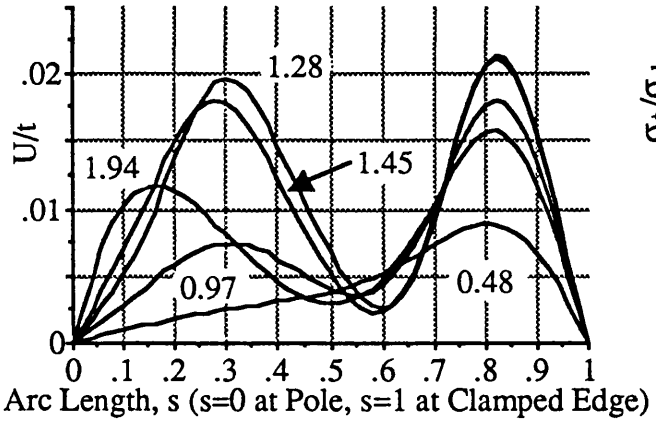
(b) Fundamental Path, P -vs- ΔV



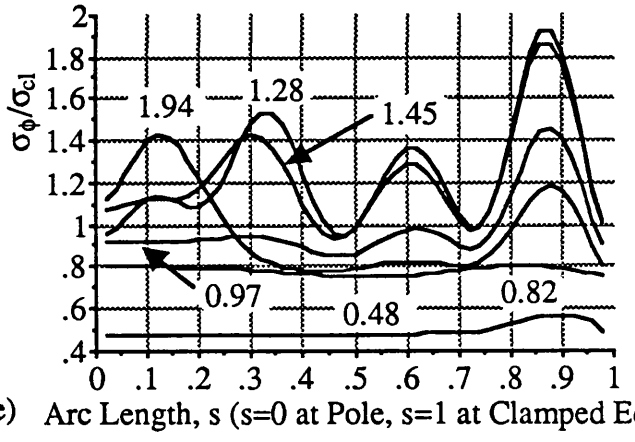
(c) Fundamental Path, P -vs- δ



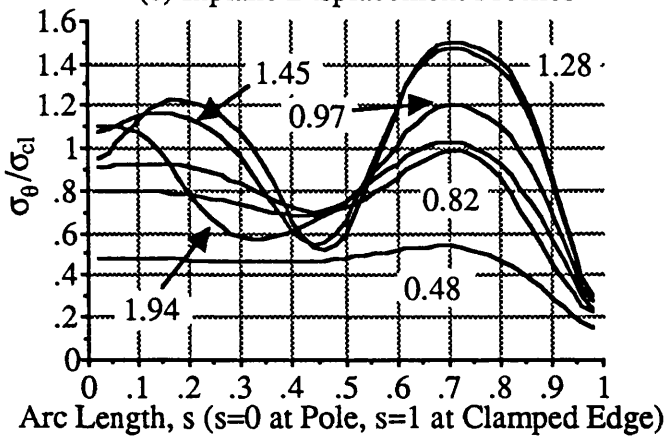
(d) Radial Displacement Profiles



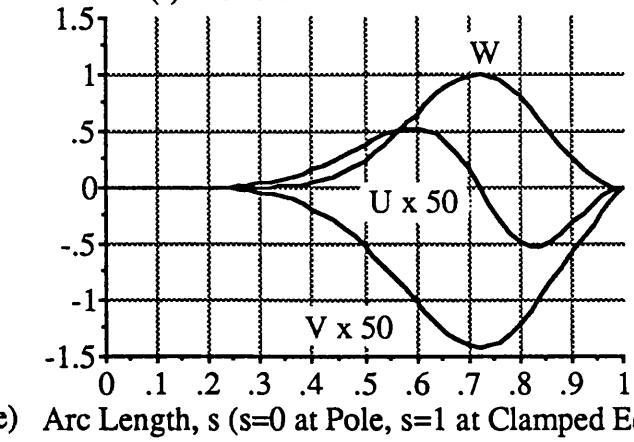
(e) Inplane Displacement Profiles



(f) Meridional Stress Profiles

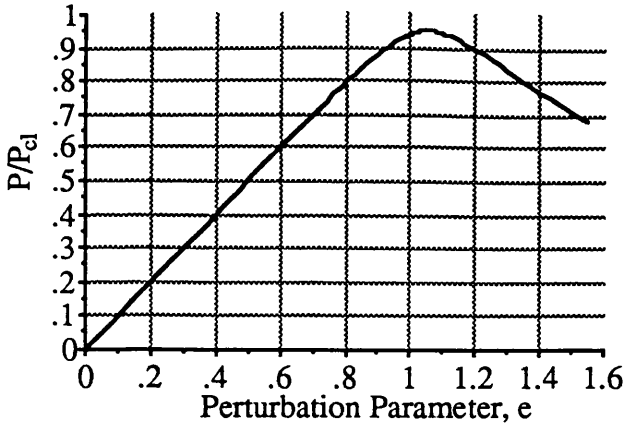


(g) Circumferential Stress Profiles

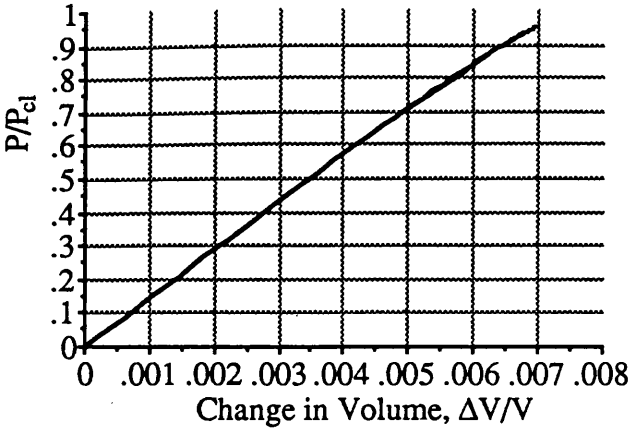


(h) Critical, i=7, Displacement Profiles

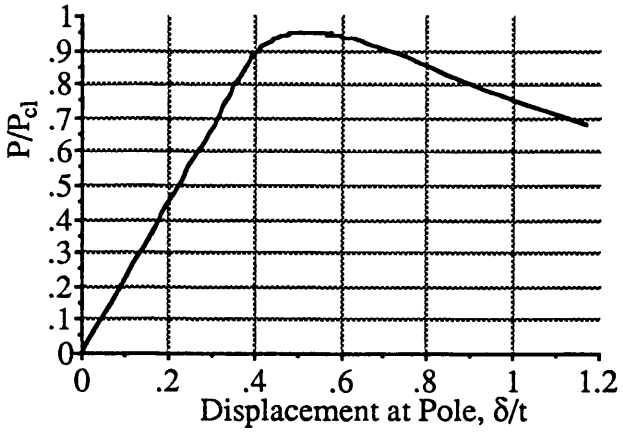
Figure 7.7 Fundamental Path and Shell Profiles for Perfect Shell, λ=12, r/t=500. P_{bk}/P_{cl}=0.959 at e=1.28, P_{cr}/P_{cl}=0.772, i_{cr}=7 at e=0.82.



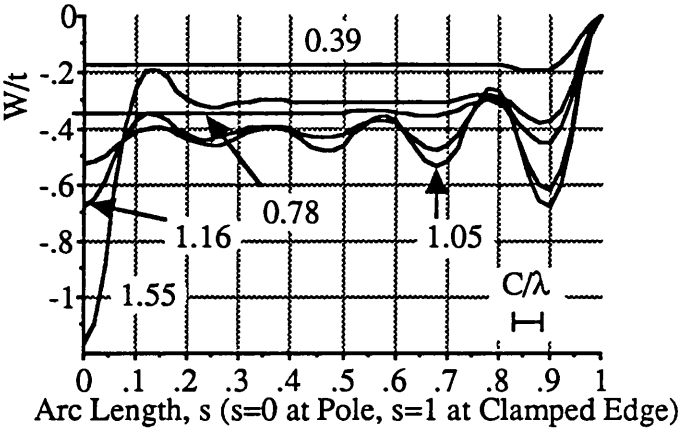
(a) Fundamental Path, P -vs- e



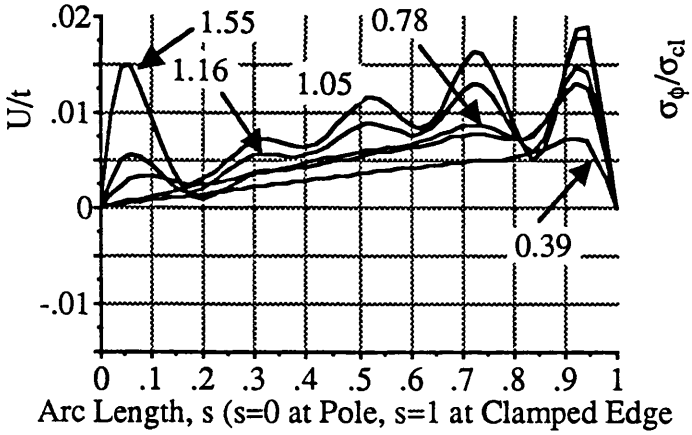
(b) Fundamental Path, P -vs- ΔV



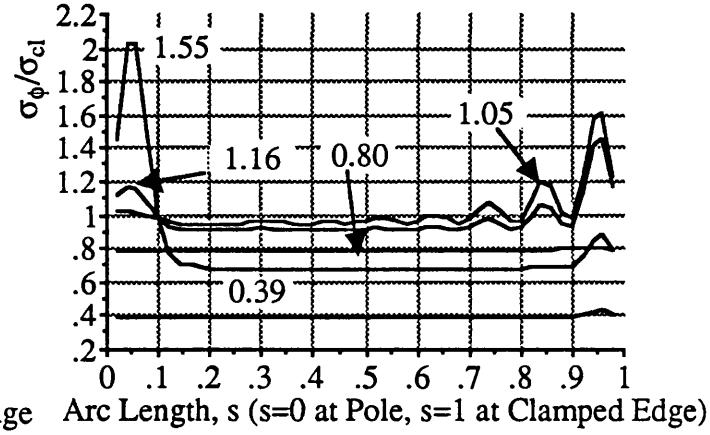
(c) Fundamental Path, P -vs- δ



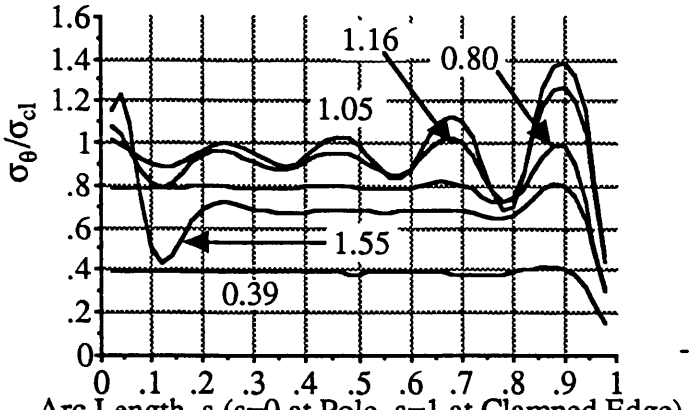
(d) Radial Displacement Profiles



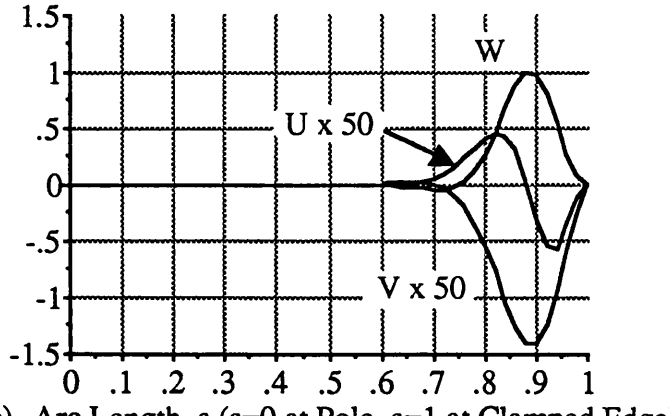
(e) Inplane Displacement Profiles



(f) Meridional Stress Profiles



(g) Circumferential Stress Profiles



(h) Critical, i=21 Displacement Profiles

Figure 7.8 Fundamental Path and Shell Profiles for Perfect Shell, λ=30, r/t=500. P_{bk}/P_{cl}=0.967 at e=1.05, P_{cr}/P_{cl}=0.794, i_{cr}=7 at e=0.80.

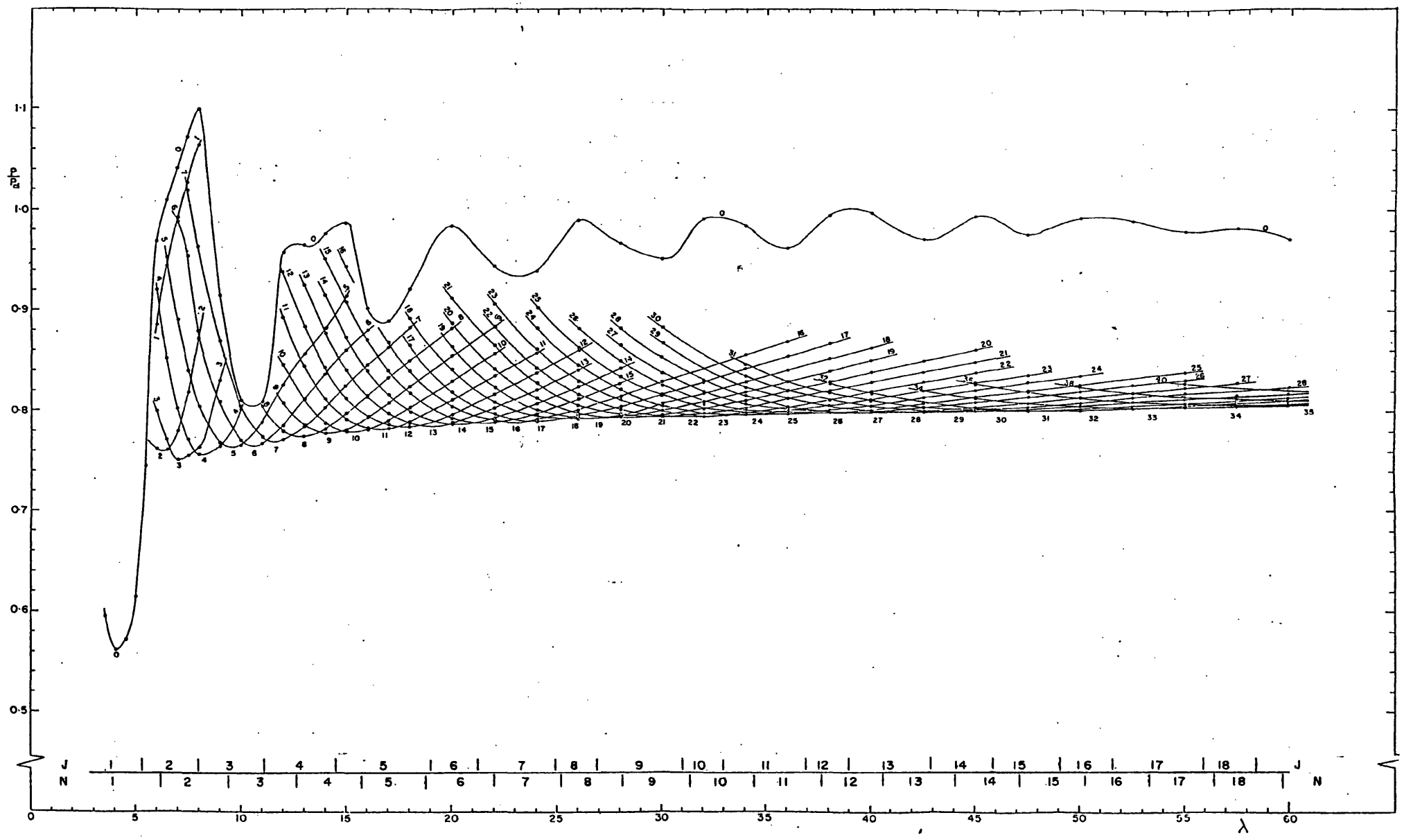
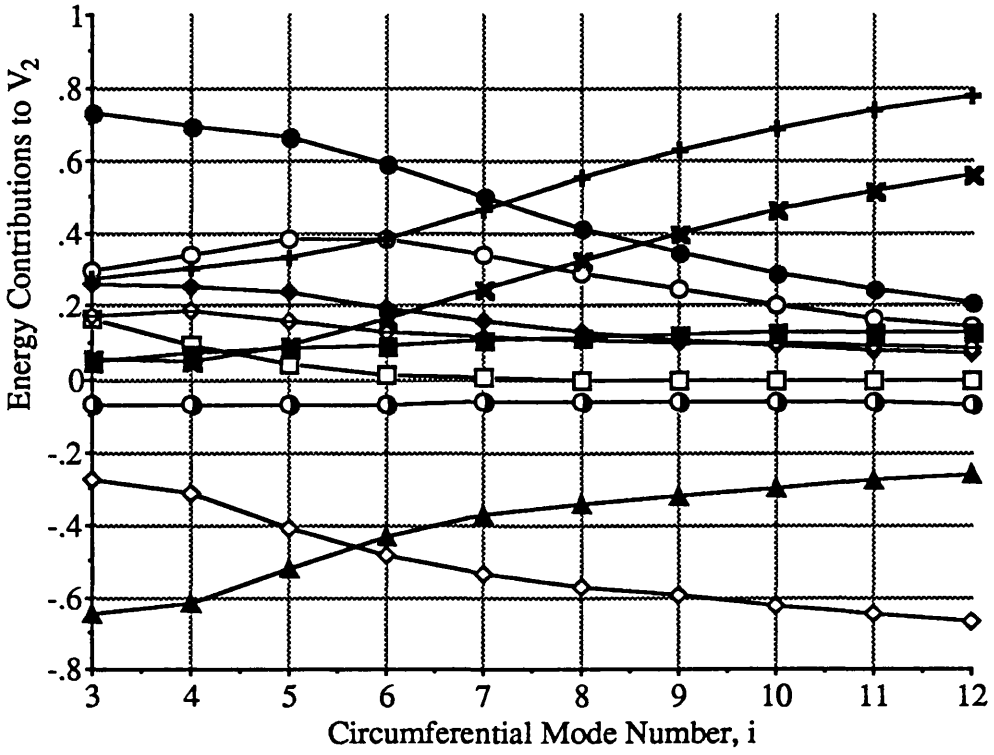


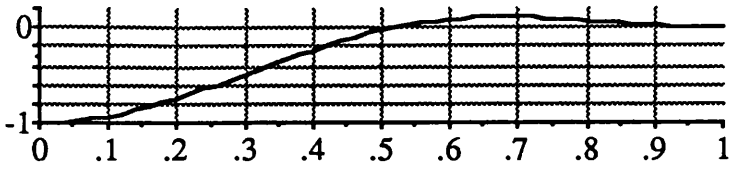
Figure 7.9 Axisymmetric and Periodic Buckling Pressures for Clamped Perfect Spherical Caps, $3.5 \leq \lambda \leq 60$, N - Legendre Polynomial for Complete Sphere, J - Mode Shape for Spherical Cap.



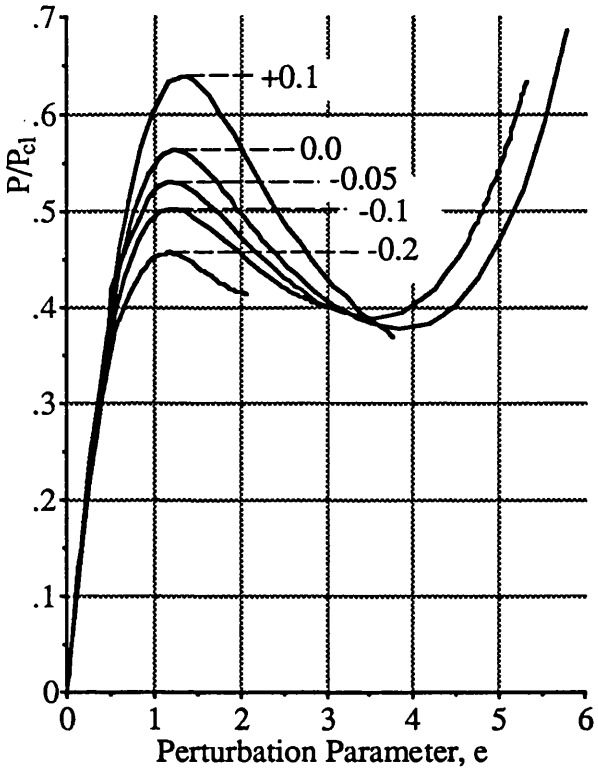
LEGEND

Contributions to the Quadratic Terms of the Total Potential Energy, V_2 { notation defined by equation (4.51) }		
"Stabilising Energy" Secondary Path Membrane Energy	"Stabilising Energy" Secondary Path Bending Energy	"Destabilising Energy" Interaction Between Fundamental and Secondary Path Displacements
● $\dot{N}_\phi\dot{\varepsilon}_\phi + \dot{N}_\theta\dot{\varepsilon}_\theta + \dot{N}_{\theta\phi}\dot{\varepsilon}_{\theta\phi}$ ○ $\dot{N}_\phi\dot{\varepsilon}_\phi$ □ $\dot{N}_\theta\dot{\varepsilon}_\theta$ ◆ $\dot{N}_{\theta\phi}\dot{\varepsilon}_{\theta\phi}$	✚ $U_{2,B_\phi} + U_{2,B_\theta} + U_{2,B_{\theta\phi}}$ ◈ U_{2,B_ϕ} ✖ U_{2,B_θ} ■ $U_{2,B_{\theta\phi}}$	▲ $\overset{f'}{N}_\phi\overset{f'}{\varepsilon}_\phi + \overset{f}{N}_\phi\overset{''}{\varepsilon}_\phi + \overset{''}{N}_\phi\overset{f}{\varepsilon}_\phi$ ◇ $\overset{f'}{N}_\theta\overset{f'}{\varepsilon}_\theta + \overset{f}{N}_\theta\overset{''}{\varepsilon}_\theta + \overset{''}{N}_\theta\overset{f}{\varepsilon}_\theta$ ◑ $\overset{f'}{N}_{\theta\phi}\overset{f'}{\varepsilon}_{\theta\phi}$

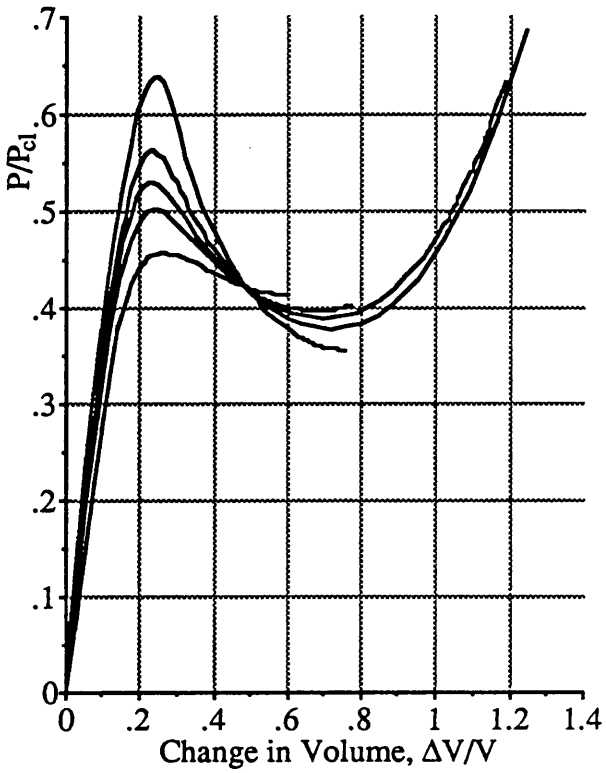
Figure 7.10 Relative Contributions to the Quadratic Terms of the Total Potential Energy as a Function of the Circumferential Mode number, for λ=12.



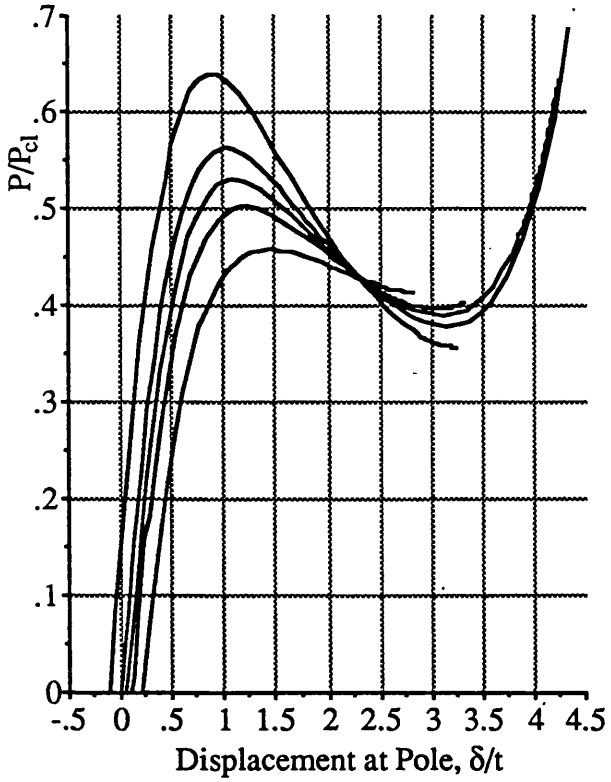
(a) Imperfection Shape, W*



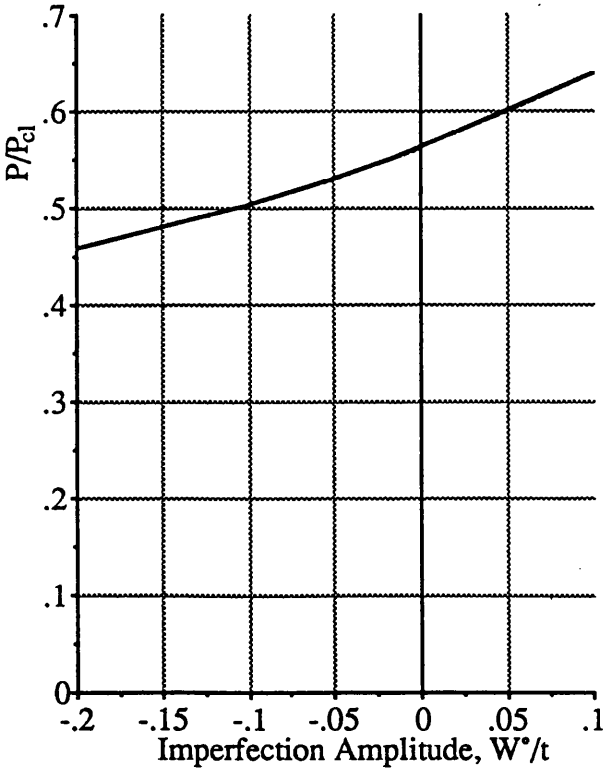
(b) Fundamental Path, P -vs- e



(c) Fundamental Path, P -vs- ΔV

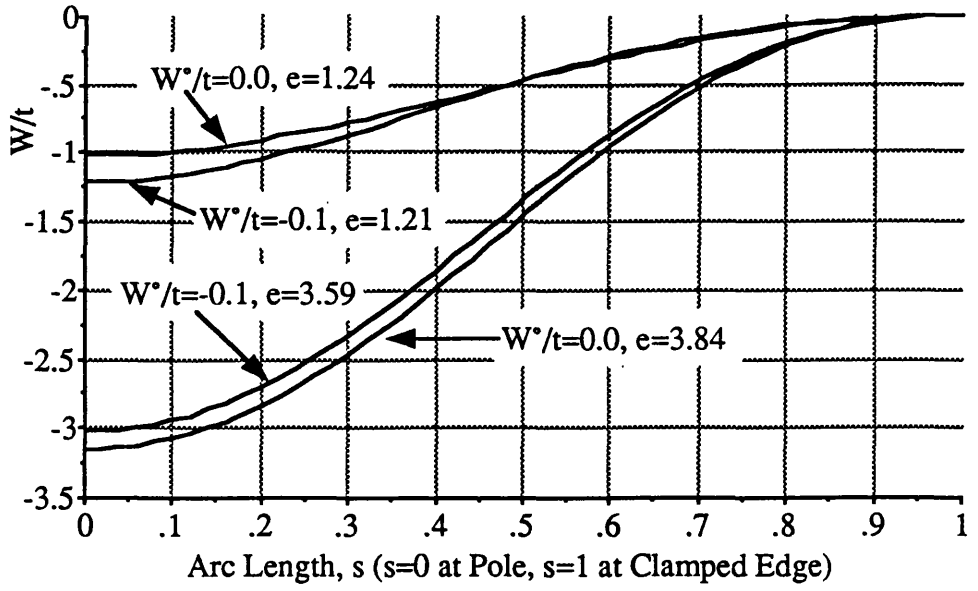


(d) Fundamental Path, P -vs- δ

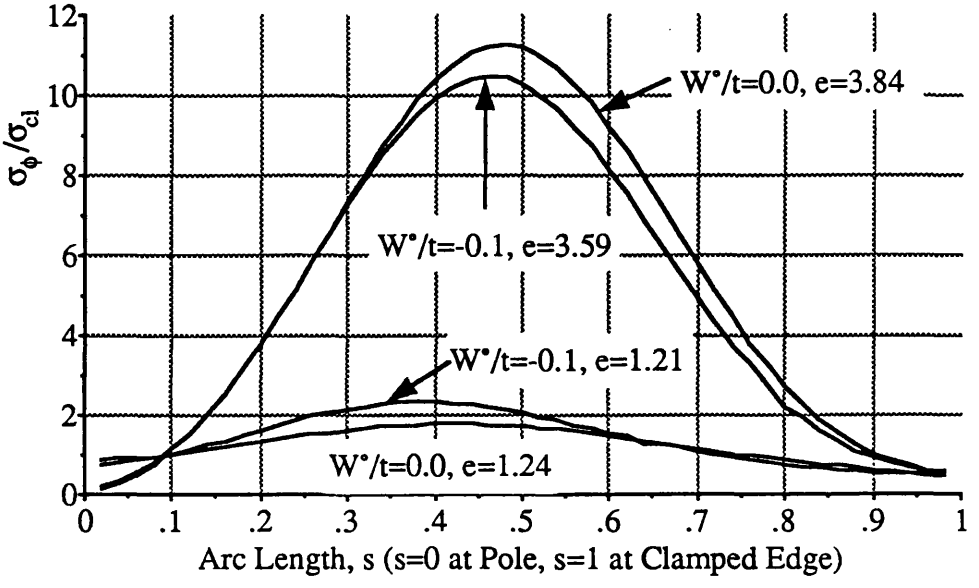


(e) Imperfection Sensitivity, W*

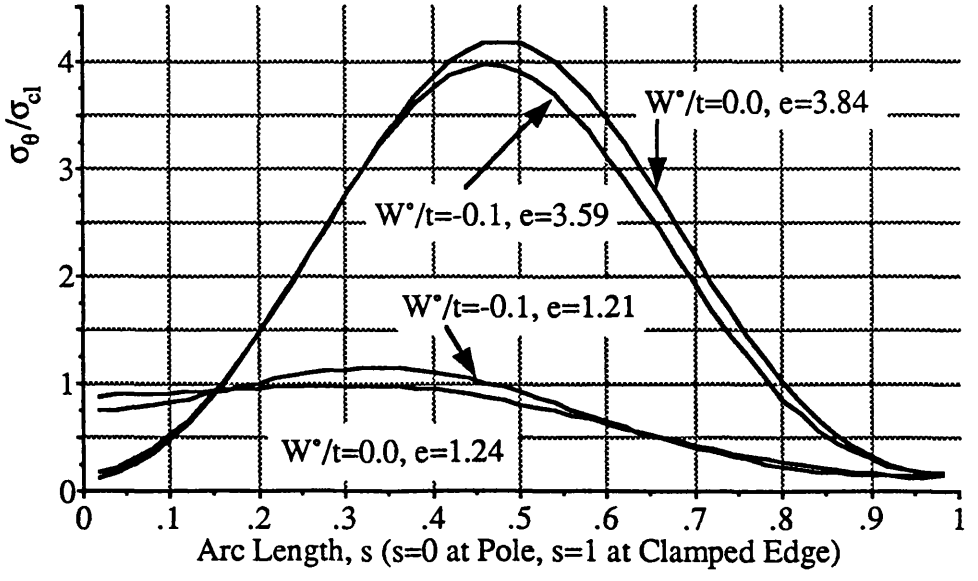
Figure 7.11 Fundamental Path and Shell Profiles for Imperfect Shells, λ=4, r/t=500.



(f) Radial Displacement Profiles

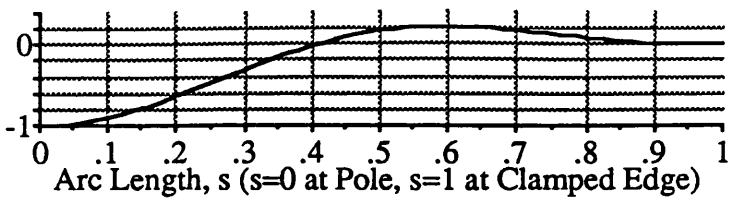


(g) Meridional Stress Profiles

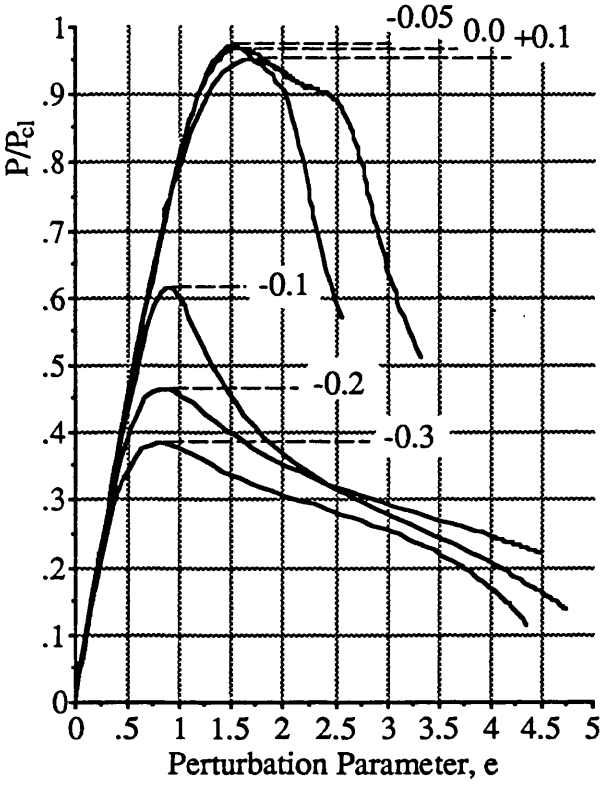


(h) Circumferential Stress Profiles

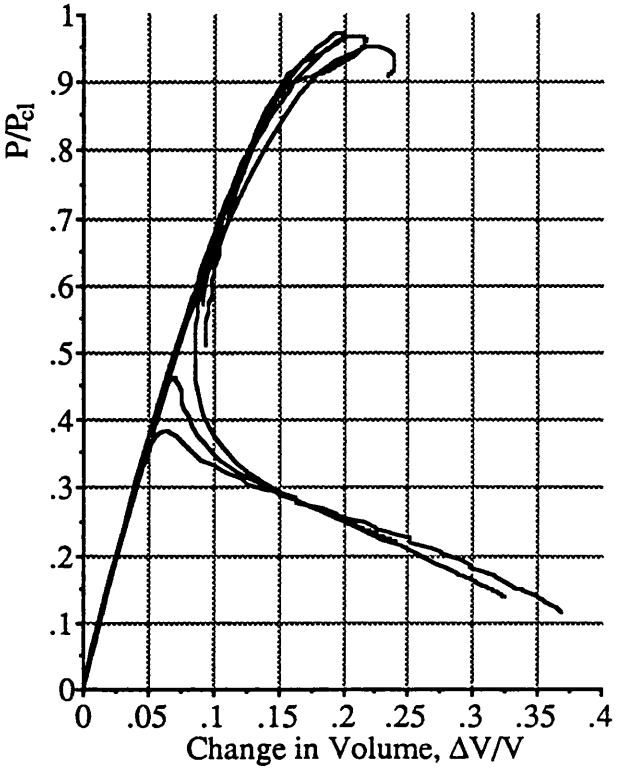
Figure 7.11 Fundamental Path and Shell Profiles for Imperfect Shells, λ=4, r/t=500.



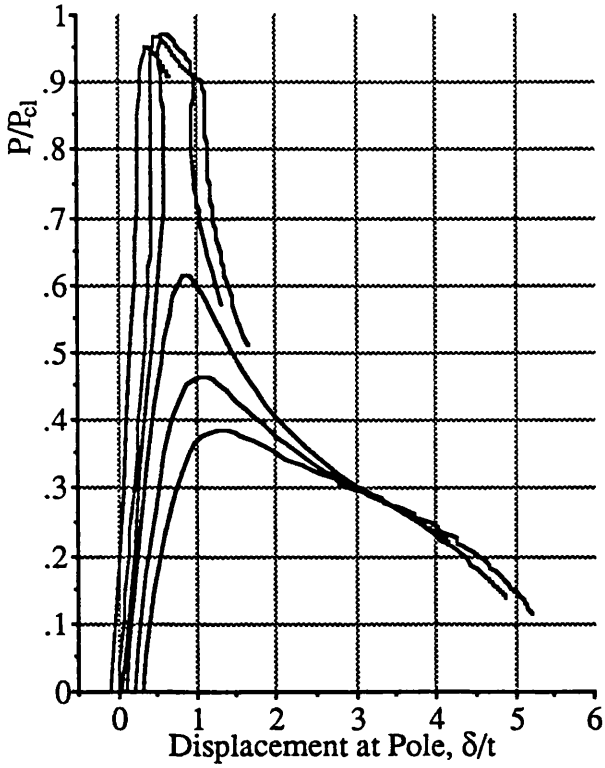
(a) Imperfection Shape, W*



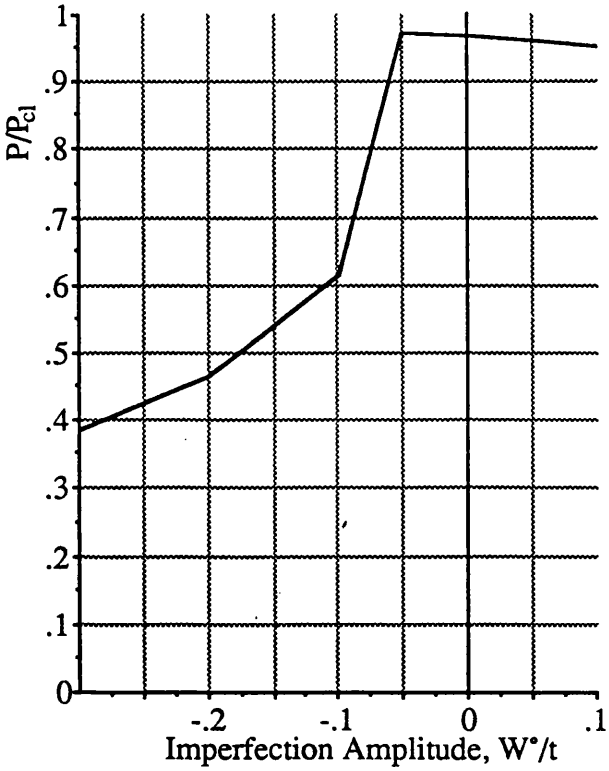
(b) Fundamental Path, P -vs- e



(c) Fundamental Path, P -vs- ΔV

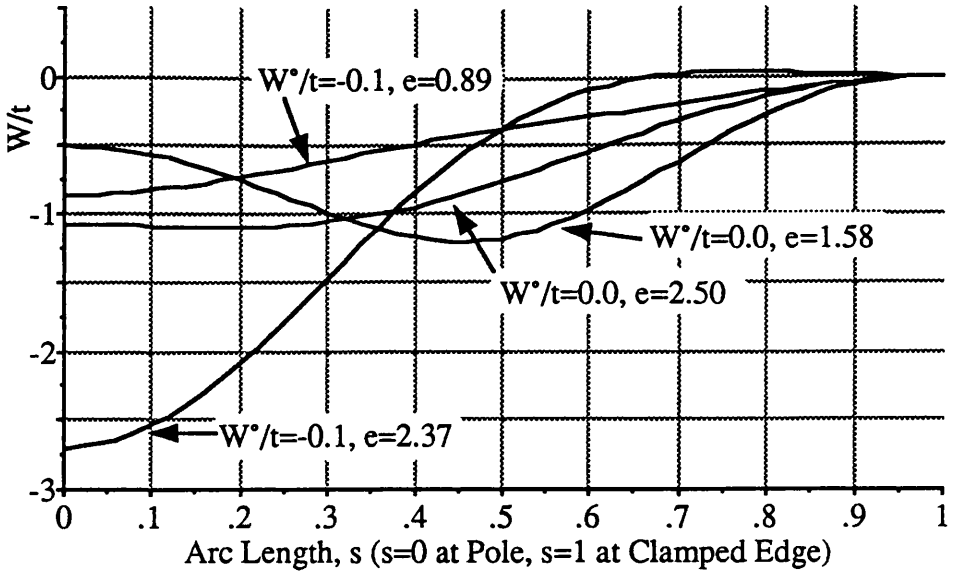


(d) Fundamental Path, P -vs- δ

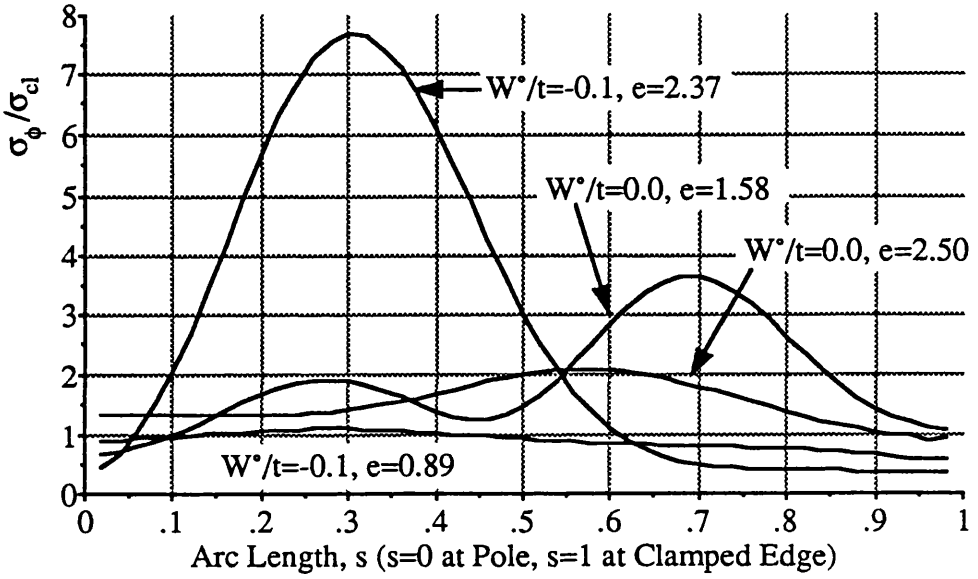


(e) Imperfection Sensitivity, W*

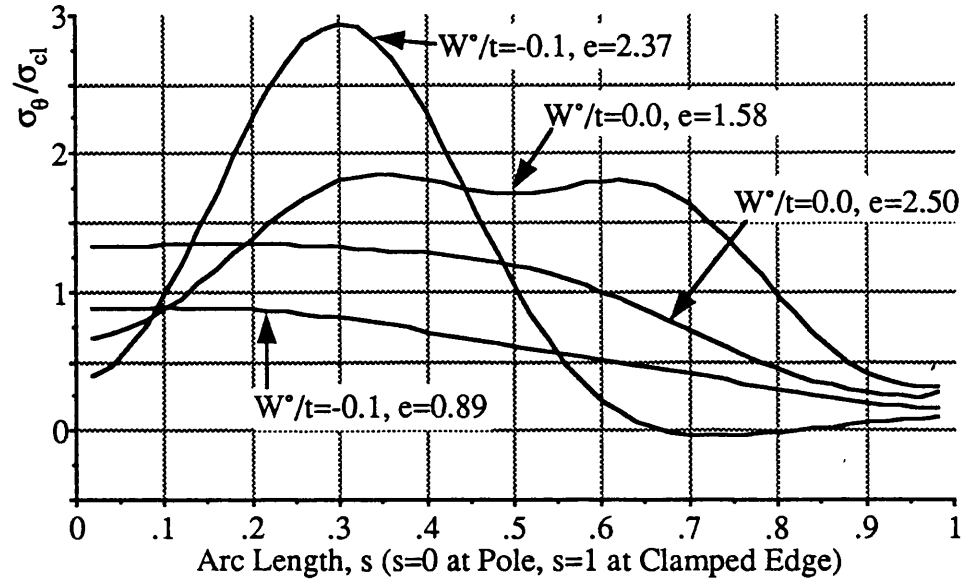
Figure 7.12 Fundamental Path and Shell Profiles for Imperfect Shells, λ=6, r/t=500.



(f) Radial Displacement Profiles

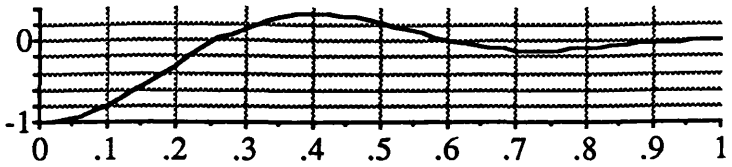


(g) Meridional Stress Profiles

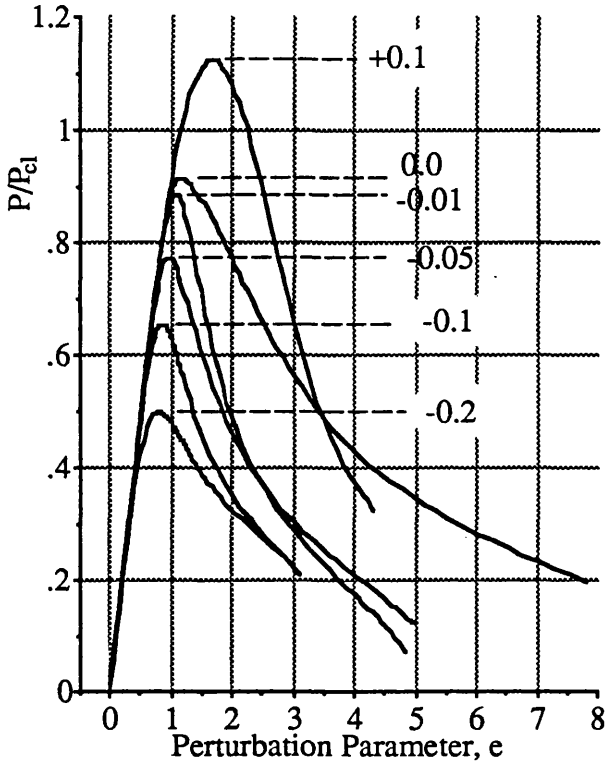


(h) Circumferential Stress Profiles

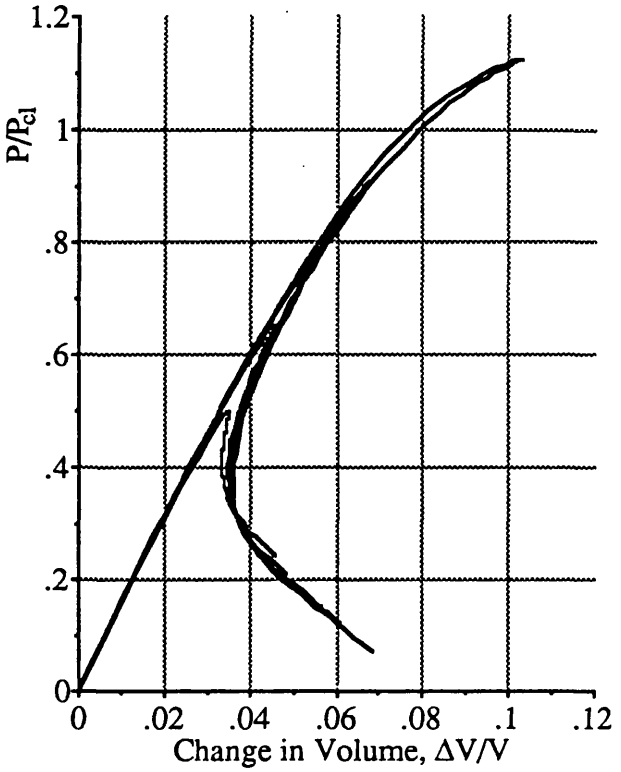
Figure 7.12 Fundamental Path and Shell Profiles for Imperfect Shells, λ=6, r/t=500.



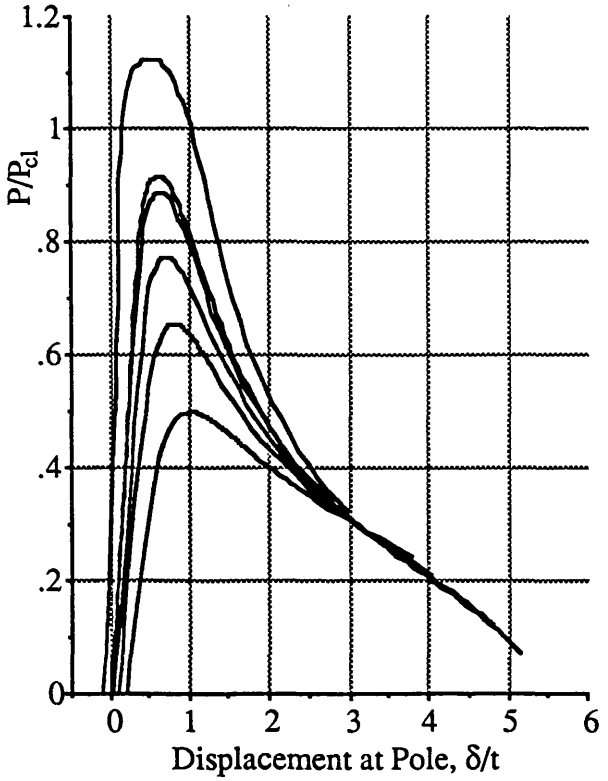
(a) Imperfection Shape, W*



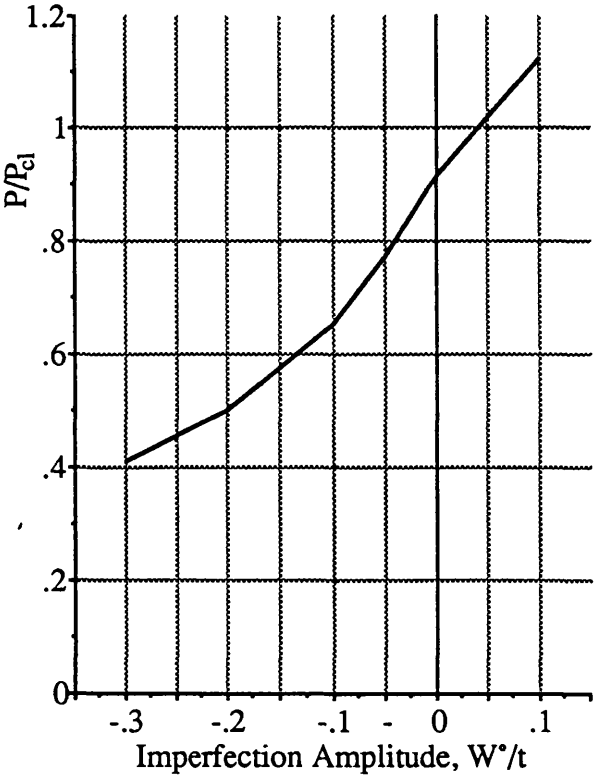
(b) Fundamental Path, P -vs- e



(c) Fundamental Path, P -vs- ΔV

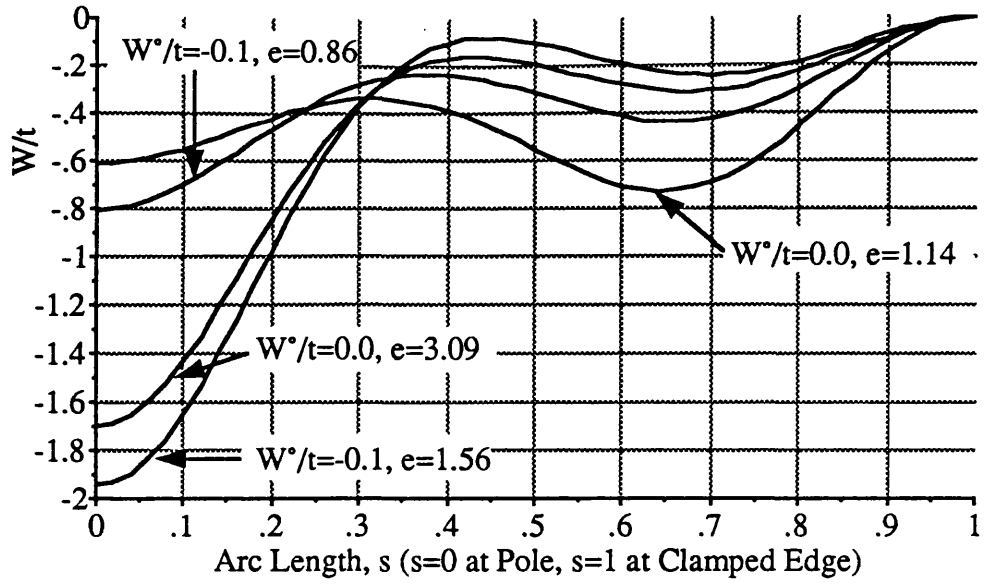


(d) Fundamental Path, P -vs- δ

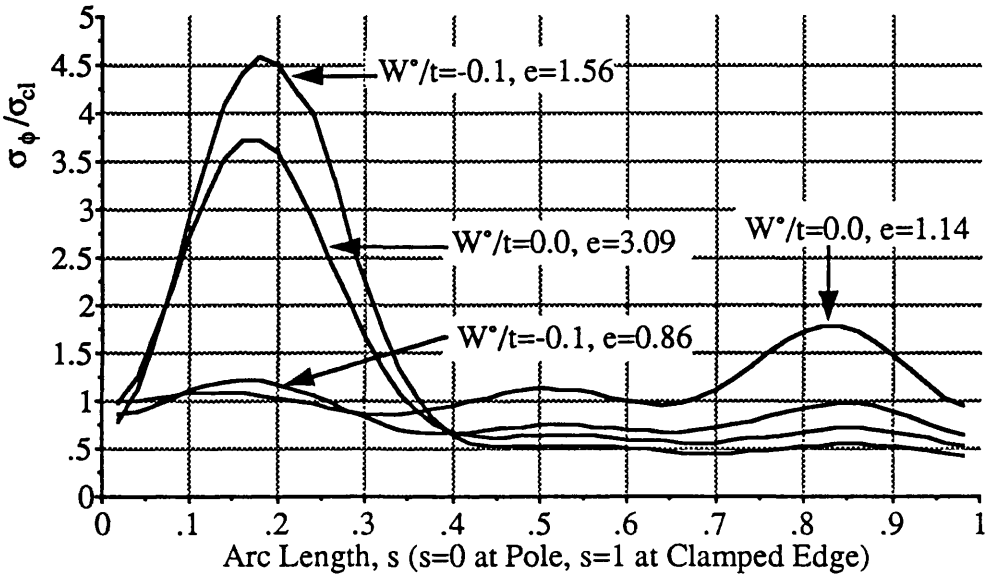


(e) Imperfection Sensitivity, W*

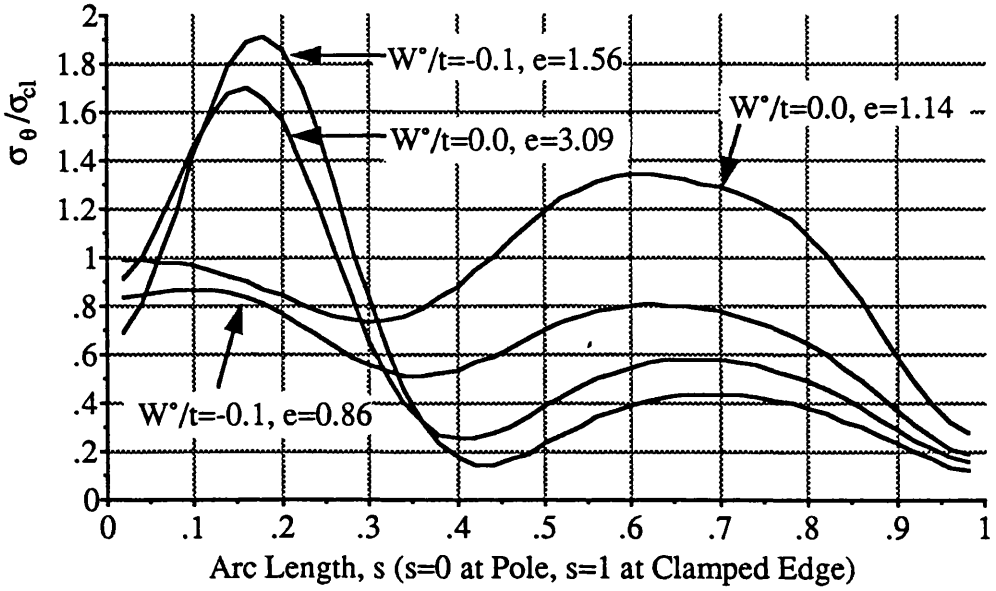
Figure 7.13 Fundamental Path and Shell Profiles for Imperfect Shells, λ=9, r/t=500.



(f) Radial Displacement Profiles

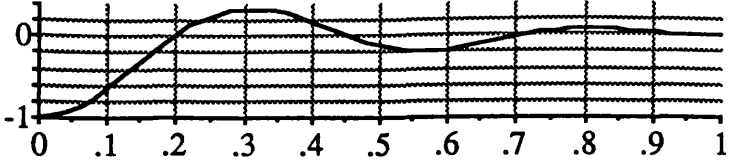


(g) Meridional Stress Profiles



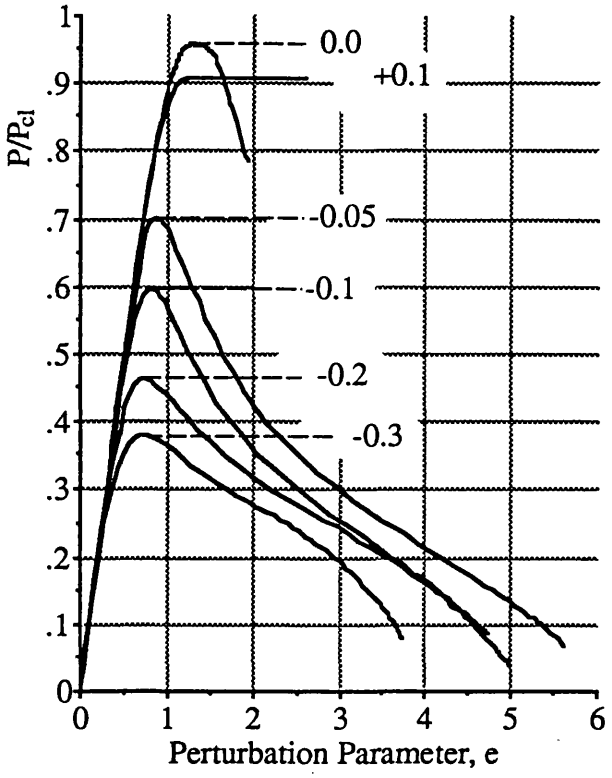
(h) Circumferential Stress Profiles

Figure 7.13 Fundamental Path and Shell Profiles for Imperfect Shells, λ=9, r/t=500.

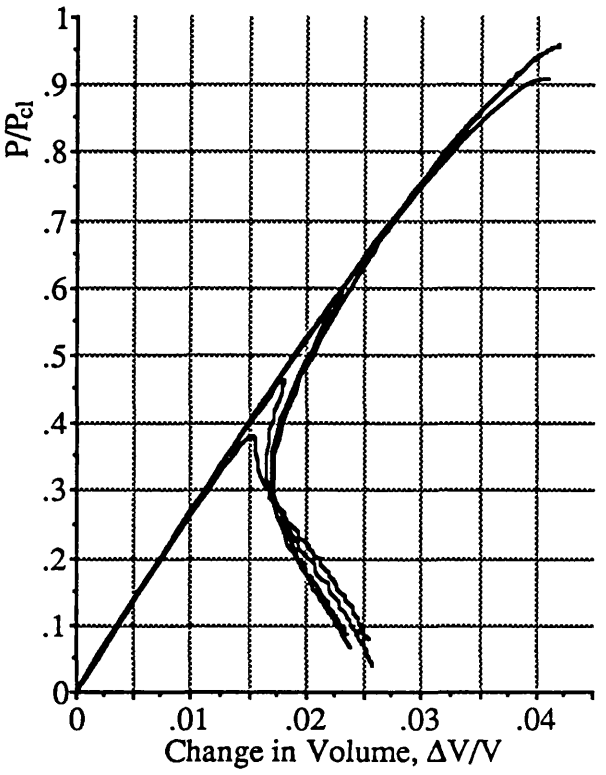


Arc Length, s (s=0 at Pole, s=1 at Clamped Edge)

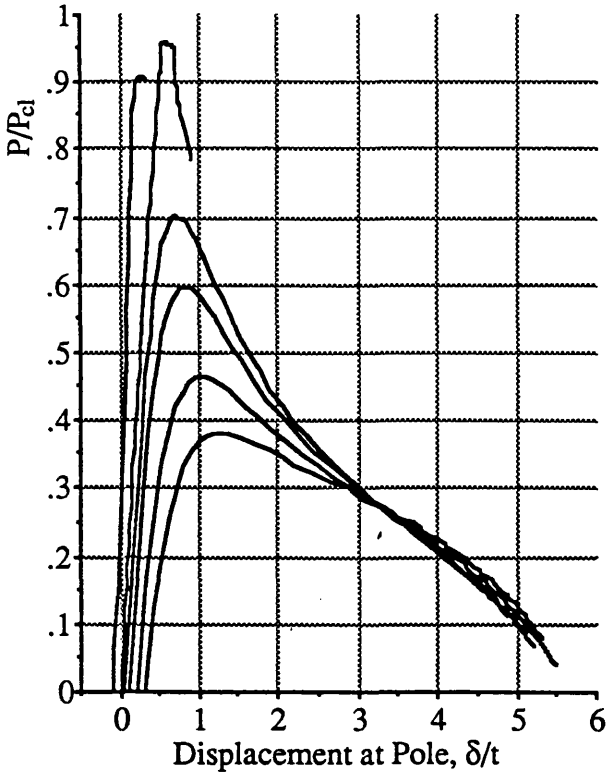
(a) Imperfection Shape, W*



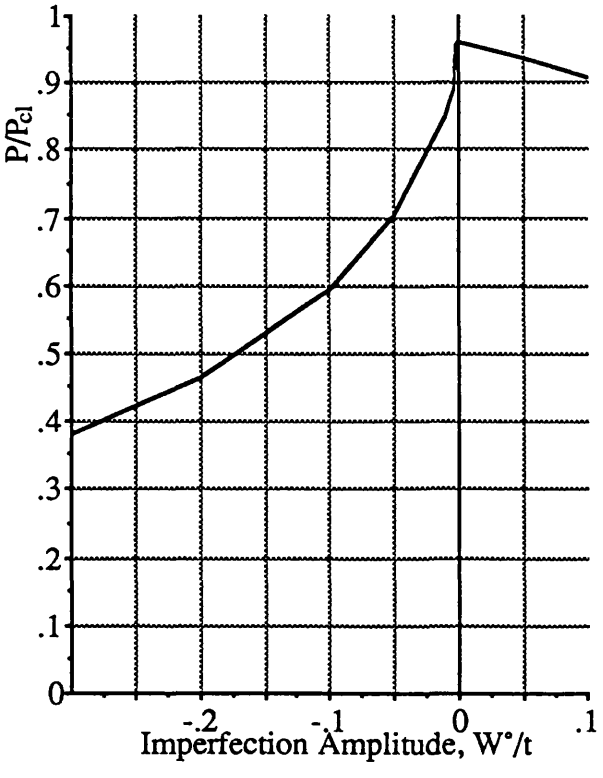
(b) Fundamental Path, P -vs- e



(c) Fundamental Path, P -vs- ΔV

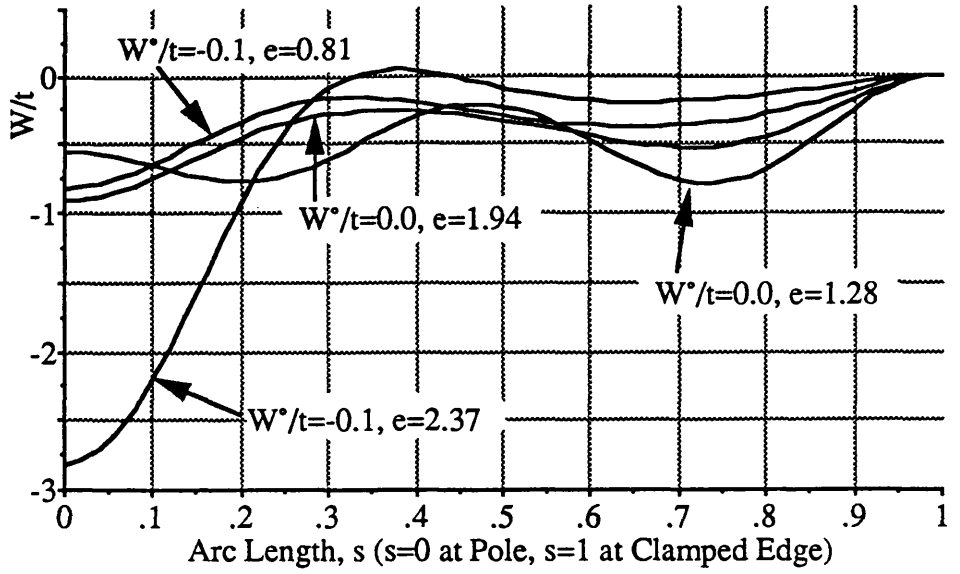


(d) Fundamental Path, P -vs- δ

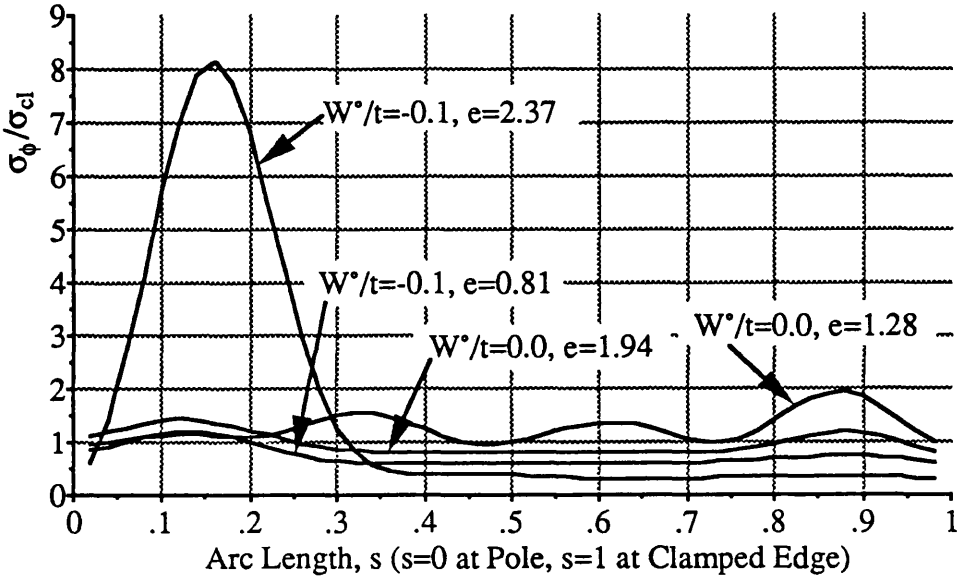


(e) Imperfection Sensitivity, W*

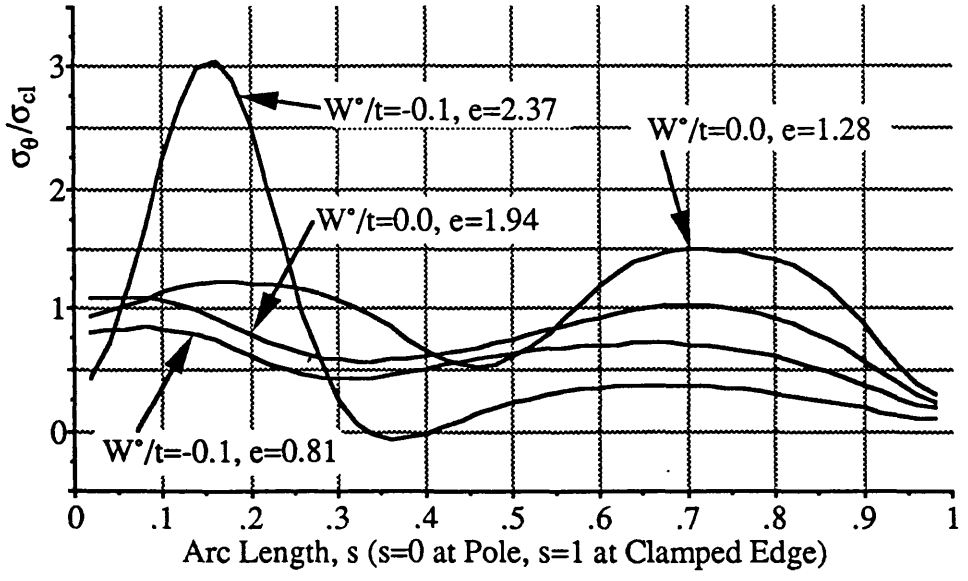
Figure 7.14 Fundamental Path and Shell Profiles for Imperfect Shells, λ=12, r/t=500.



(f) Radial Displacement Profiles

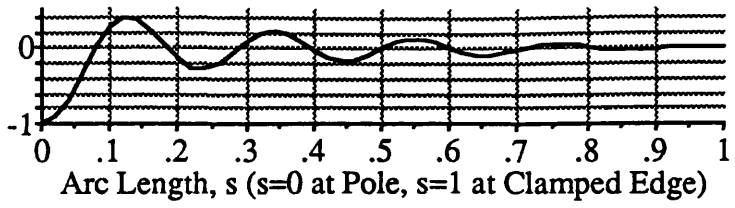


(g) Meridional Stress Profiles

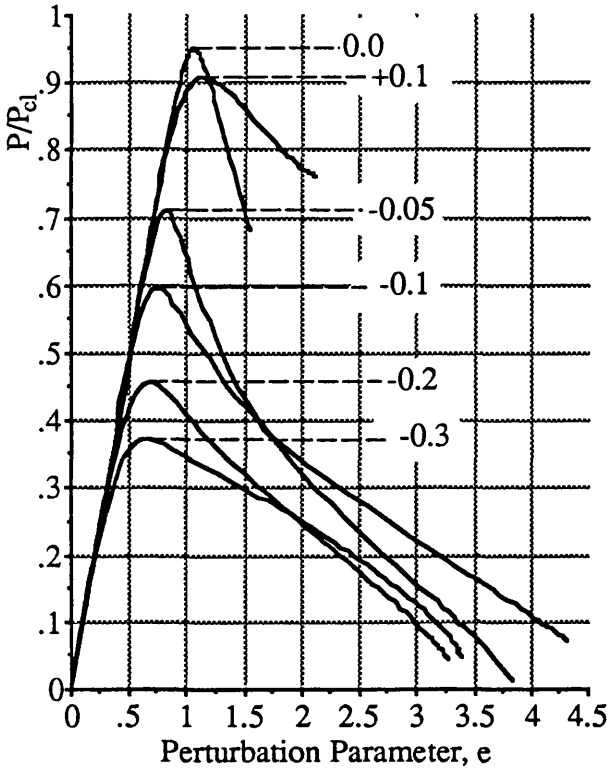


(h) Circumferential Stress Profiles

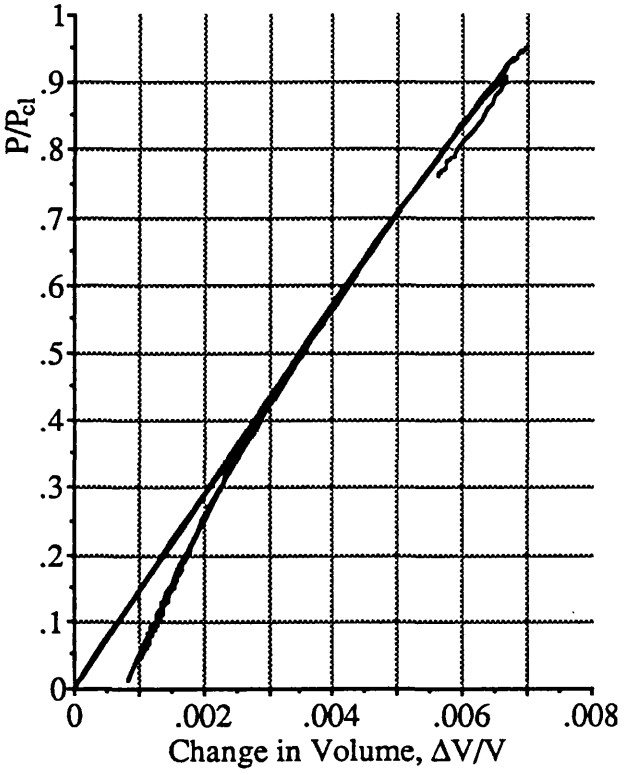
Figure 7.14 Fundamental Path and Shell Profiles for Imperfect Shells, λ=12, r/t=500.



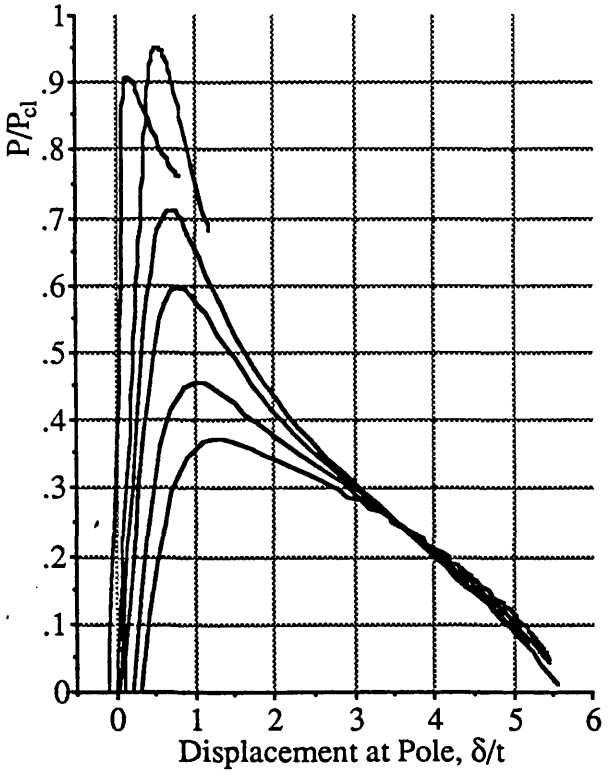
(a) Imperfection Shape, W*



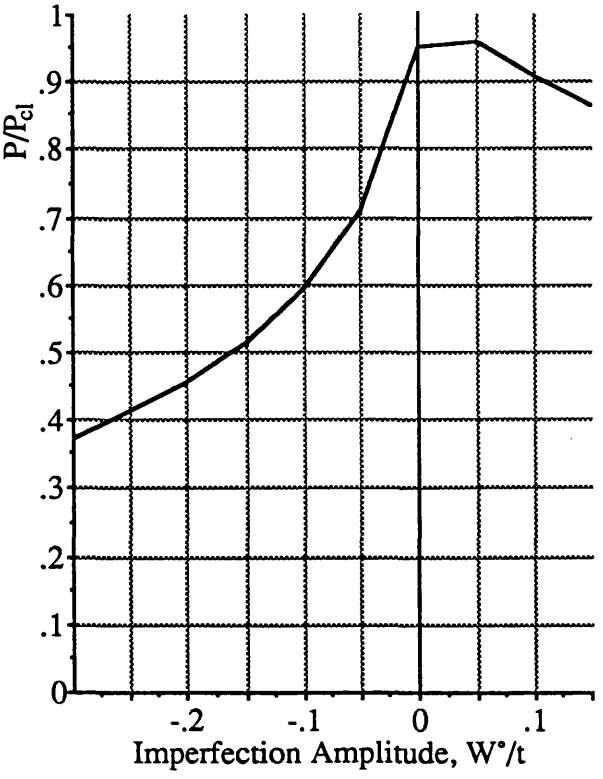
(b) Fundamental Path, P -vs- e



(c) Fundamental Path, P -vs- ΔV

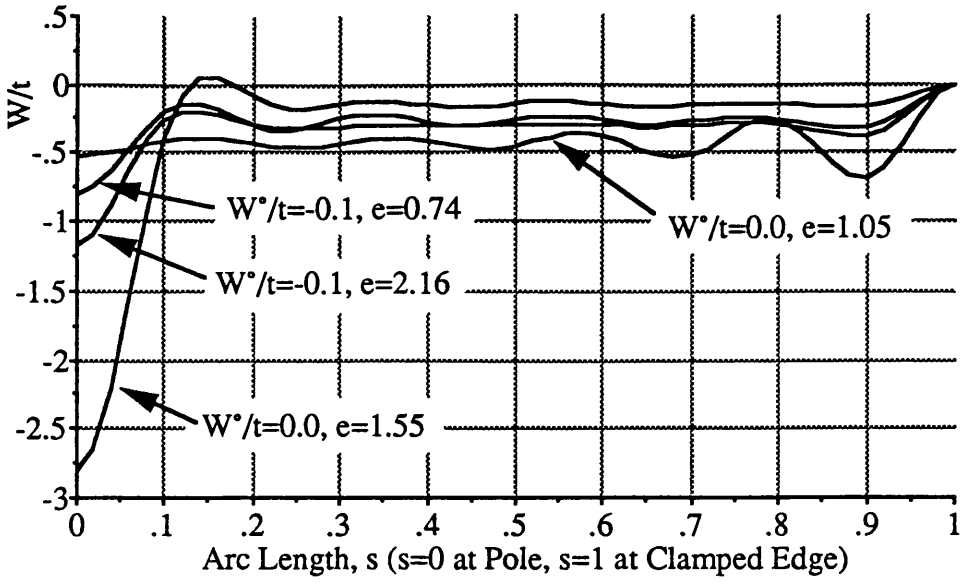


(d) Fundamental Path, P -vs- δ

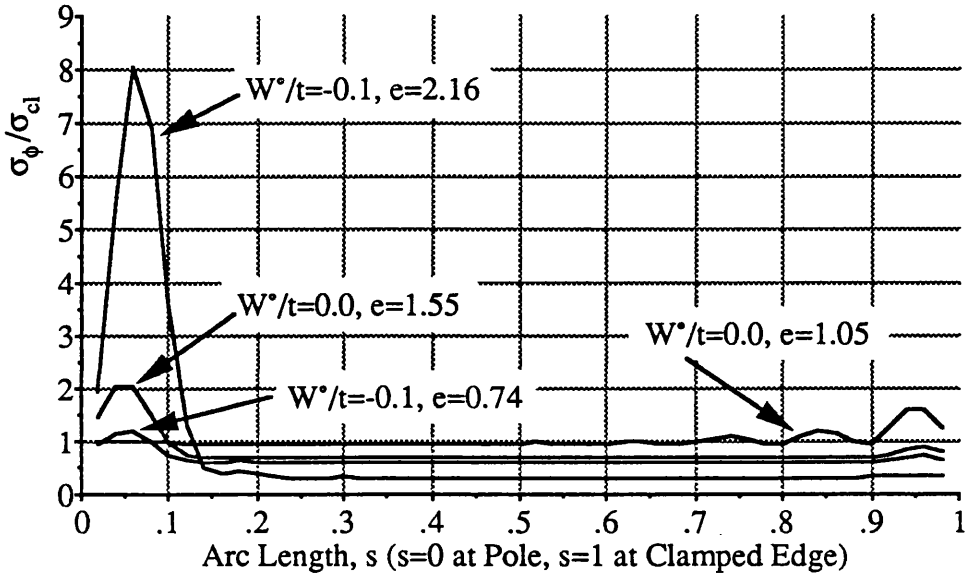


(e) Imperfection Sensitivity, W*

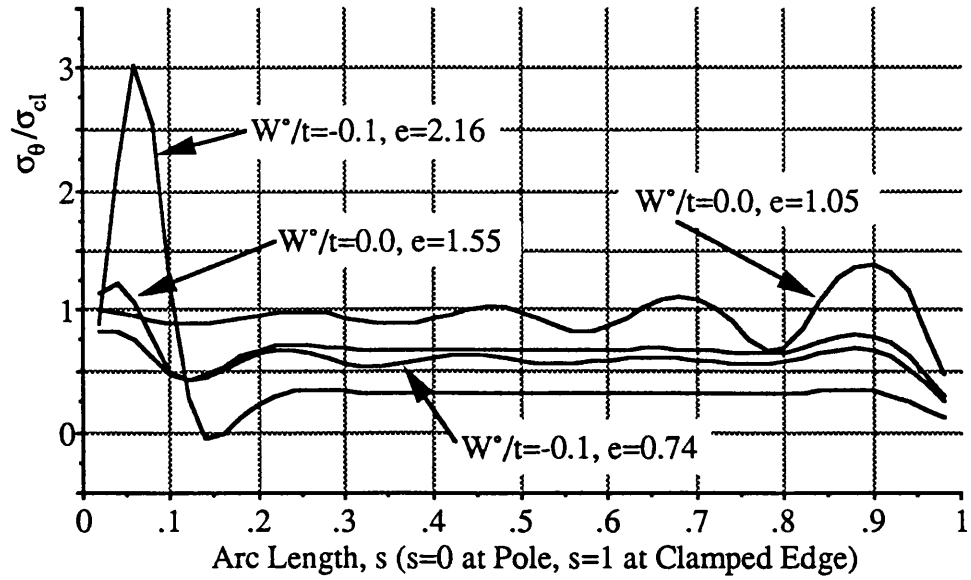
Figure 7.15 Fundamental Path and Shell Profiles for Imperfect Shells, λ=30, r/t=500.



(f) Radial Displacement Profiles



(g) Meridional Stress Profiles



(h) Circumferential Stress Profiles

Figure 7.15 Fundamental Path and Shell Profiles for Imperfect Shells, λ=30, r/t=500.

CHAPTER EIGHT

CONCLUSIONS

CHAPTER EIGHT

CONTENTS

CONCLUSIONS

8.1 ELASTIC ANALYSIS OF SPHERICAL SHELLS

In the classical analysis of complete spherical shells the bifurcation, or buckling, pressures are calculated from a linear eigenvalue problem in which the fundamental path consists of a pure radial contraction of the sphere, and the secondary path is axisymmetric. The total displacement of a complete perfect sphere under uniform external pressure loading at the lowest point of bifurcation on the fundamental path is approximately 42 per cent of the thickness of the shell, and the load carrying capacity of the shell on the secondary paths fall rapidly for small amplitudes of the secondary path displacements. In most cases the load carrying capacity of the shell drops to zero for secondary path displacements with an amplitude of about 20 per cent of the thickness of the shell. Despite the relatively small amplitude of the secondary path displacements, it is at least necessary to model the secondary path using a cubic expression, that is the total potential energy must contain at least quartic terms.

The axisymmetric nature of the secondary path displacements in the classical analysis implies that these displacements will cause circumferential bands of tension and compression to develop in the shell. The ability of the shell to develop circumferential bands of membrane compression will be sensitive to the presence of imperfections. When the circumferential membrane energy is completely eliminated from the classical analysis the bifurcation, or buckling, wavelength increases and the bifurcation, or buckling, loads decrease. For shells with radius to thickness ratios of more than 250 the critical buckling load is reduced to less than one quarter of the critical classical value, and the corresponding critical buckling wavelength doubles by comparison to the critical classical wavelength.

The comparisons for isotropic spherical caps clamped at the boundary under uniform pressure load, presented in Chapter Seven, show that the nonlinear fundamental path and bifurcation analysis are, when suitably nondimensionalised, dependent on the geometric slenderness parameter alone.

For initially perfect spherical caps with values of the geometric slenderness parameter greater than approximately 5.5, bifurcation into periodic modes occur at a lower pressures than axisymmetric buckling. The maximum radial displacement of the periodic critical mode and the maximum radial displacement of the axisymmetric fundamental path displacements at the critical load occur at approximately the same position on the shell. As the geometric slenderness parameter increases the periodic buckling displacements affect a smaller area of the shell adjacent to the clamped boundary, and the critical circumferential mode number increases. The number of periodic bifurcation modes and their proximity to the critical mode also increases as the geometric slenderness parameter increases.

For all initially perfect spherical caps with values of the geometric slenderness parameter of more than 4 the lowest buckling pressure, axisymmetric buckling or periodic bifurcation, occurs when the fundamental path circumferential stress first reaches the classical buckling stress for the complete sphere with the same radius to thickness ratio.

For perfect slender caps under uniform pressure loading periodic bifurcation occurs before axisymmetric buckling, and the initial displacements of the critical periodic buckling mode are large close to the boundary of the cap, and small in the central part of the cap. The periodic post bifurcation paths have not been calculated in the present work. However, if we consider the central part of the cap as a separate cap, with a smaller slenderness value and less restraint at the boundary, then it is reasonable to suppose that this part of the cap will have a lower bifurcation or axisymmetric buckling pressure than the entire cap. It may be imagined that the elastic buckling of perfect slender spherical caps under uniform external pressure, will be initiated by periodic bifurcation, in which the initial secondary path displacements near the boundary will be relatively large, the central portion of the spherical cap which has a lower critical pressure will then bifurcate or buckle axisymmetrically, and in this way the 'buckle' will propagate toward the pole.

The influence of one particular type of initial stress-free axisymmetric imperfection on the elastic behaviour of spherical caps has been examined in Chapter Seven. In all cases when the amplitude of the initial imperfection was greater than one tenth of the thickness of the spherical cap all periodic bifurcation was eliminated and the behaviour of the cap became entirely axisymmetric. Spherical caps with initial axisymmetric imperfections, of the type considered in Chapter Seven, buckle axisymmetrically with the radial displacement at the pole 'leading' the buckle.

For spherical caps with geometric slenderness values greater than 6 the results presented in Chapter Seven indicate that the postbuckling path becomes negative before the cap turns 'inside out'. In some cases, but not all, a small outward initial imperfection, of the type considered, resulted in a small increase in the axisymmetric buckling load.

8.2 GENERAL CONCLUSIONS

The derivation of the nonlinear strain-displacement expressions presented in Chapter Two makes use of the first Kirchhoff assumption and the resulting strain-displacement expressions may be applied to thin shells of arbitrary shape. By extending the derivation of the strain-displacement expressions to include the terms that are quartic in displacements, the limitations present in linear thin shell theory may be lifted and for all technical applications 'exact' partial differential equations governing the nonlinear elastic behaviour of thin shells are available.

The conditions that are required at the boundaries of incomplete shells in order that the displacements within the shell are continuous have been considered. In general, thin shell theory does not allow the reference surface displacements and their derivatives to be uniquely determined at the boundary by the displacements of the boundary surface alone. Five boundary conditions are required for the unique definition of the reference surface displacements and their derivatives. The 'natural' boundary conditions of the variational calculus correspond to boundaries at which no work is done. If work is done at the boundary supports, then the appropriate energy components expressed in terms of boundary integrals must be included in the total potential energy of the system. Particular boundary conditions for the spherical shells considered in this thesis are introduced in Chapter Two.

In the development of the total potential energy functional contained in Chapter Two, the use of the second Kirchhoff assumption and of Hooke's Law for an isotropic linear elastic material, allow the development of the expression for the strain energy of a spherical shell. The particular expression given in Chapter Two contains all the terms in the displacements up to and including the fifth order terms in the membrane energy, and up to and including the cubic terms in the bending energy. The expression for the load potential energy for a uniform external pressure load acting on a spherical shell is also derived in Chapter Two, allowing the total potential energy functional to be expressed in terms of the middle surface displacements and their derivatives. The methods used in the development of the total potential energy functional for pressure loaded spherical shells contained in Chapter Two are applicable to shell structures composed of one or more incomplete shells of arbitrary shape under arbitrary loading.

For a system to be in a state of equilibrium it is necessary that the total potential energy of the system be stationary with respect to any small kinematically admissible displacement function. The calculus of variations provides the necessary and sufficient conditions for equilibrium of the system. The total potential energy functional provides the common starting point for both the fundamental and secondary path analyses. Application of the stationary conditions yield the nonlinear differential equations that govern equilibrium of the fundamental and secondary paths.

A perturbation method is developed for solving the nonlinear fundamental path problem. The linear terms of the secondary path equation give rise to a nonlinear eigenvalue problem. The eigenvalue problem is nonlinear in terms of the fundamental path displacements and load. A method for solving the nonlinear eigenvalue problem is presented in Chapter Five. The solution of the nonlinear eigenvalue problem, in which the eigenvalue is the bifurcation load and the eigenvector is the initial displacement vector of the secondary path at the point of bifurcation, presented in Chapter Five is identical to solving for the first term in the secondary path perturbation series.

The fundamental and secondary path equations that are the subject of the solution algorithms are both equilibrium equations, and the fundamental and secondary path solutions, obtained by solving the equilibrium equations, have been used to calculate various contributions to the strain and load terms of the total potential energy. For a conservative system the sum of these various energy contributions must be zero, therefore the error in the sum of these energy contributions may be used to provide a check on the solution method. This check, back substituting the displacement vector and load into the total potential energy functional, is significant as it is independent of the solution methods used.

The methods used to solve the nonlinear partial differential equation, the equilibrium equation, for pressure loaded spherical shells presented in Chapters Four and Five take advantage of the axisymmetric nature of the problem to develop efficient solution routines. However, the solution methods and techniques developed in Chapters Four and Five may also be used for shell structures composed of one or more incomplete thin elastic shells of arbitrary shape under arbitrary loading.

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APPENDIX A

AXISYMMETRIC ANALYSIS OF COMPLETE SPHERICAL SHELLS

APPENDIX A

AXISYMMETRIC ANALYSIS OF COMPLETE SPHERICAL SHELLS

A.1 POST BUCKLING ANALYSIS

Using the strain displacement relations

$$\varepsilon_\phi = \dot{u} - (w_f + w) + \tfrac{1}{2}\dot{w}^2 \qquad (A.1)$$

$$\varepsilon_\theta = u\cot\phi - (w_f + w)$$

$$\chi_\phi = \frac{1}{r}\ddot{w}$$

$$\chi_\theta = \frac{1}{r}\dot{w}\cot\phi$$

The strain energy, U, is then given by

$$U = \int_0^\pi 2\pi r^2 K \sin\phi \, d\phi \qquad (A.2)$$

where

$$K = \frac{Et}{2(1-\mu^2)}(\varepsilon_\phi^2 + \varepsilon_\theta^2 + 2\mu\varepsilon_\phi\varepsilon_\theta) + \frac{Et^3}{24(1-\mu^2)}(\chi_\phi^2 + \chi_\theta^2 + 2\mu\chi_\phi\chi_\theta) \qquad (A.3)$$

The load potential energy, J_L, is given by

$$J_L = -P2\pi r^3 \int_0^\pi w \sin\phi \, d\phi \qquad (A.4)$$

The total potential energy functional may now be written as,

$$V(u, w) = V_0 + V_1(u, w) + V_2(u, w) + V_3(u, w) + \ldots \qquad (A.5)$$

and the fundamental state is given by $V_1(u, w) = 0$, which yields

$$w_f = \frac{\lambda}{1+\mu} \qquad \text{where} \qquad \lambda = \frac{Pr(1-\mu^2)}{2Et} \qquad (A.6)$$

The total potential energy may now be written as

$$V(u, w, \lambda) = V_2(u, w, \lambda) + V_3(u, w) + V_4(w) \qquad (A.7)$$

where

$$V_2(u,w,\lambda) = \frac{\pi E t r^2}{1-\mu^2} \int_0^{\pi} [\,(\dot{u}+w)^2 + (u\cot\phi - w)^2 + 2\mu(\dot{u}-w)(u\cot\phi - w) + \alpha(\ddot{w}^2 + \dot{w}^2\cot^2\phi + 2\mu\ddot{w}\dot{w}\cot\phi) - \lambda\dot{w}^2]\,\sin\phi d\phi \qquad (A.8)$$

$$V_3(u,\ w) = \frac{\pi E t r^2}{1-\mu^2} \int_0^{\pi} [\dot{w}^2(\dot{u} - \mu u\cot\phi - (1+\mu)w)]\sin\phi d\phi$$

$$V_4(w) = \frac{\pi E t r^2}{1-\mu^2} \int_0^{\pi} \tfrac{1}{4}\dot{w}^4 \sin\phi d\phi$$

in which $\alpha = \dfrac{t^2}{12r^2}$

The displacements of the middle surface, u and w, may be expressed in terms of the Legendre polynomials $P_i(x)$.

$w(x) = a_i P_i(x)$
$u(x) = b_i z P_i(x)$

Where repeated subscripts imply summation, and using the following notation,

$x = \cos\phi, \qquad z = (1-x^2)^{\frac{1}{2}} = \sin\phi, \qquad \beta_i = i(i+1)$

$(\dot{\ }) = \dfrac{d(\ )}{dx}, \qquad V^*(u,w) = \dfrac{1-\mu^2}{\pi E t r^2} V(u,w)$

equation (A.8) may be written as

$$V_2^*(u,\ w,\ \lambda) = \int_{-1}^{+1} b_i^2[\beta_i^2 P_i^2 + 2(1-\mu)\{z^2\ddot{P}_i - x\dot{P}_i\}\dot{P}_i x] + a_i b_i[-2(1+\mu)\beta_i P_i^2) + a_i^2(2(1+\mu)P_i^2] + \alpha a_i^2[\beta_i P_i^2 + 2(1-\mu)(z^2\ddot{P}_i - x\dot{P}_i)\dot{P}_i x] - \lambda a_i^2[z^2\dot{P}_i^2]\,dx \qquad (A.9)$$

$$V_3^*(u,\ w) = \int_{-1}^{+1} [\,(1+\mu)xz^2\dot{P}_i\dot{P}_j\dot{P}_k b_i a_j a_k - z^4\ddot{P}_i\dot{P}_j\dot{P}_k b_i a_j a_k - (1+\mu)z^2 P_i\dot{P}_j\dot{P}_k a_i a_j a_k]\,dx$$

$$V_4^*(w) = \int_{-1}^{+1} \tfrac{1}{4} z^4 \dot{P}_i\dot{P}_j\dot{P}_k\dot{P}_\ell a_i a_j a_k a_\ell\,dx$$

Using the Legendre integrals given in Section A.2 we may write the total potential energy as follows,

$$V^*(a,\ b,\ \lambda) = V_2^*(a,\ b,\ \lambda) + V_3^*(a,\ b) + V_4^*(a)$$

and

$$V^*(a_i,\ b_i,\ \lambda) = \sum_{m=0,1}^{\infty} ({}^mC_1 a_m^2 + {}^mC_2 a_m b_m + {}^mC_3 b_m^2 + \lambda\, {}^mC_4 a_m^2) \qquad (A.10)$$
$$+ {}^1H_{ijk}\, b_i\, a_j\, a_k + {}^2H_{ijk} a_i a_k a_k + H_{ijk\ell} a_i a_j a_k a_\ell$$

where

$${}^mC_1 = \frac{2}{2m+1}\{2(1+\mu) + \alpha(\beta_m - (1-\mu)\beta_m)\}$$
$${}^mC_2 = \frac{2}{2m+1}\{-2(1+\mu)\beta_m\}$$
$${}^mC_3 = \frac{2}{2m+1}\{\beta_m^2 - (1-\mu)\beta_m\}$$
$${}^mC_4 = \frac{2}{2m+1}\{-\beta_m\}$$

The use of the condition $\delta V^*/\delta b_r = 0$ allows the elimination of the passive coordinate,

$$\frac{\delta V^*}{\delta b_r} = 0 \Rightarrow b_r = \frac{1}{2\,{}^rC_3}({}^rC_2 a_r + {}^1H_{rjk} a_j a_k) \qquad (A.11)$$

and

$$V^*(a_i,\ \lambda) = \sum_{m=0}^{\infty} ({}^m\bar{C}_1 + \lambda\, {}^m\bar{C}_2) a_m^2 + \tfrac{1}{2}\,{}^i\bar{C}_3\, {}^1H_{ijk} a_i a_j a_k \qquad (A.12)$$
$$+ \tfrac{1}{4}\,{}^i\bar{C}_4\, {}^1H_{ijk}\, {}^1H_{i\ell p}\, a_j a_k a_\ell a_p + {}^2H_{ijk} a_i a_j a_k + H_{ijk\ell}\, a_i a_j a_k a_\ell$$

where

$${}^m\bar{C}_1 = {}^mC_1 - \frac{({}^mC_2)^2}{4\,{}^mC_3}, \qquad {}^m\bar{C}_2 = {}^mC_4$$
$${}^i\bar{C}_3 = -\frac{{}^iC_2}{{}^iC_3}, \qquad {}^i\bar{C}_4 = -\frac{1}{{}^iC_3}$$

The condition $\delta V^*/\delta a_r = 0$ yields

$$2({}^r\bar{C}_1 + {}^r\bar{C}_2\lambda)a_r + \tfrac{1}{2}\,{}^r\bar{C}_3\, {}^1H_{rjk}\, a_j a_k + {}^i\bar{C}_3\, {}^1H_{irk}\, a_i a_k \qquad (A.13)$$
$$+ {}^2H_{rjk} a_j a_k + 2\,{}^2H_{irk}\, a_i a_k + {}^i\bar{C}_4\, {}^1H_{irk}\, {}^1H_{i\ell p}\, a_k a_\ell a_p + 4H_{rjk\ell}\, a_j a_k a_\ell = 0$$

Introducing the perturbation series

$$a_i(s) = (\dot{a}_N)s + \tfrac{1}{2}\ddot{a}_j s^2 + \tfrac{1}{6}\dddot{a}_j s^3 + \ldots \qquad \text{for } j \neq N \tag{A.14}$$

$$\lambda(s) = \lambda_{cr} + \dot{\lambda}s + \tfrac{1}{2}\ddot{\lambda}s^2 + \tfrac{1}{6}\dddot{\lambda}s^3 + \ldots$$

and identifying s with a_N:

$$s = a_N \quad \Rightarrow \quad \dot{a}_N = 1, \quad \ddot{a}_N = \dddot{a}_N = \ldots = 0; \quad \dot{a}_i = 0 \quad \text{for } i \neq N$$

By comparing coefficients of the powers of s, s^2, s^3, and s^4 we have

From the s term, or $\frac{\delta V^*}{\delta s} = 0$.

$$\lambda_{cr} = -\frac{{}^N\bar{C}_1}{{}^N\bar{C}_2} = \frac{\alpha\beta_N^2 + (1-\mu^2)}{\beta_N} \tag{A.15a}$$

where

$$\beta_N = N(N+1) = \sqrt{\frac{1-\mu^2}{\alpha}} \tag{A.15b}$$

then

$$\frac{\delta\lambda_{cr}}{\delta\beta_N} = 0 \quad \Rightarrow \quad (\lambda_{cr})_{min} = 2\sqrt{\alpha(1-\mu^2)} \tag{A.15c}$$

From the s^2 term, or $\frac{\delta^2 V^*}{\delta s^2} = 0$

$$\dot{\lambda}\Big|_{a_i=0}^{\lambda=\lambda_{cr}} = \frac{-1}{4\,{}^N\bar{C}_2}(3\,{}^N\bar{C}_3\,{}^1H_{NNN} + 6\,{}^2H_{NNN}) \tag{A.16a}$$

$$\ddot{a}_r\Big|^{s=0} = -\frac{{}^r\bar{C}_3\,{}^1H_{rNN} + 2\,{}^N\bar{C}_3\,{}^1H_{NrN} + 2\,{}^2H_{rNN} + 4\,{}^2H_{NrN}}{2({}^r\bar{C}_1 + {}^r\bar{C}_2\lambda_{cr})} \tag{A.16b}$$

From the s^3 term, or $\frac{\delta^3 V^*}{\delta s^3} = 0$

$$\ddot{\lambda}\Big|^{s=0} = \frac{1}{2\,{}^N\bar{C}_2}(2\,{}^N\bar{C}_3\,{}^1H_{NjN}\ddot{a}_j + {}^i\bar{C}_3\,{}^1H_{iNN}\ddot{a}_i + 2\,{}^i\bar{C}_4\,{}^1H_{iNN}\,{}^1H_{iNN} \tag{A.17a}$$

$$+ 2\,{}^2H_{iNN}\,\ddot{a}_i + 4\,{}^2H_{NNk}\ddot{a}_k + 8H_{NNNN})$$

and

$$\left.\dddot{a}_r\right|^{s=0} = \frac{-3}{2(\bar{C}_1 + \bar{C}_2\lambda_{cr})} \left(2\,\bar{C}_2\dot{\lambda}\ddot{a}_r + \bar{C}_3\,{}^1H_{rjN}\ddot{a}_j + \bar{C}_3\,{}^1H_{irN}\ddot{a}_i + \bar{C}_3\,{}^1H_{Nrk}\ddot{a}_k \right. \quad \text{(A.17b)}$$

$$+ 2\,\bar{C}_4\,{}^1H_{irN}\,{}^1H_{iNN} + 2\,{}^2H_{rjN}\ddot{a}_j + 2\,{}^2H_{irN}\ddot{a}_i$$

$$\left. + 2\,{}^2H_{Nrk}\ddot{a}_k + 8H_{rNNN} \right)$$

From the s^4 term, or $\frac{\delta^4 V^*}{\delta s^4} = 0$

$$\left.\ddot{\lambda}\right|^{s=0} = \frac{1}{8\bar{C}_2} \left(8\,\bar{C}_3\,{}^1H_{NjN}\dddot{a}_j + 3\,\bar{C}_3\,{}^1H_{Njk}\ddot{a}_j\ddot{a}_k + 4\,\bar{C}_3\,{}^1H_{iNN}\dddot{a}_i \right. \quad \text{(A.18)}$$

$$+ 6\,\bar{C}_3\,{}^1H_{iNk}\ddot{a}_i\ddot{a}_k + 12\,\bar{C}_4\,{}^1H_{iNk}\,{}^1H_{iNN}\ddot{a}_k + 24\,\bar{C}_4\,{}^1H_{iNN}\,{}^1H_{iLN}\ddot{a}_L$$

$$\left. + 16\,{}^2H_{NjN}\dddot{a}_j + 6\,{}^2H_{Njk}\ddot{a}_j\ddot{a}_k + 8\,{}^2H_{iNN}\dddot{a}_i + 12\,{}^2H_{iNk}\ddot{a}_i\ddot{a}_k + 144\,H_{jNNN}\ddot{a}_j \right)$$

A.2 THE LEGENDRE INTEGRAL FORMULAE

The Legendre equation and other related formulae are given by Byerly [19] The Legendre equation is,

$$z^2\ddot{P} - 2x\dot{P} + \beta P = 0\,, \quad \text{where} \quad z^2 = (1 - x^2) \tag{A.19}$$

and the Legendre polynomial of order N is given by,

$$P_N = \sum_{m=0}^{M} (-1)^m \frac{(2N-2m)!\, x^{(N-2m)}}{2^N m!(N-m)!(N-2m)!} \qquad \begin{cases} M = \frac{N}{2} & , \text{N even} \\ M = \frac{N-1}{2} & , \text{N odd} \end{cases} \tag{A.20}$$

Also,

$$\dot{P}_N = (2N-1)P_{N-1} + (2N-5)P_{N-3} + (2N-9)P_{N-5} + \ldots\ldots + \begin{cases} +3P_1 & , \text{N even} \\ +1 & , \text{N odd} \end{cases} \tag{A.21}$$

and,

$$(x^2 - 1)\dot{P}_N = NxP_N - NP_{N-1} \tag{A.22}$$

hence $x\dot{P}_N$ may be written in terms of P_j where $j \le N$, by using

$$x\dot{P}_N = NP_N + \dot{P}_{N-1} \tag{A.23}$$

The integral of the product of Legendre polynomials is given by

$$\int_{-1}^{+1} P_m P_n = \begin{cases} \frac{2}{2m+1} & , \ m = n \\ 0 & , \ m \neq n \end{cases} \tag{A.24}$$

And the integrals of the quadratic terms of the total potential energy may be evaluated using the following formulae,

$$\int_{-1}^{+1} (z^2\ddot{P}_i - x\dot{P}_i)x\dot{P}_i\, dx = \int_{-1}^{+1} (x^2 - 1)\dot{P}_i^2\, dx + \int_{-1}^{+1} \dot{P}_i^2\, dx - \int_{-1}^{+1} \beta_i x\dot{P}_i P_i\, dx \tag{A.25}$$

and

$$\int_{-1}^{+1} (x^2 - 1)\dot{P}_N^2 dx = \frac{-2}{2N+1}\beta_N\,, \qquad \int_{-1}^{+1} \dot{P}_N^2 dx = \beta_N\,, \qquad \int_{-1}^{+1} \beta_N x\dot{P}_N P_N dx = \left(\frac{2N}{2N+1}\right)\beta_N \tag{A.26}$$

Hence

$$\int_{-1}^{+1} (z^2\ddot{P}_i - x\dot{P}_i)x\dot{P}_i dx = \int_{-1}^{+1} (x\dot{P}_i - \beta_i P_i)x\dot{P}_i dx = \int_{-1}^{+1} -\tfrac{1}{2}\beta_i P_i^2\, dx = \int_{-1}^{+1} -\tfrac{1}{2}z^2\dot{P}_i^2 dx = \frac{-\beta_i}{2i+1} \tag{A.27}$$

In order to evaluate the integrals that are cubic and quartic in the Legendre polynomials P we will need to make use of the following product formulae given by Adams[20].

$$P_\alpha P_\beta = \sum_{s=0,1}^{s=t} \frac{A(\alpha-s)A(s)A(\beta-s)}{A(\alpha+\beta-s)} \frac{(2\alpha+2\beta-4s+1)}{(2\alpha+2\beta-2s+1)} P_{\alpha+\beta-2s} \tag{A.28}$$

where $t=\alpha$ if $\beta \geq \alpha$, or $t=\beta$ if $\alpha > \beta$, and

$$A(m) = \frac{1.3.5...(2m-1)}{1.2.3...m}; \qquad \text{with} \quad A(0)=1, \text{and} \quad A(-m)=0$$

Also

$$\dot{P}_\alpha \dot{P}_\beta = \sum_{r=0,1}^{r=u} \frac{B(\alpha-r-1)B(r)B(\beta-r-1)}{A(\alpha+\beta-r)} \frac{(2\alpha+2\beta-4r-1)}{(\alpha+\beta-2r-1)(\alpha+\beta-2r)} \dot{P}_{\alpha+\beta-2r-1} \tag{A.29}$$

where $u=\alpha-1$ if $\beta \geq \alpha$, or $u=\beta-1$ if $\alpha > \beta$, and

$$B(m) = \frac{1.3.5....(2m+1)}{1.2.3....m}; \qquad \text{with} \quad B(0)=1, \text{ and } B(-m)=0$$

The cubic integrals ${}^1H_{ijk}$ and ${}^2H_{ijk}$ may be written as follows,

$${}^1H_{ijk}={}^1H_{ikj} = \int_{-1}^{+1} [(1+\mu)xz^2\dot{P}_i\dot{P}_j\dot{P}_k - z^4\ddot{P}_i\dot{P}_j\dot{P}_k]dx \tag{A.30}$$

$$= \frac{j(j+1)}{(2j+1)} \int_{-1}^{+1} \{i(i+\mu)[P_iP_{j-1}\dot{P}_k - P_iP_{j+1}\dot{P}_k] + (1-\mu)[\dot{P}_{i-1}\dot{P}_kP_{j+1} - \dot{P}_{i-1}\dot{P}_kP_{j-1}]\}\, dx$$

and

$${}^2H_{ijk}={}^2H_{ikj} = \int_{-1}^{+1} -(1+\mu)z^2\dot{P}_i\dot{P}_j\dot{P}_k\, dx = \frac{j(j+1)}{(2j+1)} \int_{-1}^{+1} (1+\mu)[P_iP_{j+1}\dot{P}_k - P_iP_{j-1}\dot{P}_k]\, dx \tag{A.31}$$

The quartic integral, $H_{ijk\ell}$ may be written as,

$$H_{ijk\ell} = \int_{-1}^{+1} z^4\dot{P}_i\dot{P}_j\dot{P}_k\dot{P}_\ell dx \tag{A.32}$$

$$= \frac{k\ell(k+1)(\ell+1)}{(2k+1)(2\ell+1)} \int_{-1}^{+1} [\dot{P}_i\dot{P}_j(P_{k-1}P_{\ell-1} - P_{k-1}P_{\ell+1} + P_{k+1}P_{\ell+1} - P_{k+1}P_{\ell-1})]\, dx$$

Finally the integrals of the cubic and quartic Legendre polynomials may be evaluated as follows.

$$\int_{-1}^{+1} P_\alpha P_\beta \dot{P}_\delta dx = \sum_{s=0,1}^{t} \Omega \frac{A(\alpha-s)A(s)A(\beta-s)}{A(\alpha+\beta-s)} \frac{(2\alpha+2\beta-4s+1)}{(2\alpha+2\beta-2s+1)} \tag{A.33}$$

$\Omega = 2$ if $\alpha \geq 0$ and $\beta \geq 0$ and $\delta \geq 0$ and $\alpha+\beta+\delta = 1,3,5,\ldots$ (odd) and $\alpha+\beta-2s < \delta$

$\Omega = 0$ if $\alpha < 0$ or $\beta < 0$ or $\delta < 1$ or $\alpha+\beta+\delta = 2,4,6\ldots$(even) or $\alpha+\beta-2s \geq \delta$

$$\int_{-1}^{+1} \dot{P}_\alpha \dot{P}_\beta P_\delta dx = \sum_{r=0,1}^{r=u} \Omega \frac{B(\alpha-r-1)B(r)B(\beta-r-1)}{A(\alpha+\beta-r)} \frac{(2\alpha+2\beta-4r-1)}{(\alpha+\beta-2r-1)(\alpha+\beta-2r)} \tag{A.34}$$

$\Omega = 2$ if $\alpha \geq 1$ and $\beta \geq 1$ and $\delta \geq 0$ and $\alpha+\beta+\delta = 2,4,6\ldots$(even) and $\alpha+\beta-2r-1 > \delta$

$\Omega = 0$ if $\alpha < 1$ or $\beta < 1$ or $\delta < 0$ or $\alpha+\beta+\delta = 1,3,5\ldots$(odd) or $\alpha+\beta-2r-1 \leq \delta$

$$\int_{-1}^{+1} \dot{P}_\alpha \dot{P}_\beta P_\delta P_\gamma dx = \sum_{r=0,1}^{r=u} \sum_{s=0,1}^{s=t} \Omega \frac{B(\alpha-r-1)B(r)B(\beta-r-1)A(\delta-s)A(s)A(\gamma-s)}{A(\alpha+\beta-r)A(\delta+\gamma-s)} \otimes \tag{A.35}$$

$$\otimes \frac{(2\alpha+2\beta-4r-1)(2\delta+2\gamma-4s+1)}{(\alpha+\beta-2r+1)(\alpha+\beta-2r)(2\delta+2\gamma-2s+1)}$$

$\Omega = 2$ if $\alpha \geq 1$ and $\beta \geq 1$ and $\delta \geq 0$ and $\gamma \geq 0$ and $\alpha+\beta+\delta+\gamma = 0, 2, 4\ldots$(even) and $\alpha+\beta-2r-1 > \delta+\gamma-2s$

$\Omega = 0$ if $\alpha < 1$ or $\beta < 1$ or $\delta < 0$ or $\gamma < 0$ or $\alpha+\beta+\gamma+\delta = 1,3,5\ldots$ (odd) or $\alpha+\beta-2r-1 \leq \delta+\gamma-2s$

A.3 ENERGY CONTRIBUTIONS TO THE QUADRATIC TERMS OF THE TOTAL POTENTIAL ENERGY

The energy contributions to $V_2^*(u, w, \lambda)$ may be evaluated as follows.

$$U_{2M_\phi} = \frac{2}{2m+1}\left\{(1+\mu)+[2(m-\beta_m)-\mu\beta_m]\left(-\frac{{}^mC_2}{2^mC_3}\right)+(\beta_m^2-\frac{2m+1}{2}\beta_m+\tfrac{1}{2}\mu\beta_m)\left(\frac{{}^mC_2}{2^mC_3}\right)\right\}a_m^2$$

$$U_{2M_\theta} = \frac{2}{2m+1}\left\{(1+\mu)+(2m+\mu\beta_m)\left(\frac{{}^mC_2}{2^mC_3}\right)+(\frac{2m-1}{2}\beta_m+\tfrac{1}{2}\mu\beta_m)\left(\frac{{}^mC_2}{2^mC_3}\right)^2\right\}a_m^2$$

$$U_{2B_\phi} = \frac{2\alpha}{2m+1}\left\{\beta_m^2-\frac{2m+1}{2}\beta_m+\tfrac{1}{2}\mu\beta_m\right\}a_m^2 \qquad \text{(A.36)}$$

$$U_{2B_\theta} = \frac{2\alpha}{2m+1}\left\{\frac{2m-1}{2}\beta_m+\tfrac{1}{2}\mu\beta_m\right\}a_m^2$$

$$J_{2L} = \lambda\frac{2}{2m+1}\{-\beta_m\}a_m^2$$

APPENDIX B

COMPUTER PROGRAM, SOURCE AND OUTPUT LISTINGS

```
C      DAVID SMITH  SPHERICAL SHELL FINITE DIFF. ANALYSIS VAR. F.D. EXPRESSIONS   SPHERE      4
C      DAVID SMITH  SPHERICAL SHELL FINITE DIFF. ANALYSIS VAR. F.D. EXPRESSIONS   SPHERE      5
       DIMENSION HA(147,62),A(147,41),UF(147,160),UFP(147,160),U(147,7),          SPHERE      6
      CX(159),V(147),V2(147),V3(147),V4(147),P(7),AP(7,4),AR(12,27),Z3(14         SPHERE      7
      C7),Z4(159),PF(160),UFC(106,160),Z5(106),Z6(106)                           SPHERE      8
       COMMON/COM1/ AN(7,4,12),ITYPE,ILN(7),IMN(7),JSN(7),JFN(7),KPCF(7)         SPHERE      9
       COMMON/COM2/ D0,D1,D2,D3,D4,AL,THETA,PCI,N,AI,IER,ION                     SPHERE     10
       COMMON /COM3/ JAD,JAW,JABW,IE,ID,IC,FT,IZAP                               SPHERE     11
       COMMON /COM4/ IN,N1,N2                                                    SPHERE     12
       COMMON /COM5/ VF(147,20),VFB(147,20),VFC(106,20),PFC(20),EFC(20)          SPHERE     13
      C,SF(147)                                                                  SPHERE     14
       COMMON UF,UFB,UFC                                                         SPHERE     15
       LEVEL 2,UF,UFB,UFC                                                        SPHERE     16
                        CALL SECOND(AAA)                                         SPHERE     17
                        WRITE (6,8000) AAA                                       SPHERE     18
       DO 1 I=1,20                                                               SPHERE     19
       EFC(I)=0.0                                                                SPHERE     20
     1 PFC(I)=0.0                                                                SPHERE     21
       I87=147                                                                   SPHERE     22
       199=159                                                                   SPHERE     23
       KA=41                                                                     SPHERE     24
       KHA=62                                                                    SPHERE     25
       KUF=160                                                                   SPHERE     26
       KU=7                                                                      SPHERE     27
       KAP=27                                                                    SPHERE     28
       NUFC=106                                                                  SPHERE     29
       IFT=0                                                                     SPHERE     30
       READ (5,1000) N,NMB,NO,ITYPE,IPP,AI,D0,ION                                SPHERE     31
 1000 FORMAT (5I10,2F10.2,I10)                                                   SPHERE     32
       READ (5,1005) R,DIV                                                       SPHERE     33
 1005 FORMAT (2F10.2)                                                            SPHERE     34
       DO 20 I=1,ITYPE                                                           SPHERE     35
         DO 10 J=1,4                                                             SPHERE     36
           READ (5,1010) AP(I,J),(AN(I,J,K),K=1,7)                               SPHERE     37
           READ (5,1020) (AN(I,J,K),K=8,12)                                      SPHERE     38
   10    CONTINUE                                                                SPHERE     39
         READ(5,1030) IMN(I),ILN(I),JSN(I),JFN(I),KPCF(I)                        SPHERE     40
   20 CONTINUE                                                                   SPHERE     41
 1010 FORMAT(8F10.1)                                                             SPHERE     42
 1020 FORMAT(5F10.1)                                                             SPHERE     43
 1030 FORMAT (5I10)                                                              SPHERE     44
       READ (5,1040) IAD,IABW,ISTEP                                              SPHERE     45
 1040 FORMAT (3I10)                                                              SPHERE     46
       THETA=3.141592654/DIV                                                     SPHERE     47
       ALAMB=SQRT(12.0*(1.0-D0*D0))                                              SPHERE     48
       ALAMB=SQRT(ALAMB*R)*THETA                                                 SPHERE     49
       AL=1.0/(12.0*(R**2))                                                      SPHERE     50
       D1=1.0+AL                                                                 SPHERE     51
       D2=1.0-D0                                                                 SPHERE     52
       D3=1.0+D0                                                                 SPHERE     53
       D4=3.0-D0                                                                 SPHERE     54
       BN=N                                                                      SPHERE     55

       IN=N                                                                      SPHERE     56
       H=THETA/(BN-1.0)                                                          SPHERE     57
       PCI=-(1.0-D0*D0)/(SQRT(3.0*(1.0-D0*D0))*R)                                SPHERE     58
       NX=3*(N-2)                                                                SPHERE     59
       NY=NX+12                                                                  SPHERE     60
       N1=2*(N-2)                                                                SPHERE     61
       N2=N1+8                                                                   SPHERE     62
       JAD=(IAD*2)/3                                                             SPHERE     63
       JAW=2*JAD-1                                                               SPHERE     64
       JABW=(IABW*2)/3                                                           SPHERE     65
       IF=1                                                                      SPHERE     66
       ID=4                                                                      SPHERE     67
       IC=2                                                                      SPHERE     68
       WRITE (6,2000)                                                            SPHERE     69
 2000 FORMAT (1H1,10X,32H******** PARTIAL SPHERE ********  )                    SPHERE     70
       WRITE (6,2020) N,R,DO,THETA,ALAMB,IER                                     SPHERE     71
 2020 FORMAT (1H0,2X,16HNUMBER OF NODES=,I3,3X,4HR/T=,F8.3,3X,15HPOISSON         SPHERE     72
      CS RATIO=,F5.3,3X,11HCOPER ANGLE=,F7.3,3X,7HLAMBDA=,F7.3,3X,11HPERT        SPHERE     73
      CORDER=,I3)                                                                SPHERE     74
       DO 50 I=1,ITYPE                                                           SPHERE     75
         ITCI=ILN(I)                                                             SPHERE     76
         DO 50 J=1,4                                                             SPHERE     77
           AIT=AI**J                                                             SPHERE     78
           DO 50 K=1,ITCL                                                        SPHERE     79
             AN(I,J,K)=AN(I,J,K)/(AP(I,J)*AIT)                                   SPHERE     80
   50 CONTINUE                                                                   SPHERE     81
       AIT=AI                                                                    SPHERE     82
       AI=0.0                                                                    SPHERE     83
       ASTEP=ISTEP                                                               SPHERE     84
       CALL IMPRFT(Z5,NUFC,P,HA,A,KHA,KA,I87,AP,KAP,UF,UFB,KUF,Z6,V2,V3,         SPHERE     85
      CV4,V)                                                                     SPHERE     86
       CALL ENDETH (A,I87,KA,HA,KHA,X,I99,UF,KUF,AP,KAP,UFB,PF,Z4,ESN,ELM        SPHERE     87
      CT,E,V,UFC,NUFC,IFT,Z3,V2,V3,LNO,V4,Z5,Z6,U,KU)                            SPHERE     88
       AI=AIT                                                                    SPHERE     89
       AI=AI-ASTEP                                                               SPHERE     90
       KEEP=NO                                                                   SPHERE     91
       DO 370 L=1,NMB                                                            SPHERE     92
         NO=KEEP                                                                 SPHERE     93
         AI=AI+ASTEP                                                             SPHERE     94
         WRITE (6,2030)                                                          SPHERE     95
 2030 FORMAT (1H0, 25X,30H------------------------------ )                       SPHERE     96
         WRITE (6,2040) AI                                                       SPHERE     97
 2040 FORMAT(1H0,25X,10H----   AI=,F4.1,7H   ---- )                              SPHERE     98
         WRITE (6,2030)                                                          SPHERE     99
         AI2=AI**2                                                               SPHERE    100
         AI3=AI**3                                                               SPHERE    101
         AI4=AI**4                                                               SPHERE    102
         IF (AI.EQ.0.0) GO TO 60                                                 SPHERE    103
         IF (N1.EQ.NX) GO TO 60                                                  SPHERE    104
         N1=NX                                                                   SPHERE    105
         N2=NY                                                                   SPHERE    106
         JAD=IAD                                                                 SPHERE    107
         JAW=2*IAD-1                                                             SPHERE    108
         JABW=IABW                                                               SPHERE    109
         IE=2                                                                    SPHERE    110
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      DO 120 I=1,N1                                                     SPHERE     114
        DO 120 J=1,JAW                                                  SPHERE     115
120       A(I,J)=0.0                                                    SPHERE     116
      DO 140 I=1,12                                                     SPHERE     117
        DO 140 J=1,JAPW                                                 SPHERE     118
140       AP(I,J)=0.0                                                   SPHERE     119
      PA=0.0                                                            SPHERE     120
    AK=1.0                                                              SPHERE     121
    AJ=1.0                                                              SPHERE     122
    IF (ICN.EQ.20.OR.ICN.EQ.30) AK=2.0                                  SPHERE     123
    IF (ICN.EQ.20.OR.ICN.EQ.0) AJ=0.0                                   SPHERE     124
      DO 290 KK=1,ITYPE                                                 SPHERE     125
      CALL EDDATA (KK,ITCL,MID,ISTART,IFIN)                             SPHERE     126
        DO 280 IJ=ISTART,IFIN                                           SPHERE     127
          PA=PA+1.0                                                     SPHERE     128
         PHI=PA*H                                                       SPHERE     129
          S=SIN(PHI)                                                    SPHERE     130
          C=COS(PHI)                                                    SPHERE     131
          CT=C/S                                                        SPHERE     132
         K1=IC*IJ-IE                                                    SPHERE     133
         K2=K1+IE/2                                                     SPHERE     134
         K3=K1+IE                                                       SPHERE     135
    K4=3*IJ-2                                                           SPHERE     136
    E0=VE(K4,1)+(1.0-VE(K4,1))*(VE(K4+2,1)**2)/2.0+D0*VE(K4+1,1)        SPHERE     137
    EC=VE(K4+1,1)+D0*(VE(K4,1)+(1.0-VE(K4,1))*(VE(K4+2,1)**2)/2.0)      SPHERE     138
    E01=VEP(K4,1)+(1.0-VE(K4,1))*(VEP(K4+2,1)*VE(K4+2,1))-VEP(K4,1)*(V  SPHERE     139
   1E(K4+2,1)**2)/2.0+D0*VEP(K4+1,1)                                    SPHERE     140
    E0=E0*AK/2.0-AJ*SE(K4)/2.0                                          SPHERE     141
    E01=E01*AK/2.0-AJ*SE(K4+2)/2.0                                      SPHERE     142
    EC=E0*AK/2.0-AJ*SE(K4+1)/2.0                                        SPHERE     143
          DO 200 I=1,ITCL                                               SPHERE     144
          MA=JAD+IC*(I-MID)                                             SPHERE     145
          A(K1,MA)=A(K1,MA)+AN(KK,1,I)*D1*CT+AN(KK,2,I)*D1              SPHERE     146
          A(K1,MA+IE)=A(K1,MA+IE)+AN(KK,1,I)*(D3+AI*(D0+CT*CT)+AI2*AL/  SPHERE     147
C         (S*S))-AN(KK,2,I)*AL*CT-AN(KK,3,I)*AL                         SPHERE     148
          A(K3,MA-IE)=A(K3,MA-IE)+AN(KK,1,I)*(D3+AL*(D0+1.0/(S*S))+AI2  SPHERE     149
C         *AL/(S*S))-2.0*AL*AN(KK,2,I)*CT-AL*AN(KK,3,I)                 SPHERE     150
          A(K3,MA)=A(K3,MA)+AN(KK,1,I)*AI*CT*(1.0/(S*S)+D2+2.0*AI2/(S*  SPHERE     151
C         S))-AN(KK,2,I)*AL*(D3+CT*CT+2.0*AI2/(S*S))+2.0*AL*CT*AN(KK,3  SPHERE     152
C         ,I)+AL*AN(KK,1,I)                                             SPHERE     153
     IF(ION.EQ.-10) GO TO 150                                           SPHERE     154
      A(K1,MA+IE)=A(K1,MA+IE)-AN(KK,1,I)*E0                             SPHERE     155
      A(K3,MA-IE)=A(K3,MA-IE)-AN(KK,1,I)*E0                             SPHERE     156
      A(K3,MA)=A(K3,MA)+AN(KK,1,I)*(E01+E0*CT)+AN(KK,2,I)*E0            SPHERE     157
150 CONTINUE                                                            SPHERE     158
          IF(AI.EQ.0.0) GO TO 200                                       SPHERE     159
          A(K1,MA+1)=A(K1,MA+1)+AN(KK,1,I)*AI*D1*D3/(2.0*S)             SPHERE     160
          A(K2,MA-1)=A(K2,MA-1)-AN(KK,1,I)*AI*D1*D3/(2.0*S)             SPHERE     161
          A(K2,MA)=A(K2,MA)+AN(KK,1,I)*D1*D2*CT/2.0+AN(KK,2,I)*D1*D2/2  SPHERE     162
C         .0                                                            SPHERE     163
          A(K2,MA+1)=A(K2,MA+1)+AN(KK,1,I)*AI*AL*CT/S+AN(KK,2,I)*AI*AL  SPHERE     164
C         /S                                                            SPHERE     165
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          A(K3,MA-1)=A(K3,MA-1)+AN(KK,1,I)*AI*AL*CT/S-AN(KK,2,I)*AI*AL  SPHERE     166
C         /S                                                            SPHERE     167
      IF (ION.EQ.-10) GO TO 200                                         SPHERE     168
       A(K2,MA)=A(K2,MA)-AN(KK,1,I)*(E01+E0*CT)-AN(KK,2,I)*E0           SPHERE     169
200       CONTINUE                                                      SPHERE     170
         A(K1,JAD)=A(K1,JAD)-D1*(D0+CT*CT)-AI2*D1*D2/(S*S*2.0)          SPHERE     171
          A(K1,JAD+IE)=A(K1,JAD+IE)-2.0*AI2*AL*CT/(S*S)                 SPHERE     172
          A(K3,JAD-IE)=A(K3,JAD-IE)-AL*(CT/(S*S)+D2*CT)+D3*CT+AI2*AL*CT SPHERE     173
C         /(S*S)                                                        SPHERE     174
          A(K3,JAD)=A(K3,JAD)+2.0*D3-AL*AI2*(4.0*((CT/S)**2)+D4/(S*S))+ SPHERE     175
C         AL*AI4/(S**4)                                                 SPHERE     176
      IF (ION.EQ.-10) GO TO 210                                         SPHERE     177
       A(K1,JAD)=A(K1,JAD)+E0                                           SPHERE     178
       A(K3,JAD-IE)=A(K3,JAD-IE)-E01-E0*CT                              SPHERE     179
210 CONTINUE                                                            SPHERE     180
          IF(AI.EQ.0.0) GO TO 280                                       SPHERE     181
          A(K1,JAD+1)=A(K1,JAD+1)-AI*D1*D1*CT/(2.0*S)                   SPHERE     182
          A(K2,JAD-1)=A(K2,JAD-1)-AI*D1*D4*CT/(2.0*S)                   SPHERE     183
          A(K2,JAD)=A(K2,JAD)+D1*D2*(1.0-CT*CT)/2.0-AI2*D1/(S*S)        SPHERE     184
          A(K2,JAD+1)=A(K2,JAD+1)-AI*(D3-AL*D2)/S-AI3*AL/(S**3)         SPHERE     185
          A(K3,JAD-1)=A(K3,JAD-1)-AL*AI*(1.0/(S**3)+D2/S)+AI*D3/S+AL*AI SPHERE     186
C         3/(S**3)                                                      SPHERE     187
      IF (ION.EQ.-10) GO TO 280                                         SPHERE     188
       A(K1,JAD)=A(K1,JAD)+AI2*EC/(S*S)                                 SPHERE     189
       A(K1,JAD+1)=A(K1,JAD+1)+AI*EC*CT/S                               SPHERE     190
       A(K2,JAD-1)=A(K2,JAD-1)+AI*EC*CT/S                               SPHERE     191
       A(K2,JAD)=A(K2,JAD)+EC/(S*S)                                     SPHERE     192
       A(K2,JAD+1)=A(K2,JAD+1)+AI*EC/S                                  SPHERE     193
       A(K3,JAD-1)=A(K3,JAD-1)-AI*EC/S                                  SPHERE     194
       A(K3,JAD)=A(K3,JAD)-AI2*EC/(S*S)                                 SPHERE     195
280       CONTINUE                                                      SPHERE     196
290     CONTINUE                                                        SPHERE     197
C       THE BOUNDARY CONDITIONS                                         SPHERE     198
C       THE BOUNDARY CONDITIONS                                         SPHERE     199
        IWILL=AI                                                        SPHERE     200
        CALL BCOND (AB,KAB,IWILL,THETA,AL,H)                            SPHERE     201
C         END OF BOUNDARY CONDITIONS                                    SPHERE     202
C         END OF BOUNDARY CONDITIONS                                    SPHERE     203
                    CALL SECOND(AAA)                                    SPHERE     204
                    AAB=AAA                                             SPHERE     205
                    WRITE (6,8001) AAA                                  SPHERE     206
C                                                                       SPHERE     207
C                                                                       SPHERE     208
C         CONTROL OF SOLUTION AND ENERGY ROUTINES                       SPHERE     209
C         CONTROL OF SOLUTION AND ENERGY ROUTINES                       SPHERE     210
        IZAP=0                                                          SPHERE     211
      CALL REDUCE(A,AB,I87,KA,KAB,N1,AI,JAW,JAPW,IZAP,V,I87)           SPHERE     212
                    CALL SECOND(AAA)                                    SPHERE     213
                    AAC=AAA-AAB                                         SPHERE     214
                    AAB=AAA                                             SPHERE     215
                    WRITE (6,8002) AAA,AAC                              SPHERE     216
        IFAIL=0                                                         SPHERE     217
      CALL SLVITR(A,HA,I87,KA,KHA,UE,I87,KUE,UEP,I87,KUP,U,I87,KU,V2,V3,SPHERE     218
     CI87,X,I99,V,I87,E,KU,NC,IFAIL,AH,KAB,NIK,Z3,Z4,V4,PE,ESN,EIMI,E,  SPHERE     219
     CUEC,NUEC,IFT,LNC)                                                 SPHERE     220
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                      WRITE (6,8004) AAA,AAC                                    SPHERE      224
      IF (IFAIL.EQ.0) GO TO 360                                                 SPHERE      225
      WRITE (6,2050) L,IFAIL                                                    SPHERE      226
      IF(NIK.EQ.0) GO TO 370                                                    SPHERE      227
  360 CONTINUE                                                                  SPHERE      228
      CALL CONTRL(A,IB7,KA,V,IB7,X,I99,AR,KAR,UF,IB7,KUF,UFR,IB7,KUF,V2,        SPHERE      229
     CV3,IB7,V1,IB7,P,KU,NIK,U,IB7,KU,PF,Z3,UFC,NUFC,IFT,LNC,FIM1,Z4)           SPHERE      230
                      CALL SECOND(AAA)                                          SPHERE      231
                      AAC=AAA-AAB                                               SPHERE      232
                      AAB=AAA                                                   SPHERE      233
                      WRITE (6,8005) AAA,AAC                                    SPHERE      234
  370 CONTINUE                                                                  SPHERE      235
 2050 FORMAT (1H ,5X,24HSLVITR HAS FAILED FOR L=  ,I3,3X,6HIFAIL= ,I3)          SPHERE      236
 8000 FORMAT(1H ,2X,14HELAPSED TIME =  ,F10.4)                                  SPHERE      237
 8001 FORMAT(1H ,75X,26HEND OF SET UP, TOTAL TIME=  ,F9.3)                      SPHERE      238
 8002 FORMAT(1H ,60X,5HT.T.=,F7.2,2X,10HT. REDUCE=,F6.3)                        SPHERE      239
 8004 FORMAT(1H ,60X,5HT.T.=,F7.2,2X,10HT. SLVITR=,F6.3)                        SPHERE      240
 8005 FORMAT(1H ,60X,5HT.T.=,F7.2,2X,10HT. CONTRL=,F6.3)                        SPHERE      241
      STOP                                                                      SPHERE      242
      END                                                                       SPHERE      243
**    *DECK ENDPTH                                                              ENDPTH        1
      SUBROUTINE ENDPTH (A,NA,KA,HA,KHA,EV,NEV,U,KU,AR,KAR,UFR,PF,V,ESN,        ENDPTH        2
     CFIM1,F,VR,UFC,NUFC,IFT,Z3,V2,V3,LNC,V4,VIMP,Z6,XX,KX)                     ENDPTH        3
      DIMENSION A(NA,KA),HA(NA,KHA),EV(NEV),U(NA,KU),AR(12,KAR),UFR(NA,         ENDPTH        4
     CKU),PF(KU),V(NEV),VR(NA),UFC(NUFC,KU),Z3(NA),V2(NA),V3(NA),V4(NA)         ENDPTH        5
     C,VIMP(NUFC),Z6(NUFC),XX(NA,KX)                                            ENDPTH        6
      COMMON/COM1/ AN(7,4,12),ITYPE,ILN(7),IMI(7),JSN(7),JFN(7),KHOE(7)         ENDPTH        7
      COMMON /COM2/ D0,D1,D2,D3,D4,AL,THETA,PCL,M,AI,IER,ION                    ENDPTH        8
      COMMON /COM3/ JAC,JAW,JARW,IC,IE,IF,PI,IZAP                               ENDPTH        9
      COMMON /COM4/ IN,N1,N2                                                    ENDPTH       10
      COMMON /COM5/ VF(147,20),VFR(147,20),VFC(106,20),PFC(20),EFC(20)          ENDPTH       11
     C,SF(147)                                                                  ENDPTH       12
      LEVEL 2,U,UFR,UFC                                                         ENDPTH       13
      IFT=0                                                                     ENDPTH       14
      LNC=10                                                                    ENDPTH       15
      IPR1=30                                                                   ENDPTH       16
      NNN=3*(IN-2)                                                              ENDPTH       17
      MMM=IN+2                                                                  ENDPTH       18
      PA=0.0                                                                    ENDPTH       19
      DO 30 I=1,N1                                                              ENDPTH       20
        DO 30 J=1,JAW                                                           ENDPTH       21
   30     A(I,J)=0.0                                                           ENDPTH       22
      DO 40 I=1,12                                                              ENDPTH       23
        DO 40J=1,JARW                                                           ENDPTH       24
   40     AR(I,J)=0.0                                                          ENDPTH       25
      IF (ION.EQ.-10.OR.ION.EQ.-20) GO TO 65                                    ENDPTH       26
      I=1                                                                       ENDPTH       27
      CALL STEPCP(VIMP,NUFC,U,NA,KU,UFR,NA,KU,I)                                ENDPTH       28
      DO 50 I=1,N2                                                              ENDPTH       29
      VFC(I,1)=VIMP(I)                                                          ENDPTH       30
   50 VIMP(I)=0.0                                                               ENDPTH       31
      DO 60 I=1,NNN                                                             ENDPTH       32

      VF(I,1)=U(I,1)                                                            ENDPTH       33
   60 VFR(I,1)=UFR(I,1)                                                         ENDPTH       34
      GO TO 85                                                                  ENDPTH       35
   65 DO 70 I=1,N2                                                              ENDPTH       36
      VFC(I,1)=0.0                                                              ENDPTH       37
   70 VIMP(I)=0.0                                                               ENDPTH       38
      DO 80 I=1,NNN                                                             ENDPTH       39
      VF(I,1)=0.0                                                               ENDPTH       40
   80 VFR(I,1)=0.0                                                              ENDPTH       41
   85 CONTINUE                                                                  ENDPTH       42
      AK=1.0                                                                    ENDPTH       43
      AJ=1.0                                                                    ENDPTH       44
      IF (ION.EQ.20.OR.ION.EQ.30) AK=2.0                                        ENDPTH       45
      IF (ION.EQ.20.OR.ION.EQ.0) AJ=0.0                                         ENDPTH       46
      DO 110 KK=1,ITYPE                                                         ENDPTH       47
        CALL ECDATA (KK,ITCL,MID,ISTART,IFIN)                                   ENDPTH       48
        DO 100 IJ=ISTART,IFIN                                                   ENDPTH       49
          PA=PA+1.0                                                             ENDPTH       50
          PHI=PA*H                                                              ENDPTH       51
          S=SIN(PHI)                                                            ENDPTH       52
          C=COS(PHI)                                                            ENDPTH       53
          CT=C/S                                                                ENDPTH       54
        M1=3*IJ-2                                                               ENDPTH       55
        M2=M1+1                                                                 ENDPTH       56
        M3=M1+2                                                                 ENDPTH       57
          K=2*IJ-1                                                              ENDPTH       58
          K1=K+1                                                                ENDPTH       59
      F0=AK*(VF(M1,1)+0.5*VF(M3,1)**2+D0*VF(M2,1))-AJ*SF(M1)                    ENDPTH       60
      F01=AK*(VFR(M1,1)+VFR(M3,1)*VF(M3,1)+D0*VFR(M2,1))-AJ*SF(M3)              ENDPTH       61
          DO 90 I=1,ITCL                                                        ENDPTH       62
            MA=JAD+2*(I-MID)                                                    ENDPTH       63
          A(K,MA)=A(K,MA)+AN(KK,1,I)*D1*CT+AN(KK,2,I)*D1                        ENDPTH       64
          A(K,MA+1)=A(K,MA+1)+AN(KK,1,I)*(D3+AL*(D0+CT*CT))-AL*AN(KK,2,I        ENDPTH       65
     C     )*CT-AN(KK,3,I)*AL                                                   ENDPTH       66
          A(K1,MA-1)=A(K1,MA-1)+AN(KK,1,I)*(D3+AL*(D0+1.0/(S*S)))-2.0*AL        ENDPTH       67
     C*AN(KK,2,I)*CT-AL*AN(KK,3,I)                                              ENDPTH       68
          A(K1,MA)=A(K1,MA)+AN(KK,1,I)*(AL*(D2*CT+CT/(S*S)))-AL*AN(KK,2,        ENDPTH       69
     C     I)*(D0+1.0/(S*S))+2.0*AL*AN(KK,3,I)*CT+AL*AN(KK,4,I)                 ENDPTH       70
        IF (ION.EQ.-10) GO TO 90                                                ENDPTH       71
         A(K,MA+1)=A(K,MA+1)-AN(KK,1,I)*E0/2.0                                  ENDPTH       72
        A(K1,MA-1)=A(K1,MA-1)-AN(KK,1,I)*E0/2.0                                 ENDPTH       73
        A(K1,MA)=A(K1,MA)+AN(KK,1,I)*(E01+E0*CT)/2.0+AN(KK,2,I)*E0/2.0          ENDPTH       74
   90       CONTINUE                                                           ENDPTH       75
          A(K,JAD)=A(K,JAD)-D1*(CT*CT+D0)                                       ENDPTH       76
          A(K1,JAD-1)=A(K1,JAD-1)+CT*(D3-AL*(D2+1.0/(S*S)))                     ENDPTH       77
          A(K1,JAD)=A(K1,JAD)+2.0*D3                                            ENDPTH       78
      IF(ION.EQ.-10) GO TO 100                                                  ENDPTH       79
        A(K,JAD)=A(K,JAD)+EC/2.0                                                ENDPTH       80
        A(K1,JAD-1)=A(K1,JAD-1)-(E0*CT+E01)/2.0                                 ENDPTH       81
      IF (ION.EQ.-20) GO TO 100                                                 ENDPTH       82
      VIMP(K)=VF(M3,1)*(0.5*D2*VF(M3,1)*CT+VFR(M3,1)-VF(M1,1)-D0*VF(M2,1        ENDPTH       83
     1))+VF(M3,1)*(-VF(M3,1)*(0.5*VF(M3,1)+(1.C-D0/2.0)*VF(M1,1)*CT+D0*V        ENDPTH       84
     2F(M2,1)*CT/2.0+D0*VFR(M2,1)/2.0+VFR(M1,1))+VF(M1,1)*(-2.0*VFR(M3,1        ENDPTH       85
     3)+VF(M1,1)+D0*VF(M2,1))-D0*VF(M2,1)*VFR(M3,1))                            ENDPTH       86
      VIMP(K1)=VF(M3,1)*(0.5*D3*VF(M3,1)+VF(M1,1)*CT+D0*VF(M2,1)*CT+VFR(        ENDPTH       87
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     4,1))=F0*VFB(M1,1)*VF(M2,1))-VFB(M3,1)*VF(M1,1)*(VF(M1,1)+D0*VF(M2,
     51))
      E0=AK*(VF(M1,1)+D0*VF(M2,1))-AJ*SF(M1)
      E01=AK*(VFB(M1,1)+D0*VFB(M2,1))-AJ*SF(M3)
      VIMP(K)=VIMP(K)+VF(M3,1)*(VF(M3,1)*(E01+E0*CT)/4.0-VF(M1,1)*E0/2.0
     1+VFB(M3,1)*E0/2.0)
      VIMP(K1)=VIMP(K1)+VF(M3,1)*(VF(M3,1)*E0/4.0+VF(M1,1)*(E0*CT+E01)/2
     1.0+VFB(M1,1)*E0/2.0)+VFB(M3,1)*VF(M1,1)*E0/2.0
100       CONTINUE
110     CONTINUE
      IJ=0
      CALL PCOND (AB,KAB,IJ,THETA,AL,H)
      PT=1.0
      IZAP=0
      CALL REDUCE (A,AB,NA,KA,KAB,N1,AI,JAW,JABW,IZAP,V,NFV)
      IZAP=1
      PE=PCL
      I1=0
241   IFT=IFT+1
      I=1
      I1=I1+1
      PT=1.0
      DO 250 J=1,N1
       DO 250 K=1,JAW
250    HA(J,K)=A(J,K)
      CALL FNDLHS(HA,NA,KHA,V,NFV,IFT)
      CALL REDUCE (HA,AB,NA,KHA,KAB,N1,AI,JAW,[illegible],IZAP,V,NFV)
      CALL MAC7B (HA,V,NA,N1,JAW,PT,KHA,NFV)
      PT=0.0
      CALL BCKSUB (V,NFV,AB,KAB,AI,IN,JABW)
      CALL PERTBN (I,V,NFV,UFC,NUFC,PF,KU,PE,IFT,[illegible])
      CALL STRNCP (V,NFV,U,NA,KU,UFB,NA,KU,I1)
261   I=I+1
      I1=I1+1
      IF(I.GT.LNO) GO TO 270
      CALL FNDRHS (U,NA,KU,UFB,I,FV,NFV,PF,IFT,[illegible])
      CALL FNDPH2 (U,NA,KU,UFB,I,FV,NFV,PF,IFT,[illegible])
      CALL MAC7B (HA,FV,NA,N1,JAW,PT,KHA,NFV)
      CALL BCKSUB (FV,NFV,AB,KAB,AI,IN,JABW)
      CALL PERTBN (I,FV,NFV,UFC,NUFC,PF,KU,PE,IFT,[illegible])
      CALL STRNCP (FV,NFV,U,NA,KU,UFB,NA,KU,I1)
      GO TO 261
270   CONTINUE
      I1=I1-1
      CALL CNVRGE (UFC,PF,NUFC,KU,U,UFB,NA,IFT,[illegible],VB,NA,Z3,NA
     C,V2,NA,V3,NA,V4,NA,A,NA,KA,VIMP)
      EHLD=E
      EFC(IFT)=E
      IF(IFT.NE.1) EFC(IFT)=E+EFC(IFT-1)
      ELMT=EFC(IFT)
      IK=(IFT-1)*LNO
      P=PFC(IFT)+PF(1+IK)*E+PF(2+IK)*E**2
      PE=PF(1+IK)+2.0*PF(2+IK)*E
      PMX=2.0*PF(2+IK)
      DO 280 I=3,LNO
      K=I-1
      K1=I-2
      AJ=I
      AK=K
      P=P+PF(I+IK)*(E**I)
      PE=PE+AJ*PF(I+IK)*(E**K)
280   PMX=PMX+AJ*AK*PF(I+IK)*(E**K1)
      ITMP=LNO*(IFT+1)
      IF(ITMP.GT.IPR.OR.IFT.GE.20) GO TO 308
      PFC(IFT+1)=P
      TEMP=ABS(P/PCL)
      IF(TEMP.GT.1.25) GO TO 308
      IF(P.GT.0.0) GO TO 308
      E=EFC(IFT)
      EMN=0.0
      ITMP=0
      CALL FUNVEC(U,UFB,KU,NA,UFC,NUFC,V,FV,NFV,[illegible],IFT,LNO,E,E
     CMN,IN,P,ITMP,IFAIL,ELMT)
      DO 290 J=1,NNN
      VF(J,IFT+1)=V(J)
290   VFB(J,IFT+1)=FV(J)
      DO 300 J=1,N2
300   VFC(J,IFT+1)=VB(J)
      CALL IMPRVE (VB,V,FV,NA,NFV,Z3,V2,V3,V4,[illegible],IFT,[illegible],KAB,
     CU,UFB,KU,UFC,NUFC,PF,LNO,LNC,F,EMN,IFAIL,[illegible])
      E=EFC(IFT)
      IF(IFT.GT.1) E=EFC(IFT)-EFC(IFT-1)
      EHLD=E
      ELMT=EFC(IFT)
      IK=(IFT-1)*LNO
      P=PFC(IFT)+PF(1+IK)*E+PF(2+IK)*E**2
      PE=PF(1+IK)+2.0*PF(2+IK)*E
      PMX=2.0*PF(2+IK)
      DO 305 I=3,LNO
      K=I-1
      K1=I-2
      AJ=I
      AK=K
      P=P+PF(I+IK)*(E**I)
      PE=PE+AJ*PF(I+IK)*(E**K)
305   PMX=PMX+AJ*AK*PF(I+IK)*(E**K1)
308   WRITE (6,2000) IFT,EHLD,EFC(IFT),P,PE,PMX
      ITMP=LNO*(IFT+1)
      IF(ITMP.GT.IPR.OR.IFT.GE.20) GO TO 320
      TEMP=ABS(P/PCL)
      IF(TEMP.GT.1.15) GO TO 340
      IF (P.GT.0.0) GO TO 335
      GO TO 241
C     ****   LACK OF STORAGE    *****
320   WRITE (6,2020)
      GO TO 350
C     **** P.GT.0.0  *****
```

```
  340 WRITE (6,2050)                                                    FNDPTH      200
C  ******  EVALUATE F AND CHECK FOR TURNING POINTS  ***                 FNDPTH      201
  350 AK=1.0                                                            FNDPTH      202
      SAV=0.0                                                           FNDPTH      203
      K=0                                                               FNDPTH      204
      WRITE (6,2060)                                                    FNDPTH      205
      ITMP=0                                                            FNDPTH      206
      PMN=0.0                                                           FNDPTH      207
      DO 390 I=1,IFT                                                    FNDPTH      208
      IF(I.EQ.1) GO TO 360                                              FNDPTH      209
      EMN=EFC(I)-EFC(I-1)                                               FNDPTH      210
      EMX=EFC(I-1)                                                      FNDPTH      211
      GO TO 370                                                         FNDPTH      212
  360 EMN=EFC(1)                                                        FNDPTH      213
      EMX=0.0                                                           FNDPTH      214
  370 DO 380 J=1,8                                                      FNDPTH      215
      AJ=J                                                              FNDPTH      216
      E=EMX+AJ*EMN/8.0                                                  FNDPTH      217
      CALL FUNVEC (U,UFB,KU,NA,UFC,NUFC,V,FV,NFV,VB,NA,PF,IQ,IFT,LNO,E,  FNDPTH      218
     CPMN,IN,P,ITMP,IFAIL,ELMT)                                         FNDPTH      219
      Z3(10+J)=P                                                        FNDPTH      220
      Z3(J)=P/PCL                                                       FNDPTH      221
      Z3(20+J)=E                                                        FNDPTH      222
      CALL VOLDIS(VB,NA,V,NFV,AJ,EMX)                                   FNDPTH      223
      Z3(30+J)=AJ                                                       FNDPTH      224
      Z3(40+J)=PMX                                                      FNDPTH      225
      TEMP=AK*Z3(J)                                                     FNDPTH      226
      IF(TEMP.GT.SAV) SAV=TEMP                                          FNDPTH      227
      IF(TEMP.EQ.SAV) GO TO 380                                         FNDPTH      228
      K=K+1                                                             FNDPTH      229
      Z3(100+K)=E-EMN/8.0                                               FNDPTH      230
      Z3(50+K)=EMN/8.0                                                  FNDPTH      231
      AK=-AK                                                            FNDPTH      232
      SAV=-TEMP                                                         FNDPTH      233
  380 CONTINUE                                                          FNDPTH      234
      WRITE (6,2070) I,(Z3(J),J=21,28)                                  FNDPTH      235
      WRITE (6,2080) (Z3(J),J=11,18)                                    FNDPTH      236
      WRITE (6,2090) (Z3(J),J=1,8)                                      FNDPTH      237
      WRITE (6,2091) (Z3(J),J=31,38)                                    FNDPTH      238
      WRITE (6,2092) (Z3(J),J=41,48)                                    FNDPTH      239
  390 CONTINUE                                                          FNDPTH      240
      ITMP=-1                                                           FNDPTH      241
      IQ=LNC                                                            FNDPTH      242
      IF(K.EQ.0) GO TO 490                                              FNDPTH      243
C  ***    WE HAVE FOUND TURNING POINTS *****                            FNDPTH      244
      AK=1.0                                                            FNDPTH      245
      DO 460 I=1,K                                                      FNDPTH      246
      AK=-AK                                                            FNDPTH      247
      PP=Z3(50+I)                                                       FNDPTH      248
      E=Z3(100+I)                                                       FNDPTH      249
  440 EMN=E-PP/2.0                                                      FNDPTH      250
      EMX=E+PP/2.0                                                      FNDPTH      251
                                                                        FNDPTH      252
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```
      CALL FUNVEC (U,UFB,KU,NA,UFC,NUFC,V,FV,NFV,VB,NA,PF,IQ,IFT,LNO,EMN  FNDPTH      253
     C,PMN,IN,P,ITMP,IFAIL,ELMT)                                        FNDPTH      254
      V(1)=P*AK                                                         FNDPTH      255
      CALL FUNVEC (U,UFB,KU,NA,UFC,NUFC,V,FV,NFV,VB,NA,PF,IQ,IFT,LNO,E,P  FNDPTH      256
     CMN,IN,P,ITMP,IFAIL,ELMT)                                          FNDPTH      257
      V(2)=P*AK                                                         FNDPTH      258
      CALL FUNVEC (U,UFB,KU,NA,UFC,NUFC,V,FV,NFV,VB,NA,PF,IQ,IFT,LNO,EMX  FNDPTH      259
     C,PMN,IN,P,ITMP,IFAIL,ELMT)                                        FNDPTH      260
      V(3)=P*AK                                                         FNDPTH      261
      IF(V(2).GT.V(1).AND.V(2).GT.V(3)) GO TO 450                       FNDPTH      262
      IF(V(2).LT.V(1)) E=EMN                                            FNDPTH      263
      IF(V(2).LT.V(3)) E=EMX                                            FNDPTH      264
      GO TO 440                                                         FNDPTH      265
  450 TEMP=ABS((V(2)-V(3))/PCL)                                         FNDPTH      266
      BEMP=ABS((V(2)-V(1))/PCL)                                         FNDPTH      267
      PP=PP/2.0                                                         FNDPTH      268
      IF(BEMP.LT.TEMP) BEMP=TEMP                                        FNDPTH      269
      IF(BEMP.GT.0.01) GO TO 440                                        FNDPTH      270
      V(50+I)=P                                                         FNDPTH      271
  460 CONTINUE                                                          FNDPTH      272
      WRITE (6,3000)                                                    FNDPTH      273
      ITMP=0                                                            FNDPTH      274
      DO 480 I=1,K                                                      FNDPTH      275
      E=V(50+I)                                                         FNDPTH      276
      CALL FUNVEC(U,UFB,KU,NA,UFC,NUFC,V2,Z3,NA,VB,NA,PF,IQ,IFT,LNO,E,P  FNDPTH      277
     CMN,IN,P,ITMP,IFAIL,ELMT)                                          FNDPTH      278
      PE=P/PCL                                                          FNDPTH      279
      ESN=E                                                             FNDPTH      280
      DO 461 JK=1,20                                                    FNDPTH      281
       IK=21-JK                                                         FNDPTH      282
  461  IF(E.LE.EFC(IK)) IKK=IK                                          FNDPTH      283
      IF(IKK.GT.1) ESN=ESN-EFC(IKK-1)                                   FNDPTH      284
      IK=LNC*(IKK-1)                                                    FNDPTH      285
      DO 462 JK=1,N2                                                    FNDPTH      286
  462  XX(JK,I)=UFC(JK,IK+1)                                            FNDPTH      287
      DO 463 JK=2,LNO                                                   FNDPTH      288
       AJ=JK                                                            FNDPTH      289
      A1=ESN**(JK-1)                                                    FNDPTH      290
       DO 463 J=1,N2                                                    FNDPTH      291
  463   XX(J,I)=XX(J,I)+AJ*UFC(J,IK+JK)*A1                              FNDPTH      292
      TEMP=0.0                                                          FNDPTH      293
      DO 464 J=1,N2                                                     FNDPTH      294
       SAV=ABS(XX(J,I))                                                 FNDPTH      295
       IF (SAV.LE.TEMP) GO TO 464                                       FNDPTH      296
       IK=J                                                             FNDPTH      297
       TEMP=SAV                                                         FNDPTH      298
  464 CONTINUE                                                          FNDPTH      299
      DO 465 J=1,N2                                                     FNDPTH      300
  465  XX(J,I)=XX(J,I)/TEMP                                             FNDPTH      301
      WRITE (6,3010) I,E,P,PE                                           FNDPTH      302
      WRITE (6,3020)                                                    FNDPTH      303
      DO 470 J=1,MMM                                                    FNDPTH      304
      JK=J-1                                                            FNDPTH      305
      IU=2*J-1                                                          FNDPTH      306
      IW=IU+1                                                           FNDPTH      307
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      IF(E.LE.EFC(J)) IK=J                                              FNDPTH      311
  471 CONTINUE                                                          FNDPTH      312
      KK=(IK-1)*LNO                                                     FNDPTH      313
      A1=E                                                              FNDPTH      314
      IF(IK.GT.1) A1=E-EFC(IK-1)                                        FNDPTH      315
      DO 472 J=1,NNN                                                    FNDPTH      316
      V3(J)=U(J,KK+1)                                                   FNDPTH      317
  472 V4(J)=UFB(J,KK+1)                                                 FNDPTH      318
      DO 473 JK=2,LNO                                                   FNDPTH      319
      AK=JK                                                             FNDPTH      320
      KJ=JK-1                                                           FNDPTH      321
      AJ=AK**(A1**KJ)                                                   FNDPTH      322
      DO 473 J=1,NNN                                                    FNDPTH      323
      V3(J)=V3(J)+U(J,KK+JK)*AJ                                         FNDPTH      324
  473 V4(J)=V4(J)+UFB(J,KK+JK)*AJ                                       FNDPTH      325
      MRK=0                                                             FNDPTH      326
      CALL TFEV01(V3,V4,NA,V2,Z3,NA,FV,NFV,P,MRK)                       FNDPTH      327
      WRITE (6,4000)                                                    FNDPTH      328
      WRITE (6,4015)  (FV(J),J=1,5)                                     FNDPTH      329
      WRITE (6,4020) (FV(J),J=6,10)                                     FNDPTH      330
      WRITE (6,4030)                                                    FNDPTH      331
      WRITE (6,4010) (FV(J),J=51,55),FV(61)                             FNDPTH      332
      WRITE(6,4020) (FV(J),J=56,60)                                     FNDPTH      333
      DO 474 J=1,10                                                     FNDPTH      334
  474 FV(20+J)=FV(J)                                                    FNDPTH      335
      MRK=-1                                                            FNDPTH      336
      A1=E-0.05                                                         FNDPTH      337
      CALL FUNVEC (U,UFB,KU,NA,UFC,NUFC,V2,Z3,NA,VB,NA,PF,IQ,IFT,LNO,A1,FNDPTH      338
     CMN,IN,F,ITMP,IFAIL,ELMT)                                          FNDPTH      339
      CALL TFEV01 (V3,V4,NA,V2,Z3,NA,FV,NFV,P,MRK)                      FNDPTH      340
      EMN=0.0                                                           FNDPTH      341
      PE=0.0                                                            FNDPTH      342
      DO 476 J=1,5                                                      FNDPTH      343
      FV(40+J)=FV(20+J)-FV(J)                                           FNDPTH      344
      IF(FV(40+J).LE.0.0) GO TO 475                                     FNDPTH      345
      PE=PE+FV(40+J)                                                    FNDPTH      346
      GO TO 476                                                         FNDPTH      347
  475 EMN=EMN+FV(40+J)                                                  FNDPTH      348
  476 CONTINUE                                                          FNDPTH      349
      PE=(PE-EMN)/2.0                                                   FNDPTH      350
      DO 477 J=1,5                                                      FNDPTH      351
  477 FV(45+J)=FV(40+J)/PE                                              FNDPTH      352
      WRITE (6,4040)                                                    FNDPTH      353
      WRITE (6,4015) (FV(J),J=41,45)                                    FNDPTH      354
      WRITE(6,4020) (FV(J),J=46,50)                                     FNDPTH      355
  480 CONTINUE                                                          FNDPTH      356
      ESN=V(51)                                                         FNDPTH      357
      GO TO 495                                                         FNDPTH      358
  490 WRITE (6,3040)                                                    FNDPTH      359
      ESN=2.0*ELMT                                                      FNDPTH      360
  495 ITMP=0                                                            FNDPTH      361
      MRK=-1                                                            FNDPTH      362
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      EMN=ELMT/4.0                                                      FNDPTH      363
      E=0.0                                                             FNDPTH      364
      E=E+EMN                                                           FNDPTH      365
      CALL FUNVEC(U,UFB,KU,NA,UFC,NUFC,V4,FV,NA,VB,NA,PF,IQ,IFT,LNO,E,P FNDPTH      366
     CMN,IN,F,ITMP,IFAIL,ELMT)                                          FNDPTH      367
      V(1)=F                                                            FNDPTH      368
      V(5)=P                                                            FNDPTH      369
      V(9)=P/PCL                                                        FNDPTH      370
      F=F+EMN                                                           FNDPTH      371
      CALLTFEV01 (Z3,VB,NA,V4,FV,NA,Z6,NUFC,P,MRK)                      FNDPTH      372
      DO 496 J=1,10                                                     FNDPTH      373
  496 Z6(60+J)=Z6(J)                                                    FNDPTH      374
      CALL FUNVEC(U,UFB,KU,NA,UFC,NUFC,V4,FV,NA,Z3,NA,PF,IQ,IFT,LNO,E,P FNDPTH      375
     CMN,IN,F,ITMP,IFAIL,ELMT)                                          FNDPTH      376
      V(2)=E                                                            FNDPTH      377
      V(6)=P                                                            FNDPTH      378
      V(10)=P/PCL                                                       FNDPTH      379
      E=E+EMN                                                           FNDPTH      380
      CALL TFEV01 (Z3,VB,NA,V4,FV,NA,Z6,NUFC,P,MRK)                     FNDPTH      381
      DO 497 J=1,10                                                     FNDPTH      382
  497 Z6(40+J)=Z6(J)                                                    FNDPTH      383
      CALL FUNVEC (U,UFB,KU,NA,UFC,NUFC,V4,FV,NA,V2,NA,PF,IQ,IFT,LNO,E,P FNDPTH      384
     CMN,IN,F,ITMP,IFAIL,ELMT)                                          FNDPTH      385
      V(3)=E                                                            FNDPTH      386
      V(7)=P                                                            FNDPTH      387
      V(11)=P/PCL                                                       FNDPTH      388
      E=E+EMN                                                           FNDPTH      389
      CALL TFEV01 (Z3,VB,NA,V4,FV,NA,Z6,NUFC,P,MRK)                     FNDPTH      390
      DO 498 J=1,10                                                     FNDPTH      391
  498 Z6(20+J)=Z6(J)                                                    FNDPTH      392
      CALL FUNVEC (U,UFB,KU,NA,UFC,NUFC,V4,FV,NA,V3,NA,PF,IQ,IFT,LNO,E,P FNDPTH      393
     CMN,IN,F,ITMP,IFAIL,ELMT)                                          FNDPTH      394
      CALL TFEV01 (Z3,VB,NA,V4,FV,NA,Z6,NUFC,P,MRK)                     FNDPTH      395
      V(4)=E                                                            FNDPTH      396
      V(8)=P                                                            FNDPTH      397
      V(12)=P/PCL                                                       FNDPTH      398
      WRITE (6,3050) (V(J),J=1,4)                                       FNDPTH      399
      WRITE (6,3051) (V(J),J=5,8)                                       FNDPTH      400
      WRITE (6,3052) (V(J),J=9,12)                                      FNDPTH      401
      WRITE (6,3060)                                                    FNDPTH      402
      DO 500 I=1,NNN                                                    FNDPTH      403
      JK=I-1                                                            FNDPTH      404
      IU=2*I-1                                                          FNDPTH      405
      IW=IU+1                                                           FNDPTH      406
  500 WRITE (6,3070) JK,VB(IU),VB(IW),Z3(IU),Z3(IW),V2(IU),V2(IW),V3(IU)FNDPTH      407
     C,V3(IW),JK                                                        FNDPTH      408
      F=0.0                                                             FNDPTH      409
      WRITE (6,5070)                                                    FNDPTH      410
      WRITE (6,5000) Z6(61),Z6(41),Z6(21),Z6(1)                         FNDPTH      411
      WRITE (6,5010) Z6(66),Z6(46),Z6(26),Z6(6)                         FNDPTH      412
      WRITE(6,5020) Z6(62),Z6(42),Z6(22),Z6(2)                          FNDPTH      413
      WRITE(6,5010) Z6(67),Z6(47),Z6(27),Z6(7)                          FNDPTH      414
      WRITE (6,5030) Z6(63),Z6(43),Z6(23),Z6(3)                         FNDPTH      415
      WRITE (6,5010) Z6(68),Z6(48),Z6(28),Z6(8)                         FNDPTH      416
      WRITE (6,5040) Z6(64),Z6(44),Z6(24),Z6(4)                         FNDPTH      417
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      RETURN                                                            ENDPTH     420
 2000 FORMAT (1H0,5X,14HPERT PERT PARM ,I2,13H MAX LOCAL E= ,E10.4,9H TOT  ENDPTH     421
     CAL E=,E10.4,12H AT WHICH P=,E10.4,6H PDIF=,E10.4,7H PDIF2=,E10.4)     ENDPTH     422
 2020 FORMAT (1H ,5X,14HPERT METHOD COMPLETE,  ** LACK OF STORAGE **  )     ENDPTH     423
 2040 FORMAT (1H ,5X,36HPERT METHOD COMPLETE,  ** P.GT.0.0 **      )        ENDPTH     424
 2050 FORMAT (1H ,5X,39HPERT METHOD COMPLETE,  ** P TO LARGE **    )        ENDPTH     425
 2060 FORMAT (1H0,45X,6HE-VS-E  )                                           ENDPTH     426
 2070 FORMAT (1H0,2X,14HPERT PERT PARM= ,I2,5X,2HE=,8(2X,E10.4))            ENDPTH     427
 2080 FORMAT (1H ,22X,2HE=,8(2X,E10.4))                                     ENDPTH     428
 2090 FORMAT (1H ,18X,6HE/PCL=,8(2X,E10.4))                                 ENDPTH     429
 2091 FORMAT (1H ,20X,4HVCL=,8(2X,F10.4))                                   ENDPTH     430
 2092 FORMAT (1H ,20X,4HDIS=,8(2X,E10.4))                                   ENDPTH     431
 3000 FORMAT (1H0,5X,42HWE HAVE FOUND THE FOLLOWING TURNING POINTS  )       ENDPTH     432
 3010 FORMAT (1H0,5X,4HNO. ,I2,6H AT E=,E10.4,3H P=,E10.4,7H P/PCL=,E10.    ENDPTH     433
     C4,15X,13HBUCKLING MODE )                                              ENDPTH     434
 3020 FORMAT (1H ,4X,2(4HNODE,10X,1HU,14X,1HW,10X,4HNODE,10X))              ENDPTH     435
 3030 FORMAT (1H ,5X,2(I2,5X,E12.6,4X,E12.6,5X,I2,10X))                     ENDPTH     436
 3040 FORMAT (1H0,5X,31HWE HAVE FOUND NO TURNING POINTS  )                  ENDPTH     437
 3050 FORMAT (1H0,15X,4(2HE=,E12.6,11X))                                    ENDPTH     438
 3051 FORMAT (1H ,15X,4(2HE=,E12.6,11X))                                    ENDPTH     439
 3052 FORMAT (1H ,12X,4(6HE/PCL=,E12.6,6X))                                 ENDPTH     440
 3060 FORMAT (1H ,5X,4HNODE,6X,3(1HU,11X,1HW,11X),1HU,11X,1HW,5X,4HNODE)    ENDPTH     441
 3070 FORMAT (1H ,6X,I2,2X,8(F10.4,2X),I2)                                  ENDPTH     442
 4000 FORMAT(1H0,3X,16HT.E.E. VO    UMO,10X,3HUMO,10X,3HUBO,10X,3HUBO,10    ENDPTH     443
     CX,4HL.E. )                                                            ENDPTH     444
 4010 FORMAT (1H ,16X,5(E10.4,3X),4X,13HENERGY ERROR= ,F7.3,1X,8HPER CEN    ENDPTH     445
     CT)                                                                    ENDPTH     446
 4015 FORMAT (1H ,16X,5(E10.4,3X))                                          ENDPTH     447
 4020 FORMAT (1H ,16X,5(E10.4,3X))                                          ENDPTH     448
 4030 FORMAT (1H ,3X,16HT.E.E. V1    UMO,10X,3HUMO,10X,3HUBO,10X,3HUBC,1    ENDPTH     449
     C0X,4HL.E. )                                                           ENDPTH     450
 4040 FORMAT(1H0,1X,18HTEE DELTA VO   UMO,10X,3HUMO,10X,3HUBO,10X,3HUBC,    ENDPTH     451
     "10X,4HL.E.)                                                           ENDPTH     452
 5000 FORMAT(1H0,13X,4(5HUMO= ,E10.4,9X))                                   ENDPTH     453
 5010 FORMAT(1H 18X,4(E10.4,14X))                                           ENDPTH     454
 5020 FORMAT (1H0,13X,4(5HUMO= ,E10.4,9X))                                  ENDPTH     455
 5030 FORMAT (1H0,13X,4(5HUBO= ,E10.4,9X))                                  ENDPTH     456
 5040 FORMAT (1H0,13X,4(5HUBC= ,E10.4,9X))                                  ENDPTH     457
 5050 FORMAT(1H0,12X,4(6HL.E.= ,E10.4,8X))                                  ENDPTH     458
 5070 FORMAT (1H0,10X,36H---- TOTAL POTENTIAL ENERGY, VO ----  )            ENDPTH     459
      END                                                                   ENDPTH     460
      END                                                                   ENDPTH     461
*    *DECK IMPRFT                                                          IMPRFT       1
      SUBROUTINE IMPRFT(V,NUEC,R,HA,A,KHA,KA,NA,AB,KAPW,UF,UFB,KU,B,V2,     IMPRFT       2
     CV3,V4,V5)                                                             IMPRFT       3
      DIMENSION V(NUEC),HA(NA,KHA),A(NA,KA),AB(12,KAPW),UF(NA,KU),UFB(NA    IMPRFT       4
     *,KU),B(NUEC),V2(NA),V3(NA),V4(NA),V5(NA)                              IMPRFT       5
      COMMON /COM1/ AN(7,4,12),ITYPE,ION(7),IMA(7),JSN(7),JEN(7),KBOE(7)    IMPRFT       6
      COMMON/COM2/ DO,D1,D2,D3,D4,AL,THETA,PCL,H,AI,IPR,ION                 IMPRFT       7
      COMMON/COM3/ JAC,JAW,JABW,IF,ID,IC,PT,IZAP                            IMPRFT       8
      COMMON/COM4/ IN,N1,N2                                                 IMPRFT       9
      COMMON /COM5/ VE(147,20),VFB(147,20),VFC(106,20),PFC(20),EFC(20)      IMPRFT      10
     C,SF(147)                                                              IMPRFT      11
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      LEVEL 2,UF,UFB                                                    IMPRFT      12
      IDID=0                                                            IMPRFT      13
      NNN=3*(IN-2)                                                      IMPRFT      14
      DO 5 I=1,NNN                                                      IMPRFT      15
    5  SF(I)=0.0                                                        IMPRFT      16
      DO 10 I=1,N2                                                      IMPRFT      17
   10  V(I)=0.0                                                         IMPRFT      18
      IF (ION.LT.0) GO TO 1000                                          IMPRFT      19
      IF(ION.LT.20) GO TO 2000                                          IMPRFT      20
      IF(ION.LT.40) GO TO 3000                                          IMPRFT      21
C      ION .GE. 40 --PERFECT SHELL --                                   IMPRFT      22
      IF (ION.EQ.41) ION=-10                                            IMPRFT      23
      IF (ION.EQ.42) ION=-20                                            IMPRFT      24
      WRITE (6,8000)                                                    IMPRFT      25
 8000 FORMAT(1H0,20X,23H***  PERFECT SHELL  ***  )                      IMPRFT      26
C     -- ** IF ION.EQ.-20 SET SF(I) TO STRESS FIELD  ** --              IMPRFT      27
      RETURN                                                            IMPRFT      28
 1000 CONTINUE                                                          IMPRFT      29
C     ION.LT.0 IMPERFECTION PROFILE WILL BE READ IN IJU=0 ONLY W TERMS.  IMPRFT      30
      IF (ION.EQ.-1) ION=20                                             IMPRFT      31
      IF (ION.EQ.-2) ION=0                                              IMPRFT      32
      IF (ION.EQ.-11) ION=30                                            IMPRFT      33
      IF (ION.EQ.-12) ION=10                                            IMPRFT      34
      READ (5,7000) WT,IJU                                              IMPRFT      35
 7000 FORMAT(F10.6,I10)                                                 IMPRFT      36
      DO 1010 I=1,6                                                     IMPRFT      37
       K=(I-1)*16+2                                                     IMPRFT      38
      K2=K+14                                                           IMPRFT      39
       J=K-1                                                            IMPRFT      40
       J2=K2-1                                                          IMPRFT      41
       IF(IJU.NE.0) READ(5,7010) (V(L),L=J,J2,2)                        IMPRFT      42
 1010  READ(5,7010) (V(L),L=K,K2,2)                                     IMPRFT      43
 7010 FORMAT(8F10.6)                                                    IMPRFT      44
      IF(IJU.NE.0) READ(5,7020) (V(J),J=97,105,2)                       IMPRFT      45
      READ (5,7020) (V(J),J=98,106,2)                                   IMPRFT      46
 7020 FORMAT(5F10.6)                                                    IMPRFT      47
      WRITE (6,8010) WT,R                                               IMPRFT      48
 8010 FORMAT(1H0,10X,45H***  IMPERFECT SHELL, PROFILE READ IN.   W/T=  ,  IMPRFT      49
     CF8.4,2X,4HR/T=,F9.3)                                              IMPRFT      50
      GO TO 5000                                                        IMPRFT      51
 2000 CONTINUE                                                          IMPRFT      52
C     ION.LT.20 IMPERFECTION PROFILE OF LEGENDRE TYPE                   IMPRFT      53
      IDID=1                                                            IMPRFT      54
      IF (ION.EQ.1) ION=20                                              IMPRFT      55
      IF (ION.EQ.2) ION=0                                               IMPRFT      56
      IF (ION.EQ.11) ION=30                                             IMPRFT      57
      IF (ION.EQ.12) ION=10                                             IMPRFT      58
      RT=R*(THETA/3.141592654)**2                                       IMPRFT      59
      ALPHA=1.0/(12.0*RT*RT)                                            IMPRFT      60
      ALP=-(1.0+3.0*D0)+SQRT((1.0+3.0*D0)**2+(1.0-D0*D0)/ALPHA)         IMPRFT      61
      AN1=-0.5+0.5*SQRT(1.0+4.0*(ALP+2.0))                              IMPRFT      62
      NN1=AN1                                                           IMPRFT      63
      NN2=NN1+1                                                         IMPRFT      64
      ALP=NN1*NN2-2                                                     IMPRFT      65
      ALP2=NN2*(NN2+1)-2                                                IMPRFT      66
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     1*D0)                                                                    IMPRFT      70
      IF(PCR1.LT.PCR2) GO TO 2010                                             IMPRFT      71
      PCR1=PCR2                                                               IMPRFT      72
      NN1=NN2                                                                 IMPRFT      73
      ALB=ALB2                                                                IMPRFT      74
 2010 CONTINUE                                                                IMPRFT      75
      RN=-(ALPHA*ALB+1.0+DO+PCR1)/(ALB+1.0+D0+ECR1)                           IMPRFT      76
      P=PCR1*SQRT(3.0*(1.0-D0*D0))*RT/(1.0-D0*D0)                             IMPRFT      77
      CALL LEGEND (NN1,IN,PN,V,NUFC)                                          IMPRFT      78
      READ (5,7000) WT,IJU                                                    IMPRFT      79
      IF(IJU.NE.0) GO TO 2030                                                 IMPRFT      80
      DO 2020 I=1,N2,2                                                        IMPPFT      81
 2020  V(I)=0.0                                                               IMPRFT      82
 2030 WRITE (6,8020) NN1,E,NT,R                                               IMPRFT      83
 8020 FORMAT (1H0,10X,53H***  IMPERFECT SHELL, PROFILE OF LEGENDRE TYPE       IMPRFT      84
     CORDER= ,I2,14H   ***  P/PCL= ,E10.4,5H W/T=,F8.4,2X,4HR/T=,F9.3)        IMPRFT      85
      GO TO 5000                                                              IMPRFT      86
 3000 CONTINUE                                                                IMPRFT      87
C     ION.LT.40 EDGE DISPLACEMENT TYPE IMPERFECTION PROFILE                   IMPRFT      88
      IF (ICN.EQ.21) ICN=20                                                   IMPRFT      89
      IF (ICN.EQ.22) ICN=0                                                    IMPRFT      90
      IF (ICN.EQ.31) ICN=30                                                   IMPRFT      91
      IF (ICN.EQ.32) ICN=10                                                   IMPRFT      92
      READ (5,7000) WT,IJU                                                    IMPRFT      93
      DO 3010 I=1,N2                                                          IMPRFT      94
 3010 P(I)=0.0                                                                IMPRFT      95
      DO 3020 I=1,N1                                                          IMPRFT      96
       DO 3020 J=1,JAW                                                        IMPRFT      97
 3020   A(I,J)=0.0                                                           IMPRFT      98
      DO 3030 I=1,12                                                          IMPRFT      99
       DO 3030 J=1,JABW                                                       IMPRFT     100
 3030 AB(I,J)=0.0                                                             IMPRFT     101
      PA=0.0                                                                  IMPRFT     102
      DO 3060 KK=1,ITYPE                                                      IMPRFT     103
      CALL FCDATA (KK,ITCL,MID,ISTART,IFIN)                                   IMPRFT     104
       DO 3050 IJ=ISTART,IFIN                                                 IMPRFT     105
        PA=PA+1.0                                                             IMPRFT     106
        PHI=PA*H                                                              IMPRFT     107
        S=SIN(PHI)                                                            IMPRFT     108
        C=COS(PHI)                                                            IMPRFT     109
        CT=C/S                                                                IMPRFT     110
        K=2*IJ-1                                                              IMPRFT     111
        K1=K+1                                                                IMPRFT     112
        DO 3040 I=1,ITCL                                                      IMPRFT     113
         MA=JAD+2*(I-MID)                                                     IMPRFT     114
          A(K,MA)=A(K,MA)+AN(KK,1,I)*D1*CT+AN(KK,2,I)*D1                      IMPRFT     115
          A(K,MA+1)=A(K,MA+1)+AN(KK,1,I)*(D3+AL*(D0+CT*CT))-AL*AN(KK,2,I      IMPRFT     116
     C     )*CT-AN(KK,3,I)*AL                                                IMPRFT     117
          A(K1,MA-1)=A(K1,MA-1)+AN(KK,1,I)*(D3+AL*(D0+1.0/(S*S)))-2.0*AL      IMPRFT     118
     C*AN(KK,2,I)*CT-AL*AN(KK,3,I)                                            IMPRFT     119
          A(K1,MA)=A(K1,MA)+AN(KK,1,I)*(AL*(D2*CT+CT/(S*S)))-AL*AN(KK,2,      IMPRFT     120
     C     I)*(D0+1.0/(S*S))+2.0*AL*AN(KK,3,I)*CT+AL*AN(KK,4,I)               IMPRFT     121
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 3040 CONTINUE                                                                IMPRFT     122
        A(K,JAD)=A(K,JAD)-D1*(CT*CT+D0)                                       IMPRFT     123
        A(K1,JAD-1)=A(K1,JAD-1)+CT*(D3-AL*(D2+1.0/(S*S)))                     IMPRFT     124
        A(K1,JAD)=A(K1,JAD)+2.0*D3                                            IMPRFT
 3050 CONTINUE                                                                IMPRFT     126
 3060 CONTINUE                                                                IMPRFT     127
      B(N1+6)=1.0                                                             IMPRFT     128
      IJ=0                                                                    IMPRFT     129
      CALL BCOND (AB,KAB,IJ,THETA,AL,H)                                       IMPRFT     130
      PT=1.0                                                                  IMPRFT     131
      IZAP=10                                                                 IMPRFT     132
      CALL REDUCE(A,AB,NA,KA,KAB,N1,AT,JAW,JABW,IZAP,B,NUFC)                  IMPRFT     133
      DO 3070 I=1,N1                                                          IMPRFT     134
       V(I)=B(I+4)                                                            IMPRFT     135
       DO 3070 J=1,JAW                                                        IMPRFT     136
 3070   PA(I,J)=A(I,J)                                                       IMPRFT     137
      CALL MAGTR (HA,V,NA,N1,JAW,PT,KHA,NUFC)                                 IMPRFT     138
      AJ=-3.0                                                                 IMPRFT     139
      CALL BCKSUB(V,NUFC,AB,KAB,AJ,IN,JABW)                                   IMPRFT     140
      IF (IJU.NE.0) GO TO 3170                                                IMPRFT     141
      DO 3160 I=1,N2,2                                                        IMPRFT     142
 3160 V(I)=0.0                                                                IMPRFT     143
 3170 WRITE (6,8030) WT,R                                                     IMPRFT     144
 8030 FORMAT (1H0,10X,58H***  IMPERFECT SHELL PROFILE DUE TO EDGE DISPLA      IMPRFT     145
     CEMENT  *** ,2X,4HW/T=F8.4,2X,4HR/T=,F9.3,3X,15HLINEAR SOLUTION )        IMPRFT     146
 5000 TEMP=0.0                                                                IMPRFT     147
      IF (ICN.EQ.20.OR.ICN.EQ.30) WRITE (6,9020)                              IMPRFT     148
 9020 FORMAT(1H ,60X,23HSTRESS FREE DEFORMATION  )                            IMPRFT     149
      IF (ICN.EQ.0.OR.ICN.EQ.10) WRITE (6,9030)                               IMPRFT     150
 9030 FORMAT (1H ,60X,21HSTRESSED DEFORMATION  )                              IMPRFT     151
      DO 5010 I=1,N2                                                          IMPRFT     152
      HOLD=ABS(V(I))                                                          IMPRFT     153
      IF(HOLD.LE.TEMP) GO TO 5010                                             IMPRFT     154
      BEMP=V(I)                                                               IMPRFT     155
      TEMP=HOLD                                                               IMPRFT     156
 5010 CONTINUE                                                                IMPRFT     157
      DO 5020 I=1,N2                                                          IMPRFT     158
 5020  V(I)=V(I)*WT/(BEMP*R)                                                  IMPRFT     159
      WRITE (6,9000)                                                          IMPRFT     160
 9000 FORMAT(1H ,10X,4HNODE,8X,1HU,14X,1HW,8X,4HNODE)                         IMPRFT     161
      NNN=IN+2                                                                IMPRFT     162
      DO 5030 I=1,NNN                                                         IMPRFT     163
      J=I-1                                                                   IMPRFT     164
      K=2*I-1                                                                 IMPRFT     165
      K2=K+1                                                                  IMPRFT     166
 5030 WRITE (6,9010) J,V(K),V(K2),J                                           IMPRFT     167
 9010 FORMAT(1H ,11X,I2,3X,E12.6,2X,E12.6,3X,I2)                              IMPRFT     168
      IF (ICID.EQ.0) RETURN                                                   IMPRFT     169
      K=0                                                                     IMPRFT     170
      AJ=1.0                                                                  IMPRFT     171
       IF(V(2).GT.V(4)) AJ=-1.0                                               IMPRFT     172
      NN2=IN+1                                                                IMPRFT     173
      DO 5040 II=1,NN2                                                        IMPRFT     174
       I=II*2+2                                                               IMPRFT     175
       TEMP=(V(I)-V(I-2))*AJ                                                  IMPRFT     176
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      AJ=-AJ                                                            IMPREI      179
      V2(K)=I-2                                                         IMPREI      180
 5040 CONTINUE                                                          IMPREI      181
      I1=(V2(K-1)-2)/2                                                  IMPREI      182
      AJ=IN-I1                                                          IMPREI      183
      AJ=AJ/2                                                           IMPREI      184
      NN2=IN-2                                                          IMPREI      185
      DO 5050 I=I1,NN2                                                  IMPREI      186
       J=I*2+3                                                          IMPREI      187
       J1=J+1                                                           IMPREI      188
       AK=I-I1+1                                                        IMPREI      189
       TEMP=4.0*(AK-AJ)/(AJ-1.0)                                        IMPREI      190
       TEMP=0.5-0.5*TANH(TEMP)                                          IMPREI      191
       V(J)=V(J)*TEMP                                                   IMPREI      192
 5050  V(J1)=V(J1)*TEMP                                                 IMPREI      193
      V(N2-3)=0.0                                                       IMPREI      194
      V(N2-2)=0.0                                                       IMPREI      195
      V(N2-1)=-V(N2-5)                                                  IMPREI      196
      V(N2)=V(N2-4)                                                     IMPREI      197
      IDID=0                                                            IMPREI      198
      GO TO 5000                                                        IMPREI      199
C     -- ** IF (ICN.EQ.30.OR.ICN.EQ.10) SET UP SF(I) FOR STRESS FIELD    IMPREI      200
      END                                                               IMPREI      201
**    *DECK LEGEND                                                      IMPREI      202
      SUBROUTINE LEGEND(N,IN,BN,V,NV)                                   LEGEND        1
      DIMENSION V(NV)                                                   LEGEND        2
      DO 1 I=1,NV                                                       LEGEND        3
    1  V(NV)=0.0                                                        LEGEND        4
      ANN=IN-1                                                          LEGEND        5
      NMM=IN*2                                                          LEGEND        6
      NTERM=N/2+1                                                       LEGEND        7
      IJ=2*N-1                                                          LEGEND        8
      COEFF=1.0                                                         LEGEND        9
      DO 10 J=3,IJ,2                                                    LEGEND       10
       AJ=J                                                             LEGEND       11
   10  COEFF=COEFF*AJ                                                   LEGEND       12
      DO 20 J=2,N                                                       LEGEND       13
       AJ=J                                                             LEGEND       14
   20  COEFF=COEFF/AJ                                                   LEGEND       15
      COEFF=2.0*COEFF/(2.0**N)                                          LEGEND       16
      ANN=N                                                             LEGEND       17
      THET=3.141592654/ANN                                              LEGEND       18
      SBCF=1.0                                                          LEGEND       19
      K=-2                                                              LEGEND       20
      DO 40 J=1,NTERM                                                   LEGEND       21
       AJ=J                                                             LEGEND       22
       K=K+2                                                            LEGEND       23
       SUBN=N-K                                                         LEGEND       24
       DO 30 I=1,NMM,2                                                  LEGEND       25
       AI=(I+1)/2-1                                                     LEGEND       26
        V(I+2)=V(I+2)-BN*COEFF*SBCF*SUBN*SIN(SUBN*AI*THET)              LEGEND       27
        V(I+3)=V(I+3)+COEFF*SBCF*COS(SUBN*AI*THET)                      LEGEND       28
                                                                        LEGEND       29
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   30 CONTINUE                                                          LEGEND       30
       IF(J.EQ.NTERM) GO TO 40                                          LEGEND       31
       SBCF=SBCF*(2.0*AJ-1.0)*(ANN+1.0-AJ)/(AJ*(2.0*ANN+1.0-2.0*AJ))    LEGEND       32
       IF(N.EQ.2) SBCF=1.0/3.0                                          LEGEND       33
   40 CONTINUE                                                          LEGEND       34
      V(1)=-V(5)                                                        LEGEND       35
      V(2)=V(6)                                                         LEGEND       36
      V(NMM+3)=-V(NMM-1)                                                LEGEND       37
      V(NMM+4)=V(NMM)                                                   LEGEND       38
      RETURN                                                            LEGEND       39
      END                                                               LEGEND       40
**    *DECK REDUCE                                                      REDUCE        1
      SUBROUTINE REDUCE (A,AB,NX,KAW,KABW,N,AI,IAW,IABW,IZAP,B,NB)      REDUCE        2
      DIMENSION A(NX,KAW),AB(12,KABW),B(NB)                             REDUCE        3
C     TO REDUCE BOTH AB AND A MATRIX WHEN IZAP=0, WHEN IZAP.NE.0 AB MUST BE IN  REDUCE  4
C     REDUCED FORM ALLREADY                                             REDUCE        5
C     N=ORDER OF MATRIX A,AI=CIRCUM MODE NO. IAW=BANDWIDTH OF A IABW=BANDWIDTH  REDUCE  6
C     OF AB                                                             REDUCE        7
C          TO REDUCE AB(I,J) TO TRIANG. FORM, UPPER & LOWER             REDUCE        8
      IAB=(IAW+1)/2                                                     REDUCE        9
      IF(AI.EQ.0.0) GO TO 10                                            REDUCE       10
      IC=6                                                              REDUCE       11
      IE=14                                                             REDUCE       12
      GO TO 20                                                          REDUCE       13
   10 CONTINUE                                                          REDUCE       14
      IC=4                                                              REDUCE       15
      IE=10                                                             REDUCE       16
   20 CONTINUE                                                          REDUCE       17
      IAB=IABW-IC                                                       REDUCE       18
      ID=IAB+1                                                          REDUCE       19
      IF(IZAP.EQ.10) GO TO 25                                           REDUCE       20
      IF (IZAP.NE.0) GO TO 35                                           REDUCE       21
   25 IF (IZAP.EQ.10) IZAP=20                                           REDUCE       22
      DO 30 I=2,IC                                                      REDUCE       23
        IKI=I-1                                                         REDUCE       24
        IKP=IE-I                                                        REDUCE       25
      IKC=N+2*IC+2-I                                                    REDUCE       26
        TEMP=1.0/AB(IKI,IC)                                             REDUCE       27
        BEMP=1.0/AB(IKP,ID)                                             REDUCE       28
        J=0                                                             REDUCE       29
        DO 30 JJ=I,IC                                                   REDUCE       30
          J=J+1                                                         REDUCE       31
          JB=IE-1-JJ                                                    REDUCE       32
      JC=N+2*IC+1-JJ                                                    REDUCE       33
          HCLD=AB(JJ,IC-J)*TEMP                                         REDUCE       34
          BCLD=AB(JB,ID+J)*BEMP                                         REDUCE       35
      IF(IZAP.EQ.20) B(JJ)=B(JJ)-B(IKI)*HCLD                            REDUCE       36
      IF(IZAP.EQ.20) B(JC)=B(JC)-B(IKC)*BCLD                            REDUCE       37
          DO 30 K=1,IAB                                                 REDUCE       38
            AB(JJ,IC-J+K)=AB(JJ,IC-J+K)-AB(IKI,IC+K)*HCLD               REDUCE       39
   30       AB(JB,ID+J-K)=AB(JB,IC+J-K)-AB(IKP,ID-K)*BCLD               REDUCE       40
C       TO REMOVE FIRST AND LAST 4/6 COLS. OF A(I,J)                    REDUCE       41
   35 CONTINUE                                                          REDUCE       42
      DO 50 I=1,IC                                                      REDUCE       43
        IL=IAB-IC-1+I                                                   REDUCE       44
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      TEMP=1.0/AR(I,IC)                                                 REDUCE      47
      PEMP=1.0/AR(IB,ID)                                                REDUCE      48
      DO 40 J=1,IA                                                      REDUCE      49
        JB=N+1-J                                                        REDUCE      50
      JC=JH+IC                                                          REDUCE      51
          HOLD=A(J,IAD-IC+I-J)*TEMP                                     REDUCE      52
          BOLD=A(JB,IAD+IC-I+J)*BEMP                                    REDUCE      53
      IF(IZAP.EQ.20) B(J+IC)=B(J+IC)-B(I)*HOLD                          REDUCE      54
      IF(IZAP.EQ.20) B(JC)=B(JC)-B(JD)*BOLD                             REDUCF      55
          A(J,IAD-IC+I-J)=0.0                                           REDUCE      56
          A(JB,IAD+IC-I+J)=0.0                                          REDUCE      57
          DO 40 K=1,IAP                                                 REDUCE      58
            KT=IAD-IC+I-J+K                                             REDUCE      59
            KB=IAD+IC-I+J-K                                             REDUCE      60
            A(J,KT)=A(J,KT)-AR(I,IC+K)*HOLD                             REDUCE      61
   40       A(JB,KB)=A(JB,KB)-AR(IB,ID-K)*BOLD                          REDUCE      62
   50 CONTINUE                                                          REDUCE      63
      RETURN                                                            REDUCE      64
      END                                                               REDUCE      65
***   *DECK PERTBN                                                      REDUCE      66
      SUBROUTINE PERTBN (IRR,X,NEV,UFC,NUFC,PF,KU,PF,IFT,LNO)           PERTBN       1
      DIMENSION X(NEV),UFC(NUFC,KU),PF(KU)                              PERTBN       2
      COMMON /COM2/ D0,D1,D2,D3,D4,AL,THETA,PCL,H,AI,IER,IDN            PERTBN       3
      COMMON /COM4/ IN,N1,N2                                            PERTBN       4
      LEVEL 2,UFC                                                       PERTBN       5
      KK=0                                                              PERTBN       6
      IF(IFT.NE.0) KK=(IFT-1)*LNO                                       PERTBN       7
      IF(IRR.EQ.1) GO TO 100                                            PERTBN       8
      AJ=2*IRR                                                          PERTBN       9
      HOLD=0.0                                                          PERTBN      10
      DO 20 I=1,N2                                                      PERTBN      11
   20  HOLD=HOLD+UFC(I,KK+1)*X(I)*AJ                                    PERTBN      12
      IPR1=(IRR+1)/2                                                    PERTBN      13
      IF(IPR1.LT.2) GO TO 50                                            PERTBN      14
      DO 30 I=2,IPR1                                                    PERTBN      15
       J=IRR+1-I                                                        PERTBN      16
       K=(I*J)*2                                                        PERTBN      17
       AK=K                                                             PERTBN      18
       IF (I.EQ.J) AK=AK/2.0                                            PERTBN      19
       HOLD=HOLD+AK*PF(I+KK)*PF(J+KK)                                   PERTBN      20
       DO 30 L=1,N2                                                     PERTBN      21
   30   HOLD=HOLD+AK*UFC(L,J+KK)*UFC(L,I+KK)                            PERTBN      22
   50 CONTINUE                                                          PERTBN      23
      PF(IRR+KK)=-HOLD/(AJ*D1)                                          PERTBN      24
      HOLD=PF(IRR+KK)/PF(1+KK)                                          PERTBN      25
      DO 10 I=1,N2                                                      PERTBN      26
      X(I)=X(I)+HOLD*UFC(I,1+KK)                                        PERTBN      27
   10 UFC(I,IRR+KK)=X(I)                                                PERTBN      28
      RETURN                                                            PERTBN      29
  100 PF(1+KK)=PE                                                       PERTBN      30
      DO 120 I=1,N2                                                     PERTBN      31
      X(I)=X(I)*PE                                                      PERTBN      32
  120  UFC(I,KK+1)=X(I)                                                 PERTBN      33
      SAV=0.0                                                           PERTBN      34
      DO 125 I=1,N2                                                     PERTBN      35
  125 SAV=SAV+X(I)*X(I)                                                 PERTBN      36
      B1=PE+SAV/PE                                                      PERTBN      37
      RETURN                                                            PERTBN      38
      END                                                               PERTBN      39
***   *DECK CNVRGE                                                      PERTBN      40
      SUBROUTINE CNVRGE(UFC,PF,NUFC,KU,UF,UFB,NUF,IFT,LNO,F,P,NP,U,NU,X,CNVRGE       1
     CNX,Y,NY,V2,NV2,V,NV,A,NA,KA,VIMP)                                 CNVRGE       2
      DIMENSION UFC(NUFC,KU),PF(KU),UF(NUF,KU),UFB(NUF,KU),P(NP),U(NU), CNVRGE       3
     CX(NX),Y(NY),V2(NV2),V(NV),A(NA,KA),VIMP(NUFC)                     CNVRGE       4
      COMMON/COM4/ IN,N1,N2                                             CNVRGE       5
      COMMON /COM5/ VF(147,20),VFB(147,20),VFC(106,20),PFC(20),EFC(20)  CNVRGE       6
     C,SF(147)                                                          CNVRGE       7
      LEVEL 2,UF,UFB,UFC                                                CNVRGE       8
      CHK1=0.005                                                        CNVRGE       9
      CHK2=1.0/(10.0**9)                                                CNVRGE      10
      KK=0                                                              CNVRGE      11
      IF(IFT.NE.0) KK=(IFT-1)*LNO                                       CNVRGE      12
      NNN=3*(IN-2)                                                      CNVRGE      13
      FMX=0.0                                                           CNVRGE      14
      FMN=0.0                                                           CNVRGE      15
      E=1.0                                                             CNVRGE      16
    1 IF(IFT.EQ.0) GO TO 220                                            CNVRGE      17
      DO 200 I=1,N1                                                     CNVRGE      18
  200  U(I)=VFC(I+4,IFT)-VFC(I+4,1)                                     CNVRGE      19
      DO 210 I=1,NNN                                                    CNVRGE      20
       X(I)=VF(I,IFT)                                                   CNVRGE      21
  210  Y(I)=VFB(I,IFT)                                                  CNVRGE      22
      PL=PFC(IFT)                                                       CNVRGE      23
      GO TO 250                                                         CNVRGE      24
  220 DO 230 I=1,N1                                                     CNVRGE      25
  230  U(I)=0.0                                                         CNVRGE      26
      DO 240 I=1,NNN                                                    CNVRGE      27
       X(I)=0.0                                                         CNVRGE      28
  240  Y(I)=0.0                                                         CNVRGE      29
      PL=0.0                                                            CNVRGE      30
  250 CONTINUE                                                          CNVRGE      31
      TEMP=0.0                                                          CNVRGE      32
      DO 20 J=1,LNO                                                     CNVRGE      33
       A1=E**J                                                          CNVRGE      34
       PL=PL+PF(KK+J)*A1                                                CNVRGE      35
       DO 10 I=1,N1                                                     CNVRGE      36
   10   U(I)=U(I)+UFC(I+4,KK+J)*A1                                      CNVRGE      37
       DO 20 I=1,NNN                                                    CNVRGE      38
         X(I)=X(I)+UF(I,KK+J)*A1                                        CNVRGE      39
   20    Y(I)=Y(I)+UFB(I,KK+J)*A1                                       CNVRGE      40
      CALL FVALFN(U,NU,X,NX,Y,NY,V2,NV2,V,NV,A,NA,KA,PL)                CNVRGE      41
      DO 30 I=1,N1                                                      CNVRGE      42
       BEMP=ABS(V(I))+ABS(V2(I))+ABS(VIMP(I))                           CNVRGE      43
      PEMP=BEMP/3.0                                                     CNVRGE      44
       V(I)=V(I)-VIMP(I)                                                CNVRGE      45
      IF(BEMP.LE.CHK2) GO TO 30                                         CNVRGE      46
      HOLD=ABS((V2(I)+V(I))/BEMP)                                       CNVRGE      47
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```
      IF(IEMF.LE.CHK1) GO TO 40                                         CNVRGE    51
      EMX=E                                                             CNVRGE    52
      IF(EMN.NE.0.0) GO TO 50                                           CNVRGE    53
      E=EMX/2.0                                                         CNVRGE    54
      GO TO 1                                                           CNVRGE    55
   40 EMN=E                                                             CNVRGE    56
      IF(EMX.NE.0.0) GO TO 50                                           CNVRGE    57
      E=EMN+EMN/2.0                                                     CNVRGE    58
      GO TO 1                                                           CNVRGE    59
   50 DO 100 K=1,6                                                      CNVRGE    60
       E=(EMN+EMX)/2.0                                                  CNVRGE    61
       IF (IFT.EQ.0) GO TO 320                                          CNVRGE    62
       DO 300 I=1,N1                                                    CNVRGE    63
  300   U(I)=VEC(I+4,IFT)-VEC(I+4,1)                                     CNVRGE    64
       DO 310 I=1,NNN                                                   CNVRGE    65
        X(I)=VE(I,IFT)                                                  CNVRGE    66
  310   Y(I)=VEB(I,IFT)                                                  CNVRGE    67
       PL=PEC(IFT)                                                      CNVRGE    68
       GO TO 350                                                        CNVRGE    69
  320  DO 330 I=1,N1                                                    CNVRGE    70
  330   U(I)=0.0                                                        CNVRGE    71
       DO 340 I=1,NNN                                                   CNVRGE    72
        X(I)=0.0                                                        CNVRGE    73
  340   Y(I)=0.0                                                        CNVRGE    74
       PL=0.0                                                           CNVRGE    75
  350 CONTINUE                                                          CNVRGE    76
       TEMP=0.0                                                         CNVRGE    77
       DO 70 J=1,LNO                                                    CNVRGE    78
        A1=E**J                                                         CNVRGE    79
        PL=PL+PF(KK+J)*A1                                               CNVRGE    80
        DO 60 I=1,N1                                                    CNVRGE    81
   60    U(I)=U(I)+UFC(I+4,KK+J)*A1                                     CNVRGE    82
        DO 70 I=1,NNN                                                   CNVRGE    83
         X(I)=X(I)+UE(I,KK+J)*A1                                        CNVRGE    84
   70    Y(I)=Y(I)+UFB(I,KK+J)*A1                                       CNVRGE    85
      CALL EVALFN(U,NU,X,NX,Y,NY,V2,NV2,V,NV,A,NA,KA,PI)                CNVRGE    86
      DO 80 I=1,N1                                                      CNVRGE    87
       BEMP=ABS(V(I))+ABS(V2(I))+ABS(VIMP(I))                           CNVRGE    88
      BEMP=BEMP/3.0                                                     CNVRGE    89
       V(I)=V(I)-VIMP(I)                                                CNVRGE    90
      IF (BEMP.LE.CHK2) GO TO 80                                        CNVRGE    91
      HOLD=ABS((V2(I)+V(I))/BEMP)                                       CNVRGE    92
      IF (HOLD.LE.TEMP) GO TO 80                                        CNVRGE    93
      TEMP=HOLD                                                         CNVRGE    94
   80 CONTINUE                                                          CNVRGE    95
       IF(TEMP.LE.CHK1) GO TO 90                                        CNVRGE    96
       EMX=E                                                            CNVRGE    97
       GO TO 100                                                        CNVRGE    98
   90  EMN=E                                                            CNVRGE    99
  100 CONTINUE                                                          CNVRGE   100
      RETURN                                                            CNVRGE   101
      END                                                               CNVRGE   102
                                                                        CNVRGE   103
```

```
**    *DECK IMPRVE                                                      IMPRVE     1
      SUBROUTINE IMPRVE (U,X,Y,NA,NEV,V1,V2,V3,V4,P,A,KA,HA,KHA,IFT,AP,K IMPRVE     2
     CAP,UE,UFB,KU,UFC,NUFC,PF,IIO,LNO,E,EMN,IFAIL,FIMT,VIMP,Z6)        IMPRVE     3
      DIMENSION U(NA),X(NEV),Y(NEV),V1(NA),V2(NA),V3(NA),V4(NA),A(NA,KA) IMPRVE     4
     C,HA(NA,KHA),AB(12,KAB),UE(NA,KU),UFB(NA,KU),UFC(NUFC,KU),PF(KU)   IMPRVE     5
     C,VIMP(NUFC),Z6(NUFC)                                              IMPRVE     6
      COMMON /COM2/ DO,D1,D2,D3,D4,AL,THETA,PCL,H,AT,IFR,ION            IMPRVE     7
      COMMON /COM3/ JAE,JAW,JABW,IC,IE,ID,FT,IZAP                       IMPRVE     8
      COMMON /COM4/ IN,N1,N2                                            IMPRVE     9
      COMMON /COM5/ VE(147,20),VEB(147,20),VFC(106,20),PFC(20),EFC(20)  IMPRVE    10
     C,SF(147)                                                          IMPRVE    11
      LEVEL 2,UE,UFB,UFC                                                IMPRVE    12
      ICC=0                                                             IMPRVE    13
      M1=0                                                              IMPRVE    14
      PI=1.0                                                            IMPRVE    15
      NNN=3*(IN-2)                                                      IMPRVE    16
      IGT=IFT+1                                                         IMPRVE    17
      LMT=15                                                            IMPRVE    18
      IL=IPR                                                            IMPRVE    19
      CHK1=1.0/(10.0**6)                                                IMPRVE    20
      CHK2=1.0/(10.0**9)                                                IMPRVE    21
   21 M1=M1+1                                                           IMPRVE    22
      IF (M1.GT.LMT) GO TO 90                                           IMPRVE    23
      DO 5 I=1,N2                                                       IMPRVE    24
       Z6(I)=U(I)                                                       IMPRVE    25
    5  U(I)=U(I)-VFC(I,1)                                               IMPRVE    26
      DO 10 I=1,NNN                                                     IMPRVE    27
       V1(I)=VE(I,IGT)                                                  IMPRVE    28
   10  V2(I)=VEB(I,IGT)                                                 IMPRVE    29
      DO 30 I=1,N1                                                      IMPRVE    30
       U(I)=U(I+4)                                                      IMPRVE    31
       DO 30 J=1,JAW                                                    IMPRVE    32
   30   HA(I,J)=A(I,J)                                                  IMPRVE    33
      CALL EVALFN(U,NA,V1,NA,V2,NA,V3,NA,V4,NA,A,NA,KA,P)               IMPRVE    34
      TEMP=0.0                                                          IMPRVE    35
      DO 35 I=1,N1                                                      IMPRVE    36
       BEMP=ABS(V3(I))+ABS(V4(I))+ABS(VIMP(I))                          IMPRVE    37
      BEMP=BEMP/3.0                                                     IMPRVE    38
       V4(I)=V4(I)-VIMP(I)                                              IMPRVE    39
      IF(BEMP.LE.CHK2) GO TO 35                                         IMPRVE    40
      HOLD=ABS((V3(I)+V4(I))/BEMP)                                      IMPRVE    41
       IF (HOLD.LE.TEMP) GO TO 35                                       IMPRVE    42
       TEMP=HOLD                                                        IMPRVE    43
   35 CONTINUE                                                          IMPRVE    44
      IF (TEMP.LE.CHK1) GO TO 60                                        IMPRVE    45
      DO 40 I=1,N1                                                      IMPRVE    46
   40 V3(I)=-(V3(I)+V4(I))                                              IMPRVE    47
      CALL FACTHS(HA,NA,KHA,V4,NA,IGT)                                  IMPRVE    48
      CALL REDUCE (HA,AP,NA,KHA,KAB,N1,AT,JAW,JABW,IZAP,VIMP,NUFC)      IMPRVE    49
      CALL MAQ7B (HA,V3,NA,N1,JAW,PI,KHA,NA)                            IMPRVE    50
      CALL BCKSUB(V3,NA,AP,KAB,AT,IN,JABW)                              IMPRVE    51
      DO 50 I=1,N2                                                      IMPRVE    52
   50  U(I)=Z6(I)+V3(I)                                                 IMPRVE    53
      CALL STRNCP(U,NA,UF,NA,KU,UFB,NA,KU,IL)                           IMPRVE    54
      DO 55 I=1,NNN                                                     IMPRVE    55
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      DO 70 I=1,N2                                                      IMPRVE       59
   70  U(I)=Z6(I)                                                       IMPRVE       60
       WRITE (6,2010) M1                                                IMPRVE       61
       DO 80 I=1,N2                                                     IMPRVE       62
   80  VFC(I,IGT)=U(I)                                                  IMPRVE       63
       RETURN                                                           IMPRVE       64
   90  IF(ICC.GT.1) GO TO 140                                           IMPRVE       65
       WRITE (6,2010) M1                                                IMPRVE       66
       ICC=ICC+1                                                        IMPRVE       67
       IF (IFT.LE.1) GO TO 100                                          IMPRVE       68
       EFC(IFT)=0.75*(EFC(IFT)-EFC(IFT-1))+EFC(IFT-1)                   IMPRVE       69
       E=EFC(IFT)                                                       IMPRVE       70
       GO TO 110                                                        IMPRVE       71
  100  E=0.75*E                                                         IMPRVE       72
       IF (IFT.EQ.1) EFC(IFT)=E                                         IMPRVE       73
  110  CONTINUE                                                         IMPRVE       74
       ITMP=0                                                           IMPRVE       75
       CALL RUNVFC(UF,UFP,KU,NA,UFC,NUFC,X,Y,NEV,U,NA,EF,ILO,IFT,INO,E,  IMPRVE       76
      CEMN,IN,P,ITMP,IFAIL,FLMT)                                        IMPRVE       77
       PFC(IGT)=P                                                       IMPRVE       78
       DO 120 J=1,NNN                                                   IMPRVE       79
       VF(J,IGT)=X(J)                                                   IMPRVE       80
  120  VFP(J,IGT)=Y(J)                                                  IMPRVE       81
       DO 130 J=1,N2                                                    IMPRVE       82
  130  VFC(J,IGT)=U(J)                                                  IMPRVE       83
       M1=0                                                             IMPRVE       84
       GO TO 21                                                         IMPRVE       85
  140  WRITE (6,2000)                                                   IMPRVE       86
       DO 150 I=1,NNN                                                   IMPRVE       87
       VF(I,IGT)=X(I)                                                   IMPRVE       88
  150  VFP(I,IGT)=Y(I)                                                  IMPRVE       89
       RETURN                                                           IMPRVE       90
 2000  FORMAT (1H ,10X,17HIMPRVE HAS FAILED  )                          IMPRVE       91
 2010  FORMAT (1H ,5X,3HM1=,I3)                                         IMPRVE       92
       END                                                              IMPRVE       93
**   *DECK TPEV01                                                       TPEV01        1
       SUBROUTINE TPEV01 (XX,YY,NXX,X,Y,NX,V,NV,P,MRK)                  TPEV01        2
       DIMENSION XX(NXX),YY(NXX),X(NX),Y(NX),V(NV)                      TPEV01        3
       COMMON /COM2/ D0,D1,D2,D3,D4,AL,THETA,PCL,H,AI,IER,ICN           TPEV01        4
       COMMON /COM4/ IN,N1,N2                                           TPEV01        5
       COMMON /COM5/ VF(147,20),VFP(147,20),VFC(106,20),PFC(20),EFC(20)  TPEV01        6
      C,SF(147)                                                         TPEV01        7
       AK=0.0                                                           TPEV01        8
       AJ=1.0                                                           TPEV01        9
       IF(ICN.EQ.20.OR.ICN.EQ.30) AK=1.0                                TPEV01       10
       IF(ICN.EQ.20.OR.ICN.EQ.0) AJ=0.0                                 TPEV01       11
       IM=IN-2                                                          TPEV01       12
       IF(MRK.GT.0) GO TO 150                                           TPEV01       13
       DO 10 I=1,5                                                      TPEV01       14
   10  V(I)=0.0                                                         TPEV01       15
       PA=0.0                                                           TPEV01       16
       DO 100 IJ=1,IM                                                   TPEV01       17

       K=IJ*3-2                                                         TPEV01       18
       K1=K+1                                                           TPEV01       19
       K2=K+2                                                           TPEV01       20
       IKT=IJ/2                                                         TPEV01       21
       IKT=IJ-2*IKT                                                     TPEV01       22
       AKT=IKT+1                                                        TPEV01       23
       PA=PA+1.0                                                        TPEV01       24
       PHI=PA*H                                                         TPEV01       25
       S=SIN(PHI)                                                       TPEV01       26
       C=COS(PHI)                                                       TPEV01       27
       CT=C/S                                                           TPEV01       28
       EOI=AK*(VF(K,1)*(1.0-0.5*VF(K2,1)**2)+0.5*VF(K2,1)**2+D0*VF(K1,1))  TPEV01       29
      1-AJ*SF(K)                                                        TPEV01       30
       EOI=AK*(VF(K1,1)+D0*(VF(K,1)*(1.0-0.5*VF(K2,1)**2)+0.5*VF(K2,1)**2  TPEV01       31
      1))-AJ*SF(K1)                                                     TPEV01       32
       XOI=AK*(VFP(K2,1)+D0*VF(K2,1)*CT)                                TPEV01       33
       XOI=AK*(VF(K2,1)*CT+D0*VFP(K2,1))                                TPEV01       34
       EOS=X(K)+0.5*(X(K2)**2)*(1.0-X(K))                               TPEV01       35
       E0=VF(K,1)+0.5*(VF(K2,1)**2)*(1.0-VF(K,1))                       TPEV01       36
       V(1)=V(1)+AKT*(EOS*(EOS+D0*X(K1)-E0-EOI)-D0*X(K1)*E0+E0*EOI)*S/2.0  TPEV01       37
       V(2)=V(2)+AKT*(X(K1)*(X(K1)+D0*EOS-VF(K1,1)-ECI)-D0*EOS*VF(K1,1)+  TPEV01       38
      1VF(K1,1)*EOI)*S/2.0                                              TPEV01       39
       V(3)=V(3)+AKT*AL*(Y(K2)*(Y(K2)+D0*X(K2)*CT-VFP(K2,1)-XOI)-D0*X(K2)  TPEV01       40
      1*CT*VFP(K2,1)+VFP(K2,1)*XOI)*S/2.0                               TPEV01       41
       V(4)=V(4)+AKT*AL*(X(K2)*CT*(X(K2)*CT+D0*Y(K2)-VF(K2,1)*CT-XOI)-D0*  TPEV01       42
      1Y(K2)*VF(K2,1)*CT+VF(K2,1)*CT*XOI)*S/2.0                         TPEV01       43
  100  CONTINUE                                                         TPEV01       44
       EOI=0.0                                                          TPEV01       45
       ECI=0.0                                                          TPEV01       46
       DO 120 I=1,5                                                     TPEV01       47
       IF(V(I).LE.0.0) GO TO 110                                        TPEV01       48
       EOI=EOI+V(I)                                                     TPEV01       49
       GO TO 120                                                        TPEV01       50
  110  ECI=ECI+V(I)                                                     TPEV01       51
  120  CONTINUE                                                         TPEV01       52
       V(5)=-(EOI+EOI)                                                  TPEV01       53
       IF(V(5).LE.0.0) EOI=EOI+V(5)                                     TPEV01       54
       IF(V(5).GE.0.0) EOI=EOI+V(5)                                     TPEV01       55
       EOS=(EOI-EOI)/2.0                                                TPEV01       56
       DO 130 I=1,5                                                     TPEV01       57
       V(I+5)=V(I)/EOS                                                  TPEV01       58
  130  V(I)=V(I)*H/3.0                                                  TPEV01       59
       IF(MRK.LT.0) RETURN                                              TPEV01       60
  150  CONTINUE                                                         TPEV01       61
       DO 280 I=51,55                                                   TPEV01       62
  280  V(I)=0.0                                                         TPEV01       63
       PA=0.0                                                           TPEV01       64
       DO 300 IJ=1,IM                                                   TPEV01       65
       K=IJ*3-2                                                         TPEV01       66
       K1=K+1                                                           TPEV01       67
       K2=K+2                                                           TPEV01       68
       IKT=IJ/2                                                         TPEV01       69
       IKT=IJ-2*IKT                                                     TPEV01       70
       AKT=IKT+1                                                        TPEV01       71
       PA=PA+1.0                                                        TPEV01       72
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      EOI=AK*(VF(K,1)*(1.0-0.5*VF(K2,1)**2)+0.5*VF(K2,1)**2+D0*VF(K1,1))  TPEV01   77
     1 -AJ*SF(K)                                                          TPEV01   78
      ECI=AK*(VF(K1,1)+D0*(VF(K,1)*(1.0-0.5*VF(K2,1)**2)+0.5*VF(K2,1)**2  TPEV01   79
     1))-AJ*SF(K1)                                                        TPEV01   80
      XOI=AK*(VFB(K2,1)+D0*VF(K2,1)*CT)                                   TPEV01   81
      XCI=AK*(VF(K2,1)*CT+D0*VFB(K2,1))                                   TPEV01   82
      EOS=X(K)+0.5*(X(K2)**2)*(1.0-X(K))                                  TPEV01   83
      E0=VF(K,1)+0.5*(VF(K2,1)**2)*(1.0-VF(K,1))                          TPEV01   84
      V(51)=V(51)+AKT*(XX(K)*(2.0*EOS+D0*X(K1))*(1.0-0.5*X(K2)**2)+XX(K2  TPEV01   85
     1)*(2.0*EOS+D0*X(K1))*X(K2)*(1.0-X(K))+D0*XX(K1)*EOS-(XX(K)*(1.0-0.  TPEV01   86
     25*X(K2)**2)+XX(K2)*X(K2)*(1.0-X(K)))*(E0+EOI)-D0*XX(K1)*E0)*S/2.0   TPEV01   87
      V(52)=V(52)+AKT*(XX(K1)*(2.0*X(K1)+D0*EOS)+D0*X(K1)*(XX(K)*(1.0-0.  TPEV01   88
     15*X(K2)**2)+XX(K2)*X(K2)*(1.0-X(K)))-(XX(K1)*(VF(K1,1)+ECI)+D0*VF(  TPEV01   89
     2K1,1)*(XX(K)*(1.0-0.5*X(K2)**2)+XX(K2)*X(K2)*(1.0-X(K))))*S/2.0     TPEV01   90
      V(53)=V(53)+AKT*AL*(YY(K2)*(2.0*Y(K2)+D0*X(K2)*CT)+XX(K2)*D0*Y(K2)  TPEV01   91
     1*CT-YY(K2)*(VFB(K2,1)+XOI)-D0*XX(K2)*VFB(K2,1)*CT)*S/2.0            TPEV01   92
     1*CT-YY(K2)*(Y(K2)+XOI)-D0*XX(K2)*VFB(K2,1)*CT)*S/2.0                TPEV01   93
      V(54)=V(54)+AKT*AL*(YX(K2)*CT*(2.0*X(K2)*CT+D0*Y(K2))+YY(K2)*D0*X(  TPEV01   94
     1K2)*CT-XX(K2)*CT*(VF(K2,1)*CT+XOI)-YY(K2)*D0*VF(K2,1)*CT)*S/2.0     TPEV01   95
      A1=XX(K)-XX(K1)+XX(K2)*CT                                           TPEV01   96
      A2=X(K)-X(K1)+X(K2)*CT                                              TPEV01   97
      E0=VF(K,1)-VF(K1,1)+VF(K2,1)*CT                                     TPEV01   98
      V(55)=V(55)-AKT*P*(XX(K1)*(3.0+X(K1)*(6.0+3.0*X(K1)+2.0*A2)+2.0*A2  TPEV01   99
     1-VF(K1,1)*(6.0+6.0*X(K1)-3.0*VF(K1,1)+2.0*A2-2.0*E0)-E0*(2.0+2.0*X  TPEV01  100
     2(K1)))+A1*(1.0+X(K1)*(2.0+X(K1))-VF(K1,1)*(2.0+2.0*X(K1)-VF(K1,1))  TPEV01  101
     3))*(S**3)                                                           TPEV01  102
  300 CONTINUE                                                            TPEV01  103
      EOI=0.0                                                             TPEV01  104
      ECI=0.0                                                             TPEV01  105
      DO 320 I=51,55                                                      TPEV01  106
      IF(V(I).LE.0.0) GO TO 310                                           TPEV01  107
      EOI=EOI+V(I)                                                        TPEV01  108
      GO TO 320                                                           TPEV01  109
  310 EOI=EOI+V(I)                                                        TPEV01  110
  320 CONTINUE                                                            TPEV01  111
      V(61)=200.0*(EOI+ECI)/(EOI-ECI)                                     TPEV01  112
      EOS=(EOI-ECI)/2.0                                                   TPEV01  113
      DO 330 I=51,55                                                      TPEV01  114
      V(I+5)=V(I)/EOS                                                     TPEV01  115
  330 V(I)=V(I)*H/3.0                                                     TPEV01  116
      RETURN                                                              TPEV01  117
      END                                                                 TPEV01  118
***   *DECK VOLDIS                                                        VOLDIS    1
      SUBROUTINE VOLDIS(U,IU,X,IX,VOL,DELTA)                              VOLDIS    2
      DIMENSION U(IU),X(IX)                                               VOLDIS    3
      COMMON /COM2/ D0,D1,D2,D3,D4,AL,THETA,PCL,H,AT,IPR,ION              VOLDIS    4
      COMMON /COM4/ IN,N1,N2                                              VOLDIS    5
      VOL=0.0                                                             VOLDIS    6
      PA=0.0                                                              VOLDIS    7
      IM=IN-2                                                             VOLDIS    8
      DO 100 IJ=1,IM                                                      VOLDIS    9
      K=IJ*3-2                                                            VOLDIS   10
      K1=K+1                                                              VOLDIS   11
      K2=K+2                                                              VOLDIS   12
      IKT=IJ/2                                                            VOLDIS   13
      IKT=IJ-2*IKT                                                        VOLDIS   14
      AKT=IKT+1                                                           VOLDIS   15
      PA=PA+1.0                                                           VOLDIS   16
      PHI=PA*H                                                            VOLDIS   17
      S=SIN(PHI)                                                          VOLDIS   18
      C=COS(PHI)                                                          VOLDIS   19
      CT=C/S                                                              VOLDIS   20
  100 VOL=VOL+AKT*(3.0*X(K1)*(1.0+X(K1))+X(K1)**3+(X(K)-X(K1)+X(K2)*CT)   VOLDIS   21
     1 *((1.0+X(K1))**2))*S**3                                            VOLDIS   22
      PHI=PHI+H                                                           VOLDIS   23
      C=COS(PHI)                                                          VOLDIS   24
      S=SIN(PHI)                                                          VOLDIS   25
      CT=C/S                                                              VOLDIS   26
      E=U(N1+6)+U(N1+5)*CT                                                VOLDIS   27
      VOL=(H/3.0)*(2.0*VOL+(3.0*E*(1.0+E)+E**3)*S**3)                     VOLDIS   28
      VOL=-VOL/((2.0-3.0*C+C**3)/3.0)                                     VOLDIS   29
      DELTA=U(4)                                                          VOLDIS   30
      RETURN                                                              VOLDIS   31
      END                                                                 VOLDIS   32
***   *DECK EVALFN                                                        EVALFN    1
      SUBROUTINE EVALFN(U,NU,X,NX,Y,NY,V2,NV2,V,NV,A,NA,KA,P)             EVALFN    2
      DIMENSION U(NU),X(NX),Y(NY),V2(NV2),V(NV),A(NA,KA)                  EVALFN    3
      COMMON/COM2/ D0,D1,D2,D3,D4,AL,THETA,PCL,H,AT,IPR,ION               EVALFN    4
      COMMON/COM3/ JAC,JAW,JARW,IE,ID,IC,PI,I2AP                          EVALFN    5
      COMMON/COM4/IN,N1,N2                                                EVALFN    6
      COMMON/COM5/ VF(147,20),VFB(147,20),VFC(106,20),EFC(20),EFC(20)     EVALFN    7
     C,SF(147)                                                            EVALFN    8
      NNN=IN-2                                                            EVALFN    9
      CALL PRODUC (NA,KA,A,V2,U,N1,NV2,JAW)                               EVALFN   10
      PA=0.0                                                              EVALFN   11
      AK=1.0                                                              EVALFN   12
      AJ=1.0                                                              EVALFN   13
      IF (ION.EQ.20.OR.ION.EQ.30) AK=2.0                                  EVALFN   14
      IF (ION.EQ.20.OR.ION.EQ.0) AJ=0.0                                   EVALFN   15
      DO 50 I=1,NNN                                                       EVALFN   16
      PA=PA+1.0                                                           EVALFN   17
      PHI=PA*H                                                            EVALFN   18
      S=SIN(PHI)                                                          EVALFN   19
      C=COS(PHI)                                                          EVALFN   20
      CT=C/S                                                              EVALFN   21
      K1=I*2-1                                                            EVALFN   22
      K2=K1+1                                                             EVALFN   23
      L=I*3-2                                                             EVALFN   24
      L1=L+1                                                              EVALFN   25
      L2=L+2                                                              EVALFN   26
      V(K1)=X(L2)*(X(L2)*D2*CT/2.0+Y(L2)-X(L)-D0*X(L1))                   EVALFN   27
      V(K2)=X(L2)*(D3*X(L2)/2.0+X(L)*CT+D0*X(L1)*CT+Y(L)+D0*Y(L           EVALFN   28
     1 1))+Y(L2)*(X(L)+D0*X(L1))                                          EVALFN   29
      V(K1)=V(K1)+2.0*P*X(L2)                                             EVALFN   30
      V(K2)=V(K2)-2.0*P*(1.0+X(L)+X(L1))                                  EVALFN   31
      V(K1)=V(K1)+X(L2)*(X(L2)*(-X(L2)/2.0-(1.0-0.5*D0)*X(L)*CT-D0*CT     EVALFN   32
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     1  (L)-D0*X(L1)/2.0)+X(L)*(-CT*X(L)-D0*CT*X(L1)-D0*Y(L1)-2.0*Y(L))-D      EVALFN     36
     2  0*Y(I)*X(L1))+Y(L2)*X(L)*(-X(L)-D0*X(L1))                             EVALFN     37
        V(K1)=V(K1)+2.0*P*X(I2)*X(L1)                                         EVALFN     38
        V(K2)=V(K2)-2.0*P*X(L)*X(L1)                                          EVALFN     39
       E0=AK*(VF(L,1)+D0*VF(L1,1))-AJ*SF(L)                                   EVALFN     40
       E01=AK*(VFR(L,1)+D0*VFR(L1,1))-AJ*SF(L2)                               EVALFN     41
       V(K1)=V(K1)+X(L2)*(X(I2)*(E01+E0*CT)/4.0-X(L)*E0/2.0+Y(I2)*E0/2.0)     EVALFN     42
       V(K2)=V(K2)+X(L2)*(X(I2)*E0/4.0+X(L)*(E0*CT+E01)/2.0+Y(I)*E0/2.0)      EVALFN     43
     1+Y(I2)*X(L)*E0/2.0                                                      EVALFN     44
        V(K1)=V(K1)-2.0*P*((1.0-VF(L1,1))*VF(L2,1)+X(I2)*VF(L1,1)+X(L1)*V     EVALFN     45
     1 F(L2,1))                                                               EVALFN     46
        V(K2)=V(K2)+2.0*P*(VF(L,1)+VF(L1,1)-VF(L1,1)*VF(L,1)+X(L1)*(VF(L,     EVALFN     47
     1 1)+VF(L1,1))+VF(L1,1)*(X(L)-X(L1)))                                    EVALFN     48
  50 CONTINUE                                                                 EVALFN     49
     RETURN                                                                   EVALFN     50
     END                                                                      EVALFN     51
***  *DECK FNDRHS                                                            FNDRHS      1
     SUBROUTINE FNDRHS (UF,NUF,KUF,UFR,IRR,V,NV,PF,IFT,LNO)                  FNDRHS      2
     DIMENSION UF(NUF,KUF),UFR(NUF,KUF),V(NV),PF(KUF)                        FNDRHS      3
     COMMON/COM1/ AN(7,4,12),ITYPE,ILN(7),IMN(7),JSN(7),JEN(7),KBOE(7)       FNDRHS      4
     COMMON/COM2/ D0,D1,D2,D3,D4,AL,THETA,PCL,H,AI,IRR,ION                   FNDRHS      5
     COMMON/COM4/ LN,L1,L2                                                   FNDRHS      6
     COMMON/COM5/ VF(147,20),VFR(147,20),VFC(106,20),PFC(20),FFC(20)         FNDRHS      7
    C,SF(147)                                                                FNDRHS      8
     LEVEL 2,UF,UFR                                                          FNDRHS      9
     IRR1=IRR-1                                                              FNDRHS     10
     IRR2=IRR-2                                                              FNDRHS     11
     JJ=0                                                                    FNDRHS     12
     IF(IFT.GT.1) JJ=(IFT-1)*LNO                                             FNDRHS     13
     PA=0.0                                                                  FNDRHS     14
     AK=1.0                                                                  FNDRHS     15
     AJ=1.0                                                                  FNDRHS     16
     IF (ION.EQ.20.OR.ION.EQ.30) AK=2.0                                      FNDRHS     17
     IF (ION.EQ.20.OR.ION.EQ.0) AJ=0.0                                       FNDRHS     18
     DO 10 I=1,L1                                                            FNDRHS     19
  10   V(I)=0.0                                                             FNDRHS     20
     DO 110 KK=1,ITYPE                                                       FNDRHS     21
       CALL FDDATA (KK,ITCL,MID,ISTART,IFIN)                                 FNDRHS     22
       DO 100 IJ=ISTART,IFIN                                                 FNDRHS     23
         PA=PA+1.0                                                           FNDRHS     24
         PHI=PA*H                                                            FNDRHS     25
         S=SIN(PHI)                                                          FNDRHS     26
         C=COS(PHI)                                                          FNDRHS     27
         CT=C/S                                                              FNDRHS     28
         K=2*IJ-1                                                            FNDRHS     29
         K1=K+1                                                              FNDRHS     30
         N1=3*IJ-2                                                           FNDRHS     31
         N2=N1+1                                                             FNDRHS     32
         N3=N1+2                                                             FNDRHS     33
       E0=AK*(VF(N1,1)+D0*VF(N2,1))-AJ*SF(N1)                                FNDRHS     34
       E01=AK*(VFR(N1,1)+D0*VFR(N2,1))-AJ*SF(N3)                             FNDRHS     35
       DO 20 I=1,IRR1                                                        FNDRHS     36
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         I1=I+JJ                                                             FNDRHS     37
         I2=IRR-I+JJ                                                         FNDRHS     38
            V(K)=V(K)-UF(N3,I1)*(UF(N3,I2)*D2*CT/2.0+UFR(N3,I2)-UF(N1,I2)    FNDRHS     39
        1     -UF(N2,I2)*D0)                                                 FNDRHS     40
            V(K1)=V(K1)-UF(N3,I1)*(UF(N3,I2)*D3/2.0+UF(N1,I2)*CT+UF(N2,I2    FNDRHS     41
        1     )*D0*CT+UFR(N1,I2)+UFR(N2,I2)*D0)-UFR(N3,I1)*(UF(N1,I2)+UF(N2    FNDRHS     42
        2     ,I2)*D0)                                                       FNDRHS     43
         V(K)=V(K)-2.0*UF(N3,I1)*PF(I2)                                      FNDRHS     44
         V(K1)=V(K1)+2.0*(UF(N1,I1)+UF(N2,I1))*PF(I2)                        FNDRHS     45
         IF (ION.EQ.-10) GO TO 20                                            FNDRHS     46
         V(K)=V(K)-UF(N3,I1)*(UF(N3,I2)*(E01+E0*CT)/4.0-UF(N1,I2)*E0/2.0+UF  FNDRHS     47
        1R(N3,I2)*E0/2.0)                                                    FNDRHS     48
         V(K1)=V(K1)-UF(N3,I1)*(UF(N3,I2)*E0/4.0+UF(N1,I2)*(E0*CT+E01)/2.0+  FNDRHS     49
        1UFR(N1,I2)*E0/2.0)-UF(N1,I1)*UFR(N3,I2)*E0/2.0                      FNDRHS     50
          V(K)=V(K)+2.0*PF(I1)*(UF(N2,I2)*VF(N3,1)+UF(N3,I2)*VF(N2,1))       FNDRHS     51
          V(K1)=V(K1)-2.0*PF(I1)*(UF(N2,I2)*(VF(N1,1)+VF(N2,1))+VF(N2,1)*(U  FNDRHS     52
        1 F(N1,I2)-UF(N2,I2)))                                               FNDRHS     53
  20       CONTINUE                                                          FNDRHS     54
           IF(IRR.LT.3) GO TO 100                                            FNDRHS     55
       DO 40 I=1,IRR2                                                        FNDRHS     56
         IHD=IRR-I-1                                                         FNDRHS     57
         DO 30 J=1,IHD                                                       FNDRHS     58
           I1=I+JJ                                                           FNDRHS     59
           I2=J+JJ                                                           FNDRHS     60
           I3=IRR-I-J+JJ                                                     FNDRHS     61
            V(K)=V(K)-UF(N3,I1)*(UF(N3,I2)*(-UF(N3,I3)/2.0-UF(N1,I3)*CT*(    FNDRHS     62
        1     1.0-D0/2.0)-UFR(N2,I3)*D0/2.0-UFR(N1,I3)-UF(N2,I3)*D0*CT/2.0)  FNDRHS     63
        2     +UF(N1,I2)*(-UFR(N3,I3)*2.0+UF(N1,I3)+UF(N2,I3)*D0)-UF(N2,I2)  FNDRHS     64
        3     *UFR(N3,I3)*D0)                                                FNDRHS     65
            V(K1)=V(K1)-UF(N3,I1)*(UF(N3,I2)*(UF(N3,I3)*CT/2.0+UFR(N3,I3)    FNDRHS     66
        1     *3.0/2.0-UF(N1,I3)*(1.0+D0/2.0)-UF(N2,I3)*D0/2.0)+UF(N1,I2)*(  FNDRHS     67
        2     -UF(N1,I3)*CT-(F(N2,I3)*D0*CT-UFR(N1,I3)*2.0-UFR(N2,I3)*D0)-U  FNDRHS     68
        3     F(N2,I2)*UFR(N1,I3)*D0)+UF(N1,I1)*(UF(N1,I2)*UFR(N3,I3)+D0*UF  FNDRHS     69
        4     (N2,I2)*UFR(N3,I3))                                            FNDRHS     70
         V(K)=V(K)-2.0*UF(N3,I1)*UF(N2,I2)*PF(I3)                            FNDRHS     71
         V(K1)=V(K1)+2.0*UF(N1,I1)*UF(N2,I2)*PF(I3)                          FNDRHS     72
  30       CONTINUE                                                          FNDRHS     73
  40       CONTINUE                                                          FNDRHS     74
 100    CONTINUE                                                             FNDRHS     75
 110 CONTINUE                                                                FNDRHS     76
     RETURN                                                                  FNDRHS     77
     END                                                                     FNDRHS     78
***  *DECK FNDRH2                                                            FNDRH2      1
     SUBROUTINE FNDRH2 (UF,NUF,KUF,UFR,IRR,V,NV,PF,IFT,LNO)                  FNDRH2      2
     DIMENSION UF(NUF,KUF),UFR(NUF,KUF),V(NV),PF(KUF)                        FNDRH2      3
     COMMON/COM1/ AN(7,4,12),ITYPE,ILN(7),IMN(7),JSN(7),JEN(7),KBOE(7)       FNDRH2      4
     COMMON /COM2/ D0,D1,D2,D3,D4,AL,THETA,PCL,H,AI,IRR,ION                  FNDRH2      5
     COMMON /COM5/ VF(147,20),VFR(147,20),VFC(106,20),PFC(20),FFC(20)        FNDRH2      6
    C,SF(147)                                                                FNDRH2      7
     LEVEL 2,UF,UFR                                                          FNDRH2      8
     IRR1=IRR-1                                                              FNDRH2      9
     JJ=(IFT-1)*LNO                                                          FNDRH2     10
     PA=0.0                                                                  FNDRH2     11
     DO 110 KK=1,ITYPE                                                       FNDRH2     12
       CALL FDDATA (KK,ITCL,MID,ISTART,IFIN)                                 FNDRH2     13
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      S=SIN(PHI)                                                        FNDRH2     17
      C=COS(PHI)                                                        FNDRH2     18
      CT=C/S                                                            FNDRH2     19
      K=2*IJ-1                                                          FNDRH2     20
      K1=K+1                                                            FNDRH2     21
      L=3*IJ-2                                                          FNDRH2     22
      L1=L+1                                                            FNDRH2     23
      L2=L+2                                                            FNDRH2     24
      DO 20 I=1,IPR1                                                    FNDRH2     25
     I1=I+JJ                                                            FNDRH2     26
     I2=IRR-I+JJ                                                        FNDRH2     27
      V(K)=V(K)-VF(L2,IFT)*(UF(L2,I1)*(-1.5*UF(L2,I2)-(2.0-D0)*CT*UF(I  FNDRH2     28
     1,I2)-D0*CT*UF(L1,I2)-D0*UFB(L1,I2)-2.0*UFB(L,I2))+UF(L,I1)*(-2.0  FNDRH2     29
     2*UFB(L2,I2)+UF(L,I2)+D0*UF(L1,I2))-D0*UFB(L2,I1)*UF(L1,I2))-VF(L  FNDRH2     30
     3,IFT)*UF(L2,I1)*(-(1.0-0.5*D0)*UF(I2,I2)*CT-2.0*UFB(L2,I2)+2.0*U  FNDRH2     31
     4F(I,I2)+D0*UF(I1,I2))-VF(L1,IFT)*UF(L2,I1)*(-D0*CT*UF(L2,I2)/2.0  FNDRH2     32
     5+D0*(UF(L,I2)-UFB(L2,I2)))+VFB(L2,IFT)*UF(L2,I1)*(2.0*UF(L,I2)+D0  FNDRH2     33
     6*UF(L1,I2))+VFB(L,IFT)*UF(L2,I1)*UF(L2,I2)+VFB(L1,IFT)*UF(L2,I1)  FNDRH2     34
     7*UF(L2,I2)*D0/2.0-2.0*(PFC(IFT)*UF(L2,I1)*UF(I1,I2)+VF(L2,IFT)*P  FNDRH2     35
     8F(I1)*UF(L1,I2)+VF(L1,IFT)*PF(I1)*UF(L2,I2))                      FNDRH2     36
      V(K1)=V(K1)-VF(L2,IFT)*(UF(L2,I1)*(1.5*UF(L2,I2)*CT+3.0*UFB(L2,I  FNDRH2     37
     1 2)-(2.0+D0)*UF(L,I2)-D0*UF(L1,I2))+UF(L,I1)*(-CT*UF(L,I2)-D0*UF(  FNDRH2     38
     2 L1,I2)*CT-2.0*UFB(L,I2)-D0*UFB(L1,I2))-D0*UF(L1,I1)*UFB(L,I2))-V  FNDRH2     39
     3F(I,IFT)*(UF(L2,I1)*(-(1.0+D0/2.0)*UF(L2,I2)-CT*(2.0*UF(L,I2)+D0  FNDRH2     40
     4*UF(L1,I2))-2.0*UFB(L1,I2))-UFB(L2,I1)*(2.0*UF(L,I2  FNDRH2     41
     5 )+D0*UF(L1,I2)))-VF(L1,IFT)*(UF(L2,I1)*(-D0*(UF(L2,I2)/2.0+UF(L,  FNDRH2     42
     6 I2)*CT+UFB(L,I2)))-UFB(L2,I1)*UF(L,I2)*D0)-VFB(L2,IFT)*(1.5*UF(L  FNDRH2     43
     7 2,I1)*UF(L2,I2)-UF(L,I1)*(UF(L,I2)+D0*UF(L1,I2)))+VFB(L,IFT)*UF(  FNDRH2     44
     8 L2,I1)*(2.0*UF(L,I2)+D0*UF(L1,I2))+VFB(L1,IFT)*D0*UF(L2,I1)*UF(L  FNDRH2     45
     9 ,I2)+2.0*(PFC(IFT)*UF(L,I1)*UF(L1,I2)+VF(L,IFT)*PF(I1)*UF(L1,I2)  FNDRH2     46
     A +VF(L1,IFT)*PF(I1)*UF(L,I2))                                     FNDRH2     47
   20  CONTINUE                                                         FNDRH2     48
  100  CONTINUE                                                         FNDRH2     49
  110 CONTINUE                                                          FNDRH2     50
      RETURN                                                            FNDRH2     51
      END                                                               FNDRH2     52
****  *DECK FNDLHS                                                      FNDLHS      1
      SUBROUTINE FNDLHS (A,NA,KHA,V,NV,IK)                              FNDLHS      2
      DIMENSION A(NA,KHA),V(NV)                                         FNDLHS      3
      COMMON/COM1/ AN(7,4,12),ITYPE,ILN(7),IMN(7),JSN(7),JEN(7),KHUE(7) FNDLHS      4
      COMMON /COM2/ D0,D1,D2,D3,D4,AL,THETA,PCL,H,AT,IFR,ION            FNDLHS      5
      COMMON /COM3/ JAC,JAM,JAPW,IC,IE,ID,FT,IZAP                       FNDLHS      6
      COMMON /COM4/ IN,N1,N2                                            FNDLHS      7
      COMMON /COM5/ VF(147,20),VFB(147,20),VFC(106,20),PFC(20),PFC(20)  FNDLHS      8
     C,SF(147)                                                          FNDLHS      9
      P=PFC(IK)                                                         FNDLHS     10
      PA=0.0                                                            FNDLHS     11
      AK=1.0                                                            FNDLHS     12
      AJ=1.0                                                            FNDLHS     13
      IF (ICN.EQ.20.OR.ICN.EQ.30) AK=2.0                                FNDLHS     14
      IF (ICN.EQ.20.OR.ICN.EQ.0) AJ=0.0                                 FNDLHS     15
      DO 110 KK=1,ITYPE                                                 FNDLHS     16
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      CALL FDDATA (KK,ITOL,MID,ISTART,IFIN)                             FNDLHS     17
      DO 100 IJ=ISTART,IFIN                                             FNDLHS     18
       PA=PA+1.0                                                        FNDLHS     19
       PHI=PA*H                                                         FNDLHS     20
       S=SIN(PHI)                                                       FNDLHS     21
       C=COS(PHI)                                                       FNDLHS     22
       CT=C/S                                                           FNDLHS     23
       K=2*IJ-1                                                         FNDLHS     24
       K1=K+1                                                           FNDLHS     25
       L=3*IJ-2                                                         FNDLHS     26
       L1=L+1                                                           FNDLHS     27
       L2=L+2                                                           FNDLHS     28
     F0=AK*(VF(L,1)+D0*VF(L1,1))-AJ*SF(L)                               FNDLHS     29
     F01=AK*(VFB(L,1)+D0*VFB(L1,1))-AJ*SF(L2)                           FNDLHS     30
       DO 70 I=1,ITOL                                                   FNDLHS     31
        MA=JAC+2*(I-MID)                                                FNDLHS     32
       A(K,MA+1)=A(K,MA+1)+AN(KK,1,I)*(-2.0*P+VF(L,IK)+D0*VF(L1,IK)-VFB      2   FNDLHS     33
    1  (L2,IK)-D2*VF(L2,IK)*CT)+AN(KK,2,I)*(-VF(L2,IK))                2   FNDLHS     34
       A(K1,MA-1)=A(K1,MA-1)+AN(KK,1,I)*(-2.0*P+D3*VF(L2,IK)*CT+VFB(L2,      2   FNDLHS     35
    1  IK)+VF(L,IK)+D0*VF(L1,IK))+AN(KK,2,I)*VF(L2,IK)                  2   FNDLHS     36
       A(K1,MA)=A(K1,MA)+AN(KK,1,I)*(-VF(L,IK)*CT-VFB(L,IK)-D0*(VF(L1,1      2   FNDLHS     37
    1  K)*CT+VFB(L1,IK)))+AN(KK,2,I)*(-VF(L,IK)-D0*VF(L1,IK))           2   FNDLHS     38
       A(K,MA)=A(K,MA)-AN(KK,1,I)*VF(L2,IK)*(VF(L2,IK)*CT+VFB(L2,IK)*2       FNDLHS     39
    1  .0)-AN(KK,2,I)*VF(L2,IK)**2                                          FNDLHS     40
       A(K,MA+1)=A(K,MA+1)+AN(KK,1,I)*(-2.0*P*VF(L1,IK)+VF(L2,IK)*(D2*      3   FNDLHS     41
    1  VF(L2,IK)/2.0+(2.0-D0)*VF(L,IK)*CT+D0*VFB(L,IK)+D0*VFB(L1,IK)      3   FNDLHS     42
    2  +2.0*VFB(L,IK))+VF(L,IK)*(2.0*VFB(L2,IK)-VF(L,IK)-D0*VF(L1,IK))+   3   FNDLHS     43
    3  D0*VFB(L2,IK)*VF(L1,IK))+AN(KK,2,I)*(VF(L2,IK)*(2.0*VF(L,IK)+D0*   3   FNDLHS     44
    4  VF(L1,IK)))                                                       3   FNDLHS     45
       A(K1,MA-1)=A(K1,MA-1)+AN(KK,1,I)*(-2.0*P*VF(L1,IK)+VF(L2,IK)*(D2      3   FNDLHS     46
    1  *VF(L2,IK)/2.0-(2.0+D0)*VF(L,IK)*CT-D0*(VF(L),IK)*CT+VFB(L),IK))   3   FNDLHS     47
    2  -2.0*VFB(L,IK))-VF(L,IK)*(2.0*VFB(L2,IK)+VF(L,IK)+D0*VF(L1,IK))-   3   FNDLHS     48
    3  D0*VFB(L2,IK)*VF(L1,IK))+AN(KK,2,I)*VF(L2,IK)*(-2.0*VF(L,IK)-D0*   3   FNDLHS     49
    4  VF(L1,IK))                                                        3   FNDLHS     50
       A(K1,MA)=A(K1,MA)+AN(KK,1,I)*(-VF(L2,IK)*(VF(I2,IK)*1.5*CT+3.0*      3   FNDLHS     51
    1  VFB(L2,IK))+VF(L,IK)*(VF(L,IK)*CT+2.0*VFB(L,IK)+D0*(VF(L1,IK)*CT   3   FNDLHS     52
    2  +VFB(L1,IK)))+D0*VF(L1,IK)*VFB(L,IK))+AN(KK,2,I)*(+VF(L,IK)*(VF(   3   FNDLHS     53
    3  L,IK)+D0*VF(L1,IK))-1.5*VF(L2,IK)**2)                            3   FNDLHS     54
       IF (ICN.EQ.-10) GO TO 70                                         FNDLHS     55
       A(K,MA+1)=A(K,MA+1)+AN(KK,1,I)*(-E0*(VF(L2,IK)*CT-VF(L,IK)+VFB(L2,       FNDLHS     56
      1IK))-VF(L2,IK)*E01)/2.0-AN(KK,2,I)*VF(L2,IK)*E0/2.0              FNDLHS     57
       A(K1,MA-1)=A(K1,MA-1)+AN(KK,1,I)*(E0*(VF(L2,IK)*CT+VFB(L2,IK)+VF(L       FNDLHS     58
      1,IK))+VFB(L2,IK)*E01)/2.0+AN(KK,2,I)*VF(L2,IK)*E0/2.0            FNDLHS     59
       A(K1,MA)=A(K1,MA)-AN(KK,1,I)*(E0*(VF(L,IK)*CT+VFB(L,IK))+VF(L,IK)*       FNDLHS     60
      1E01)/2.0-AN(KK,2,I)*VF(L,IK)*E0/2.0                              FNDLHS     61
       A(K,MA+1)=A(K,MA+1)+AN(KK,1,I)*2.0*P*VF(L1,1)                    FNDLHS     62
       A(K1,MA-1)=A(K1,MA-1)+AN(KK,1,I)*2.0*P*VF(L1,1)                  FNDLHS     63
   70  CONTINUE                                                         FNDLHS     64
       V(K)=-2.0*VF(L2,IK)                                              FNDLHS     65
       V(K1)=2.0*(1.0+VF(L,IK)+VF(L1,IK))                               FNDLHS     66
       A(K,JAC)=A(K,JAC)+2.0*P+VFB(L2,IK)-VF(L,IK)-D0*VF(L1,IK)+(1.0-2.0      2   FNDLHS     67
    1 *D0)*VF(L2,IK)*CT                                                 2   FNDLHS     68
       A(K,JAC+1)=A(K,JAC+1)-D3*VF(L2,IK)                               2   FNDLHS     69
       A(K1,JAC-1)=A(K1,JAC-1)-2.0*P*CT+VF(L2,IK)+VF(L,IK)*CT+VFB(L,IK)      2   FNDLHS     70
    1 +D0*(VFB(L2,IK)*CT+VF(L1,IK)*CT+VFB(L1,IK))                       2   FNDLHS     71
```

```
      A(K,JAD)=A(K,JAD)+2.0*P*(VF(L1,IK)+VF(L2,IK)*CT)-VF(L2,IK)*(0.5*D     3   FNDLHS     75
     1 4*VF(L2,IK)+2.0*D2*CT*VF(L,IK)+DO*(VF(L1,IK)*CT+VFB(L1,JK)+VFB(L2     3   FNDLHS     76
     2 ,IK)*CT)+2.0*VFB(L,IK))+VF(L,IK)*(VF(L,IK)-2.0*VFB(L2,IK)+DO*VF(L     3   FNDLHS     77
     3 1,IK))-DO*VF(L1,IK)*VFB(L2,IK)                                        3   FNDLHS     78
      A(K,JAD+1)=A(K,JAD+1)+2.0*P*VF(L2,IK)+VF(L2,IK)*(-VF(L2,IK)*CT+2.     3   FNDLHS     79
     1 0*(VF(L,IK)-VFB(L2,IK))+DO*(VF(L1,IK)+VF(L,IK)-VFB(L2,IK)))           3   FNDLHS     80
      A(K1,JAD-1)=A(K1,JAD-1)-2.0*P*VF(L,IK)*CT+VF(L2,IK)*(D4*CT*VF(L2,     3   FNDLHS     81
     1 IK)/2.0-2.0*VF(L,IK)+3.0*VFB(L2,IK)-DO*(VF(L1,IK)+VFB(L,IK)*CT))-     FNDLHS     82
     2 VF(L,IK)*(VF(L,IK)*CT+2.0*VFB(L,IK)+DO*(VF(L1,IK)*CT+VFB(L1,IK)+V     FNDLHS     83
     3 FB(L2,IK)*CT))-DO*VFB(L,IK)*VF(L1,IK)                                 FNDLHS     84
      A(K1,JAD)=A(K1,JAD)-2.0*P*(VF(L,IK)+VF(L1,IK))+VF(L2,IK)*(-D3*VF(     3   FNDLHS     85
     1 L2,IK)-(2.0+DO)*(VF(L,IK)*CT+VFB(L,IK))-DO*(VF(L1,IK)*C1+VFB(L1,I     3   FNDLHS     86
     2 K)))-VF(L,IK)*(2.0+DO)*VFB(L2,IK)-DO*VFB(L2,JK)*VF(L1,IK)             3   FNDLHS     87
       IF (ICN.EQ.-10) GO TO 100                                             FNDLHS     88
      A(K,JAD)=A(K,JAD)+EO*(VF(L2,IK)*CT-VF(L,IK)+VFB(L2,IK))/2.0+VF(L2,     FNDLHS     89
     1IK)*E01/2.0                                                            FNDLHS     90
      A(K,JAD+1)=A(K,JAD+1)-VF(L2,IK)*EO/2.0                                 FNDLHS     91
      A(K1,JAD-1)=A(K1,JAD-1)+EO*(VF(L2,IK)+VF(L,IK)*CT+VFB(L,IK))/2.0+V     FNDLHS     92
     1F(L,IK)*E01/2.0                                                        FNDLHS     93
      A(K1,JAD)=A(K1,JAD)+EO*(VF(L2,IK)*CT+VFB(L2,IK))/2.0+VF(L2,IK)*E01     FNDLHS     94
     1/2.0                                                                   FNDLHS     95
       A(K,JAD)=A(K,JAD)-2.0*P*(VF(L1,1)+VF(L2,1)*CT)                        FNDLHS     96
       A(K,JAD+1)=A(K,JAD+1)-2.0*P*VF(L2,1)                                  FNDLHS     97
       A(K1,JAD-1)=A(K1,JAD-1)+2.0*P*VF(L,1)*CT                              FNDLHS     98
       A(K1,JAD)=A(K1,JAD)+2.0*P*(VF(L,1)+VF(L1,1))                          FNDLHS     99
       V(K)=V(K)+2.0*((1.0-VF(L1,1))*VF(L2,1)+VF(L1,IK)*VF(L2,1)+VF(L2,I     FNDLHS    100
     1 K)*VF(L1,1))                                                          FNDLHS    101
       V(K1)=V(K1)-2.0*(VF(L,1)+VF(L1,1)-VF(L,1)*VF(L1,1)+VF(L1,IK)*(VF(     FNDLHS    102
     1 L,1)+VF(L1,1))+VF(L1,1)*(VF(L,IK)-VF(L1,IK)))                         FNDLHS    103
  100  CONTINUE                                                              FNDLHS    104
  110 CONTINUE                                                              FNDLHS    105
      RETURN                                                                FNDLHS    106
      END                                                                   FNDLHS    107
**    *DECK STRNCP                                                          STRNCP      1
      SUBROUTINE STRNCP (X,NX,UF,NUF,KUF,UFB,NUFB,KUFB,IKK)                 STRNCP      2
      DIMENSION X(NX),UF(NUF,KUF),UFB(NUFB,KUFB)                            STRNCP      3
      COMMON /COM1/ AN(7,4,12),ITYPE,ILN(7),IMB(7),JSN(7),JEN(7),KBOE(7)    STRNCP      4
      COMMON /COM2/ DO,D1,D2,D3,D4,AL,THETA,PCL,H,AT,IBR,ION                STRNCP      5
      LEVEL 2,UF,UFB                                                        STRNCP      6
      PA=0.0                                                                STRNCP      7
      DO 100 KK=1,ITYPE                                                     STRNCP      8
        CALL FRDATA (KK,ITCL,MID,ISTART,IFIN)                               STRNCP      9
        DO 90 IJ=ISTART,IFIN                                                STRNCP     10
          PA=PA+1.0                                                         STRNCP     11
          PHI=PA*H                                                          STRNCP     12
          S=SIN(PHI)                                                        STRNCP     13
          C=COS(PHI)                                                        STRNCP     14
          CT=C/S                                                            STRNCP     15
          K1=3*IJ-2                                                         STRNCP     16
          K2=K1+1                                                           STRNCP     17
          K3=K1+2                                                           STRNCP     18
          IW=IJ*2+3                                                         STRNCP     19
          IW=IW+1                                                           STRNCP     20
          ISW=IW-2*MID                                                      STRNCP     21
          U1=0.0                                                            STRNCP     22
          U2=0.0                                                            STRNCP     23
          W1=0.0                                                            STRNCP     24
          W2=0.0                                                            STRNCP     25
          DO 80 K=1,ITCL                                                    STRNCP     26
            ICU=ISW+2*K                                                     STRNCP     27
            ICW=ICU+1                                                       STRNCP     28
            U1=U1+AN(KK,1,K)*X(ICU)                                         STRNCP     29
            U2=U2+AN(KK,2,K)*X(ICU)                                         STRNCP     30
            W1=W1+AN(KK,1,K)*X(ICW)                                         STRNCP     31
            W2=W2+AN(KK,2,K)*X(ICW)                                         STRNCP     32
   80     CONTINUE                                                          STRNCP     33
          UF(K1,IKK)=U1+X(IW)                                               STRNCP     34
          UF(K2,IKK)=X(IW)*CT+X(IW)                                         STRNCP     35
          UF(K3,IKK)=X(IW)-W1                                               STRNCP     36
          UFB(K1,IKK)=U2+W1                                                 STRNCP     37
          UFB(K2,IKK)=W1+U1*CT-X(IW)/(S*S)                                  STRNCP     38
          UFB(K3,IKK)=U1-W2                                                 STRNCP     39
   90   CONTINUE                                                            STRNCP     40
  100 CONTINUE                                                              STRNCP     41
      RETURN                                                                STRNCP     42
      END                                                                   STRNCP     43
**    *DECK BCKSUB                                                          BCKSUB      1
      SUBROUTINE BCKSUB (FV,NFV,AB,KAB,AI,IN,IABW)                          BCKSUB      2
      DIMENSION FV(NFV),AB(12,KAB)                                          BCKSUB      3
C     TO FIND THE FIRST AND LAST 4/6 NODAL VALUES FROM THE BOUNDARY CONDITIONS  BCKSUB  4
C     ON ENTRY FV CONTAINS THE REDUCED VECTOR STARTING WITH FV(1)           BCKSUB      5
C     ON EXIT FV CONTAINS THE FULL VECTOR STARTING WITH FV(1)               BCKSUB      6
      IF(AI.LT.0.0) GO TO 10                                                BCKSUB      7
      IF (AI.EQ.0.0) GO TO 10                                               BCKSUB      8
      ID=6                                                                  BCKSUB      9
      IC=3                                                                  BCKSUB     10
      GO TO 20                                                              BCKSUB     11
   10 ID=4                                                                  BCKSUB     12
      IC=2                                                                  BCKSUB     13
   20 CONTINUE                                                              BCKSUB     14
      IAB=IABW-ID                                                           BCKSUB     15
      IG=IAB+1                                                              BCKSUB     16
      NNA=IC*(IN-2)                                                         BCKSUB     17
      KJ=NNA+1                                                              BCKSUB     18
      DO 30 JJ=1,NNA                                                        BCKSUB     19
        J=KJ-JJ                                                             BCKSUB     20
   30   FV(J+ID)=FV(J)                                                      BCKSUB     21
      DO 50 IJ=1,ID                                                         BCKSUB     22
        BEMP=0.0                                                            BCKSUB     23
        TEMP=0.0                                                            BCKSUB     24
        DO 40 J=1,IAB                                                       BCKSUB     25
          BEMP=BEMP+AB(IC+IJ,IG-J)*FV(NNA+ID+IJ-J)                          BCKSUB     26
   40     TEMP=TEMP+AB(IC+1-IJ,ID+J)*FV(IC+1-IJ+J)                          BCKSUB     27
      IHC=NNA+ID+IJ                                                         BCKSUB     28
      IF(IHC.EQ.104.AND.AI.LT.0.0) BEMP=BEMP-1.0                            BCKSUB     29
        FV(NNA+ID+IJ)=-BEMP/AB(IC+IJ,IG)                                    BCKSUB     30
   50   FV(IC+1-IJ)=-TEMP/AB(ID+1-IJ,ID)                                    BCKSUB     31
```

```
      SUBROUTINE FCDATA (KK,ITOL,MID,ISTART,IFIN)                       FCDATA     2
      COMMON/COM1/ AN(7,4,12),ITYPE,ILN(7),IMN(7),JSN(7),JEN(7),KBOE(7) FCDATA     3
      COMMON /COM4/ IN,N1,N2                                            FCDATA     4
      ITOL=ILN(KK)                                                      FCDATA     5
      MID=IMN(KK)                                                       FCDATA     6
      IF(KBOE(KK)) 10,20,30                                             FCDATA     7
   10 ISTART=JSN(KK)                                                    FCDATA     8
      IFIN=JEN(KK)                                                      FCDATA     9
      RETURN                                                            FCDATA    10
   20 ISTART=JSN(KK)                                                    FCDATA    11
      IFIN=IN-2-JEN(KK)                                                 FCDATA    12
      RETURN                                                            FCDATA    13
   30 ISTART=IN-2-JSN(KK)                                               FCDATA    14
      IFIN=IN-2-JEN(KK)                                                 FCDATA    15
      RETURN                                                            FCDATA    16
      END                                                               FCDATA    17
***   *DECK FUNVEC                                                      FUNVEC     1
      SUBROUTINE FUNVEC (UF,UFB,KUF,NUF,UFC,NUFC,X,Y,NY,Z,NZ,FF,IPR,IFT, FUNVEC     2
     CLNC,E,FD,IN,PL,ICHS,IFAIL,ELMT)                                   FUNVEC     3
      DIMENSION UF(NUF,KUF),UFB(NUF,KUF),UFC(NUFC,KUF),PF(KUF),X(NY),Y(N FUNVEC     4
     CY),Z(NZ)                                                          FUNVEC     5
      COMMON /COM5/ VF(147,20),VFB(147,20),VFC(106,20),PFC(20),EFC(20)  FUNVEC     6
     C,SF(147)                                                          FUNVEC     7
      LEVEL 2,UF,UFB,UFC                                                FUNVEC     8
      CK2=1.0/(10.0**6)                                                 FUNVEC     9
      A1=E+FF                                                           FUNVEC    10
      IF(A1.GT.ELMT.OR.A1.LT.CK2) GO TO 130                             FUNVEC    11
      N=3*(IN-2)                                                        FUNVEC    12
      N2=2*(IN+2)                                                       FUNVEC    13
      IK=0                                                              FUNVEC    14
      KK=0                                                              FUNVEC    15
      IF(IFT.EQ.0) GO TO 40                                             FUNVEC    16
      DO 1 IJ=1,20                                                      FUNVEC    17
      I=21-IJ                                                           FUNVEC    18
      IF (A1.LE.EFC(I)) IK=I                                            FUNVEC    19
    1 CONTINUE                                                          FUNVEC    20
      KK=(IK-1)*LNC                                                     FUNVEC    21
      PL=PFC(IK)                                                        FUNVEC    22
      IF(IK.GT.1) A1=A1-EFC(IK-1)                                       FUNVEC    23
      IF(ICHS.LT.0) GO TO 80                                            FUNVEC    24
      IF (ICHS.EQ.2) GO TO 20                                           FUNVEC    25
      DO 10 I=1,N                                                       FUNVEC    26
       X(I)=VF(I,IK)                                                    FUNVEC    27
   10  Y(I)=VFB(I,IK)                                                   FUNVEC    28
      IF (ICHS.EQ.1) GO TO 80                                           FUNVEC    29
   20 DO 30 I=1,N2                                                      FUNVEC    30
   30  Z(I)=VFC(I,IK)                                                   FUNVEC    31
      GO TO 80                                                          FUNVEC    32
   40 PL=0.0                                                            FUNVEC    33
      IF(ICHS.LT.0) GO TO 80                                            FUNVEC    34
      IF (ICHS.EQ.2) GO TO 60                                           FUNVEC    35
      DO 50 I=1,N                                                       FUNVEC    36
       X(I)=0.0                                                         FUNVEC    37
   50  Y(I)=0.0                                                         FUNVEC    38
      IF (ICHS.EQ.1) GO TO 80                                           FUNVEC    39
   60 DO 70 I=1,N2                                                      FUNVEC    40
   70  Z(I)=0.0                                                         FUNVEC    41
   80 DO 120 I=1,IPR                                                    FUNVEC    42
       AF=A1**I                                                         FUNVEC    43
       PL=PL+PF(I+KK)*AF                                                FUNVEC    44
       IF (ICHS.LT.0) GO TO 120                                         FUNVEC    45
       IF (ICHS.EQ.2) GO TO 100                                         FUNVEC    46
       DO 90 J=1,N                                                      FUNVEC    47
        X(J)=X(J)+UF(J,I+KK)*AF                                         FUNVEC    48
   90   Y(J)=Y(J)+UFB(J,I+KK)*AF                                        FUNVEC    49
       IF (ICHS.EQ.1) GO TO 120                                         FUNVEC    50
  100  DO 110 J=1,N2                                                    FUNVEC    51
  110   Z(J)=Z(J)+UFC(J,I+KK)*AF                                        FUNVEC    52
  120 CONTINUE                                                          FUNVEC    53
      RETURN                                                            FUNVEC    54
  130 IF(A1.GT.ELMT) IFAIL=1                                            FUNVEC    55
      IF(A1.LT.CK2) IFAIL=2                                             FUNVEC    56
      RETURN                                                            FUNVEC    57
      END                                                               FUNVEC    58
***   *DECK SLVITR                                                      SLVITR     1
      SUBROUTINE SLVITR (A,HA,NA,KA,KHA,UF,NUF,KUF,UFB,NUFB,KUFB,U,NU,KU SLVITR     2
     C,Z1,Z2,NZ,X,NX,V,NV,F,NP,NUM,IFAIL,AP,KAP,NIK,Z3,Z4,V4,FF,ESN,EPCL SLVITR     3
     C,E,UFC,NUFC,IFT,LNC)                                              SLVITR     4
      DIMENSION A(NA,KA),HA(NA,KHA),UF(NUF,KUF),UFB(NUFB,KUFB),U(NU,KU), SLVITR     5
     CZ1(NZ),Z2(NZ),X(NX),V(NV),P(NP),AP(12,KAP),Z3(NZ),Z4(NZ),V4(NZ),FF SLVITR     6
     C(KUF),UFC(NUFC,KUF)                                               SLVITR     7
      LEVEL 2,UF,UFB,UFC                                                SLVITR     8
C     SLVITR CONTROLS THE ITERATIVE ROUTINES DIRITR AND INVITR WHICH SOLVE THE SLVITR     9
C     SECONDARY PATH EIGEN VALUE PROBLEM. THE A MATRIX IS UNCHANGED HA IS USED SLVITR    10
C     TO STORE THE A MATRIX BEING WORKED ON                             SLVITR    11
      IFNV=0                                                            SLVITR    12
      DO 40 KKK=1,NUM                                                   SLVITR    13
      NIK=NKK                                                           SLVITR    14
      CALL DIRITR (A,HA,NA,KA,KHA,X,NX,UF,NUF,KUF,UFB,NUFB,KUFB,U,NU,KU, SLVITR    15
     CZ1,Z2,NZ,V,NV,IFAIL,NIK,Y,AP,KAP,FF,ESN,EPCL,E,UFC,NUFC,IFT,LNC,IF SLVITR    16
     CNV)                                                               SLVITR    17
      IF(IFAIL.NE.0) GO TO 20                                           SLVITR    18
   10 CALL INVITR (A,HA,NA,KA,KHA,X,NX,Z1,Z2,NZ,V,NV,IFAIL,NIK,Y,U,NU,KU SLVITR    19
     C,UF,NUF,KUF,UFB,NUFB,KUFB,F,NP,AP,KAP,Z3,Z4,V4,FF,ESN,EPCL,E,UFC, SLVITR    20
     CNUFC,IFT,LNC,IFNV)                                                SLVITR    21
      IF(IFAIL.NE.0) GO TO 30                                           SLVITR    22
      GO TO 40                                                          SLVITR    23
   20 WRITE (6,8001)                                                    SLVITR    24
      IFAIL=0                                                           SLVITR    25
      GO TO 10                                                          SLVITR    26
   30 NIK=NIK-1                                                         SLVITR    27
      WRITE (6,8002) NIK                                                SLVITR    28
      RETURN                                                            SLVITR    29
   40 CONTINUE                                                          SLVITR    30
 8001 FORMAT (1H ,10X,61HDIRECT ITERATION HAS FAILED, INVERSE ITERATION SLVITR    31
     CWILL BE TRIED.)                                                   SLVITR    32
 8002 FORMAT (1H ,10X,34HINVERSE ITERATION HAS FAILED WITH ,I3,1X,15HE-V SLVITR    33
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```
***     *DECK DIRITR
              SUBROUTINE DIRITR (A,HA,NA,KA,KHA,X,NX,UF,NUF,KUF,UFB,NUFB,KUFB,U,
             CNU,KU,Z1,Z2,NZ,V,NV,IFAIL,NIK,Y,AB,KAB,FF,ESN,ELMT,E,UFC,NUFC,IFT,
             CLNC,IFNV)
              DIMENSION A(NA,KA),HA(NA,KHA),X(NX),UF(NUF,KUF),UFB(NUFB,KUFB),
             CU(NU,KU),Z1(NZ),Z2(NZ),V(NV),AB(12,KAB),FF(KUF),UFC(NUFC,KUF)
              COMMON/COM2/ D0,D1,D2,D3,D4,AL,THETA,PCL,H,AT,IPR,ION
              COMMON /COM3/ JAC,JAW,JABW,IE,ID,IC,FT,IZAP
              COMMON /COM4/ IN,N1,N2
              LEVEL 2,UF,UFB,UFC
C             TO PERFORM DIRECT ITERATION WHEN THE LEFT SIDE IS A BANDED MATRIX A (HA)
C             AND THE RHS IS CALCULATED AS A SINGLE VECTOR V BY SNDRHS SUB-DOMINANT E-VA
C             LUES/VECTORS ARE FOUND BY ORTHOGONALIZATION THE SMALLEST E-VAL IS FOUND
C             FIRST
C             ARGUMENTS ******
C             ARGUMENTS ******
C             A- LHS MATRIX(UNCHNGD), HA-WRKSPC(TRIANG DECOMP OF A ON EXIT), UF- FUND
C             PATH STRESS COMPONENTS (UNCHNGD) U- PREVIOUSLY FOUND VECTORS USED BY
C             ORTHOG. (UNCHNGD), V-WRKSPC,X- UNDEFINED (APPROX E-VECT,NORM) Y- UNDEFINED
C             (APPROX E-VAL) NIK- NO OF E-VALS BEING FOUND. OTHERS AS DEFINED BY MAIN
C             PROGRAM
C             THE FOLLOWING COULD BE DEFINED ELSEWHERE
              CK2=1.0/(10.0**6)
              IO=LNC
              IF(IFT.EQ.0) IO=IPR
              ICHS=1
              IFFV=0
              YZ=0.0
              LMT=30
              CHK1=1.0/(10.0**3)
              ISTEP=1
              IWILL=3
C             LMT - NO. OF ITERATIONS THAT WILL BE ALLOWED ON FIRST TRY,CHK1 - A
C             CCURACY REQUIRED FOR E-VAL ISTEP - NO OF ITERATIONS PERFORMED BET
              ICOUNT=ISTEP-1
              PT=1.0
C             WEEN ORTHOGS. IWILL - NO OF ATTEMPTS I.E.40-10-10--FAIL.
              IDID=0
              Y=0.295*(E+0.05)
              M1=0
              DO 10 I=1,N1
                DO 10 J=1,JAW
                  HA(I,J)=A(I,J)
           10 CONTINUE
              DO 30 I=1,N2
                X(I)=1.0
           30 CONTINUE
C             START OF DIRECT ITERATION
           40 M1=M1+1
              ICOUNT=ICOUNT+1
              IF(M1.GT.LMT) GO TO 70
              CALL FINVEC (UF,UFB,KUF,NUF,UFC,NUFC,Z1,Z2,NZ,V,NV,PF,IO,IFT,LNC,Y
             C,YZ,IN,FL,ICHS,IFNV,ELMT)
              IF(IFNV.NE.0) GO TO 70
              CALL SNDRHS(Z1,Z2,NZ,V,NV,X,NX,PL)
              DO 45 I=1,N1
           45   V(I)=V(I)/Y
              CALL MAC7D (HA,V,NA,N1,JAW,FT,KHA,NV)
              FT=0.0
C             IF SUB-DOMINAHANT E-VECT ORTHOG
              IF (NIK.GT.1.AND.ICOUNT.GE.ISTEP) CALL ORTHOG (NIK,ISTEP,ICOUNT,N1
             C,V,U,KU,NV,KU)
C             FIND LARGEST ELEMENT OF V, V(IJJ)
              SAV=0.0
              DO 50 I=1,N1
                TEMP=ABS(V(I))
                IF(TEMP.LE.SAV) GO TO 50
                SAV=TEMP
                IJJ=I
           50 CONTINUE
              TEMP=V(IJJ)
              DO 60 I=1,N1
                X(I)=V(I)/TEMP
           60 CONTINUE
C             TEST FOR CONVERGENCE ON E-VALUE
              HOLD=1.0/TEMP
              SAV=ABS((Y-HOLD)/HOLD)
              Y=HOLD
              CALL BCKSUB (X,NX,AB,KAB,AT,IN,JABW)
              IF(SAV.GT.CHK1) GO TO 40
              WRITE (6,8001) CHK1,M1,Y
              RETURN
           70 CONTINUE
              IF(IFNV.EQ.0) GO TO75
              IFFV=IFFV+1
              TH1=ELMT-ELMT/50.0
              IF (Y.LE.CK2) TH1=0.6*ELMT
              ESN=Y
              Y=TH1
              IFNV=0
              IF(IFFV.LE.4) GO TO 40
              WRITE (6,8003) ELMT,ESN
              IFAIL=IFAIL+1
              RETURN
           75 CONTINUE
              WRITE (6,8002) CHK1,LMT,Y
              IDID=IDID+1
              IF(IDID.GE.IWILL) GO TO 80
              LMT=10
              IDO=2*(IDID/2)
              IDO=IDID-IDO
              LMT=LMT+IDO
              M1=0
              GO TO 40
           80 IFAIL=IFAIL+1
              RETURN
         8001 FORMAT (1H ,28HCONVERGENCE ON E-VALUE (ERR< ,E10.3,8H) AFTER ,I3,1
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 8003 FORMAT (1H ,5X,29HY OUTSIDE OF FUNVEC RANGE, 0-    ,E10.4,2X,2HY=,      DIRITR     110
     1E10.4)                                                                  DIRITR     111
      END                                                                     DIRITR     112
      END                                                                     DIRITR     113
***   *DECK INVITR                                                            INVITR       1
      SUBROUTINE INVITR (A,HA,NA,KA,KHA,X,NX,Z1,Z2,NZ,V,NV,IFAIL,NIK,Y,       INVITR       2
     CU,NU,KU,UF,NUF,KUF,UFB,NUFB,KUFB,P,NP,AB,KAB,Z3,Z4,V4,PF,ESN,EPCL,      INVITR       3
     CF,UFC,NUFC,IFT,LNC,IFNV)                                                INVITR       4
      DIMENSION A(NA,KA),HA(NA,KHA),X(NX),Z1(NZ),Z2(NZ),V(NV),U(NU,KU),       INVITR       5
     CUF(NUF,KUF),UFB(NUFB,KUFB),P(NP),AB(12,KAB),Z3(NZ),Z4(NZ),V4(NZ)        INVITR       6
     C,PF(KUF),UFC(NUFC,KUF)                                                  INVITR       7
      COMMON/COM2/ DO,D1,D2,D3,D4,AL,THETA,PCL,H,AT,IPR,ION                   INVITR       8
      COMMON /COM3/ JAC,JAW,JAWW,IE,ID,IC,FT,IZAP                             INVITR       9
      COMMON /COM4/ IN,N1,K2                                                  INVITR      10
      LEVEL 2,UF,UFB,UFC                                                      INVITR      11
C     TO PERFORM INVERSE ITERATION WITH MULTIPLE SHIFTS OF ORIGIN, WHEN THE LHS  INVITR    12
C     IS A BANDED MATRIX A AND THE APPROX E-VAL IS GIVEN BY Y AND INITIAL E-VECT INVITR    13
C     BY X. THE RHS IS CALCULATED BY SNDRHS AS A SINGLE VECT V AND THE NEW LHS  INVITR     14
C     BANDED MATRIX BY SNDLHS. ORTHOG IS USED TO FIND THE SUB-DOM E-VECT INITIAL INVITR    15
C     LY AND DROPPED (AS IT MAY INTRODUCE SIGNIFICANT ERRORS) FOR THE FINALSET  INVITR     16
C     OF ITERATIONS. E.G. SHIFT OF ORIGIN BY SNDLHS TO Y ITERATION TO 10**-6    INVITR     17
C     ACCURACY USING ORTHOG THEN NEW SHIFT OF ORIGIN BY SNDLHS AND ITERATION TO INVITR     18
C     10**-9 WITHOUT ORTHOG.                                                    INVITR     19
C     ARGUMENTS                                                                 INVITR     20
C     ARGUMENTS                                                                 INVITR     21
C     A- LHS MATRIX (UNCHANGD) HA-WRKSPCE, UF-FUND PATH STRESS COMPONENTS (UNCHG) INVITR    22
C     U- E-VECTORS (NEW VECT ADDEDAS U( ,NIK), V-WRKSPC, X-APPROX E-VECTOR (FINA INVITR    23
C     L E-VECT), Y- APPROX E-VAL, P- E-VALS (NEW E-VAL ADDED AS P(NIK))         INVITR     24
C     OTHERS AS DEFINED IN THE CALLING PROGRAM                                  INVITR     25
C     THE FOLLOWING COULD BE DEFINED ELSEWHERE.                                 INVITR     26
      CK2=1.0/(10.0**6)                                                       INVITR      27
      IO=LNC                                                                  INVITR      28
      IF(IFT.EQ.0) IO=IPR                                                     INVITR      29
      ICHS=1                                                                  INVITR      30
      IFFV=0                                                                  INVITR      31
      IZAP=1                                                                  INVITR      32
      LMT=40                                                                  INVITR      33
      CHK1=1.0/(10.0**5)                                                      INVITR      34
      CHK2=1.0/(10.0**12)                                                     INVITR      35
      ISTEP=1                                                                 INVITR      36
      IWILL = 5                                                               INVITR      37
      KK12=1                                                                  INVITR      38
      NJK=NIK                                                                 INVITR      39
      PT=1.0                                                                  INVITR      40
      ITEST=0                                                                 INVITR      41
      ICOUNT=ISTEP-1                                                          INVITR      42
      ICID=1                                                                  INVITR      43
      M1=0                                                                    INVITR      44
      Z=Y                                                                     INVITR      45
      PC=0.0                                                                  INVITR      46
C     CHK1-ALLOWED ERR IN E-VALUE AND E-VECTOR , CHK2 ELEMENT OF E-VECT       INVITR      47
C     LESS NOT CHECKED, ISTEPNO OF ITERATIVE STEPS BETWEEN ORTHOG.,IWILL      INVITR      48
C     NO. OF ORIGIN SHIFTS TO BE TRIED. LMTNO OF ITERATIONS BEFORE SHIFT      INVITR      49
C     OF ORIGIN                                                               INVITR      50
C     SHIFT OF ORIGIN I.E. SET UP HA=A+Y*B                                    INVITR      51
    1 DO 10 I=1,N1                                                            INVITR      52
        DO 10 J=1,JAW                                                         INVITR      53
   10     HA(I,J)=A(I,J)                                                      INVITR      54
      CALL FUNVEC (UF,UFB,KUF,NUF,UFC,NUFC,Z3,Z4,NZ,V,NV,PF,IO,IFT,LNC,Z      INVITR      55
     C,PC,IN,PL,ICHS,IFNV,EPCL)                                               INVITR      56
      CALL SNDLHS (HA,NA,KHA,Z3,Z4,NZ,PL)                                     INVITR      57
      CALL REDUCE (HA,AB,NA,KHA,KAB,N1,AI,JAW,JAWW,IZAP,Z2,NZ)                INVITR      58
      PC=Z*CHK1*50.0                                                          INVITR      59
      Z=Y+PC                                                                  INVITR      60
C     START OF INVERSE ITERATION                                              INVITR      61
   20 M1=M1+1                                                                 INVITR      62
      ICOUNT=ICOUNT+1                                                         INVITR      63
      IF(M1.GT.LMT) GO TO 90                                                  INVITR      64
      CALL FUNVEC (UF,UFB,KUF,NUF,UFC,NUFC,Z1,Z2,NZ,V,NV,PF,IO,IFT,LNC,Y      INVITR      65
     C,PC,IN,PL2,ICHS,IFNV,EPCL)                                              INVITR      66
      IF(IFNV.NE.0) GO TO 90                                                  INVITR      67
      CALL SNDRHS(Z1,Z2,NZ,V,NV,X,NX,PL2)                                     INVITR      68
      CALL SNDRHS(Z3,Z4,NZ,V4,NV,X,NX,PL)                                     INVITR      69
      DO 25 I=1,N1                                                            INVITR      70
   25   V(I)=(V(I)-V4(I))/PC                                                  INVITR      71
      CALL MAC7B (HA,V,NA,N1,JAW,PT,KHA,NV)                                   INVITR      72
      P1=0.0                                                                  INVITR      73
C     IF SUB-DOMINANANT E-VAL/VECT ORTOG                                      INVITR      74
      IF (NJK.GT.1.AND.ICOUNT.GE.ISTEP) CALL ORTHOG (NIK,ISTEP,ICOUNT,N1      INVITR      75
     C,V,U,KU,NV,KU)                                                          INVITR      76
C     FIND LARGEST ELEMENT OF V, V(IJJ)                                       INVITR      77
      SAV=0.0                                                                 INVITR      78
      DO 30 I=1,N1                                                            INVITR      79
        TEMP=ABS(V(I))                                                        INVITR      80
        IF (TEMP.LE.SAV) GO TO 30                                             INVITR      81
        SAV=TEMP                                                              INVITR      82
        IJJ=I                                                                 INVITR      83
   30   CONTINUE                                                              INVITR      84
      TEMP=V(IJJ)                                                             INVITR      85
      PC=1.0/TEMP                                                             INVITR      86
C     TEST FOR CONVRGENCE ON E-VAL                                            INVITR      87
      KLL=0                                                                   INVITR      88
      KLM=0                                                                   INVITR      89
      IKK=ITEST                                                               INVITR      90
      HOLD=Y+1.0/TEMP                                                         INVITR      91
      SAV=ABS((Z-HOLD)/HOLD)                                                  INVITR      92
      Z=HOLD                                                                  INVITR      93
      IF (SAV.GT.CHK1) KLL=1                                                  INVITR      94
C     TEST, FOR CONVERGENCE ON E-VECTOR, AND COPY V ONTO X                    INVITR      95
      DO 40 I=1,N1                                                            INVITR      96
        V(I)=V(I)/TEMP                                                        INVITR      97
        HOLD=ABS(V(I))                                                        INVITR      98
        IF (HOLD.LE.CHK2) GO TO 40                                            INVITR      99
        HOLD=ABS((X(I+ID)-V(I))/V(I))                                         INVITR     100
        IF (HOLD.GT.CHK1) KLM=2                                               INVITR     101
   40 X(I)=V(I)                                                               INVITR     102
      IKK=IKK+KLL+KLM                                                         INVITR     103
      CALL BCKSUB (X,NX,AB,KAB,AI,IN,JAWW)                                    INVITR     104
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      IF(IKK.EQ.1) WRITE (6,8001) CHK1,M1,Z                              INVITR     108
      IF( IKK.EQ.2) WRITE(6,8002) CHK1,M1                                INVITR     109
      ITEST=10                                                           INVITR     110
      GO TO 20                                                           INVITR     111
   50 CONTINUE                                                           INVITR     112
C,    WILL DROP ORIOG KCK AND PERFORM FINAL ITERATIONS WITHOUT            INVITR     113
      IF (KK12.EQ.2) GO TO 60                                            INVITR     114
      WRITE (6,8003) CHK1,M1,Z                                           INVITR     115
      NJK=1                                                              INVITR     116
      KK12=KK12+1                                                        INVITR     117
      CHK1=CHK1/(10.0**3)                                                INVITR     118
      CHK2=CHK2/(10.0**3)                                                INVITR     119
      PC=0.0                                                             INVITR     120
      PT=1.0                                                             INVITR     121
      ITEST=0                                                            INVITR     122
      M1=0                                                               INVITR     123
      Y=Z                                                                INVITR     124
      LMT=40                                                             INVITR     125
      IDID=1                                                             INVITR     126
      GO TO 1                                                            INVITR     127
   60 CONTINUE                                                           INVITR     128
C     RECORD NEW E-VAL/VECT                                              INVITR     129
      E(NIK)=Z                                                           INVITR     130
      DO 70 I=1,N1                                                       INVITR     131
   70   U(I,NIK)=V(I)                                                    INVITR     132
      WRITE (6,8003) CHK1,M1,Z                                           INVITR     133
C     TO COMPUTE AND STORE EUCLIDEAN NORM OF NEW VECTOR                  INVITR     134
      IF(NIK.EQ.KU) RETURN                                               INVITR     135
      U(N1-NIK,KU)=0.0                                                   INVITR     136
      DO 80 I=1,N1                                                       INVITR     137
   80   U(N1-NIK,KU)=U(N1-NIK,KU)+V(I)*V(I)                              INVITR     138
      RETURN                                                             INVITR     139
   90 CONTINUE                                                           INVITR     140
      IF(IENV.EQ.0) GO TO 95                                             INVITR     141
      ESN=Y+PC                                                           INVITR     142
      PC=EPCL-EPCL/25.0-Y                                                INVITR     143
      IF (Z.LE.CK2) PC=0.5*EPCL-Y                                        INVITR     144
      IENV=0                                                             INVITR     145
      IFFV=IFFV+1                                                        INVITR     146
      IF(IFFV.LE.4) GO TO 20                                             INVITR     147
      WRITE (6,8006) EPCL,ESN                                            INVITR     148
      IFAIL=IFAIL+2                                                      INVITR     149
      RETURN                                                             INVITR     150
   95 WRITE (6,8004) LMT,Z                                               INVITR     151
      IF (IDID.EQ.IWILL) GO TO 100                                       INVITR     152
      PT=1.0                                                             INVITR     153
      ITEST=0                                                            INVITR     154
      M1=0                                                               INVITR     155
      Y=Z                                                                INVITR     156
      LMT=20                                                             INVITR     157
      IDO=2*(IDID/2)                                                     INVITR     158
      IDO=IDID-IDO                                                       INVITR     159
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      LMT=LMT+IDO                                                        INVITR     160
      IDID=IDID+1                                                        INVITR     161
      WRITE (6,8005) Z                                                   INVITR     162
      GO TO 1                                                            INVITR     163
  100 IFAIL=IFAIL+2                                                      INVITR     164
      RETURN                                                             INVITR     165
 8001 FORMAT(1H ,10X,29HCONVERGENCE ON E-VALUE (ERR< ,E10.3,8H) AFTER ,I INVITR     166
     C3,1X,12HITERATIONS   ,E12.5)                                       INVITR     167
 8002 FORMAT(1H ,10X,30HCONVERGENCE ON E-VECTOR (ERR< ,E10.3,8H) AFTER ,  INVITR     168
     CI3,1X,12HITERATIONS   )                                            INVITR     169
 8003 FORMAT (1H ,10X,39HCOMPLETE CONVERGENCE, E-VAL/VECT (ERR< ,E10.3,8  INVITR     170
     CH) AFTER ,I3,1X,12HITERATIONS  ,E12.5)                             INVITR     171
 8004 FORMAT(1H ,10X,50HFAILED TO CONVERGE USING INVERSE ITERATION WITHI  INVITR     172
     CN  ,I3,1X,17HITERATIONS E-VAL= ,E12.5)                             INVITR     173
 8005 FORMAT(1H ,10X,36HWILL TRY ANOTHER SHIFT OF ORIGIN TO  ,E12.5)      INVITR     174
 8006 FORMAT(1H ,5X,29HY OUTSIDE OF FUNVEC RANGE, 0-  ,E10.4,2X,2HY=,E10  INVITR     175
     1.4)                                                                INVITR     176
      END                                                                INVITR     177
*DECK SNDRHS
      SUBROUTINE SNDRHS (Z1,Z2,NZ,V,NV,X,NX,P)                           SNDRHS       1
      DIMENSION Z1(NZ),Z2(NZ),V(NV),X(NX)                                SNDRHS       2
      COMMON/COM1/ AN(7,4,12),ITYPE,ILN(7),IMN(7),JSN(7),JFN(7),KBOF(7)  SNDRHS       3
      COMMON/COM2/ DO,D1,D2,D3,D4,AL,THETA,PCL,H,AI,TPP,ION              SNDRHS       4
      COMMON /COM3/ JAD,JAW,JAPW,IE,ID,IC,ET,IZAP                        SNDRHS       5
      COMMON /COM4/ IN,N1,N2                                             SNDRHS       6
      COMMON/COM5/ VF(147,20),VFP(147,20),VFC(106,20),EFC(20),EFC(20)    SNDRHS       7
     C,SF(147)                                                           SNDRHS       8
      AI2=AI*AI                                                          SNDRHS       9
      PA=0.0                                                             SNDRHS      10
      DO 30 I=1,N1                                                       SNDRHS      11
   30   V(I)=0.0                                                        SNDRHS      12
      AK=1.0                                                             SNDRHS      13
      AJ=1.0                                                             SNDRHS      14
      IF (ICN.EQ.20.OR.ICN.EQ.30) AK=2.0                                 SNDRHS      15
      IF (ICN.EQ.20.OR.ICN.EQ.0) AJ=0.0                                  SNDRHS      16
      DO 120 KK=1,ITYPE                                                  SNDRHS      17
        CALL FDDATA (KK,TICL,MID,ISTART,IFIN)                            SNDRHS      18
        DO 110 IJ=ISTART,IFIN                                            SNDRHS      19
          PA=PA+1.0                                                      SNDRHS      20
          PHI=PA*H                                                       SNDRHS      21
          S=SIN(PHI)                                                     SNDRHS      22
          C=COS(PHI)                                                     SNDRHS      23
          CT=C/S                                                         SNDRHS      24
          K=IC*IJ-IE                                                     SNDRHS      25
          K1=K+IE/2                                                      SNDRHS      26
          K2=K+IE                                                        SNDRHS      27
          L=3*IJ-2                                                       SNDRHS      28
          L1=L+1                                                         SNDRHS      29
          L2=L+2                                                         SNDRHS      30
      F0=VF(L,1)+(1.0-VF(L,1))*(VF(L2,1)**2)/2.0+D0*VF(L1,1)             SNDRHS      31
      E0=VF(L1,1)+D0*(VF(L,1)+(1.0-VF(L,1))*(VF(L2,1)**2)/2.0)           SNDRHS      32
      E01=VFP(L,1)+(1.0-VF(L,1))+VFP(L2,1)*VF(L2,1)-VFP(L,1)*(VF(L2,1)** SNDRHS      33
     12)/2.0+D0*VFP(L1,1)                                                SNDRHS      34
      F0=(AK*E0-AJ*SF(L))/2.0                                            SNDRHS      35
      F0=(AK*E0-AJ*SF(L1))/2.0                                           SNDRHS      36
                                                                         SNDRHS      37
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      M1=N1+IE*(I-MID)
      M2=M1+IE/2
      M3=M1+IE
      V(K)=V(K)-AN(KK,1,I)*X(M3)*(Z1(L)+D0*Z1(L1)-D2*Z1(L2)*CT-Z2(L2)
     1-2.0*P)+AN(KK,2,I)*X(M3)*Z1(L2)
      V(K2)=V(K2)-AN(KK,1,I)*(X(M1)*(Z1(L)+D0*Z1(L1)+D3*Z1(L2)*CT+Z2(
     1L2)-2.0*P)-X(M3)*((Z1(L)+D0*Z1(L1))*CT+Z2(L)+D0*Z2(L1)))-AN(KK,2
     2,I)*(X(M1)*Z1(L2)-X(M3)*(Z1(L)+D0*Z1(L1)))
      V(K)=V(K)-AN(KK,1,I)*(X(M1)*(-Z1(L2)*(2.0*Z2(L2)+Z1(L2)*CT))+X(
     1M3)*(Z2(L2)*(2.0*Z1(L)+D0*Z1(L1))+Z1(L2)*((2.0-D0)*Z1(L)+D0*Z1(
     2L1))*CT+Z1(L2)*(Z1(L2)*D2/2.0+2.0*Z2(L)+D0*Z2(L1))-Z1(L)*(Z1(L)
     3+D0*Z1(L1))-2.0*P*Z1(L1)))-AN(KK,2,I)*(X(M1)*(-Z1(L2)**2)+X(M3)*
     4Z1(L2)*(2.0*Z1(L)+D0*Z1(L1)))
      V(K2)=V(K2)-AN(KK,1,I)*(X(M1)*(-Z2(L2)*(2.0*Z1(L)+D0*Z1(L1))+Z1
     1(L2)*(D2*Z1(L2)/2.0-2.0*Z2(L)-D0*Z2(L1))-Z1(L)*(Z1(L)+D0*Z1(L1)
     2)-Z1(L2)*((2.0+D0)*Z1(L)+D0*Z1(L1))*CT-2.0*P*Z1(L1))+X(M3)*(-3.
     30*Z2(L2)*Z1(L2)+2.0*Z1(L)*Z2(L)+D0*(Z2(L)*Z1(L1)+Z1(L)*Z2(L1))+
     4(-1.5*Z1(L2)**2+Z1(L)**2+D0*Z1(L)*Z1(L1))*CT))-AN(KK,2,I)*(X(M1
     5)*(-Z1(L2)*(2.0*Z1(L)+D0*Z1(L1)))+X(M3)*(-1.5*Z1(L2)**2+Z1(L)**
     62+D0*Z1(L)*Z1(L1)))
      IF (ICN.EQ.-10) GO TO 50
      V(K)=V(K)-AN(KK,1,I)*X(M3)*(E0*Z1(L)-Z1(L2)*(E0*CT+E01)-Z2(L2)*E0
     1)+AN(KK,2,I)*Z1(L2)*E0*X(M3)
      V(K2)=V(K2)-AN(KK,1,I)*(X(M1)*(Z1(L2)*(E0*CT+E01)+E0*(Z1(L)+Z2(L2
     1)))-X(M3)*(Z1(L)*(E0*CT+E01)+Z2(L)*E0))-AN(KK,2,I)*(X(M1)*Z1(L2)*
     2E0-X(M3)*Z1(L)*E0)
      V(K)=V(K)-AN(KK,1,I)*2.0*X(M3)*P*VF(L1,1)
      V(K2)=V(K2)-AN(KK,1,I)*2.0*X(M1)*P*VF(L1,1)
   50 CONTINUE
      IF(AI.EQ.0.0) GO TO 80
      V(K)=V(K)-AI*AN(KK,1,I)*X(M2)*(-D2*(Z1(L)+Z1(L1))/(S*2.0)-P*S)
      V(K2)=V(K2)-AI*AN(KK,1,I)*X(M2)*(D3*Z1(L2)/(2.0*S)-P*C)
      V(K1)=V(K1)-AN(KK,1,I)*(X(M1)*AI*(D2*(Z1(L)+Z1(L1))/(2.0*S)+P*S
     1)+X(M2)*(D0*(Z2(L)+Z2(L1))+D0*(Z1(L)+Z1(L1))*CT)+X(M3)*AI*(D3*Z
     21(L2)/(2.0*S)-P*C))-AN(KK,2,I)*X(M2)*D0*(Z1(L)+Z1(L1))
      V(K)=V(K)-AI*AN(KK,1,I)*X(M2)*(-Z1(L2)*Z1(L2)/(2.0*S)+D2*(Z1(L)
     1**2+Z1(L1)**2+Z1(L)*Z1(L1))/(2.0*S)-P*Z1(L1)*S)
      V(K2)=V(K2)+AI*AN(KK,1,I)*X(M2)*(Z1(L2)*(Z1(L)+D2*Z1(L1)/2.0)/S
     1+P*Z1(L1)*C)
      V(K1)=V(K1)-AN(KK,1,I)*(-AI*X(M1)*(-Z1(L2)*Z1(L2)/(2.0*S)+D2*(Z
     11(L)**2+Z1(L1)**2+Z1(L)*Z1(L1))/(2.0*S)-P*Z1(L1)*S)+X(M2)*(D0*Z
     21(L2)*Z2(L2)+(1.0-3.0*D0)*Z1(L)*Z2(L)-D0*(Z2(L)*Z1(L1)+Z1(L)*Z2
     3(L1))+(D0*Z1(L2)**2+(1.0-3.0*D0)*Z1(L)**2-2.0*D0*Z1(L)*Z1(L1))*
     4CT/2.0)-AI*X(M3)*(Z1(L2)*(Z1(L)+D2*Z1(L1)/2.0)/S+P*Z1(L1)*C))-AN
     5(KK,2,I)*(0.5*D0*Z1(L2)**2+0.5*(1.0-3.0*D0)*Z1(L)**2-D0*Z1(L)*Z
     61(L1))*X(M2)
      IF(ICN.EQ.-10) GO TO 80
      V(K1)=V(K1)-AN(KK,1,I)*(X(M2)*((E0*CT+E01)*Z1(L)+E0*Z2(L)))-AN(K
     1K,2,I)*X(M2)*E0*Z1(L)
      V(K)=V(K)-AN(KK,1,I)*X(M2)*AI*P*VF(L1,1)*S
      V(K1)=V(K1)+AN(KK,1,I)*(X(M1)*AI*P*VF(L1,1)*S-X(M3)*AI*P*VF(L1,1)
     1*C)
      V(K2)=V(K2)-AN(KK,1,I)*X(M2)*AI*P*VF(L1,1)*C

   80 CONTINUE
      V(K)=V(K)-X(K+ID)*(Z2(L2)+Z1(L2)*(1.0-2.0*D0)*CT-Z1(L)-D0*Z1(L1)
     1+2.0*P)+X(K2+ID)*D3*Z1(L2)
      V(K2)=V(K2)-X(K+ID)*(Z1(L2)+Z2(L)+D0*Z2(L1)+(D0*Z2(L2)+Z1(L)+D0
     1*Z1(L1))*CT-2.0*P*CT)-X(K2+ID)*(D3*Z2(L2)+D3*Z1(L2)*CT-4.0*P)
      V(K)=V(K)-X(K+ID)*(-0.5*D1*Z1(L2)**2-D0*Z2(L2)*Z1(L2)*CT-Z1(L2)
     1*(2.0*D2*Z1(L)+D0*Z1(L1))*CT-Z2(L2)*(2.0*Z1(L)+D0*Z1(L1))-Z1(L2
     2)*(2.0*Z2(L)+D0*Z2(L1))+Z1(L)**2+D0*Z1(L)*Z1(L1)+P*2.0*(Z1(L1)-
     3Z2(L1)*C*S+(Z1(L)-Z1(L1)+Z1(L2)*CT)*C*C))-X(K2+ID)*(Z1(L2)*((2.0
     4+D0)*(-Z2(L2)+Z1(L))+D0*Z1(L1)-Z1(L2)*CT)-2.0*P*(Z2(L1)*S*S-(Z1(
     5L)-Z1(L1)+Z1(L2)*CT)*C*S))
      V(K2)=V(K2)-X(K+ID)*(Z1(L2)*(3.0*Z2(L2)-2.0*Z1(L)-D0*Z1(L1)+D4*
     1Z1(L2)*CT/2.0-D0*Z2(L)*CT)+Z1(L)*(-2.0*Z2(L)-D0*Z2(L1)-Z1(L)*CT
     2-D0*Z2(L2)*CT-D0*Z1(L1)*CT)-D0*Z2(L)*Z1(L1)-2.0*P*(Z1(L1)*CT+Z2
     3(L1)*C*C+(Z1(L)-Z1(L1)-Z1(L2)*CT)*C*S))-X(K2+ID)*(Z1(L2)*(-D3*Z
     41(L2)-(2.0+D0)*(Z2(L)+Z1(L)*CT)-D0*Z2(L1)-D0*Z1(L1)*CT)-Z2(L2)*
     5((2.0+D0)*Z1(L)+D0*Z1(L1))-2.0*P*(2.0*Z1(L1)+Z2(L1)*S*C+(Z1(L)-
     6Z1(L1)+Z1(L2)*CT)*S*S))
      IF(ICN.EQ.-10) GO TO 100
      V(K)=V(K)+X(K+ID)*(E0*Z1(L)-Z1(L2)*(E01+E0*CT)-Z2(L2)*E0)+X(K2+ID
     1)*Z1(L2)*E0
      V(K2)=V(K2)-X(K+ID)*(Z1(L)*(E0*CT+E01)+E0*(Z2(L)+Z1(L2)))-X(K2+ID
     1)*(Z1(L2)*(E0*CT+E01)+Z2(L2)*E0)
      V(K)=V(K)+2.0*X(K+ID)*(VF(L1,1)+VF(L2,1)*CT)*P+2.0*X(K2+ID)*P*VF(
     1L2,1)
      V(K2)=V(K2)-2.0*X(K+ID)*P*VF(L,1)*CT-2.0*X(K2+ID)*P*(VF(L,1)+VF(L
     11,1))
  100 CONTINUE
      IF(AI.EQ.0.0) GO TO 110
      V(K)=V(K)+AI2*X(K+ID)*D0*(Z1(L)+Z1(L1))/(S*S)-AI*X(K1+ID)*((1.0-
     13.0*D0)*Z1(L2)/(2.0*S)-D0*(Z1(L)+Z1(L1))*CT/S)-AI2*X(K2+ID)*D2*Z
     21(L2)/(S*S*2.0)
      V(K2)=V(K2)+AI2*X(K+ID)*D2*Z1(L2)/(2.0*S*S)-AI*X(K1+ID)*(D0*Z2(
     1L2)/S+(Z1(L1)+D0*Z1(L))/S-D2*Z1(L2)*CT/(2.0*S)-P/S)-AI2*X(K2+ID)
     2*(Z1(L1)+D0*Z1(L))/(S*S)
      V(K1)=V(K1)+AI*X(K+ID)*((3.0*D0-1.0)*Z1(L2)/(2.0*S)-D2*(Z2(L)+Z2
     1(L1))/(2.0*S)+D0*(Z1(L)+Z1(L1))*CT/S-2.0*P*C)-X(K1+ID)*(D2*Z2(L2
     2)/2.0-(3.0-D0)*Z1(L1)/2.0-D3*Z1(L)/2.0+D2*(3.0*Z1(L2)+Z2(L)+Z2(L
     31))*CT/2.0-D0*(Z1(L)+Z1(L1))*CT*CT)+AI*X(K2+ID)*(-D2*(Z2(L2)+Z1(
     4L2)*CT)/(2.0*S)+(Z1(L1)+D0*Z1(L))/S-2.0*P*S)
      V(K)=V(K)+AI2*X(K+ID)*(-(1.0-2.0*D0)*Z1(L2)**2+Z1(L1)*((1.0-3.0*
     1D0)*Z1(L1)-2.0*D0*Z1(L)))/(2.0*S*S)-AI*X(K1+ID)*(-Z1(L2)*(D0*Z2(
     2L2)+(1.0-3.0*D0)*Z1(L)/2.0+D2*Z1(L1))/S+(0.5*(1.0-2.0*D0)*Z1(L2)
     3**2-(0.5-1.5*D0)*Z1(L1)**2+D0*Z1(L)*Z1(L1))*CT/S-P*(Z2(L1)*S+(Z1
     4(L)-Z1(L1)+Z1(L2)*CT)*C))+AI2*X(K2+ID)*D2*Z1(L2)*(Z1(L)+2.0*Z1(L
     51))/(2.0*S*S)
      V(K1)=V(K1)+AI*X(K+ID)*(Z1(L2)*(-D2*Z2(L2)+((1.0-3.0*D0)*Z1(L)+2
     1.0*D2*Z1(L1))/2.0)/S+D2*(Z1(L)*(Z2(L)+0.5*Z2(L1))+Z1(L1)*(Z2(L1)
     2+0.5*Z2(L)))/S+(-(0.5-D0)*Z1(L2)**2+(0.5-1.5*D0)*Z1(L1)**2-D0*Z1
     3(L)*Z1(L1))*CT/S-P*(Z1(L1)*C*2.0+(Z1(L)-Z1(L1)+Z1(L2)*CT)*C))-X(
     4K1+ID)*(Z1(L2)*(-(2.0-D0)*Z1(L2)-D2*(2.0*Z2(L)+Z2(L1))+2.0*D2*Z2
     5(L2)*CT-D2*CT*(5.0*Z1(L1)+4.0*Z1(L))+(1.0-2.0*D0)*Z1(L2)*CT*CT)/
     62.0+(-D2*Z2(L2)*(2.0*Z1(L)+Z1(L1))+D2*Z1(L)**2+D4*Z1(L1)**2+D3*Z1
     7(L)*Z1(L1)-2.0*D2*CT*(Z1(L)*(Z2(L)+0.5*Z2(L1))+Z1(L1)*(Z2(L1)+0.
     85*Z2(L)))-Z1(L1)*((1.0-3.0*D0)*Z1(L1)-D0*Z1(L))*CT*CT)/2.0)+AI*X
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      V(K2)=V(K2)-AI*X(K1+IC)*(Z1(L2)*(D2*Z1(L2)/2.0-D0*Z2(L)+D2*(Z1(L   2  SNDRHS  151
     1  1)+0.5*Z1(L))*CT)/S-Z1(L)*(Z1(L1)+Z2(L2))*D0/S-Z1(L1)**2/S-P*(   2  SNDRHS  152
     2  Z1(L1)/S+Z2(L1)*C+(Z1(L)-Z1(L1)+Z1(L2)*CT)*S))+AI2*X(K2+ID)*(-0   2  SNDRHS  153
     3  5*Z1(L2)**2+Z1(L1)**2+D0*Z1(L)*Z1(L1))/(S*S)-AI2*X(K+ID)*D2*Z1(L   2  SNDRHS  154
     4  2)*(Z1(L)+2.0*Z1(L1))/(2.0*S*S)                                    2  SNDRHS  155
       IF (ICN.EQ.-10) GO TO 110                                             SNDRHS  156
        V(K)=V(K)+X(K+IC)*AI2*E0*Z1(L1)/(S*S)+X(K1+IC)*AI*E0*Z1(L1)*CT/S     SNDRHS  157
        V(K1)=V(K1)+X(K+IC)*AI*E0*Z1(L)*C1/S+X(K1+IC)*F0*Z1(L1)/(S*S)+X(     SNDRHS  158
     1  K2+IC)*AI*F0*Z1(L1)/S                                                 SNDRHS  159
        V(K2)=V(K2)-X(K1+IC)*AI*E0*Z1(L1)/S-X(K2+ID)*AI2*E0*Z1(L1)/(S*S)     SNDRHS  160
        V(K)=V(K)+X(K+IC)*AI*P*VF(L2,1)/S                                    SNDRHS  161
        V(K1)=V(K1)+(X(K+IC)*C+X(K2+ID)*S)*AI*P*(VF(L1,1)+VF(L,1)+VF(L2,1    SNDRHS  162
     1  )*CT)                                                                 SNDRHS  163
        V(K2)=V(K2)-X(K1+IC)*AI*P*VF(L,1)/S                                  SNDRHS  164
  110     CONTINUE                                                           SNDRHS  165
  120     CONTINUE                                                           SNDRHS  166
      RETURN                                                                 SNDRHS  167
      END                                                                    SNDRHS  168
****  *DECK SNDLHS                                                           SNDLHS    1
      SUBROUTINE SNDLHS (A,NA,KA,Z1,Z2,NZ,P)                                 SNDLHS    2
      DIMENSION A(NA,KA),Z1(NZ),Z2(NZ)                                       SNDLHS    3
      COMMON/COM1/ AN(7,4,12),ITYPE,ILN(7),IMA(7),JSN(7),JEN(7),KHOF(7)      SNDLHS    4
      COMMON/COM2/ D0,D1,D2,D3,D4,AL,THETA,ECL,H,AI,IFR,ION                  SNDLHS    5
      COMMON /COM3/ JAP,JAK,JAPW,IF,ID,IC,PT,IPAP                            SNDLHS    6
      COMMON /COM4/ IE,N1,N2                                                 SNDLHS    7
      COMMON/COM5/ VF(147,20),VFR(147,20),VFC(106,20),FFC(20),FFC(20)        SNDLHS    8
     C,SF(147)                                                               SNDLHS    9
      PA=0.0                                                                 SNDLHS   10
      AI2=AI*AI                                                              SNDLHS   11
      AK=1.0                                                                 SNDLHS   12
      AJ=1.0                                                                 SNDLHS   13
      IF (ICN.EQ.20.OR.ICN.EQ.30) AK=2.0                                     SNDLHS   14
      IF (ICN.EQ.20.OR.ICN.EQ.0) AJ=0.0                                      SNDLHS   15
      DO 110 KK=1,ITYPE                                                      SNDLHS   16
        CALL FCDATA (KK,ITCL,MID,ISTART,IFIN)                                SNDLHS   17
        DO 100 IJ=ISTART,IFIN                                                SNDLHS   18
          PA=PA+1.0                                                          SNDLHS   19
          PHI=PA*H                                                           SNDLHS   20
          S=SIN(PHI)                                                         SNDLHS   21
          C=COS(PHI)                                                         SNDLHS   22
          CT=C/S                                                             SNDLHS   23
          K=IC*IJ-IE                                                         SNDLHS   24
          K1=K+IE/2                                                          SNDLHS   25
          K2=K+IE                                                            SNDLHS   26
          L=3*IJ-2                                                           SNDLHS   27
          L1=L+1                                                             SNDLHS   28
          L2=L+2                                                             SNDLHS   29
      E0=VF(L,1)+(1.0-VF(L,1))*(VF(L2,1)**2)/2.0+D0*VF(L1,1)                 SNDLHS   30
      F0=VF(L1,1)+D0*(VF(L,1)+(1.0-VF(L,1))*(VF(L2,1)**2)/2.0)               SNDLHS   31
      E01=VFR(L,1)+(1.0-VF(L,1))*VFR(L2,1)*VF(L2,1)-VFR(L,1)*(VF(L2,1)**     SNDLHS   32
     12)/2.0+D0*VFR(L1,1)                                                    SNDLHS   33
      F01=(AK*E0-AJ*SF(I))/2.0                                               SNDLHS   34
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      EC=(AK*EC-AJ*SF(L1))/2.0                                               SNDLHS   35
      E01=(AK*E01-AJ*SF(L2))/2.0                                             SNDLHS   36
          DO 70 I=1,ITCL                                                     SNDLHS   37
            MA=JAD+IC*(I-MID)                                                SNDLHS   38
        A(K,MA+IE)=A(K,MA+IE)+AN(KK,1,I)*(Z1(L)+D0*Z1(L1)-D2*Z1(L2)*CT-Z     SNDLHS   39
     1  2(L2)-2.0*P)-AN(KK,2,I)*Z1(L2)                                        SNDLHS   40
        A(K2,MA-IE)=A(K2,MA-IE)+AN(KK,1,I)*(Z1(L)+D0*Z1(L1)+D3*Z1(L2)*CT      SNDLHS   41
     1  +Z2(L2)-2.0*P)+AN(KK,2,I)*Z1(L2)                                      SNDLHS   42
        A(K2,MA)=A(K2,MA)-AN(KK,1,I)*(Z2(L)+D0*Z2(L1)+(Z1(L)+D0*Z1(L1))*      SNDLHS   43
     1  CT)-AN(KK,2,I)*(Z1(L)+D0*Z1(L1))                                      SNDLHS   44
        A(K,MA)=A(K,MA)-AN(KK,1,I)*(Z1(L2)*(2.0*Z2(L2)+Z1(L2)*CT))-AN(KK   2  SNDLHS   45
     1  ,2,I)*Z1(L2)**2                                                       SNDLHS   46
        A(K,MA+IE)=A(K,MA+IE)+AN(KK,1,I)*(Z1(L2)*(D2*Z1(L2)/2.0+(2.0*Z2(   2  SNDLHS   47
     1  L)+D0*Z2(L1))+((2.0-D0)*Z1(L)+D0*Z1(L1))*CT)-Z1(L)*(Z1(L)+D0*Z1(   2  SNDLHS   48
     2  L1))+Z2(L2)*(2.0*Z1(L)+D0*Z1(L1))-2.0*F*Z1(L1))+AN(KK,2,I)*Z1(L2   2  SNDLHS   49
     3  )*(2.0*Z1(L)+D0*Z1(L1))                                             2  SNDLHS   50
        A(K2,MA-IE)=A(K2,MA-IE)+AN(KK,1,I)*(Z1(L2)*(D2*Z1(L2)/2.0-2.0*Z2   2  SNDLHS   51
     1  (L)-D0*Z2(L1)-((2.0+D0)*Z1(L)+D0*Z1(L1))*CT)-Z1(L)*(Z1(L)+D0*Z1(   2  SNDLHS   52
     2  L1))-Z2(L2)*(2.0*Z1(L)+D0*Z1(L1))-2.0*F*Z1(L1))-AN(KK,2,I)*(Z2(L   2  SNDLHS   53
     3  2)*(2.0*Z1(L)+D0*Z1(L1)))                                           2  SNDLHS   54
        A(K2,MA)=A(K2,MA)+AN(KK,1,I)*(-3.0*Z1(L2)*Z2(L2)+2.0*Z1(L)*Z2(L)   2  SNDLHS   55
     1  +D0*(Z1(L1)*Z2(L)+Z1(L)*Z2(L1))+(-1.5*Z1(L2)**2+Z1(L)*(Z1(L)+D0*   2  SNDLHS   56
     2  Z1(L1)))*CT)+AN(KK,2,I)*(-1.5*Z1(L2)**2+Z1(L)*(Z1(L)+D0*Z1(L1)))   2  SNDLHS   57
       IF (ICN.EQ.-10) GO TO 20                                              SNDLHS   58
        A(K,MA+IE)=A(K,MA+IE)+AN(KK,1,I)*(E0*Z1(L)-Z1(L2)*(E0*CT+E01)-Z2(     SNDLHS   59
     1  L2)*F0)-AN(KK,2,I)*Z1(L2)*E0                                          SNDLHS   60
        A(K2,MA-IE)=A(K2,MA-IE)+AN(KK,1,I)*(Z1(L2)*(F0*CT+E01)+(Z1(L)+Z2(     SNDLHS   61
     1  L2))*E0)+AN(KK,2,I)*Z1(L2)*E0                                         SNDLHS   62
        A(K2,MA)=A(K2,MA)-AN(KK,1,I)*(Z1(L)*(E0*CT+E01)+Z2(L)*E0)-AN(KK,2     SNDLHS   63
     1  ,I)*Z1(L)*E0                                                          SNDLHS   64
        A(K,MA+IE)=A(K,MA+IE)+AN(KK,1,I)*2.0*P*VF(L1,1)                      SNDLHS   65
        A(K2,MA-IE)=A(K2,MA-IE)+AN(KK,1,I)*2.0*P*VF(L1,1)                    SNDLHS   66
   20 CONTINUE                                                               SNDLHS   67
      IF(AI.EQ.0.0) GO TO 70                                                 SNDLHS   68
        A(K,MA+1)=A(K,MA+1)-AI*AN(KK,1,I)*(D2*(Z1(L)+Z1(L1))/(2.0*S)+P*S     SNDLHS   69
     1  )                                                                     SNDLHS   70
        A(K1,MA-1)=A(K1,MA-1)+AI*AN(KK,1,I)*(D2*(Z1(L)+Z1(L1))/(2.0*S)+P     SNDLHS   71
     1  *S)                                                                   SNDLHS   72
        A(K1,MA)=A(K1,MA)+AN(KK,1,I)*(D0*(Z1(L)+Z1(I1))*CT+D0*(Z2(L)+        SNDLHS   73
     1  Z2(L1)))+AN(KK,2,I)*D0*(Z1(L)+Z1(L1))                                 SNDLHS   74
        A(K1,MA+1)=A(K1,MA+1)+AI*AN(KK,1,I)*(D3*Z1(L2)/(2.0*S)-P*C)          SNDLHS   75
        A(K2,MA-1)=A(K2,MA-1)+AI*AN(KK,1,I)*(D3*Z1(L2)/(2.0*S)-P*C)          SNDLHS   76
        A(K,MA+1)=A(K,MA+1)+AI*AN(KK,1,I)*(-Z1(L2)*Z1(L2)/(2.0*S)+D2*((Z   2  SNDLHS   77
     1  1(L)+Z1(L1))**2-Z1(L)*Z1(L1))/(2.0*S)-F*Z1(I1)*S)                     2  SNDLHS   78
        A(K1,MA-1)=A(K1,MA-1)-AI*AN(KK,1,I)*((D2*((Z1(L)+Z1(L1))**2-Z1(L     2  SNDLHS   79
     1  )*Z1(L1))-Z1(L2)**2)/(2.0*S)-P*Z1(I1)*S)                              2  SNDLHS   80
        A(K1,MA)=A(K1,MA)+AN(KK,1,I)*(Z1(L2)*(Z2(L2)+Z1(L2)*C1/2.0)*D0+Z     2  SNDLHS   81
     1  1(L)*((1.0-3.0*D0)*(Z2(L)+Z1(L)*C1/2.0)-D0*(Z2(L1)+Z1(L1)*CT))-D     2  SNDLHS   82
     2  0*Z2(L)*Z1(L1))+AN(KK,2,I)*(0.5*D0*Z1(L2)**2+(1.0-D0*3.0)*0.5*Z      2  SNDLHS   83
     3  1(L)**2-D0*Z1(L)*Z1(L1))                                            2  SNDLHS   84
        A(K1,MA+1)=A(K1,MA+1)-AI*AN(KK,1,I)*((2.0*Z1(L)+D2*Z1(L1))*Z1(L2     2  SNDLHS   85
     1  )/(2.0*S)+P*Z1(L1)*C)                                               2  SNDLHS   86
        A(K2,MA-1)=A(K2,MA-1)+AI*AN(KK,1,I)*((2.0*Z1(L)+D2*Z1(L1))*Z1(L2     2  SNDLHS   87
     1  )/(-2.0*S)-P*Z1(L1)*C)                                                SNDLHS   88
       IF (ICN.EQ.-10) GO TO 70                                              SNDLHS   89
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      A(K1,MA-1)=A(K1,MA-1)-AN(KK,I,I)*AI*P*VF(L1,1)*S                    SNDLHS      93
      A(K1,MA+1)=A(K1,MA+1)+AN(KK,I,I)*AI*P*VF(L1,1)*C                   SNDLHS      94
      A(K2,MA-1)=A(K2,MA-1)+AN(KK,I,I)*AI*P*VF(L1,1)*C                   SNDLHS      95
   70 CONTINUE                                                           SNDLHS      96
      A(K,JAD)=A(K,JAD)+Z2(L2)-Z1(L)-D0*Z1(L1)+(1.0-2.0*D0)*Z1(L2)*CT+   SNDLHS      97
     1 2.0*P                                                             SNDLHS      98
      A(K,JAD+IE)=A(K,JAD+IE)-D3*Z1(L2)                                  SNDLHS      99
      A(K2,JAD-IE)=A(K2,JAD-IE)+Z1(L2)+Z2(L)+D0*Z2(I1)+(D0*Z2(L2)+Z1(L)  SNDLHS     100
     1 +D0*Z1(I1))*CT-2.0*E*CT                                           SNDLHS     101
      A(K2,JAD)=A(K2,JAD)+D3*Z2(L2)+D3*Z1(L2)*CT-4.0*E                   SNDLHS     102
      A(K,JAD)=A(K,JAD)-Z1(L2)*(D4*Z1(L2)/2.0+(D0*Z2(L2)+2.0*D2*Z1(L)+D  2  SNDLHS     103
     1O*Z1(L1))*CT+2.0*Z2(L)+D0*Z2(L1))-Z2(L2)*(2.0*Z1(L)+D0*Z1(L1))+Z1(  2  SNDLHS     104
     2 L)*(Z1(L)+D0*Z1(L1))-2.0*E*(-Z1(I1)+Z2(L1)*C*S-(Z1(L)-Z1(L1)+Z1(L  2  SNDLHS     105
     3 2)*C1)*C*C)                                                       2  SNDLHS     106
      A(K,JAD+IE)=A(K,JAD+IE)+Z1(L2)*(-(2.0+D0)*(Z2(I2)-Z1(L))+D0*Z1(L1  2  SNDLHS     107
     1 )-Z1(I2)*CT)-P*2.0*(Z2(L1)*S*S-(Z1(L)-Z1(L1)+Z1(L2)*CT)*C*S)      SNDLHS     108
      A(K2,JAD-IE)=A(K2,JAD-IE)+Z1(L2)*(3.0*Z2(L2)-2.0*Z1(L)-D0*Z1(L1)+  2  SNDLHS     109
     1 (D4*Z1(L2)/2.0-D0*Z2(L))*C1)+Z1(L)*(-2.0*Z2(I)-D0*Z2(L1)+(-D0*Z2(  2  SNDLHS     110
     2 L2)-Z1(L)-D0*Z1(L1))*CT)-D0*Z2(L)*Z1(L1)-2.0*E*(Z1(L1)*CT+Z2(L1)*  2  SNDLHS     111
     3 C*C+(Z1(L)-Z1(I1)+Z1(I2)*CT)*C*S)                                 SNDLHS     112
      A(K2,JAD)=A(K2,JAD)-Z1(L2)*(D3*Z1(L2)+(2.0+D0)*(Z2(L)+Z1(L)*CT)+D  2  SNDLHS     113
     1 0*(Z2(L1)+Z1(L1)*CT))-Z2(L2)*((7.0+D0)*Z1(L)+D0*Z1(L1))-2.0*P*(2.  2  SNDLHS     114
     2 0*Z1(L1)+Z2(L))*S*C+(Z1(L)-Z1(L1)+Z1(L2)*CT)*S*S)                  2  SNDLHS     115
      IF (ICN.EQ.-10) GO TO 90                                           SNDLHS     116
      A(K,JAD)=A(K,JAD)-E0*(Z1(I)-Z2(L2)-Z1(L2)*CT)+E01*Z1(L2)           SNDLHS     117
      A(K,JAD+IE)=A(K,JAD+IE)-E0*Z1(L2)                                  SNDLHS     118
      A(K2,JAD-IE)=A(K2,JAD-IE)+E0*(Z1(L)*CT+Z2(L)+Z1(L2))+E01*Z1(L)     SNDLHS     119
      A(K2,JAD)=A(K2,JAD)+E0*(Z1(L2)*CT+Z2(I2))+E01*Z1(L2)               SNDLHS     120
      A(K,JAD)=A(K,JAD)-2.0*P*(VF(L1,1)+VF(L2,1)*CT)                     SNDLHS     121
      A(K,JAD+IE)=A(K,JAD+IE)-2.0*P*VF(L2,1)                             SNDLHS     122
      A(K2,JAD-IE)=A(K2,JAD-IE)+2.0*P*VF(L,1)*CT                         SNDLHS     123
      A(K2,JAD)=A(K2,JAD)+2.0*P*(VF(L,1)+VF(L1,1))                       SNDLHS     124
   90 CONTINUE                                                           SNDLHS     125
      IF(AI.EQ.0.0) GO TO 100                                            SNDLHS     126
      A(K,JAD)=A(K,JAD)-AI2*D0*(Z1(L)+Z1(L1))/(S*S)                      SNDLHS     127
      A(K,JAD+1)=A(K,JAD+1)+AI*((1.0-3.0*D0)*Z1(L2)/(2.0*S)-D0*(Z1(L)+Z  SNDLHS     128
     1 1(L1))*CT/S)                                                      SNDLHS     129
      A(K,JAD+2)=A(K,JAD+2)+AI2*D2*Z1(L2)/(2.0*S*S)                      SNDLHS     130
      A(K1,JAD-1)=A(K1,JAD-1)-AI*(D0*(Z1(L)+Z1(L1))*CT/S-(D2*(Z2(L)+Z2(  SNDLHS     131
     1   L1))-(1.0-3.0*D0)*Z1(L2))/(2.0*S)-2.0*E*C)                    2  SNDLHS     132
      A(K1,JAD)=A(K1,JAD)+D2*Z2(I2)/2.0-Z1(L1)*(D4/2.0+CT*CT*D0)-Z1(L)*  SNDLHS     133
     1 (D3/2.0+CT*CT*D0)+D2*CT*(3.0*Z1(L2)+Z2(L)+Z2(L1))/2.0             SNDLHS     134
      A(K1,JAD+1)=A(K1,JAD+1)+AI*(D2*(Z2(I2)+Z1(I2)*C1)/(2.0*S)-(Z1(L1)  SNDLHS     135
     1 +D0*Z1(L))/S+2.0*E*S)                                             SNDLHS     136
      A(K2,JAD-2)=A(K2,JAD-2)-AI2*D2*Z1(L2)/(2.0*S*S)                    SNDLHS     137
      A(K2,JAD-1)=A(K2,JAD-1)+AI*((D0*(Z2(L2)+Z1(L))+Z1(L1)-D2*Z1(L2)*C  SNDLHS     138
     1 T/2.0)/S-P/S)                                                     SNDLHS     139
      A(K2,JAD)=A(K2,JAD)+AI2*(Z1(L1)+D0*Z1(L))/(S*S)                    SNDLHS     140
      A(K,JAD)=A(K,JAD)-AI2*(Z1(L1)*((1.0-3.0*D0)*Z1(L1)-2.0*D0*Z1(L))-  2  SNDLHS     141
     1 (1.0-2.0*D0)*Z1(L2)**2)/(2.0*S*S)                                 SNDLHS     142
      A(K,JAD+1)=A(K,JAD+1)-AI*(Z1(L2)*(D0*Z2(L2)+(1.0-3.0*D0)*Z1(L)/2.  2  SNDLHS     143
     10+D2*Z1(L1)-(1.0-2.0*D0)*Z1(L2)*CT/2.0)/S+Z1(L1)*((0.5-1.5*D0)*Z1(  2  SNDLHS     144
```

```
     2 L1)-D0*Z1(L))*CT/S+E*(Z2(L1)*S-(Z1(L)-Z1(L1)+Z1(L2)*CT)*C))       2  SNDLHS     145
      A(K1,JAD-1)=A(K1,JAD-1)-AI*(Z1(L2)*(-D2*Z2(I2)+(0.5-1.5*D0)*Z1(I)  2  SNDLHS     146
     1 +D2*Z1(L1)-(0.5-D0)*Z1(L2)*CT)/S+Z1(L)*(D2*(Z2(L)+0.5*Z2(L1))-D0*  2  SNDLHS     147
     2 Z1(L1)*CT)/S+Z1(L1)*(D2*(Z2(L1)+0.5*Z2(L))+(0.5-1.5*D0)*Z1(L1)*CT  2  SNDLHS     148
     3 )/S-E*(2.0*Z1(L1)*C+(Z1(L)-Z1(L1)+Z1(L2)*CT)*C))                   SNDLHS     149
      A(K,JAD+2)=A(K,JAD+2)-AI2*D2*Z1(L2)*(Z1(L)+2.0*Z1(L1))/(2.0*S*S)   2  SNDLHS     150
      A(K1,JAD)=A(K1,JAD)+Z1(L2)*(-(2.0-D0)*Z1(L2)-D2*(2.0*Z2(L)+Z2(L1)  2  SNDLHS     151
     1 -(2.0*Z2(L2)-5.0*Z1(L1)-4.0*Z1(L))*CT)+(1.0-2.0*D0)*Z1(L2)*CT*CT)  2  SNDLHS     152
     2 /2.0-D2*Z2(L2)*(2.0*Z1(L)+Z1(I1))/2.0+Z1(L)*(D2*Z1(L)+D3*Z1(L1)-2  SNDLHS     153
     3 .0*D2*(Z2(L)+0.5*Z2(L1))*CT+2.0*L0*Z1(L1)*CT*CT)/2.0+Z1(L1)*(D4*Z  2  SNDLHS     154
     4 1(L1)-2.0*D2*(Z2(L1)+0.5*Z2(L))*CT-(1.0-3.0*D0)*Z1(L1)*CT*CT)/2.0  2  SNDLHS     155
      A(K1,JAD+1)=A(K1,JAD+1)-AI*(Z1(L2)*(Z1(L2)*D2+D2*(2.0*Z2(           2  SNDLHS     156
     1 L)+Z2(L1)+(Z1(L)+2.0*Z1(L1))*CT))/(2.0*S)+D2*Z2(L2)*(2.0*Z1(L)+Z1  2  SNDLHS     157
     2 (L1))/(2.0*S)-Z1(L1)*(D0*Z1(L)+Z1(I1))/S-E*(Z1(L)+Z1(L1)+Z1(L2)*C  SNDLHS     158
     3 T)*S)                                                             SNDLHS     159
      A(K2,JAD-2)=A(K2,JAD-2)+AI2*D2*Z1(I2)*(Z1(L)+2.0*Z1(L1))/(2.0*S*S  2  SNDLHS     160
     1 )                                                                 2  SNDLHS     161
      A(K2,JAD-1)=A(K2,JAD-1)+AI*(Z1(L2)*(D2*Z1(I2)-D0*Z2(L)*2.0+D2*(2.  2  SNDLHS     162
     1 0*Z1(I1)+Z1(L))*CT)/(2.0*S)-Z1(L)*(D0*Z1(L1)+D0*Z2(L2))/S-Z1(L1)*  2  SNDLHS     163
     2 Z1(L1)/S-E*(Z1(L1)/S+Z2(I1)*C+(Z1(L)-Z1(L1)+Z1(L2)*CT)*S))        SNDLHS     164
      A(K2,JAD)=A(K2,JAD)+AI2*(Z1(L2)*Z1(L2)/2.0-Z1(L1)*(Z1(L1)+D0*Z1(L  2  SNDLHS     165
     1 ))/(S*S)                                                          SNDLHS     166
      IF (ICN.EQ.-10) GO TO 100                                          SNDLHS     167
      A(K,JAD)=A(K,JAD)-AI2*E0*Z1(L1)/(S*S)                              SNDLHS     168
      A(K,JAD+1)=A(K,JAD+1)-AI*E0*Z1(L1)*CT/S                            SNDLHS     169
      A(K1,JAD-1)=A(K1,JAD-1)-AI*E0*Z1(L1)*CT/S                          SNDLHS     170
      A(K1,JAD)=A(K1,JAD)-E0*Z1(L1)/(S*S)                                SNDLHS     171
      A(K1,JAD+1)=A(K1,JAD+1)-AI*E0*Z1(L1)/S                             SNDLHS     172
      A(K2,JAD-1)=A(K2,JAD-1)+AI*E0*Z1(L1)/S                             SNDLHS     173
      A(K2,JAD)=A(K2,JAD)+AI2*E0*Z1(L1)/(S*S)                            SNDLHS     174
      A(K,JAD+1)=A(K,JAD+1)-AI*E*VF(L2,1)/S                              SNDLHS     175
      A(K1,JAD-1)=A(K1,JAD-1)-AI*E*(VF(L,1)+VF(L1,1)+VF(L2,1)*CT)*C      SNDLHS     176
      A(K1,JAD+1)=A(K1,JAD+1)-AI*E*(VF(L,1)+VF(L1,1)+VF(L2,1)*CT)*S      SNDLHS     177
      A(K2,JAD-1)=A(K2,JAD-1)+AI*E*VF(L,1)/S                             SNDLHS     178
  100    CONTINUE                                                        SNDLHS     179
  110  CONTINUE                                                          SNDLHS     180
      RETURN                                                             SNDLHS     181
      END                                                                SNDLHS     182
**    *DECK CRTHOG                                                       ORTHOG       1
      SUBROUTINE ORTHOG (NIK,ISTEP,ICOUNT,N,V,U,NUM,I87,KUW)             CRTHOG       2
      DIMENSION U(I87,KUW),V(I87)                                        CRTHOG       3
C     TO ORTHOG V W.R.T. ALL E-VECTORS ALREADY FOUND                     CRTHOG       4
C     FORMS V=V-SUM OF ((V(TRAN)*X)*X)/(X(TRAN)*X)                       ORTHOG       5
C     EXPECTS EUCLIDEAN NORM OF E-VECT TO BE STORED IN U(N-1,NUM) FOR I'TH VECT  ORTHOG  6
      ICOUNT=0                                                           ORTHOG       7
      IKN=NIK-1                                                          ORTHOG       8
      DO 20 I=1,IKN                                                      ORTHOG       9
        SAV=0.0                                                          CRTHOG      10
        DO 10 J=1,N                                                      ORTHOG      11
          SAV=SAV+V(J)*U(J,I)                                            ORTHOG      12
   10 CONTINUE                                                           ORTHOG      13
        DO 20 J=1,N                                                      ORTHOG      14
          V(J)=V(J)-SAV*U(J,I)/U(N-I,NUM)                                ORTHOG      15
   20 CONTINUE                                                           ORTHOG      16
      RETURN                                                             ORTHOG      17
```

```
      CKUFB,Z1,Z2,NZ,V,NV,F,NP,NO,U,NU,KU,PF,Z3,UFC,NUFC,IFT,LNC,ELMT,Z4)
       DIMENSION A(NA,KA),V1(NV1),X(NX),AB(12,KAB),UF(NUF,KUF),UFB(NUFB,K
      CUFB),Z1(NZ),Z2(NZ),V(NV),P(NP),U(NU,KU),PF(PUF),Z3(NZ),UFC(NUFC,KU
      CF),Z4(NX)
        COMMON/COM2/ D0,D1,D2,D3,D4,AL,THETA,PCL,H,AT,IFF,ION
        COMMON /COM3/ JAP,JAW,JABW,IE,ID,IC,FT,IZAP
        COMMON /COM4/ IN,N1,N2
        LEVEL 2,UF,UFB,UFC
C       CONTROLS CHECK AND ENERGY ROUTINES AND IN THE CASE OF AI=0.0 RE-ENTERS END
C       FTH IN ORDER TO FIND EMF & ECF (STORED IN PLACE OF U,W) AS THESE STRESS
C       COMPONENTS ARE NEEDED BY ENERGY
C       ARGUMENTS ******
C       ARGUMENTS ******
C       N1=NUMBER OF NODES (=NN IN MAIN PROGRAM) KM1=IAW OR IAWO KM3=IABW OR IABWO
C       OTHERS AS DEFINED IN MAIN PROGRAM
                     CALL SECOND(AAA)
                     AAB=AAA
        INODE=AI
        DO 90 I=1,NO
          Y=P(I)
          DO 30 J=1,N1
   30       X(J)=U(J,I)
        CALL BCKSUB (X,NUF,AB,KAB,AI,IN,JABW)
          SAV=0.0
        IFF=IC+1
        IZZ=N2-IC
          DO 70 J=IFF,IZZ
            TEMP=ABS(X(J))
            IF(TEMP.LE.SAV) GO TO 70
            SAV=TEMP
            JJJ=J
   70       CONTINUE
          TEMP=X(JJJ)
          DO 80 J=1,N2
   80       X(J)=X(J)/TEMP
          CALL NOWAVE (X,NX,IN,AI,ICOUNT)
          WRITE (6,2000)
          WRITE(6,2010) INODE,I,ICOUNT
          WRITE (6,2000)
        CALL ENERGY (X,NX,Y,V,Z1,Z2,NV,NZ,UF,NUF,KUF,UFB,NUFB,KUFB,PF,Z3,
       CUFC,NUFC,IFT,LNC,ELMT,V1,NV1,Z4)
                     CALL SECOND(AAA)
                     AAC=AAA-AAB
                    AAB=AAA
                    WRITE (6,8010)  AAA,AAC
   90   CONTINUE
        RETURN
 2000 FORMAT (1H0,5X,110H-----------------------------------------------
     C-------------------------------------------------------------     )
 2010 FORMAT (1H0,8X,25HCIRCUMFERENTIAL MODE NO.=    ,I3,3X,22HEIGEN-VALUE
     C POSITION =    ,I3,4X,37HNO. OF TURNING POINTS ON HALF SHELL = ,I4)
 8010 FORMAT(1H ,60X,5HT.T.=,F7.2,2X,10HT. ENERGY=,F6.3)
```

```
        END
***     *DECK PRODUC
        SUBROUTINE PRODUC (NA,KA,A,V2,X,N1,NUF,KM1)
        DIMENSION A(NA,KA),V2(NA),X(NUF)
C        FORMS PRODUCT V2=A(X) WHERE A IS BANDED WITH WIDTH KM1, AND ORDER N1
        NI=(KM1-1)/2
        DO 1 I=1,NI
          V2(I)=0.0
          NJ=NI+2-I
          K=0
          DO 1 J=NJ,KM1
            K=K+1
    1       V2(I)=V2(I)+A(I,J)*X(K)
        NM=NI+1
        NL=N1-NI
        DO 2 I=NM,NL
          V2(I)=0.0
          K=I-NI-1
          DO 2 J=1,KM1
            K=K+1
    2       V2(I)=V2(I)+A(I,J)*X(K)
        NM=NL+1
        M2=KM1
        DO 3 I=NM,N1
          V2(I)=0.0
          M2=M2-1
          K=I-NI-1
          DO 3 J=1,M2
            K=K+1
    3       V2(I)=V2(I)+A(I,J)*X(K)
        RETURN
        END
***     *DECK MA07B
        SUBROUTINE MA07B (A,B,IA,N,NW,PT,JOKE,NIZ)
        DIMENSION A(IA,JOKE),B(NIZ)
        EQUIVALENCE (IP,I,NFX,KIB,NCC),(NR12,NS,J,LPO),(AMAXT,TBEST,TEMP)
C       ALTERED VERSION OF HARWELL ROUTINE MA07B AND MA07C
C       PERFORMS GAUSSIAN ELIMINATION WITH PARTIAL PIVOTING ON BANDED MATRIX A
C       SEE HARWELL LIBRARY FOR FULL DETAILS
        JOKI=IA*JOKE
        NR=(NW-1)/2
        NR1=NR+1
        NR2=NR1+NR
        NR32=NR2+NR1
        IF(PT.EQ.0.) GO TO 99
        DO 4 ISET=1,N
        NR12=MIN0(NR2+1,N-ISET+NR1+1)
        DO 4 JSET=NR12,NR32
    4   A(ISET,JSET)=0.0
        DO 22 K=1,N
        IP=K
        NS=NR1
        BEST=ABS(A(IP,NS))
        DO 7 NFT=1,NR
        IPT=K+NFT
```

```
      IF(BEST.GE.TPEST) GO TO 7                                          MA07B   26
      BEST=TPEST                                                         MA07B   27
      NS=NT                                                              MA07B   28
      IP=IPT                                                             MA07B   29
    7 CONTINUE                                                           MA07B   30
      IF(K.EQ.1) PT=BEST                                                 MA07B   31
      PT=AMIN1(BEST,PT)                                                  MA07B   32
      IF(PEST.NE.0.) GO TO 5                                             MA07B   33
      WRITE (6,6)                                                        MA07B   34
    6 FORMAT (1H ,80HZERO PIVOT FOUND IN MA07B.  ATTEMPT TO SOLVE THE SY MA07B   35
     CSTEM OF EQUATIONS ABANDONED.     )                                 MA07B   36
      RETURN                                                             MA07B   37
    5 A(K,NR32)=IP                                                       MA07B   38
      IF(IP.EQ.K) GO TO 3                                                MA07B   39
      DO 8 NV=1,NR2                                                      MA07B   40
      TEMP=A(K,NR+NV)                                                    MA07B   41
      A(K,NR+NV)=A(IP,NS+NV-1)                                           MA07B   42
    8 A(IP,NS+NV-1)=TEMP                                                 MA07B   43
C       ELIMINATION AND COEFF. STORAGE                                   MA07B   44
    3 IF(K+NR.LE.N) GO TO 15                                             MA07B   45
      NL=N-K                                                             MA07B   46
      IF(NL) 22,22,16                                                    MA07B   47
   15 NL=NR                                                              MA07B   48
   16 CONTINUE                                                           MA07B   49
      CALL MA07C (A,NL,K,NR,IA,JOKI)                                     MA07B   50
   22 CONTINUE                                                           MA07B   51
C       NOW B IS PROCESSED                                               MA07B   52
   99 DO 17 KB=1,N                                                       MA07B   53
      NEX=A(KB,NR32)                                                     MA07B   54
      IF(NEX.EQ.KB) GO TO 20                                             MA07B   55
      TEMP=B(KB)                                                         MA07B   56
      B(KB)=B(NEX)                                                       MA07B   57
      B(NEX)=TEMP                                                        MA07B   58
   20 DO 17 IB=1,NR                                                      MA07B   59
      KIB=KB+IB                                                          MA07B   60
      IF(KIB.GT.N) GO TO 17                                              MA07B   61
      B(KIB)=B(KIB)-B(KB)*A(KB,IB)                                       MA07B   62
   17 CONTINUE                                                           MA07B   63
C       PACK SUBSTITUTE                                                  MA07B   64
      DO 25 NBACK=1,N                                                    MA07B   65
      NCO=N+1-NBACK                                                      MA07B   66
      BNCO=B(NCO)                                                        MA07B   67
      L2=MIN0(NR2,NBACK)                                                 MA07B   68
      IF(L2.EQ.1) GO TO 25                                               MA07B   69
      DO 31 LCO=2,L2                                                     MA07B   70
   31 BNCO=BNCO-B(LCO+NCO-1)*A(NCO,LCO+NR)                               MA07B   71
   25 B(NCO)=BNCO/A(NCO,NR1)                                             MA07B   72
      RETURN                                                             MA07B   73
      END                                                                MA07B   74
***   *DECK MA07C                                                        MA07B   75
      SUBROUTINE MA07C (A,NL,K,NR,IA,JOKI)                               MA07C    1
      DIMENSION A(JOKI)                                                  MA07C    2
      NR1=NR+1                                                           MA07C    3
      NR3=3*NR                                                           MA07C    4
      J4=K+NR*IA                                                         MA07C    5
      PIVT=A(J4)                                                         MA07C    6
      DO 9 IK=1,NL                                                       MA07C    7
      I=IK+K                                                             MA07C    8
      J1=I+(NR-IK)*IA                                                    MA07C    9
      TEMP=-A(J1)/PIVT                                                   MA07C   10
      A(K+(IK-1)*IA)=-TEMP                                               MA07C   11
      J2=I+(NR3-IK)*IA                                                   MA07C   12
      J3=J4                                                              MA07C   13
      DO 9 J=J1,J2,IA                                                    MA07C   14
      A(J)=A(J)+A(J3)*TEMP                                               MA07C   15
      J3=J3+IA                                                           MA07C   16
    9 CONTINUE                                                           MA07C   17
      RETURN                                                             MA07C   18
      END                                                                MA07C   19
***   *DECK BCOND                                                        MA07C   20
      SUBROUTINE BCOND (AB,KABW,IWILL,THETA,AL,N)                        BCOND    1
      DIMENSION AB(12,KABW)                                              BCOND    2
C     SETS UP THE BOUNDARY CONDITIONS IN AB FOR AT0.0,AI=1,3,5... AI=2,4,6....  BCOND    3
C       1ST &2ND DERIVATIVES H**6, 3RD DERIVATIVES H**4                  BCOND    4
      C=COS(THETA)                                                       BCOND    5
      S=SIN(THETA)                                                       BCOND    6
      CT=C/S                                                             BCOND    7
      AL1=AL+1.0                                                         BCOND    8
      IF(IWILL.EQ.0) GO TO 300                                           BCOND    9
      IWON= 2*(IWILL/2)                                                  BCOND   10
      IF(IWILL-IWON) 310,320,310                                         BCOND   11
C     FOR I=0                                                            BCOND   12
C     FOR I=0                                                            BCOND   13
  300 CONTINUE                                                           BCOND   14
      AB(1,4)=126.0                                                      BCOND   15
      AB(1,6)=-70.0                                                      BCOND   16
      AB(1,8)=-486.0                                                     BCOND   17
      AB(1,10)=855.0                                                     BCOND   18
      AB(1,12)=-670.0                                                    BCOND   19
      AB(1,14)=324.0                                                     BCOND   20
      AB(1,16)=-90.0                                                     BCOND   21
      AB(1,18)=11.0                                                      BCOND   22
      AB(3,4)=1.0                                                        BCOND   23
      AB(2,4)=-10.0                                                      BCOND   24
      AB(2,6)=-77.0                                                      BCOND   25
      AB(2,8)=150.0                                                      BCOND   26
      AB(2,10)=-100.0                                                    BCOND   27
      AB(2,12)=50.0                                                      BCOND   28
      AB(2,14)=-15.0                                                     BCOND   29
      AB(2,16)=2.0                                                       BCOND   30
      AB(4,2)=-469.0                                                     BCOND   31
      AB(4,4)=1818.0                                                     BCOND   32
      AB(4,6)=-2924.0                                                    BCOND   33
      AB(4,8)=2690.0                                                     BCOND   34
      AB(4,10)=-1710.0                                                   BCOND   35
      AB(4,12)=814.0                                                     BCOND   36
      AB(4,14)=-268.0                                                    BCOND   37
                                                                         BCOND   38
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C       FOR I=1,3,5...                                                  BCOND       42
C       FOR I=1,3,5...                                                  BCOND       43
  310 CONTINUE                                                          BCOND       44
      AB(1,6)=-10.0                                                     BCOND       45
      AB(1,9)=-77.0                                                     BCOND       46
      AB(1,12)=150.0                                                    BCOND       47
      AB(1,15)=-100.0                                                   BCOND       48
      AB(1,18)=50.0                                                     BCOND       49
      AB(1,21)=-15.0                                                    BCOND       50
      AB(1,24)=2.0                                                      BCOND       51
      DO 22 I=6,24,3                                                    BCOND       52
   22 AB(2,I)=AB(1,I)                                                   BCOND       53
      AB(3,6)=126.0                                                     BCOND       54
      AB(3,9)=-70.0                                                     BCOND       55
      AB(3,12)=-486.0                                                   BCOND       56
      AB(3,15)=855.0                                                    BCOND       57
      AB(3,18)=-670.0                                                   BCOND       58
      AB(3,21)=324.0                                                    BCOND       59
      AB(3,24)=-90.0                                                    BCOND       60
      AB(3,27)=11.0                                                     BCOND       61
      AB(4,6)=1.0                                                       BCOND       62
      AB(5,6)=1.0                                                       BCOND       63
      AB(6,6)=1.0                                                       BCOND       64
      GO TO 387                                                         BCOND       65
C       FOR I=2,4,6...                                                  BCOND       66
C       FOR I=2,4,6...                                                  BCOND       67
  320 CONTINUE                                                          BCOND       68
      AB(1,6)=126.0                                                     BCOND       69
      AB(1,9)=-70.0                                                     BCOND       70
      AB(1,12)=-486.0                                                   BCOND       71
      AB(1,15)=855.0                                                    BCOND       72
      AB(1,18)=-670.0                                                   BCOND       73
      AB(1,21)=324.0                                                    BCOND       74
      AB(1,24)=-90.0                                                    BCOND       75
      AB(1,27)=11.0                                                     BCOND       76
      DO 24 I=6,27,3                                                    BCOND       77
   24 AB(2,I)=AB(1,I)                                                   BCOND       78
      AB(3,6)=-10.0                                                     BCOND       79
      AB(3,9)=-77.0                                                     BCOND       80
      AB(3,12)=150.0                                                    BCOND       81
      AB(3,15)=-100.0                                                   BCOND       82
      AB(3,18)=50.0                                                     BCOND       83
      AB(3,21)=-15.0                                                    BCOND       84
      AB(3,24)=2.0                                                      BCOND       85
      AB(4,6)=1.0                                                       BCOND       86
      AB(5,6)=1.0                                                       BCOND       87
      AB(6,6)=1.0                                                       BCOND       88
      GO TO 387                                                         BCOND       89
C       FOR CLAMPED EDGE                                                BCOND       90
C       FOR CLAMPED EDGE                                                BCOND       91
  406 CONTINUE                                                          BCOND       92
      AB(5,15)=1.0                                                      BCOND       93
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      AB(6,15)=1.0                                                      BCOND       94
      AB(7,15) =(30.0*CT+126.0/H)*AL1                                   BCOND       95
      AB(7,13) =(231.0*CT-70.0/H)*AL1                                   BCOND       96
      AB(7,11) =(-450.0*CT-486.0/H)*AL1                                 BCOND       97
      AB(7,9)  =(300.0*CT+855.0/H)*AL1                                  BCOND       98
      AB(7,7)  =(-150.0*CT-670.0/H)*AL1                                 BCOND       99
      AB(7,5) =(45.0*CT+324.0/H)*AL1                                    BCOND      100
      AB(7,3) =(-6.0*CT-90.0/H)*AL1                                     BCOND      101
      AB(7,1) =11.0*AL1/H                                               BCOND      102
      AB(7,16) =(-126.0*CT-22.5*15.0/H)*AL/H                            BCOND      103
      AB(7,14) =(70.0*CT+56.0*22.5/H)*AL/H                              BCOND      104
      AB(7,12) =(486.0*CT-22.5*83.0/H)*AL/H                             BCOND      105
      AB(7,10) =(-855.0*CT+22.5*64.0/H)*AL/H                            BCOND      106
      AB(7,8)  =(670.0*CT-22.5*29.0/H)*AL/H                             BCOND      107
      AB(7,6) =(-324.0*CT+22.5*8.0/H)*AL/H                              BCOND      108
      AB(7,4) =(90.0*CT-22.5/H)*AL/H                                    BCOND      109
      AB(7,2) =-11.0*CT*AL/H                                            BCOND      110
      AB(8,15)=10.0                                                     BCOND      111
      AB(8,13)=77.0                                                     BCOND      112
      AB(8,11)=-150.0                                                   BCOND      113
      AB(8,9)=100.0                                                     BCOND      114
      AB(8,7)=-50.0                                                     BCOND      115
      AB(8,5)=15.0                                                      BCOND      116
      AB(8,3)=-2.0                                                      BCOND      117
      GO TO 340                                                         BCOND      118
  387 CONTINUE                                                          BCOND      119
      AB(7,22)=1.0                                                      BCOND      120
      AB(8,22)=1.0                                                      BCOND      121
      AB(9,22)=1.0                                                      BCOND      122
      AB(10,22)=(30.0*CT+126.0/H)*AL1                                   BCOND      123
      AB(10,19)=(231.0*CT-70.0/H)*AL1                                   BCOND      124
      AB(10,16)=(-450.0*CT-486.0/H)*AL1                                 BCOND      125
      AB(10,13)=(300.0*CT+855.0/H)*AL1                                  BCOND      126
      AB(10,10)=(-150.0*CT-670.0/H)*AL1                                 BCOND      127
      AB(10,7)=(45.0*CT+324.0/H)*AL1                                    BCOND      128
      AB(10,4)=(-6.0*CT-90.0/H)*AL1                                     BCOND      129
      AB(10,1)=11.0*AL1/H                                               BCOND      130
      AB(10,24)=(-126.0*CT-22.5*15.0/H)*AL/H                            BCOND      131
      AB(10,21)=(70.0*CT+56.0*22.5/H)*AL/H                              BCOND      132
      AB(10,18)=(486.0*CT-22.5*83.0/H)*AL/H                             BCOND      133
      AB(10,15)=(-855.0*CT+22.5*64.0/H)*AL/H                            BCOND      134
      AB(10,12)=(670.0*CT-22.5*29.0/H)*AL/H                             BCOND      135
      AB(10,9)=(-324.0*CT+22.5*8.0/H)*AL/H                              BCOND      136
      AB(10,6)=(90.0*CT-22.5/H)*AL/H                                    BCOND      137
      AB(10,3)=-11.0*CT*AL/H                                            BCOND      138
      AB(11,22)=10.0                                                    BCOND      139
      AB(11,19)=77.0                                                    BCOND      140
      AB(11,16)=-150.0                                                  BCOND      141
      AB(11,13)=100.0                                                   BCOND      142
      AB(11,10)=-50.0                                                   BCOND      143
      AB(11,7)=15.0                                                     BCOND      144
      AB(11,4)=-2.0                                                     BCOND      145
      DO 7 I=4,22,3                                                     BCOND      146
    7 AB(12,I)=AB(11,I)                                                 BCOND      147
      GO TO 340                                                         BCOND      148
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```
**     *DECK NOWAVE                                                                   NOWAVE
             SUBROUTINE NOWAVE (W,I99,IN,AI,ICOUNT)                                   NOWAVE
             DIMENSION W(I99)                                                         NOWAVE
      C      CALCULATES THE NUMBER OF TURNING POINTS IN THE W PROFILE, RETURNS THEM AS NOWAVE
      C      ICOUNT                                                                   NOWAVE
             IF(AI.EQ.0.0) GO TO 5                                                    NOWAVE
             ISTEP=3                                                                  NOWAVE
             GO TO 6                                                                  NOWAVE
           5 ISTEP=2                                                                  NOWAVE
           6 ICOUNT=0                                                                 NOWAVE
             KOLD=12                                                                  NOWAVE
             I=ISTEP                                                                  NOWAVE
             J=I*2                                                                    NOWAVE
             DO 100 IJ=2,IN                                                           NOWAVE
               I=I+ISTEP                                                              NOWAVE
               J=J+ISTEP                                                              NOWAVE
               IF(W(I)-W(J)) 20,30,40                                                 NOWAVE
          20   KK=-1                                                                  NOWAVE
             GO TO 50                                                                 NOWAVE
          30   KOLD=0                                                                 NOWAVE
             GO TO 100                                                                NOWAVE
          40   KK=1                                                                   NOWAVE
          50   CONTINUE                                                               NOWAVE
               IF(KK.EQ.KOLD) GO TO 100                                               NOWAVE
               ICOUNT=ICOUNT+1                                                        NOWAVE
               KOLD=KK                                                                NOWAVE
         100 CONTINUE                                                                 NOWAVE
             J=ISTEP*3                                                                NOWAVE
             TEMP=W(ISTEP)*W(J)                                                       NOWAVE
             IF(TEMP.LT.0.0) ICOUNT=ICOUNT-1                                          NOWAVE
             RETURN                                                                   NOWAVE
             END                                                                      NOWAVE
**     *DECK ENERGY                                                                   ENERGY
             SUBROUTINE ENERGY (U,IX,P,V,V2,V3,NV,NZ,UF,NUF,KUF,UFB,NUFB,KUFB,        ENERGY
            CPF,Z3,UFC,NUFC,IFT,LNO,ELMT,Z9,NV1,Z4)                                   ENERGY
             DIMENSION U(IX),UF(NUF,KUF),V(NV),V2(NV),V3(NZ),UFB(NUFB,KUFB),PF(       ENERGY
            CKUF),Z3(NV),UFC(NUFC,KUF),Z9(NV1),Z4(IX)                                 ENERGY
             COMMON/COM1/ AN(7,4,12),ITYPE,ILN(7),ISN(7),JSN(7),JEN(7),KBOE(7)        ENERGY
             COMMON/COM2/ D0,D1,D2,D3,D4,AL,THETA,PCL,H,AI,IPR,ION                    ENERGY
             COMMON /COM3/ JAC,JAW,JABW,IF,ID,IC,FT,IZAP                              ENERGY
             COMMON /COM4/ IN,N1,N2                                                   ENERGY
             COMMON/COM5/ VF(147,20),VFB(147,20),VFC(106,20),FFC(20),EFC(20)          ENERGY
            C,SF(147)                                                                 ENERGY
             LEVEL 2,UF,UFB,UFC                                                       ENERGY
             IO=LNC                                                                   ENERGY
             IF(IFT.EQ.0) IO=IPR                                                      ENERGY
             ICHS=1                                                                   ENERGY
             IFAIL=0                                                                  ENERGY
      C      CALCULATES THE ENERGY COMPONENTS OF THEORETICAL POTENTIAL ENERGY GIVING RISE  ENERGY
      C      TO THE SECONDARY PATH EQUATIONS. PRINTS OUT THE SOLUTION VECTOR AND THE EN ENERGY
      C      ERGY COMPONENTS                                                          ENERGY
      C      ARGUMENTS *****                                                          ENERGY
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      C      ARGUMENTS *****                                                          ENERGY
      C      U- E-VECT FULL LENGTH, P- E-VAL, UF- FUND PATH STRESS COMPONENTS(EMF &ECF  ENERGY
      C      USED), VV & V2 WRKSPC, OTHERS AS DEFINED IN CALLING PROGRAM             ENERGY
             PC=0.0                                                                   ENERGY
             MRK=-1                                                                   ENERGY
             A1=P-0.05                                                                ENERGY
             CALL FUNVEC(UF,UFB,KUF,NUF,UFC,NUFC,V2,V3,NZ,V,NV,PF,IO,IFT,LNO,A1       ENERGY
            C,PC,IN,PL,ICHS,IFAIL,ELMT)                                               ENERGY
             CALL TEEV01 (Z9,Z3,NV,V2,V3,NV,Z4,IX,PL,MRK)                             ENERGY
             DO 4 I=1,10                                                              ENERGY
           4 Z4(20+I)=Z4(I)                                                           ENERGY
             CALL FUNVEC (UF,UFB,KUF,NUF,UFC,NUFC,V2,V3,NZ,V,NV,PF,IO,IFT,LNC,P       ENERGY
            C,PC,IN,PL,ICHS,IFAIL,ELMT)                                               ENERGY
             NNN=3*(IN-2)                                                             ENERGY
             MRK=0                                                                    ENERGY
             DO 1 JK=1,20                                                             ENERGY
             I=21-JK                                                                  ENERGY
             IF(P.LE.EFC(I)) IK=I                                                     ENERGY
           1 CONTINUE                                                                 ENERGY
             KK=(IK-1)*LNC                                                            ENERGY
             A1=P                                                                     ENERGY
             IF (IK.GT.1) A1=A1-EFC(IK-1)                                             ENERGY
             DO 2 I=1,NNN                                                             ENERGY
             Z9(I)=UF(I,KK+1)                                                         ENERGY
           2 Z3(I)=UFB(I,KK+1)                                                        ENERGY
             DO 3 JK=2,LNC                                                            ENERGY
             AK=JK                                                                    ENERGY
             KJ=JK-1                                                                  ENERGY
             AJ=AK*(A1**KJ)                                                           ENERGY
             DO 3 I=1,NNN                                                             ENERGY
             Z9(I)=Z9(I)+UF(I,KK+JK)*AJ                                               ENERGY
           3 Z3(I)=Z3(I)+UFB(I,KK+JK)*AJ                                              ENERGY
             CALL TEEV01 (Z9,Z3,NV,V2,V3,NV,Z4,IX,PL,MRK)                             ENERGY
             WRITE (6,5000)                                                           ENERGY
             WRITE (6,5015) (Z4(I),I=1,5)                                             ENERGY
             WRITE (6,5020) (Z4(I),I=6,10)                                            ENERGY
             WRITE (6,5030)                                                           ENERGY
             WRITE (6,5010) (Z4(I),I=51,55),Z4(61)                                    ENERGY
             WRITE (6,5020) (Z4(I),I=56,60)                                           ENERGY
             AJ=0.0                                                                   ENERGY
             AK=0.0                                                                   ENERGY
             DO 6 I=1,5                                                               ENERGY
             Z4(40+I)=Z4(I)-Z4(20+I)                                                  ENERGY
             IF (Z4(40+I).LE.0.0) GO TO 5                                             ENERGY
             AJ=AJ+Z4(40+I)                                                           ENERGY
             GO TO 6                                                                  ENERGY
           5 AK=AK+Z4(40+I)                                                           ENERGY
           6 CONTINUE                                                                 ENERGY
             AJ=(AJ-AK)/2.0                                                           ENERGY
             DO 7 I=1,5                                                               ENERGY
           7 Z4(45+I)=Z4(40+I)/AJ                                                     ENERGY
             WRITE (6,5040)                                                           ENERGY
             WRITE (6,5015) (Z4(I),I=41,45)                                           ENERGY
             WRITE (6,5020) (Z4(I),I=46,50)                                           ENERGY
             DO 10 I=1,N1                                                             ENERGY
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```
      WRITE (6,4000)                                                    ENERGY     78
      AI2=AI*AI                                                         ENERGY     79
      PA=0.0                                                            ENERGY     80
      AK=1.0                                                            ENERGY     81
      AJ=1.0                                                            ENERGY     82
      IF (ICN.EQ.20.OR.ICN.EQ.30) AK=2.0                                ENERGY     83
      IF (ICN.EQ.20.OR.ICN.EQ.0) AJ=0.0                                 ENERGY     84
      DO 110 KK=1,ITYPE                                                 ENERGY     85
        CALL EDDATA (KK,ITCL,MID,ISTART,IFIN)                           ENERGY     86
        DO 100 IJ=ISTART,IFIN                                           ENERGY     87
          PA=PA+1.0                                                     ENERGY     88
          PHI=PA*H                                                      ENERGY     89
          S=SIN(PHI)                                                    ENERGY     90
          C=COS(PHI)                                                    ENERGY     91
          CT=C/S                                                        ENERGY     92
          U1=0.0                                                        ENERGY     93
          V1=0.0                                                        ENERGY     94
          W1=0.0                                                        ENERGY     95
          W2=0.0                                                        ENERGY     96
          IU=IC*(IJ+1)+1                                                ENERGY     97
          IV=IU+IE/2                                                    ENERGY     98
          IW=IU+IE                                                      ENERGY     99
          K1=3*IJ-2                                                     ENERGY    100
          K2=K1+1                                                       ENERGY    101
      K3=K1+2                                                           ENERGY    102
          ISU=IU-IC*MID                                                 ENERGY    103
          DO 80 K=1,ITCL                                                ENERGY    104
           ICU=ISU+IC*K                                                 ENERGY    105
           ICV=ICU+IE/2                                                 ENERGY    106
           ICW=ICU+IE                                                   ENERGY    107
           U1=U1+AN(KK,1,K)*U(ICU)                                      ENERGY    108
           W1=W1+AN(KK,1,K)*U(ICW)                                      ENERGY    109
           W2=W2+AN(KK,2,K)*U(ICW)                                      ENERGY    110
           IF(AI.EQ.0.0) GO TO 80                                       ENERGY    111
           V1=V1+AN(KK,1,K)*U(ICV)                                      ENERGY    112
   80      CONTINUE                                                     ENERGY    113
          E0=U1+U(IW)                                                   ENERGY    114
          EC=U(IW)+U(IU)*CT                                             ENERGY    115
          E0C=0.0                                                       ENERGY    116
          E20=((U(IU)-W1)**2)/2.0                                       ENERGY    117
          E2C=0.0                                                       ENERGY    118
          X0=U1-W2                                                      ENERGY    119
          XC=U(IU)*CT-W1*CT                                             ENERGY    120
          X0C=0.0                                                       ENERGY    121
          E1=U(IU)*CT+U(IW)                                             ENERGY    122
          E2=U1-W1*CT                                                   ENERGY    123
          E3=0.0                                                        ENERGY    124
      E4=0.0                                                            ENERGY    125
      E5=0.0                                                            ENERGY    126
          BU=U(IU)-W1                                                   ENERGY    127
          IF(AI.EQ.0.0) GO TO 90                                        ENERGY    128
          EC=EC+AI*U(IV)/S                                              ENERGY    129
          E0C=E0C+(V1-U(IV)*CT-AI*U(IU)/S)/2.0                          ENERGY    130
          E20=E20+V1*V1/2.0                                             ENERGY    131
          E2C=E2C+((-AI*U(IU)/S-U(IV)*CT)**2+(U(IV)+AI*U(IW)/S)**2)/2.0 ENERGY    132
          XC=X0+AI*U(IV)/S+AI2*U(IW)/(S*S)                              ENERGY    133
          X0C=X0C+(-U(IV)*CT*V1+2.0*AI*W1/S-2.0*AI*U(IW)*CT/S-AI*U(IU)/S ENERGY    134
     C    )/2.0                                                         ENERGY    135
          E3=E3+AI*U(IV)/S                                              ENERGY    136
      E4=E4-AI*U(IU)/S-U(IV)*CT                                         ENERGY    137
      E5=E5+AI*U(IW)/S+U(IV)                                            ENERGY    138
   90     CONTINUE                                                     ENERGY    139
      Z3(K1)=E0                                                         ENERGY    140
      Z3(K2)=EC                                                         ENERGY    141
      Z3(K3)=E0C                                                        ENERGY    142
          IKT=IJ/2                                                      ENERGY    143
          IKI=IJ-2*IKT                                                  ENERGY    144
          AKT=IKT+1                                                     ENERGY    145
      EA=1.0-V2(K1)*(1.0-V2(K1))-V2(K3)*V2(K3)/2.0                      ENERGY    146
      EE=V2(K3)*(-1.0+2.0*V2(K1))                                       ENERGY    147
      EX=(V1*V1+BU*BU)*EA/2.0+E0*BU*EE+(E0*E0-BU*BU)*V2(K3)*V2(K3)/2.0  ENERGY    148
      EZ=(E4*E4+E5*E5)*(1.0-V2(K2)+V2(K2)*V2(K2))/2.0                   ENERGY    149
      EA=EA*V2(K3)                                                      ENERGY    150
      EE=V2(K3)*EE                                                      ENERGY    151
      V(1)=V(1)+AKT*E0*E0*S                                             ENERGY    152
      V(2)=V(2)+AKT*EC*EC*S                                             ENERGY    153
      V(3)=V(3)+AKT*D0*2.0*E0*E0*S                                      ENERGY    154
      V(4)=V(4)+AKT*2.0*(1.0-D0)*E0C*E0C*S                              ENERGY    155
      V(5)=V(5)+AKT*(2.0*E0*BU*EA+E0*E0*EE-BU*BU*EE)*S                  ENERGY    156
      V(6)=V(6)+AKT*D0*2.0*(E0*BU*EA+E0*E0*EE/2.0)*S                    ENERGY    157
      V(7)=V(7)+AKT*(1.0-DC)*(V1*V1*(-V2(K1)-V2(K3)*V2(K3)/2.0+1.5*V2(K1 ENERGY    158
     1)*V2(K1))+E4*E4*(-V2(K2)-V2(K3)*V2(K3)/2.0+1.5*V2(K2)**2)+E5*E5   ENERGY    159
     2*V2(K3)*V2(K3)/2.0+V1*E4*(-V2(K1)-V2(K2)-V2(K3)*V2(K3)+(V2(K1)+V2( ENERGY    160
     3K2))*V2(K1)+V2(K2)*V2(K2))+V1*E5*(V2(K3)-V2(K3)*(2.0*V2(K1)+V2(K2) ENERGY    161
     4))+E4*E5*V2(K3)*(1.0-V2(K1)-2.0*V2(K2)))*S                        ENERGY    162
      V(7)=V(7)+AKT*(1.0-DC)*(V1*V1*2.0*V2(K1)*(V2(K3)**2-V2(K1)**2)+E4* ENERGY    163
     1E4*(-2.0*V2(K2)**3+V2(K3)*V2(K3)*(V2(K1)+V2(K2)))-E5*E5*V2(K3)*V2( ENERGY    164
     2K3)*(V2(K1)+V2(K2))+V1*E4*(V2(K2)*(V2(K3)**2-V2(K1)**2-V2(K2)**2)+ ENERGY    165
     3V2(K1)*(3.0*V2(K3)**2-V2(K1)**2-V2(K2)**2))+V1*E5*V2(K3)*(2.0*V2(K ENERGY    166
     41)**2-0.5*V2(K3)**2+V2(K1)*V2(K2))+E4*E5*V2(K3)*(2.0*V2(K2)**2-0.5 ENERGY    167
     5*V2(K3)**2+V2(K1)*V2(K2)))*S                                      ENERGY    168
      V(8)=V(8)+AKT*2.0*(0.5*(V1*V1+BU*BU)*(V2(K1)*(1.0-V2(K1)+V2(K1)**2 ENERGY    169
     1-1.5*V2(K3)**2)+0.5*V2(K3)**2)+E0*BU*(-V2(K3)*(V2(K1)+0.5*V2(K3)** ENERGY    170
     22-2.0*V2(K1)**2))+(E0*E0-BU*BU)*V2(K1)*V2(K3)*V2(K3)/2.0)*S       ENERGY    171
      V(9)=V(9)+AKT*2.0*EZ*V2(K2)*S                                     ENERGY    172
      V(10)=V(10)+AKT*2.0*E0*EX*V2(K2)*S                                ENERGY    173
      V(11)=V(11)+AKT*2.0*E0*(0.5*(E4*E4+E5*E5)*(V2(K1)*(1.0-V2(K2)+V2(K ENERGY    174
     12)**2-0.5*V2(K3)**2)+0.5*V2(K3)*V2(K3)*(1.0-V2(K2)))*S            ENERGY    175
      V(12)=V(12)-AKT*PI*2.0*(3.0*F1*E1+2.0*F1*(E2+E3)+E2*E3)*(S**3)    ENERGY    176
      V(13)=V(13)-AKT*PI*2.0*(E1*F1*(3.0*V2(K2)+V2(K1)-V2(K2)+V2(K3)*CT) ENERGY    177
     1+2.0*E1*(E2+E3)*V2(K2)+E2*F3*V2(K2)+E1*E3*(V2(K1)-V2(K2)+V2(K3)*CT ENERGY    178
     2))*(S**3)                                                         ENERGY    179
      V(14)=V(14)+AKT*X0*X0*S*AL                                        ENERGY    180
      V(15)=V(15)+AKT*XC*XC*S*AL                                        ENERGY    181
      V(16)=V(16)+AKT*2.0*E0*X0*XC*S*AL                                 ENERGY    182
      V(17)=V(17)+AKT*2.0*(1.0-D0)*X0C*X0C*S*AL                         ENERGY    183
      V(61)=-E0*E0*V2(K3)**2                                            ENERGY    184
                                                                        ENERGY    185
```

```
     1(K2))
      V(65)=F0*BU*(2.0*D0*V2(K3)*(1.0-V2(K1)))
      V(66)=V1*V1*((1.0-V2(K1))*(V2(K1)+D0*V2(K2))+0.5*V2(K3)**2-D2*(V2(
     1K1)+0.5*V2(K3)**2-1.5*V2(K1)**2))
      V(67)=F4*F4*((1.0-V2(K2))*(V2(K2)+D0*V2(K1))+0.5*D0*V2(K3)**2-D2*(
     1V2(K2)+0.5*V2(K3)**2-1.5*V2(K2)**2))
      V(68)=E5*E5*((1.0-V2(K2))*(V2(K2)+D0*V2(K1))+0.5*D0*V2(K3)**2+D2*V
     12(K3)*V2(K3)/2.0)
      V(69)=-V1*F4*D2*(V2(K1)*(1.0-V2(K1)-V2(K2))+V2(K2)*(1.0-V2(K2))+V2
     1(K3)**2)
      V(70)=V1*E5*D2*V2(K3)*(1.0-2.0*V2(K1)-V2(K2))
      V(71)=E1*E5*D2*V2(K3)*(1.0-V2(K1)-2.0*V2(K2))
      IF (ICN.EQ.-10) GO TO 94
      Q1=AK*(VF(K1,1)+0.5*VF(K3,1)**2-0.5*VF(K1,1)*VF(K3,1)**2)-AJ*(SF(K
     11)-D0*SF(K2))/(1.0+D0*D0)
      Q2=AK*(VF(K2,1))-AJ*(SF(K2)-D0*SF(K1))/(1.0+D0*D0)
      V(8)=V(8)-AKT*EX*Q1*S
      V(9)=V(9)-AKT*EZ*Q2*S
      V(10)=V(10)-AKT*D0*EX*Q2*S
      V(11)=V(11)-AKT*D0*EZ*Q1*S
      V(63)=V(63)+E0*BU*V2(K3)*(Q1+D0*Q2)
      V(64)=V(64)-BU*BU*(1.0-V2(K1))*(Q1+D0*Q2)/2.0
      V(66)=V(66)-V1*V1*(1.0-V2(K1))*(Q1+D0*Q2)/2.0
      V(67)=V(67)-F4*F4*(1.0-V2(K2))*(Q2+D0*Q1)/2.0
      V(68)=V(68)-E5*F5*(1.0-V2(K2))*(Q2+D0*Q1)/2.0
      V(13)=V(13)+2.0*AKT*FI*(E1*E1*(VF(K1,1)+2.0*VF(K2,1)+VF(K3,1)*CT)+
     1F1*E2*2.0*VF(K2,1)+F1*E3*(VF(K1,1)+VF(K2,1)+VF(K3,1)*CT)+F2*F3*VF(
     2K2,1))*(S**3)
   94 CONTINUE
      ILL1=ILL1+1
      IF(ILL1.EQ.4.OR.III1.LT.0) GO TO 95
      GO TO 98
   95 WRITE (6,4010) IKK1,(V(J),J=61,71),IKK1
      IKK1=IKK1+4
      IF(ILL1.LT.0) IKK1=IKK1-1
      ILL1=0
   98 DO 99 J=61,71
   99 V(J+20)=V(J+20)+AKT*V(J)*S
  100       CONTINUE
  110    CONTINUE
      EPOS=0.0
      ENEG=0.0
      DO 130 I=1,17
         IF(V(I).GT.0.0) GO TO 120
         ENEG=ENEG+V(I)
         GO TO 130
  120    EPOS=EPOS+V(I)
  130    CONTINUE
      ERR=(2.0*(EPOS+ENEG)/(EPOS-ENEG))*100.0
      TEMP=(EPOS-ENEG)/2.0
      DO 140 I=1,17
         V(I+17)=V(I)/TEMP
```

```
  140    V(I)=V(I)*H/3.0
      WRITE (6,2030)
      DO 145 I=81,91
  145 V(I+20)=V(I)/TEMP
      WRITE (6,2000)
      IF (AI.EQ.0.0) GO TO 160
      IJ=0
      WRITE (6,2010) IJ,U(1),U(2),U(3),IJ
      IJ=1
      WRITE(6,2010) IJ,U(4),U(5),U(6),IJ
      NNN=N2-6
      DO 150 I=7,NNN,3
         IJ=IJ+1
         J=I+1
         K=I+2
      K1=I-6
      K2=J-6
      K3=K-6
      TEMP=V2(K1)+0.5*V2(K3)*V2(K3)*(1.0-V2(K1)+V2(K1)*V2(K1)-0.25*V2(K3
     1)**2)
  150 WRITE (6,2011) IJ,U(I),U(J),U(K),Z3(K1),Z3(K2),Z3(K3),TEMP,V2(K2),
     CIJ
      IJ=IJ+1
      J=I+1
      K=I+2
      WRITE (6,2010) IJ,U(I),U(J),U(K),IJ
      IJ=IJ+1
      I=I+3
      J=I+1
      K=I+2
      WRITE (6,2010) IJ,U(I),U(J),U(K),IJ
      GO TO 180
  160 BEMP=0.0
      IJ=0
      WRITE (6,2010) IJ,U(1),BEMP,U(2),IJ
      IJ=1
      WRITE(6,2010) IJ,U(3),BEMP,U(4),IJ
      NNN=N2-4
      DO 170 I=5,NNN,2
      IJ=IJ+1
      J=I+1
      K1=(IJ-1)*3-2
      K2=K1+1
      K3=K1+2
      TEMP=V2(K1)+0.5*V2(K3)*V2(K3)*(1.0-V2(K1)+V2(K1)*V2(K1)-0.25*V2(K3
     1)**2)
  170 WRITE (6,2011) IJ,U(I),BEMP,U(J),Z3(K1),Z3(K2),Z3(K3),TEMP,V2(K2),
     CIJ
      IJ=IJ+1
      J=I+1
      WRITE (6,2010) IJ,U(I),BEMP,U(J),IJ
      IJ=IJ+1
      I=I+2
      J=I+1
      WRITE (6,2010) IJ,U(I),BEMP,U(J),IJ
```

```
      PNPCL=-P*PL*SQRT(1.C*(1.0-DO*DO))/(1.0-DC*DO)                      ENERGY     297
      PEMP=0.0                                                          ENERGY     298
      WRITE (6,2030)                                                    ENERGY     299
      WRITE (6,2030)                                                    ENERGY     300
      WRITE (6,2111)                                                    ENERGY     301
      WRITE (6,2110)                                                    ENERGY     302
      V(40)=V(12)+V(13)                                                 ENERGY     303
      V(41)=V(29)+V(30)                                                 ENERGY     304
      WRITE (6,2120)                                                    ENERGY     305
      WRITE (6,2130) V(12),V(13),V(40)                                  ENERGY     306
      WRITE (6,2130) V(29),V(30),V(41)                                  ENERGY     307
      WRITE (6,2030)                                                    ENERGY     308
      WRITE (6,2111)                                                    ENERGY     309
      WRITE (6,2140)                                                    ENERGY     310
      V(12)=V(1)+V(3)/2.0                                               ENERGY     311
      V(13)=V(2)+V(3)/2.0                                               ENERGY     312
      V(29)=V(18)+V(20)/2.0                                             ENERGY     313
      V(30)=V(19)+V(20)/2.0                                             ENERGY     314
      V(40)=V(12)+V(13)+V(4)                                            ENERGY     315
      V(41)=V(29)+V(30)+V(21)                                           ENERGY     316
      WRITE (6,2030)                                                    ENERGY     317
      WRITE (6,2150)                                                    ENERGY     318
      WRITE (6,2160) V(12),V(13),V(4),V(40)                             ENERGY     319
      WRITE (6,2160) V(29),V(30),V(21),V(41)                            ENERGY     320
      V(1)=V(5)+V(6)/2.0                                                ENERGY     321
      V(2)=V(6)/2.0                                                     ENERGY     322
      V(3)=V(7)                                                         ENERGY     323
      V(18)=V(22)+V(23)/2.0                                             ENERGY     324
      V(19)= V(23)/2.0                                                  ENERGY     325
      V(20)=V(24)                                                       ENERGY     326
      V(40)=V(1)+V(2)+V(3)                                              ENERGY     327
      V(41)=V(18)+V(19)+V(20)                                           ENERGY     328
      WRITE (6,2170)                                                    ENERGY     329
      WRITE(6,2180) (V(I),I=1,3),V(40)                                  ENERGY     330
      WRITE (6,2180) (V(I),I=18,20),V(41)                               ENERGY     331
      V(4)=(V(8)+V(10))/2.0                                             ENERGY     332
      V(5)=(V(8)+V(11))/2.0                                             ENERGY     333
      V(6)=(V(9)+V(11))/2.0                                             ENERGY     334
      V(7)=(V(9)+V(10))/2.0                                             ENERGY     335
      V(21)=(V(25)+V(27))/2.0                                           ENERGY     336
      V(22)=(V(25)+V(28))/2.0                                           ENERGY     337
      V(23)=(V(26)+V(28))/2.0                                           ENERGY     338
      V(24)=(V(26)+V(27))/2.0                                           ENERGY     339
      V(42)=V(4)+V(5)+V(6)+V(7)                                         ENERGY     340
      V(43)=V(21)+V(22)+V(23)+V(24)                                     ENERGY     341
      WRITE (6,2190)                                                    ENERGY     342
      WRITE (6,2200) (V(I),I=4,7),V(42)                                 ENERGY     343
      WRITE (6,2200) (V(I),I=21,24),V(43)                               ENERGY     344
      V(1)=V(1)+V(4)+V(5)                                               ENERGY     345
      V(2)=V(2)+V(6)+V(7)                                               ENERGY     346
      V(18)=V(18)+V(21)+V(22)                                           ENERGY     347
      V(19)=V(19)+V(23)+V(24)                                           ENERGY     348
      V(40)=V(40)+V(42)                                                 ENERGY     349
                                                                        ENERGY     350
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      V(41)=V(41)+V(43)                                                 ENERGY     351
      WRITE (6,2210)                                                    ENERGY     352
      WRITE (6,2220)                                                    ENERGY     353
      WRITE (6,2230) (V(I),I=1,3),V(40)                                 ENERGY     354
      WRITE (6,2230) (V(I),I=18,20),V(41)                               ENERGY     355
      WRITE (6,2210)                                                    ENERGY     356
      V(14)=V(14)+V(16)/2.0                                             ENERGY     357
      V(15)=V(15)+V(16)/2.0                                             ENERGY     358
      V(16)=V(17)                                                       ENERGY     359
      V(31)=V(31)+V(33)/2.0                                             ENERGY     360
      V(32)=V(32)+V(33)/2.0                                             ENERGY     361
      V(33)=V(34)                                                       ENERGY     362
      V(40)=V(14)+V(15)+V(16)                                           ENERGY     363
      V(41)=V(31)+V(32)+V(33)                                           ENERGY     364
      WRITE (6,2240)                                                    ENERGY     365
      WRITE (6,2250) (V(I),I=14,16),V(40)                               ENERGY     366
      WRITE (6,2250) (V(I),I=31,33),V(41)                               ENERGY     367
      WRITE (6,2030)                                                    ENERGY     368
      WRITE (6,4020)                                                    ENERGY     369
      WRITE (6,4030) (V(I),I=81,86)                                     ENERGY     370
      WRITE (6,4030) (V(I),I=101,106)                                   ENERGY     371
      WRITE (6,4040)                                                     NERGY     372
      WRITE (6,4050) (V(I),I=87,91)                                     ENERGY     373
      WRITE (6,4050) (V(I),I=107,111)                                   ENERGY     374
      WRITE (6,2030)                                                    ENERGY     375
      WRITE (6,2030)                                                    ENERGY     376
      WRITE (6,2260) ERR                                                ENERGY     377
      WRITE (6,2270) PL,PN,PNPCL                                        ENERGY     378
      WRITE (6,2030)                                                    ENERGY     379
      RETURN                                                            ENERGY     380
 2000 FORMAT (1H0,4HNODE,9X,1HU,13X,1HV,14X,1HW,12X,5HEO(S),10X,5HEO(S),  ENERGY     381
     C9X,6HEOO(S),10X,5HEO(F),10X,5HEO(F),8X,4HNODE)                    ENERGY     382
 2010 FORMAT (1H ,1X,I3,3(E13.6,2X),75X,I3)                             ENERGY     383
 2011 FORMAT (1H ,1X,I3,8(E13.6,2X),I3)                                 ENERGY     384
 2111 FORMAT (1H0)                                                      ENERGY     385
 2030 FORMAT (1H0,5X,100H- - - - - - - - - - - - - - - - - - - - - - - - ENERGY     386
     C - - - - - - - - - - - - - - - - - - - - - - - - )                 ENERGY     387
 2040 FORMAT (1H0,10X,11HEO(S)*EO(S),7X,11HEC(S)*EO(S),4X,16H2*DO*EO(S)*  ENERGY     388
     CEO(S),2X,22H2*(1-DO)*EOO(S)*EOO(S),2X,5HTOTAL )                   ENERGY     389
 2050 FORMAT (1H ,9X,3(E12.4,6X),3X,E12.4,6X,E12.4)                     ENERGY     390
 2060 FORMAT (1H0,6X,15HEO(F,S)*EO(F,S),4X,15HEO(F,S)*EO(F,S),4X,20H2*DO  ENERGY     391
     C*EO(F,S)*EO(F,S),2X,26H2*(1-DO)*EOO(F,S)*EOO(F,S),2X,5HTOTAL )    ENERGY     392
 2070 FORMAT(1H ,7X,3(E12.4,7X),3X,E12.4,16X,E12.4)                     ENERGY     393
 2080 FORMAT (1H0,8X,15H2*EO(F)*EO(F,S),4X,15H2*EO(F)*EO(F,S),4X,18H2*DO  ENERGY     394
     C*EO(F,S)*EO(F),4X,18H2*DO*EO(F,S)*EO(F),4X,5HTOTAL )              ENERGY     395
 2090 FORMAT (1H ,9X,3(E12.4,7X),4X,E12.4,7X,E12.4)                     ENERGY     396
 2100 FORMAT (1H0,10X,11HXO(S)*XO(S),7X,11HXC(S)*XO(S),4X,16H2*DO*XO(S)*  ENERGY     397
     CXO(S),2X,22H2*(1-DO)*XOO(S)*XOO(S),2X,5HTOTAL )                   ENERGY     398
 2110 FORMAT (1H0,30X,14HLOAD POTENTIAL )                               ENERGY     399
 2120 FORMAT (1H0,20X,7HP*DV(S),8X,9HP*DV(F,S),10X,5HTOTAL )            ENERGY     400
 2130 FORMAT (1H ,18X,E12.4,4X,E12.4,7X,E12.4)                          ENERGY     401
 2140 FORMAT (1H0,40X,28HSTRESS*STRAIN   MOMENT*CURV.  )                ENERGY     402
 2150 FORMAT (1H0,10X,11HNO(S)*EO(S),7X,11HNO(S)*EO(S),6X,13HNOO(S)*EOO(  ENERGY     403
     CS),5X,5HTOTAL )                                                   ENERGY     404
 2160 FORMAT (1H ,9X,4(E12.4,6X))                                       ENERGY     405
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2190 FORMAT (1H0,10X,13HNC(F)*FO(F,S),5X,13HNO(F,S)*FO(F),5X,13HNO(F)*E
     CO(F,S),5X,13HNO(F,S)*FO(F),5X,5HTOTAL)
2200 FORMAT (1H ,10X,5(E12.4,6X))
2210 FORMAT(1H0,12X,80H- - - - - - - - - - - - - - - - - - - - - - - - -
     C- - - - - - - - - - - - - - -          )
2220 FORMAT (1H0,3X,9HSUB TOTAL,3X,11HNO*FO,(F,S),6X,11HNO*EC,(F,S),6X,
     C13HNOC*EOO,(F,S),6X,5HTOTAL )
2230 FORMAT (1H ,14X,4(E12.4,6X))
2240 FORMAT(1H0,10X,11HMO(S)*XO(S),5X,11HMC(S)*XO(S),6X,13HMOC(S)*XOC(S
     C),7X,5HTOTAL )
2250 FORMAT (1H ,9X,4(E12.4,6X))
2260 FORMAT(1H ,90X,16HENERGY ERROR IS ,F7.3,9H PER CENT )
2270 FORMAT(1H0,20X,14HCRITICAL LOAD=,F16.8,3X,6HPN/F =,E16.8,3X,8HPN/P
     CCL =,E16.8)
4000 FORMAT(1H0,1X,4HNODE,3X,5HE1*E1,5X,5HE1*F2,6X,4HE1*B,7X,3HB*B,6X,4
     CHE2*B,5X,5HV1*V1,5X,5HE4*E4,5X,5HE5*F5,5X,5HV1*F4,5X,5HV1*E5,5X,5H
     CE4*F5,3X,4HNODE )
4010 FORMAT (1H ,2X,I2,2X,11(F9.3,1X),1X,I2)
4020 FORMAT (1H0,10X,5HE1*E1,10X,5HE1*F2,11X,4HE1*B,12X,3HB*B,11X,4HE2*
     CB,10X,5HV1*V1)
4030 FORMAT (1H ,6X,6(E12.6,3X))
4040 FORMAT(1H0,10X,5HE4*E4,10X,5HE5*F5,10X,5HV1*E4,10X,5HV1*F5,10X,5HE
     C4*E5 )
4050 FORMAT (1H ,6X,5(E12.6,3X))
5000 FORMAT(1H0,3X,16HT.F.F. VO     UMO,10X,3HUMO,10X,3HUBO,10X,3HUBO,10
     CX,4HL.F. )
5010 FORMAT (1H ,16X,5(F10.4,3X),4X,13HENERGY ERROR= ,F7.3,1X,8HPER CEN
     CT)
5015 FORMAT (1H ,16X,5(E10.4,3X))
5020 FORMAT (1H ,16X,5(E10.4,3X))
5030 FORMAT (1H ,3X,16HT.F.E. V1     UMO,10X,3HUMO,10X,3HUBO,10X,3HUBC,1
     C0X,4HL.P. )
5040 FORMAT(1H0,1X,18HTFE DELTA VO    UMO,10X,3HUMO,10X,3HUBO,10X,3HUBC,
     *10X,4HL.P.)
      END
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        CORRECTION IDENTS ARE LISTED IN CHRONOLOGICAL ORDER OF INSERTION

    SPHERE    ENDPTH    IMPRFT    LEGEND    REDUCE    PERTBN    CNVRGE    IMPRVE
    TPEV01    VCLDIS    EVALEN    FNDRHS    FNDRH2    FNDLHS    STRNCP    BCKSUB
    FDDATA    FUNVEC    SLVITR    DIRITR    INVITR    SNDRHS    SNDLHS    CRTHOG
    CONTRL    PRCDUC    MAC7P     MAC7C     BCCND     NOWAVE    ENERGY
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CKS ARE LISTED IN THE ORDER OF THEIR OCCURRENCE ON A NEW PROGRAM LIBRARY IF ONE IS CREATED BY THIS UPDATE

    YANKSSS   SPHERE    ENDPTH    IMPRFT    LEGEND    REDUCE    PERTBN    CNVRGE
    IMPRVE    TPEV01    VCLDIS    EVALEN    FNDRHS    FNDRH2    FNDLHS    STRNCP
    BCKSUB    FDDATA    FUNVEC    SLVITR    DIRITR    INVITR    SNDRHS    SNDLHS
    CRTHOG    CONTRL    PRCDUC    MAC7P     MAC7C     BCCND     NOWAVE    ENERGY

    DECKS WRITTEN TO COMPILE FILE

    SPHERE    ENDPTH    IMPRFT    LEGEND    REDUCE    PERTBN    CNVRGE    IMPRVE
    TPEV01    VCLDIS    EVALEN    FNDRHS    FNDRH2    FNDLHS    STRNCP    BCKSUB
    FDDATA    FUNVEC    SLVITR    DIRITR    INVITR    SNDRHS    SNDLHS    CRTHOG
    CONTRL    PRCDUC    MAC7P     MAC7C     BCCND     NOWAVE    ENERGY

THIS UPDATE REQUIRED  25100B WORDS OF SCM
               AND    07000B WORDS OF LCM.
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        ******** PARTIAL SPHERE ********
NUMBER OF NODES= 51   R/T= 100.000   POISSONS RATIO= .300   OPEN ANGLE=   .660   LAMBDA= 12.000   PERT ORDER=160
                    ***  PERFECT SHELL  ***
 M1=  2
 FOR PERT PARM  1 MAX LOCAL E= .2539E+00 TOTAL E= .2539E+00 AT WHICH P=-.1395E-02 PDIF=-.5472E-02 PDIF2= .2908E-03
 M1=  2
 FOR PERT PARM  2 MAX LOCAL E= .2695E+00 TOTAL E= .5234E+00 AT WHICH P=-.2850E-02 PDIF=-.5267E-02 PDIF2= .1543E-02
 M1=  2
 FOR PERT PARM  3 MAX LOCAL E= .1973E+00 TOTAL E= .7207E+00 AT WHICH P=-.3844E-02 PDIF=-.4717E-02 PDIF2= .4393E-02
 M1=  2
 FOR PERT PARM  4 MAX LOCAL E= .1123E+00 TOTAL E= .8330E+00 AT WHICH P=-.4341E-02 PDIF=-.4103E-02 PDIF2= .6467E-02
 M1=  2
 FOR PERT PARM  5 MAX LOCAL E= .1309E+00 TOTAL E= .9639E+00 AT WHICH P=-.4818E-02 PDIF=-.3150E-02 PDIF2= .8194E-02
 M1=  2
 FOR PERT PARM  6 MAX LOCAL E= .7715E-01 TOTAL E= .1041E+01 AT WHICH P=-.5034E-02 PDIF=-.2409E-02 PDIF2= .1127E-01
 M1=  2
 FOR PERT PARM  7 MAX LOCAL E= .7715E-01 TOTAL E= .1118E+01 AT WHICH P=-.5185E-02 PDIF=-.1493E-02 PDIF2= .1180E-01
 M1=  2
 FOR PERT PARM  8 MAX LOCAL E= .9668E-01 TOTAL E= .1215E+01 AT WHICH P=-.5277E-02 PDIF=-.4466E-03 PDIF2= .9837E-02
 M1=  3
 FOR PERT PARM  9 MAX LOCAL E= .1084E+00 TOTAL E= .1323E+01 AT WHICH P=-.5279E-02 PDIF= .2377E-03 PDIF2=-.1229E-02
 M1=  3
 FOR PERT PARM 10 MAX LOCAL E= .4736E-01 TOTAL E= .1371E+01 AT WHICH P=-.5270E-02 PDIF= .7653E-04 PDIF2=-.4624E-02
 M1=  3
 FOR PERT PARM 11 MAX LOCAL E= .4443E-01 TOTAL E= .1415E+01 AT WHICH P=-.5268E-02 PDIF= .9510E-04 PDIF2= .3439E-02
 M1=  2
 FOR PERT PARM 12 MAX LOCAL E= .5811E-01 TOTAL E= .1473E+01 AT WHICH P=-.5254E-02 PDIF= .4298E-03 PDIF2= .7401E-02
 M1=  2
 FOR PERT PARM 13 MAX LOCAL E= .7715E-01 TOTAL E= .1550E+01 AT WHICH P=-.5196E-02 PDIF= .1103E-02 PDIF2= .9782E-02
 M1=  2
 FOR PERT PARM 14 MAX LOCAL E= .9277E-01 TOTAL E= .1643E+01 AT WHICH P=-.5048E-02 PDIF= .2075E-02 PDIF2= .9626E-02
 M1=  2
 FOR PERT PARM 15 MAX LOCAL E= .7715E-01 TOTAL E= .1720E+01 AT WHICH P=-.4866E-02 PDIF= .2550E-02 PDIF2= .3048E-02
 FOR PERT PARM 16 MAX LOCAL E= .8691E-01 TOTAL E= .1807E+01 AT WHICH P=-.4639E-02 PDIF= .2645E-02 PDIF2= .7753E-03
 PERT METHOD COMPLETE,  ** LACK OF STORAGE **

                                        P-VS-E

FOR PERT PARM= 1      E=   .3174E-01   .6348E-01   .9521E-01   .1270E+00   .1587E+00   .1904E+00   .2222E+00   .2539E+00
                      P=  -.1748E-03  -.3495E-03  -.5241E-03  -.6986E-03  -.8730E-03  -.1047E-02  -.1221E-02  -.1395E-02
                  P/PCL=   .3173E-01   .6345E-01   .9516E-01   .1268E+00   .1585E+00   .1901E+00   .2217E+00   .2533E+00
                    VCL=   .1256E-02   .2517E-02   .3782E-02   .5053E-02   .6328E-02   .7609E-02   .8895E-02   .1019E-01
                    DIS=  -.1581E-03  -.3156E-03  -.4726E-03  -.6291E-03  -.7850E-03  -.9406E-03  -.1096E-02  -.1251E-02

FOR PERT PARM= 2      E=   .2876E+00   .3213E+00   .3550E+00   .3887E+00   .4224E+00   .4561E+00   .4897E+00   .5234E+00
                      P=  -.1579E-02  -.1763E-02  -.1946E-02  -.2129E-02  -.2311E-02  -.2492E-02  -.2671E-02  -.2850E-02
                  P/PCL=   .2867E+00   .3201E+00   .3534E+00   .3865E+00   .4196E+00   .4524E+00   .4850E+00   .5174E+00
                    VCL=   .1156E-01   .1294E-01   .1433E-01   .1572E-01   .1712E-01   .1851E-01   .1992E-01   .2132E-01
                    DIS=  -.1415E-02  -.1579E-02  -.1743E-02  -.1907E-02  -.2071E-02  -.2236E-02  -.2401E-02  -.2568E-02
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FOR PERT PARM= 3	E=	.5481E+00	.5728E+00	.5974E+00	.6221E+00	.6467E+00	.6714E+00	.6960E+00	.7207E+00
	P=	-.2979E-02	-.3107E-02	-.3234E-02	-.3360E-02	-.3484E-02	-.3606E-02	-.3726E-02	-.3844E-02
	P/PCL=	.5409E+00	.5642E+00	.5873E+00	.6101E+00	.6326E+00	.6547E+00	.6765E+00	.6979E+00
	VCL=	.2235E-01	.2338E-01	.2440E-01	.2543E-01	.2645E-01	.2746E-01	.2847E-01	.2948E-01
	DIS=	-.2690E-02	-.2814E-02	-.2938E-02	-.3063E-02	-.3189E-02	-.3316E-02	-.3444E-02	-.3573E-02
FOR PERT PARM= 4	E=	.7347E+00	.7488E+00	.7628E+00	.7769E+00	.7909E+00	.8049E+00	.8190E+00	.8330E+00
	P=	-.3909E-02	-.3974E-02	-.4038E-02	-.4101E-02	-.4163E-02	-.4223E-02	-.4283E-02	-.4341E-02
	P/PCL=	.7098E+00	.7216E+00	.7332E+00	.7446E+00	.7558E+00	.7668E+00	.7776E+00	.7882E+00
	VCL=	.3004E-01	.3061E-01	.3116E-01	.3172E-01	.3227E-01	.3282E-01	.3336E-01	.3389E-01
	DIS=	-.3646E-02	-.3720E-02	-.3794E-02	-.3868E-02	-.3943E-02	-.4017E-02	-.4090E-02	-.4164E-02
FOR PERT PARM= 5	E=	.9494E+00	.8657E+00	.8821E+00	.8984E+00	.9148E+00	.9312E+00	.9475E+00	.9639E+00
	P=	-.4407E-02	-.4472E-02	-.4534E-02	-.4595E-02	-.4654E-02	-.4711E-02	-.4765E-02	-.4818E-02
	P/PCL=	.8002E+00	.8119E+00	.8233E+00	.8343E+00	.8450E+00	.8553E+00	.8652E+00	.8748E+00
	VCL=	.3451E-01	.3512E-01	.3572E-01	.3632E-01	.3690E-01	.3747E-01	.3804E-01	.3859E-01
	DIS=	-.4249E-02	-.4332E-02	-.4414E-02	-.4494E-02	-.4571E-02	-.4645E-02	-.4713E-02	-.4776E-02
FOR PERT PARM= 6	E=	.9735E+00	.9832E+00	.9928E+00	.1002E+01	.1012E+01	.1022E+01	.1031E+01	.1041E+01
	P=	-.4919E-02	-.4877E-02	-.4905E-02	-.4933E-02	-.4960E-02	-.4985E-02	-.5010E-02	-.5034E-02
	P/PCL=	.8802E+00	.8855E+00	.8907E+00	.8957E+00	.9005E+00	.9052E+00	.9097E+00	.9140E+00
	VCL=	.3892E-01	.3923E-01	.3955E-01	.3986E-01	.4016E-01	.4046E-01	.4075E-01	.4104E-01
	DIS=	-.4810E-02	-.4842E-02	-.4870E-02	-.4895E-02	-.4916E-02	-.4935E-02	-.4949E-02	-.4960E-02
FOR PERT PARM= 7	E=	.1051E+01	.1060E+01	.1070E+01	.1080E+01	.1089E+01	.1099E+01	.1109E+01	.1118E+01
	P=	-.5057E-02	-.5078E-02	-.5099E-02	-.5118E-02	-.5136E-02	-.5154E-02	-.5170E-02	-.5185E-02
	P/PCL=	.9181E+00	.9220E+00	.9258E+00	.9293E+00	.9326E+00	.9357E+00	.9386E+00	.9414E+00
	VCL=	.4132E-01	.4159E-01	.4186E-01	.4211E-01	.4236E-01	.4260E-01	.4283E-01	.4305E-01
	DIS=	-.4969E-02	-.4974E-02	-.4977E-02	-.4978E-02	-.4978E-02	-.4977E-02	-.4976E-02	-.4975E-02
FOR PERT PARM= 8	E=	.1130E+01	.1142E+01	.1154E+01	.1167E+01	.1179E+01	.1191E+01	.1203E+01	.1215E+01
	P=	-.5202E-02	-.5217E-02	-.5231E-02	-.5243E-02	-.5254E-02	-.5263E-02	-.5271E-02	-.5277E-02
	P/PCL=	.9445E+00	.9473E+00	.9498E+00	.9520E+00	.9540E+00	.9556E+00	.9570E+00	.9581E+00
	VCL=	.4331E-01	.4356E-01	.4379E-01	.4401E-01	.4421E-01	.4439E-01	.4456E-01	.4471E-01
	DIS=	-.4976E-02	-.1980E-02	-.1987E-02	-.4998E-02	-.5014E-02	-.5036E-02	-.5064E-02	-.5099E-02
FOR PERT PARM= 9	E=	.1228E+01	.1242E+01	.1255E+01	.1269E+01	.1283E+01	.1296E+01	.1310E+01	.1323E+01
	P=	-.5282E-02	-.5286E-02	-.5287E-02	-.5288E-02	-.5287E-02	-.5285E-02	-.5282E-02	-.5279E-02
	P/PCL=	.9591E+00	.9597E+00	.9600E+00	.9601E+00	.9599E+00	.9595E+00	.9590E+00	.9584E+00
	VCL=	.4486E-01	.4499E-01	.4509E-01	.4517E-01	.4523E-01	.4526E-01	.4527E-01	.4525E-01
	DIS=	-.5148E-02	-.5206E-02	-.5276E-02	-.5358E-02	-.5452E-02	-.5561E-02	-.5683E-02	-.5820E-02
FOR PERT PARM=10	E=	.1329E+01	.1335E+01	.1341E+01	.1347E+01	.1353E+01	.1359E+01	.1365E+01	.1371E+01
	P=	-.5277E-02	-.5276E-02	-.5275E-02	-.5274E-02	-.5273E-02	-.5272E-02	-.5271E-02	-.5270E-02
	P/PCL=	.9582E+00	.9579E+00	.9577E+00	.9575E+00	.9573E+00	.9572E+00	.9571E+00	.9570E+00
	VCL=	.4521E-01	.4522E-01	.4520E-01	.4517E-01	.4514E-01	.4511E-01	.4508E-01	.4505E-01
	DIS=	-.5877E-02	-.5939E-02	-.6003E-02	-.6066E-02	-.6129E-02	-.6191E-02	-.6251E-02	-.6308E-02
FOR PERT PARM=11	E=	.1376E+01	.1382E+01	.1387E+01	.1393E+01	.1398E+01	.1404E+01	.1409E+01	.1415E+01
	P=	-.5270E-02	-.5270E-02	-.5269E-02	-.5269E-02	-.5269E-02	-.5269E-02	-.5268E-02	-.5268E-02
	P/PCL=	.9569E+00	.9568E+00	.9568E+00	.9567E+00	.9567E+00	.9566E+00	.9565E+00	.9564E+00
	VCL=	.4502E-01	.4499E-01	.4495E-01	.4492E-01	.4488E-01	.4485E-01	.4481E-01	.4477E-01
	DIS=	-.6351E-02	-.6396E-02	-.6437E-02	-.6477E-02	-.6514E-02	-.6548E-02	-.6581E-02	-.6610E-02
FOR PERT PARM=12	E=	.1422E+01	.1430E+01	.1437E+01	.1444E+01	.1451E+01	.1459E+01	.1466E+01	.1473E+01
	P=	-.5267E-02	-.5266E-02	-.5265E-02	-.5263E-02	-.5261E-02	-.5259E-02	-.5257E-02	-.5254E-02
	P/PCL=	.9563E+00	.9561E+00	.9559E+00	.9556E+00	.9553E+00	.9549E+00	.9544E+00	.9539E+00
	VCL=	.4472E-01	.1466E-01	.446CE-01	.4454E-01	.4446E-01	.4441E-01	.4433E-01	.4426E-01
	DIS=	-.6646E-02	-.6678E-02	-.6707E-02	-.6732E-02	-.6754E-02	-.6773E-02	-.6789E-02	-.6803E-02
FOR PERT PARM=13	E=	.1483E+01	.1492E+01	.1502E+01	.1512E+01	.1521E+01	.1531E+01	.1541E+01	.1550E+01
	P=	-.5249E-02	-.5244E-02	-.5238E-02	-.5231E-02	-.5223E-02	-.5215E-02	-.5206E-02	-.5196E-02
	P/PCL=	.9531E+00	.9521E+00	.9510E+00	.9498E+00	.9484E+00	.9469E+00	.9452E+00	.9433E+00
	VCL=	.4415E-01	.4404E-01	.4392E-01	.4379E-01	.4365E-01	.4351E-01	.4336E-01	.4320E-01
	DIS=	-.6818E-02	-.6830E-02	-.6838E-02	-.6844E-02	-.6848E-02	-.6850E-02	-.6851E-02	-.6851E-02

FOR EFRT PARM=14 E=	.1562E+01	.1573E+01	.1585E+01	.1597E+01	.1608E+01	.1620E+01	.1631E+01	.1643E+01
P=	-.5182E-02	-.5167E-02	-.5151E-02	-.5133E-02	-.5114E-02	-.5093E-02	-.5072E-02	-.5048E-02
P/PCL=	.9109E+00	.9382E+00	.9353E+00	.9321E+00	.9286E+00	.9249E+00	.9209E+00	.9166E+00
VCL=	.1299E-01	.4278E-01	.4255E-01	.4231E-01	.4206E-01	.4179E-01	.4152E-01	.4123E-01
DTS=	-.6850E-02	-.6850E-02	-.6852E-02	-.6856E-02	-.6863E-02	-.6875E-02	-.6893E-02	-.6917E-02
FOR EFRT PARM=15 E=	.1653E+01	.1662E+01	.1672E+01	.1682E+01	.1691E+01	.1701E+01	.1711E+01	.1720E+01
P=	-.5028E-02	-.5007E-02	-.4985E-02	-.4962E-02	-.4939E-02	-.4915E-02	-.4891E-02	-.4866E-02
P/PCL=	.9129E+00	.9090E+00	.9050E+00	.9009E+00	.8967E+00	.8924E+00	.8880E+00	.8836E+00
VCL=	.1099E-01	.4074E-01	.4048E-01	.4023E-01	.3997E-01	.3971E-01	.3944E-01	.3918E-01
DTS=	-.6943E-02	-.6973E-02	-.7009E-02	-.7051E-02	-.7097E-02	-.7149E-02	-.7205E-02	-.7264E-02
FOR EFRT PARM=16 E=	.1731E+01	.1742E+01	.1753E+01	.1764E+01	.1775E+01	.1785E+01	.1796E+01	.1807E+01
P=	-.4838E-02	-.4810E-02	-.4782E-02	-.4754E-02	-.4725E-02	-.4697E-02	-.4668E-02	-.4639E-02
P/PCL=	.8785E+00	.8734E+00	.8683E+00	.8631E+00	.8579E+00	.8528E+00	.8476E+00	.8423E+00
VCL=	.3889E-01	.3860E-01	.3831E-01	.3802E-01	.3774E-01	.3746E-01	.3718E-01	.3691E-01
DTS=	-.7337E-02	-.7414E-02	-.7494E-02	-.7578E-02	-.7664E-02	-.7754E-02	-.7845E-02	-.7939E-02

WE HAVE FOUND THE FOLLOWING TURNING POINTS

AC. 1 AT E= .1269E+01 P=-.5288E-02 P/PCL= .9601E+00

NODE	U	W	NODE
0	-.192802E-04	-.540719E-02	0
1	0.	-.535760E-02	1
2	.192793E-04	-.540718E-02	2
3	.391060E-04	-.555217E-02	3
4	.601286E-04	-.578146E-02	4
5	.831491E-04	-.607698E-02	5
6	.1090915E-03	-.641451E-02	6
7	.138853E-03	-.676493E-02	7
8	.173063E-03	-.709592E-02	8
9	.211769E-03	-.737434E-02	9
10	.254159E-03	-.756883E-02	10
11	.298422E-03	-.765274E-02	11
12	.341829E-03	-.760675E-02	12
13	.381067E-03	-.742114E-02	13
14	.4127785E-03	-.709711E-02	14
15	.434202E-03	-.664716E-02	15
16	.443631E-03	-.609390E-02	16
17	.440746E-03	-.546877E-02	17
18	.426545E-03	-.480866E-02	18
19	.403018E-03	-.415305E-02	19
20	.372662E-03	-.354060E-02	20
21	.337983E-03	-.300640E-02	21
22	.301137E-03	-.257966E-02	22
23	.263764E-03	-.228232E-02	23
24	.2270106E-03	-.212840E-02	24
25	.191677E-03	-.212397E-02	25
26	.158437E-03	-.226753E-02	26
27	.128036E-03	-.255064E-02	27
28	.101446E-03	-.295864E-02	28
29	.799470E-04	-.347135E-02	29
30	.651122E-04	-.406370E-02	30
31	.587039E-04	-.470642E-02	31
32	.624655E-04	-.536681E-02	32
33	.778267E-04	-.600970E-02	33
34	.1055455E-03	-.659868E-02	34
35	.145339E-03	-.709770E-02	35
36	.195582E-03	-.747294E-02	36
37	.253158E-03	-.769504E-02	37
38	.313545E-03	-.774133E-02	38
39	.371193E-03	-.759807E-02	39
40	.420173E-03	-.726235E-02	40

BUCKLING MODE

NODE	U	W	NODE
0	-.785128E-02	-.684357E+00	0
1	0.	-.674326E+00	1
2	.785075E-02	-.684355E+00	2
3	.156678E-01	-.713407E+00	3
4	.234864E-01	-.758412E+00	4
5	.314536E-01	-.814387E+00	5
6	.398224E-01	-.874675E+00	6
7	.488829E-01	-.931366E+00	7
8	.588358E-01	-.975921E+00	8
9	.696389E-01	-.100000E+01	9
10	.808817E-01	-.996406E+00	10
11	.917505E-01	-.960040E+00	11
12	.101126E+00	-.888694E+00	12
13	.107806E+00	-.783543E+00	13
14	.110806E+00	-.649217E+00	14
15	.109628E+00	-.493401E+00	15
16	.104409E+00	-.326022E+00	16
17	.958958E-01	-.158110E+00	17
18	.852365E-01	-.514752E-03	18
19	.736803E-01	.137315E+00	19
20	.622869E-01	.248286E+00	20
21	.517428E-01	.328206E+00	21
22	.423251E-01	.375874E+00	22
23	.339875E-01	.392772E+00	23
24	.265109E-01	.382494E+00	24
25	.196523E-01	.350045E+00	25
26	.132528E-01	.301152E+00	26
27	.728617E-02	.241652E+00	27
28	.185485E-02	.177027E+00	28
29	-.285104E-02	.112083E+00	29
30	-.661229E-02	.507681E-01	30
31	-.924099E-02	-.389579E-02	31
32	-.106311E-01	-.498083E-01	32
33	-.107908E-01	-.857303E-01	33
34	-.985371E-02	-.111194E+00	34
35	-.806626E-02	-.126396E+00	35
36	-.575566E-02	-.132080E+00	36
37	-.328321E-02	-.129431E+00	37
38	-.991116E-03	-.119958E+00	38
39	.847926E-03	-.105392E+00	39
40	.206934E-02	-.875768E-01	40

41	.455017E-03	-.674332E-02	41	41	.263523E-02	-.683473E-01	41
42	.471588E-03	-.606263E-02	42	42	.262800E-02	-.494178E-01	42
43	.467778E-03	-.525376E-02	43	43	.221894E-02	-.322623E-01	43
44	.443821E-03	-.436034E-02	44	44	.161795E-02	-.180068E-01	44
45	.402131E-03	-.343327E-02	45	45	.101745E-02	-.733978E-02	45
46	.346662E-03	-.252714E-02	46	46	.547136E-03	-.459685E-03	46
47	.281982E-03	-.169618E-02	47	47	.254291E-03	.292460E-02	47
48	.212347E-03	-.990450E-03	48	48	.114127E-03	.353509E-02	48
49	.141013E-03	-.452535E-03	49	49	.624563E-04	.241831E-02	49
50	.699604E-04	-.115222E-03	50	50	.352373E-04	.811345E-03	50
51	0.	0.	51	51	0.	0.	51
52	-.687948E-04	-.116385E-03	52	52	-.309428E-04	.118365E-02	52

T.F.E. V0	UM0	UMC	UB0	UBC	L.P.
	.1197E-05	.123CE-05	.2410E-06	.5980E-08	-.2674E-05
	.4476E+00	.4600E+00	.9013E-01	.2236E-02	-.1000E+01
T.F.F. V1	UM0	UMC	UB0	UBC	L.P.
	.2114E-06	.9676E-07	.2259E-06	.1687E-07	-.5481E-06
	.3848E+00	.1761E+00	.4111E+00	.3070E-01	-.9974E+00

ENERGY ERROR= .519 PER CENT

TPE DELTA V0	UM0	UMC	UB0	UBC	L.P.
	.1414E-07	.1369E-07	.1590E-07	.9826E-09	-.4471E-07
	.3163E+00	.3062E+00	.3556E+00	.2198E-01	-.1000E+01

	E= .451792E+00 P=-.246876E-02 P/PCL= .448248E+00		E= .903564E+00 P=-.461366E-02 P/PCL= .837694E+00		E= .135535E+01 P=-.527218E-02 P/PCL= .957261E+00		E= .180713E+01 P=-.463924E-02 P/PCL= .842340E+00		
NODE	U	W	U	W	U	W	U	W	NODE
0	-.4093E-05	-.2214E-02	-.1096E-04	-.4521E-02	-.2660E-04	-.6205E-02	-.4329E-04	-.7891E-02	0
1	0.	-.2215E-02	0.	-.4519E-02	0.	-.6155E-02	0.	-.7939E-02	1
2	.4093E-05	-.2214E-02	.1096E-04	-.4521E-02	.2660E-04	-.6205E-02	.4329E-04	-.7891E-02	2
3	.8176E-05	-.2213E-02	.2198E-04	-.4529E-02	.5376E-04	-.6349E-02	.8472E-04	-.7752E-02	3
4	.1224E-04	-.2211E-02	.3312E-04	-.4539E-02	.8213E-04	-.6574E-02	.1226E-03	-.7528E-02	4
5	.1627E-04	-.2209E-02	.4440E-04	-.4551E-02	.1125E-03	-.6860E-02	.1557E-03	-.7230E-02	5
6	.2026E-04	-.2205E-02	.5581E-04	-.4561E-02	.1459E-03	-.7178E-02	.1830E-03	-.6874E-02	6
7	.2419E-04	-.2200E-02	.6729E-04	-.4565E-02	.1831E-03	-.7495E-02	.2043E-03	-.6476E-02	7
8	.2806E-04	-.2194E-02	.7873E-04	-.4560E-02	.2245E-03	-.7777E-02	.2196E-03	-.6052E-02	8
9	.3184E-04	-.2188E-02	.8995E-04	-.4542E-02	.2697E-03	-.7988E-02	.2292E-03	-.5619E-02	9
10	.3554E-04	-.2180E-02	.1007E-03	-.4509E-02	.3172E-03	-.8097E-02	.2337E-03	-.5194E-02	10
11	.3912E-04	-.2172E-02	.1108E-03	-.4458E-02	.3647E-03	-.8078E-02	.2337E-03	-.4787E-02	11
12	.4259E-04	-.2162E-02	.1199E-03	-.4387E-02	.4086E-03	-.7916E-02	.2300E-03	-.4410E-02	12
13	.4594E-04	-.2152E-02	.1277E-03	-.4296E-02	.4455E-03	-.7609E-02	.2230E-03	-.4069E-02	13
14	.4916E-04	-.2142E-02	.1339E-03	-.4187E-02	.4721E-03	-.7162E-02	.2136E-03	-.3769E-02	14
15	.5224E-04	-.2132E-02	.1382E-03	-.4061E-02	.4861E-03	-.6597E-02	.2021E-03	-.3512E-02	15
16	.5520E-04	-.2122E-02	.1406E-03	-.3924E-02	.4866E-03	-.5943E-02	.1892E-03	-.3298E-02	16
17	.5804E-04	-.2113E-02	.1410E-03	-.3779E-02	.4744E-03	-.5238E-02	.1752E-03	-.3128E-02	17
18	.6080E-04	-.2105E-02	.1394E-03	-.3633E-02	.4513E-03	-.4521E-02	.1606E-03	-.2999E-02	18
19	.6349E-04	-.2100E-02	.1359E-03	-.3494E-02	.4199E-03	-.3834E-02	.1459E-03	-.2910E-02	19
20	.6616E-04	-.2097E-02	.1310E-03	-.3369E-02	.3831E-03	-.3215E-02	.1314E-03	-.2859E-02	20
21	.6886E-04	-.2099E-02	.1248E-03	-.3265E-02	.3433E-03	-.2696E-02	.1175E-03	-.2846E-02	21
22	.7164E-04	-.2104E-02	.1179E-03	-.3190E-02	.3025E-03	-.2303E-02	.1046E-03	-.2870E-02	22
23	.7458E-04	-.2114E-02	.1108E-03	-.3152E-02	.2619E-03	-.2052E-02	.9316E-04	-.2931E-02	23
24	.7776E-04	-.2130E-02	.1039E-03	-.3155E-02	.2226E-03	-.1951E-02	.8349E-04	-.3029E-02	24
25	.8125E-04	-.2152E-02	.9786E-04	-.3205E-02	.1851E-03	-.2003E-02	.7599E-04	-.3164E-02	25
26	.8513E-04	-.2179E-02	.9316E-04	-.3302E-02	.1501E-03	-.2201E-02	.7106E-04	-.3335E-02	26
27	.8950E-04	-.2213E-02	.9037E-04	-.3448E-02	.1184E-03	-.2533E-02	.6910E-04	-.3541E-02	27
28	.9443E-04	-.2251E-02	.9007E-04	-.3640E-02	.9111E-04	-.2982E-02	.7048E-04	-.3779E-02	28
29	.9999E-04	-.2294E-02	.9281E-04	-.3872E-02	.6958E-04	-.3527E-02	.7560E-04	-.4043E-02	29
30	.1062E-03	-.2339E-02	.9913E-04	-.4137E-02	.5541E-04	-.4141E-02	.8483E-04	-.4326E-02	30
31	.1131E-03	-.2385E-02	.1095E-03	-.4424E-02	.5031E-04	-.4795E-02	.9847E-04	-.4619E-02	31
32	.1207E-03	-.2428E-02	.1245E-03	-.4718E-02	.5590E-04	-.5458E-02	.1167E-03	-.4911E-02	32
33	.1289E-03	-.2466E-02	.1441E-03	-.5003E-02	.7339E-04	-.6095E-02	.1397E-03	-.5185E-02	33
34	.1375E-03	-.2495E-02	.1685E-03	-.5263E-02	.1033E-03	-.6671E-02	.1671E-03	-.5428E-02	34
35	.1465E-03	-.2511E-02	.1971E-03	-.5477E-02	.1450E-03	-.7153E-02	.1984E-03	-.5622E-02	35

36	.1554E-03	-.2509E-02	.2291E-03	-.5627E-02	.1968E-03	-.7508E-02	.2325E-03	-.5749E-02	36
37	.1640E-03	-.2486E-02	.2630E-03	-.5696E-02	.2553E-03	-.7709E-02	.2680E-03	-.5794E-02	37
38	.1718E-03	-.2435E-02	.2968E-03	-.5669E-02	.3159E-03	-.7736E-02	.3027E-03	-.5744E-02	38
39	.1783E-03	-.2355E-02	.3279E-03	-.5531E-02	.3732E-03	-.7574E-02	.3342E-03	-.5588E-02	39
40	.1829E-03	-.2242E-02	.3537E-03	-.5287E-02	.4215E-03	-.7225E-02	.3599E-03	-.5321E-02	40
41	.1849E-03	-.2095E-02	.3713E-03	-.4928E-02	.4553E-03	-.6696E-02	.3770E-03	-.4946E-02	41
42	.1838E-03	-.1913E-02	.3784E-03	-.4464E-02	.4710E-03	-.6010E-02	.3834E-03	-.4471E-02	42
43	.1789E-03	-.1698E-02	.3733E-03	-.3912E-02	.4664E-03	-.5201E-02	.3775E-03	-.3910E-02	43
44	.1697E-03	-.1456E-02	.3555E-03	-.3295E-02	.4421E-03	-.4312E-02	.3589E-03	-.3287E-02	44
45	.1561E-03	-.1193E-02	.3252E-03	-.2640E-02	.4003E-03	-.3392E-02	.3279E-03	-.2630E-02	45
46	.1380E-03	-.9204E-03	.2841E-03	-.1983E-02	.3450E-03	-.2495E-02	.2862E-03	-.1973E-02	46
47	.1158E-03	-.6515E-03	.2344E-03	-.1362E-02	.2807E-03	-.1674E-02	.2360E-03	-.1353E-02	47
48	.8986E-04	-.4034E-03	.1788E-03	-.8152E-03	.2115E-03	-.9770E-03	.1799E-03	-.8091E-03	48
49	.6121E-04	-.1964E-03	.1198E-03	-.3825E-03	.1405E-03	-.4463E-03	.1206E-03	-.3794E-03	49
50	.3088E-04	-.5352E-04	.5977E-04	-.1002E-03	.6974E-04	-.1136E-03	.6015E-04	-.9928E-04	50
51	0.	0.	0.	0.	0.	0.	0.	0.	51
52	-.3040E-04	-.6256E-04	-.5883E-04	-.1075E-03	-.6858E-04	-.1149E-03	-.5920E-04	-.1064E-03	52

---- TOTAL POTENTIAL ENERGY, VO ----

UMO= .2541E-06 .5151E+00	UMO= .8865E-06 .4955E+00	UMO= .1207E-05 .4521E+00	UMO= .9273E-06 .5070E+00
UMC= .1993E-06 .4279E+00	UMC= .8162E-06 .4562E+00	UMC= .1218E-05 .4562E+00	UMC= .8122E-06 .4441E+00
UBO= .1235E-07 .2651E-01	UBO= .8562E-07 .4785E-01	UBO= .2387E-06 .8938E-01	UBO= .8544E-07 .4672E-01
UBC= .1186E-09 .2546E-03	UBO= .9634E-09 .5384E-03	UBC= .6246E-08 .2339E-02	UBC= .3869E-08 .2116E-02
L.P.= -.4659E-06 -.1000E+01	L.P.= -.1789E-05 -.1000E+01	L.P.= -.2670E-05 -.1000E+01	L.P.= -.1829E-05 -.1000E+01

---- AI= 0.0 ----

END OF SET UP, TOTAL TIME= 23.496

T.T.= 23.50 T. REDUCE= .002

Y OUTSIDE OF FUNVEC RANGE, 0= .1807E+01 Y= .2424E+01
DIRECT ITERATION HAS FAILED, INVERSE ITERATION WILL BE TRIED
FAILED TO CONVERGE USING INVERSE ITERATION WITHIN 40 ITERATIONS E-VAL= .11166E+01
WILL TRY ANOTHER SHIFT OF ORIGIN TO .11166E+01
FAILED TO CONVERGE USING INVERSE ITERATION WITHIN 21 ITERATIONS E-VAL= .11247E+01
WILL TRY ANOTHER SHIFT OF ORIGIN TO .11247E+01
CONVERGENCE ON E-VECTOR (ERR< .100E-04) AFTER 15 ITERATIONS
FAILED TO CONVERGE USING INVERSE ITERATION WITHIN 20 ITERATIONS E-VAL= .12562E+01
WILL TRY ANOTHER SHIFT OF ORIGIN TO .12562E+01
CONVERGENCE ON E-VECTOR (ERR< .100E-04) AFTER 7 ITERATIONS
COMPLETE CONVERGENCE, E-VAL/VECT (ERR< .100E-04) AFTER 19 ITERATIONS .12562E+01
CONVERGENCE ON E-VECTOR (ERR< .100E-07) AFTER 7 ITERATIONS
COMPLETE CONVERGENCE, E-VAL/VECT (ERR< .100E-07) AFTER 9 ITERATIONS .12627E+01
Y OUTSIDE OF FUNVEC RANGE, 0= .1807E+01 Y= .2862E+01
DIRECT ITERATION HAS FAILED, INVERSE ITERATION WILL BE TRIED
FAILED TO CONVERGE USING INVERSE ITERATION WITHIN 40 ITERATIONS E-VAL= .10213E+01
WILL TRY ANOTHER SHIFT OF ORIGIN TO .10213E+01
FAILED TO CONVERGE USING INVERSE ITERATION WITHIN 21 ITERATIONS E-VAL= .10279E+01
WILL TRY ANOTHER SHIFT OF ORIGIN TO .10279E+01
Y OUTSIDE OF FUNVEC RANGE, 0= .1807E+01 Y=-.1845E+00
INVERSE ITERATION HAS FAILED WITH 1 E-VALUES FOUND

T.T.= 29.27 T. SLVITR= 5.771

SLVITR HAS FAILED FOR L= 1 IFAIL= 2

CIRCUMFRENTIAL MODE NO.= 0 EIGEN-VALUE POSITION = 1 NO. OF TURNING POINTS ON HALF SHELL = 4

T.F.E. V0	UM0	UMC	UB0	UBO	L.P.
	.1196E-05	.123CE-05	.2395E-06	.5871E-08	-.2671E-05
	.4477E+00	.4604E+00	.8969E-01	.2198E-02	-.1000E+01
T.F.E. V1	UM0	UMC	UB0	UBO	L.P.
	.2283E-06	.1407E-06	.2528E-06	.1785E-07	-.6372E-06
	.3576E+00	.2204E+00	.3960E+00	.2796E-01	-.9981E+00

ENERGY ERROR= .381 PER CENT

TPE DELTA V0	UM0	UMC	UB0	UBO	L.P.
	.1508E-07	.1592E-07	.1690E-07	.1007E-08	-.4891E-07
	.3083E+00	.3255E+00	.3456E+00	.2059E-01	-.1000E+01

NODE	E1*F1	E1*E2	E1*B	B*B	E2*B	V1*V1	E4*E4	E5*E5	V1*E4	V1*E5	E4*E5	NODE
1	-.303E-05	-.915E-06	.846E-02	-.295E-01	.255E-02	0.	0.	0.	0.	0.	0.	1
4	-.802E-06	-.750E-06	-.182E-01	-.482E+00	-.170E-01	0.	0.	0.	0.	0.	0.	4
8	-.309E-06	.938E-06	.988E-02	-.446E+00	-.300E-01	0.	0.	0.	0.	0.	0.	8
12	-.114E-04	-.132E-04	-.242E-01	-.666E-01	-.281E-01	0.	0.	0.	0.	0.	0.	12
16	-.166E-02	-.179E-04	-.107E+01	-.591E+00	-.115E-01	0.	0.	0.	0.	0.	0.	16
20	-.183E-03	.143E-04	-.242E+00	-.374E+00	-.583E-01	0.	0.	0.	0.	0.	0.	20
24	-.102E-04	.776E-05	.117E-02	-.161E-03	-.890E-03	0.	0.	0.	0.	0.	0.	24
28	-.164E-03	.506E-04	-.106E+00	-.469E-01	.326E-01	0.	0.	0.	0.	0.	0.	28
32	-.343E-04	.852E-06	-.448E-01	-.486E-01	.111E-02	0.	0.	0.	0.	0.	0.	32
36	-.358E-07	.233E-07	-.569E-03	-.120E-01	.371E-03	0.	0.	0.	0.	0.	0.	36
40	-.492E-06	-.144E-05	-.129E-02	-.206E-02	-.384E-02	0.	0.	0.	0.	0.	0.	40
44	-.328E-04	-.541E-05	-.169E-01	-.259E-02	-.275E-02	0.	0.	0.	0.	0.	0.	44
48	-.272E-07	-.143E-08	-.932E-04	-.395E-03	-.488E-05	0.	0.	0.	0.	0.	0.	48

NODE	U	V	W	E0(S)	EC(S)	EOC(S)	F0(F)	EC(F)	NODE
0	.185955E-02	0.	-.963816E-01						0
1	0.	0.	-.112364E+00						1
2	-.186024E-02	0.	-.963858E-01	-.235630E+00	-.237278E+00	0.	-.390122E-02	-.394219E-02	2
3	-.366730E-02	0.	-.490345E-01	-.183839E+00	-.187689E+00	0.	-.390595E-02	-.406479E-02	3
4	-.545203E-02	0.	.274680E-01	-.109946E+00	-.109712E+00	0.	-.391523E-02	-.425453E-02	4
5	-.739865E-02	0.	.131085E+00	-.307535E-01	-.888431E-02	0.	-.393114E-02	-.449092E-02	5
6	-.986332E-02	0.	.255735E+00	.341100E-01	.106233E+00	0.	-.395605E-02	-.474790E-02	6
7	-.134570E-01	0.	.395130E+00	.671558E-01	.225605E+00	0.	-.399238E-02	-.499644E-02	7
8	-.187546E-01	0.	.540787E+00	.589911E-01	.338431E+00	0.	-.404224E-02	-.520754E-02	8
9	-.263288E-01	0.	.682c97E+00	.135657E-01	.434346E+00	0.	-.410698E-02	-.535534E-02	9
10	-.364391E-01	0.	.809903E+00	-.498C39E-01	.504679E+00	0.	-.418683E-02	-.541972E-02	10
11	-.488573E-01	0.	.911396E+00	-.997105E-01	.543487E+00	0.	-.428060E-02	-.538817E-02	11
12	-.627676E-01	0.	.977271E+00	-.101618E+00	.548110E+00	0.	-.438571E-02	-.525643E-02	12
13	-.768296E-01	0.	.100000E+01	-.303399E-01	.519119E+00	0.	-.449839E-02	-.502819E-02	13
14	-.894203E-01	0.	.975616E+00	.118514E+00	.459742E+00	0.	-.461401E-02	-.471388E-02	14
15	-.990052E-01	0.	.904522E+00	.323418E+00	.374994E+00	0.	-.472731E-02	-.432923E-02	15
16	-.104520E+00	0.	.791702E+00	.541710E+00	.270840E+00	0.	-.483268E-02	-.389383E-02	16
17	-.105624E+00	0.	.646191E+00	.723475E+00	.153620E+00	0.	-.492438E-02	-.343004E-02	17
18	-.102727E+00	0.	.479795E+00	.828919E+00	.298085E-01	0.	-.499697E-02	-.296211E-02	18
19	-.967847E-01	0.	.305538E+00	.841277E+00	-.940325E-01	0.	-.504576E-02	-.251549E-02	19
20	-.889727E-01	0.	.135992E+00	.769818E+00	-.211327E+00	0.	-.506705E-02	-.211582E-02	20
21	-.803576E-01	0.	-.185174E-01	.643098E+00	-.315740E+00	0.	-.505819E-02	-.178778E-02	21
22	-.716891E-01	0.	-.149732E+00	.497213E+00	-.401643E+00	0.	-.501752E-02	-.155351E-02	22
23	-.633395E-01	0.	-.252773E+00	.364575E+00	-.464676E+00	0.	-.494446E-02	-.143102E-02	23
24	-.553667E-01	0.	-.325511E+00	.266699E+00	-.502238E+00	0.	-.483974E-02	-.143284E-02	24
25	-.476421E-01	0.	-.368463E+00	.211828E+00	-.513755E+00	0.	-.470554E-02	-.156498E-02	25
26	-.399900E-01	0.	-.383937E+00	.196473E+00	-.500664E+00	0.	-.454566E-02	-.182643E-02	26
27	-.322964E-01	0.	-.375760E+00	.209226E+00	-.466121E+00	0.	-.436540E-02	-.220907E-02	27

29	-.169501E-01	0.	-.307097E+00	.260141E+00	-.350841E+00	0.	-.397058E-02	-.327200E-02	29
30	-.968728E-02	0.	-.256183E+00	.272865E+00	-.280236E+00	0.	-.377141E-02	-.390460E-02	30
31	-.308619E-02	0.	-.200034E+00	.266968E+00	-.207414E+00	0.	-.358211E-02	-.456478E-02	31
32	.254713E-02	0.	-.142273E+00	.240819E+00	-.136401E+00	0.	-.341129E-02	-.521844E-02	32
33	.695657E-02	0.	-.858485E-01	.196878E+00	-.703739E-01	0.	-.326755E-02	-.582989E-02	33
34	.997390E-07	0.	-.330714E-01	.140557E+00	-.116459E-01	0.	-.315912E-02	-.636372E-02	34
35	.115422E-01	0.	.142804E-01	.790433E-01	.382428E-01	0.	-.309323E-02	-.678692E-02	35
36	.117260E-01	0.	.548538E-01	.202810E-01	.783976E-01	0.	-.307542E-02	-.707105E-02	36
37	.107101E-01	0.	.876385E-01	-.279676E-01	.108449E+00	0.	-.310887E-02	-.719431E-02	37
38	.878628E-02	0.	.111936E+00	-.591466E-01	.128468E+00	0.	-.319407E-02	-.714333E-02	38
39	.632560E-02	0.	.127381E+00	-.691004E-01	.138914E+00	0.	-.332877E-02	-.691438E-02	39
40	.373361E-02	0.	.134007E+00	-.572687E-01	.140606E+00	0.	-.350811E-02	-.651401E-02	40
41	.139138E-02	0.	.132314E+00	-.271653E-01	.134699E+00	0.	-.372457E-02	-.595887E-02	41
42	-.408834E-03	0.	.123323E+00	.123240E-01	.122643E+00	0.	-.396794E-02	-.527495E-02	42
43	-.151647E-02	0.	.108567E+00	.515923E-01	.106118E+00	0.	-.422538E-02	-.449626E-02	43
44	-.194338E-02	0.	.899833E-01	.803094E-01	.869360E-01	0.	-.448225E-02	-.366311E-02	44
45	-.183940E-02	0.	.697091E-01	.921993E-01	.669072E-01	0.	-.472375E-02	-.282008E-02	45
46	-.142876E-02	0.	.498146E-01	.866569E-01	.476995E-01	0.	-.493646E-02	-.201380E-02	46
47	-.932856E-03	0.	.320609E-01	.683115E-01	.307185E-01	0.	-.510934E-02	-.129050E-02	47
48	-.509933E-03	0.	.177358E-01	.446938E-01	.170223E-01	0.	-.523360E-02	-.693497E-03	48
49	-.229120E-03	0.	.757990E-02	.233567E-01	.726810E-02	0.	-.530221E-02	-.260764E-03	49
50	-.790459E-04	0.	.177958E-02	.954306E-02	.167494E-02	0.	-.530955E-02	-.226522E-04	50
51	C.	0.	0.						51
52	.746135E-04	0.	.144534E-02						52

LOAD POTENTIAL

P*DV(S)	P*DV(F,S)	TOTAL
-.5781E-04	.2023E-07	-.5779E-04
-.2662E-02	.9316E-06	-.2661E-02

STRESS*STRAIN MOMENT*CURV.

NO(S)*EO(S)	NO(S)*EC(S)	NOO(S)*EOC(S)	TOTAL
.6519E-02	.5504E-02	0.	.1202E-01
.3002E+00	.2535E+00	0.	.5537E+00

NO(F,S)*EO(F,S)	NO(F,S)*EO(F,S)	NOO(F,S)*EOC(F,S)	TOTAL
-.5843E-02	.1716E-03	0.	-.5671E-02
-.2691E+00	.7903E-02	0.	-.2612E+00

NO(F)*EO(F,S)	NO(F,S)*EO(F)	NO(F)*EC(F,S)	NO(F,S)*EC(F)	TOTAL
-.6885E-02	-.5639E-02	0.	-.1245E-02	-.1377E-01
-.3170E+00	-.2597E+00	0.	-.5735E-01	-.6341E+00

SUB TOTAL	NO*EO,(F,S)	NO*EO,(F,S)	NOC*EOC,(F,S)	TOTAL
	-.1837E-01	-.1074E-02	0.	-.1944E-01
	-.8458E+00	-.4945E-01	0.	-.8952E+00

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   MO(S)*XO(S)      MC(S)*XC(S)      MOC(S)*XOC(S)      TOTAL
    .7117E-02          .8265E-03      0.                  .7943E-02
    .3277E+00          .3806E-01      0.                  .3658E+00

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    E1*E1           E1*E2           E1*B            B*B             E2*B            V1*V1
 -.301741E-02    .142397E-03    -.225230E+01    -.226384E+01    .778927E-01    0.
 -.611501E-03    .288577E-04    -.456446E+00    -.458784E+00    .157856E-01    0.

    E4*E4           E5*E5           V1*E4           V1*E5           E4*E5
 0.              0.              0.              0.              0.
 0.              0.              0.              0.              0.

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                                                                                ENERGY ERROR IS   2.155 PER CENT
             CRITICAL LOAD=  -.52877566E-02   PN/E =   .11621443E-03   PN/PCL =   .96008877E+00
- - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - -
                                                    T.T.=  29.35  T. ENERGY=  .081
                                                    T.T.=  29.35  T. CCNTRL=  .082

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                      ----   AI= 1.0   ----

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                                                                 END OF SET UP, TOTAL TIME=   29.364
                                                    T.T.=  29.37  T. REDUCE=  .005
CONVERGENCE ON E-VALUE (ERR<  .100E-02) AFTER  18 ITERATIONS     .11007E+01
        CONVERGENCE ON E-VECTOR (ERR<    .100E-04) AFTER   4 ITERATIONS
        COMPLETE CONVERGENCE, E-VAL/VECT (ERR<    .100E-04) AFTER   6 ITERATICNS    .10840E+01
        CONVERGENCE ON E-VECTOR (ERR<    .100E-07) AFTER   3 ITERATIONS
        COMPLETE CONVERGENCE, E-VAL/VECT (ERR<    .100E-07) AFTER   5 ITERATICNS    .10857E+01
    Y OUTSIDE OF FUNVEC RANGE, 0- .1807E+01   Y= .2737E+01
        DIRECT ITERATION HAS FAILED, INVERSE ITERATION WILL BE TRIED
    Y OUTSIDE OF FUNVEC RANGE, 0- .1807E+01   Y= .1886E+01
        INVERSE ITERATION HAS FAILED WITH   1 E-VALUES FOUND
                                                    T.T.=  32.01  T. SLVITR= 2.637
    SLVITR HAS FAILED FOR L=  2   IFAIL=  2
-------------------------------------------------------------------------------------------
      CIRCUMFRENTIAL MODE NC.=   1   EIGEN-VALUE POSITION =  1    NO. OF TURNING POINTS ON HALF SHELL =   3
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  T.F.E. V0     UMO            UMC            UBO            UBO            L.P.
                 .1103E-05      .1098E-05      .1612E-06      .2453E-08     -.2364E-05
                 .4665E+00      .4643E+00      .6820E-01      .1037E-02     -.1000E+01
  T.F.E. V1     UMO            UMC            UBO            UBO            L.P.
                -.8669E-06      .1287E-05     -.5027E-06      .1441E-07     -.2684E-05      ENERGY ERROR=  -.471 PER CENT
                 .3238E+00      .4808E+00      .1877E+00      .5380E-02     -.1002E+01

 THE DELTA V0   UMO            UMC            UBO            UBO            L.P.
                 .4870E-07      .6997E-07      .2434E-07      .6114E-09     -.1436E-06
                 .3391E+00      .4872E+00      .1694E+00      .4257E-02     -.1000E+01
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NODE	E1*E1	E1*E2	E1*B	B*B	E2*B	V1*V1	E4*E4	E5*E5	V1*E4	V1*E5	E4*E5	NODE
1	-.481E-05	.300E-06	-.591E-01	-.931E+00	.368E-02	-.140E-02	-.244E-03	-.949E+00	-.137E-02	-.237E-01	.988E-02	1
4	-.626E-05	.364E-05	-.439E-01	-.395E+00	-.255E-01	-.656E-06	-.300E-03	-.787E+00	-.324E-04	-.149E-02	.317E-01	4
8	-.127E-06	.132E-06	.891E-03	-.947E-02	.927E-03	-.763E-04	-.637E-03	-.410E+00	.517E-03	.250E-02	.723E-02	8
12	-.395E-04	.640E-05	-.125E+00	-.525E+00	-.207E-01	-.147E-03	-.589E-03	-.828E-01	.666E-03	-.112E-01	-.220E-01	12
16	-.183E-03	.203E-04	-.275E+00	-.479E+00	.304E-01	-.474E-04	-.219E-03	-.844E-03	.217E-03	-.121E-02	-.247E-02	16
20	-.135E-04	.632E-05	-.254E-01	-.582E-01	.118E-01	-.790E-06	-.342E-04	-.357E-02	.118E-04	.201E-03	-.129E-02	20
24	-.309E-05	.210E-05	-.794E-02	-.241E-01	.538E-02	-.433E-05	-.218E-05	-.299E-02	-.702E-05	.363E-03	-.254E-03	24
28	-.269E-04	.562E-05	-.373E-01	-.466E-01	-.778E-02	-.105E-04	-.571E-06	-.464E-03	.486E-05	.521E-03	-.112E-03	28
32	-.193E-05	-.796E-06	-.594E-02	-.180E-01	-.245E-02	-.974E-05	-.152E-05	-.577E-04	.801E-05	-.127E-03	-.473E-04	32
36	-.130E-07	.145E-07	-.295E-04	-.866E-04	.329E-04	-.307E-05	-.103E-06	-.321E-03	.132E-05	.197E-04	.362E-05	36
40	-.492E-05	-.209E-05	-.457E-02	-.304E-02	-.193E-02	-.160E-06	-.855E-07	-.873E-04	-.204E-06	.360E-04	-.228E-04	40
44	-.101E-04	-.806E-06	-.446E-02	-.811E-03	-.353E-03	-.288E-08	-.260E-07	-.174E-05	.116E-07	-.150E-05	-.351E-05	44
48	-.555E-07	.294E-09	-.106E-04	-.211E-05	.558E-07	-.237E-07	-.204E-09	-.399E-11	.487E-08	-.241E-08	-.216E-09	48

NODE	U	V	W	EO(S)	EC(S)	EOO(S)	EO(F)	EC(F)	NODE
0	.428581E-01	-.761343E-01	-.178422E+00						0
1	C.	0.	C.						1
2	.692835E-02	-.111304E-01	.178052E+00	.674218E+00	-.140284E+00	-.221805E+00	-.399307E-02	-.400894E-02	2
3	.937499E-02	-.155576E-01	.351532E+00	.298042E+00	.117233E+00	.685951E-01	-.399474E-02	-.405617E-02	3
4	.770607E-02	-.179407E-01	.515018E+00	.293446E+00	.256395E+00	.461068E-01	-.399814E-02	-.412887E-02	4
5	.351608E-02	-.192782E-01	.663058E+00	.258940E+00	.364357E+00	.117476E+00	-.400416E-02	-.421866E-02	5
6	-.286689E-02	-.198389E-01	.790212E+00	.229778E+00	.446094E+00	.163683E+00	-.401382E-02	-.431486E-02	6
7	-.111663E-01	-.197214E-01	.891355E+00	.198420E+00	.501466E+00	.211667E+00	-.402819E-02	-.440559E-02	7
8	-.210254E-01	-.189504E-01	.962042E+00	.167766E+00	.529840E+00	.257521E+00	-.404822E-02	-.447883E-02	8
9	-.319465E-01	-.175464E-01	.998897E+00	.147084E+00	.531119E+00	.298975E+00	-.407454E-02	-.452352E-02	9
10	-.432750E-01	-.155538E-01	.100000E+01	.135581E+00	.506308E+00	.333383E+00	-.410737E-02	-.453055E-02	10
11	-.542354E-01	-.130534E-01	.965224E+00	.169753E+00	.457658E+00	.358068E+00	-.414639E-02	-.449353E-02	11
12	-.640108E-01	-.101636E-01	.896156E+00	.220669E+00	.388565E+00	.370910E+00	-.419072E-02	-.440928E-02	12
13	-.718536E-01	-.703317E-02	.797649E+00	.292603E+00	.303334E+00	.370786E+00	-.423889E-02	-.427798E-02	13
14	-.771991E-01	-.382646E-02	.671653E+00	.373666E+00	.206680E+00	.357834E+00	-.428895E-02	-.410318E-02	14
15	-.797548E-01	-.706520E-03	.534833E+00	.448402E+00	.104421E+00	.333471E+00	-.433849E-02	-.389150E-02	15
16	-.795384E-01	.213202E-02	.386482E+00	.501758E+00	.120310E-02	.300168E+00	-.438482E-02	-.365227E-02	16
17	-.768531E-01	.472628E-02	.238111E+00	.523042E+00	-.977402E-01	.261025E+00	-.442505E-02	-.339708E-02	17
18	-.722078E-01	.695250E-02	.977090E-01	.508735E+00	-.187603E+00	.219274E+00	-.445629E-02	-.313920E-02	18
19	-.662043E-01	.852708E-02	-.279111E-01	.463057E+00	-.265011E+00	.177803E+00	-.447576E-02	-.289307E-02	19
20	-.594250E-01	.975182E-02	-.133596E+00	.396322E+00	-.326217E+00	.138834E+00	-.448097E-02	-.267360E-02	20
21	-.523483E-01	.105556E-01	-.216103E+00	.321797E+00	-.369282E+00	.103781E+00	-.446984E-02	-.249545E-02	21
22	-.453076E-01	.109845E-01	-.274127E+00	.252198E+00	-.393203E+00	.732674E-01	-.444082E-02	-.237228E-02	22
23	-.384930E-01	.110930E-01	-.308120E+00	.196915E+00	-.398165E+00	.473897E-01	-.439297E-02	-.231592E-02	23
24	-.319857E-01	.109361E-01	-.319982E+00	.160551E+00	-.385486E+00	.257383E-01	-.432612E-02	-.233503E-02	24
25	-.258063E-01	.105654E-01	-.312677E+00	.142852E+00	-.357469E+00	.780903E-02	-.424093E-02	-.243736E-02	25
26	-.199615E-01	.100264E-01	-.289826E+00	.139738E+00	-.317155E+00	-.693073E-02	-.413898E-02	-.262317E-02	26
27	-.144784E-01	.935778E-02	-.255331E+00	.144904E+00	-.268036E+00	-.189208E-01	-.402286E-02	-.289071E-02	27
28	-.942109E-02	.859275E-02	-.213061E+00	.151548E+00	-.213738E+00	-.284675E-01	-.389609E-02	-.323296E-02	28
29	-.488885E-02	.775979E-02	-.166598E+00	.153823E+00	-.157738E+00	-.357502E-01	-.376313E-02	-.363807E-02	29
30	-.100174E-02	.688442E-02	-.119073E+00	.147829E+00	-.103133E+00	-.408574E-01	-.362924E-02	-.408947E-02	30
31	.212215E-02	.599037E-02	-.730768E-01	.132063E+00	-.524748E-01	-.438373E-01	-.350036E-02	-.456624E-02	31
32	.439105E-02	.510032E-02	-.306279E-01	.107364E+00	-.768834E-02	-.447482E-01	-.338298E-02	-.504385E-02	32
33	.576064E-02	.423622E-02	.679569E-02	.764900E-01	.299416E-01	-.437021E-01	-.328383E-02	-.549523E-02	33
34	.624828E-02	.341891E-02	.382011E-01	.434543E-01	.597245E-01	-.408953E-01	-.320965E-02	-.589223E-02	34
35	.593886E-02	.266728E-02	.630079E-01	.127783E-01	.814838E-01	-.366215E-01	-.316663E-02	-.620736E-02	35
36	.498146E-02	.199713E-02	.809820E-01	-.112553E-01	.954636E-01	-.312680E-01	-.315989E-02	-.641576E-02	36
37	.357734E-02	.141992E-02	.921871E-01	-.253048E-01	.102241E+00	-.252931E-01	-.319296E-02	-.649711E-02	37
38	.195950E-02	.941740E-03	.969577E-01	-.275965E-01	.102652E+00	-.191873E-01	-.326729E-02	-.643743E-02	38
39	.365134E-03	.562745E-03	.958981E-01	-.163977E-01	.977239E-01	-.134212E-01	-.336210E-02	-.623041E-02	39
40	-.996272E-03	.277351E-03	.898241E-01	-.193595E-03	.886266E-01	-.838849E-02	-.353429E-02	-.587830E-02	40
41	-.197415E-02	.751317E-04	.798442E-01	.226009E-01	.766092E-01	-.435500E-02	-.371836E-02	-.539215E-02	41
42	-.250005E-02	-.576001E-04	.672130E-01	.445061E-01	.629428E-01	-.142598E-02	-.392644E-02	-.479150E-02	42
43	-.259502E-02	-.135834E-03	.532990E-01	.603534E-01	.488509E-01	.456629E-03	-.414834E-02	-.410352E-02	43
44	-.235596E-02	-.173918E-03	.394526E-01	.668122E-01	.354351E-01	.148246E-02	-.437217E-02	-.336174E-02	44

45	-.192186E-02	-.184021E-03	.268622E-01	.934425E-01	.235993E-01	.191268E-02	-.458528E-02	-.260451E-02	45
46	-.143217E-02	-.175302E-03	.164198E-01	.527047E-01	.139866E-01	.200939E-02	-.477557E-02	-.187316E-02	46
47	-.989204E-03	-.153837E-03	.862949E-02	.388552E-01	.693645E-02	.197907E-02	-.493228E-02	-.120998E-02	47
48	-.640445E-03	-.123161E-03	.357443E-02	.262546E-01	.246650E-02	.195729E-02	-.504631E-02	-.656122E-03	48
49	-.382305E-03	-.850557E-04	.937196E-03	.178344E-01	.273290E-03	.188652E-02	-.510996E-02	-.249446E-03	49
50	-.180855E-03	-.396298E-04	.553926E-04	.141616E-01	-.249788E-03	.213674E-02	-.511689E-02	-.223848E-04	50
51	0.	0.	0.						51
52	.174174E-03	-.164001E-03	-.329165E-03						52

LOAD POTENTIAL

P*DV(S)	P*DV(F,S)	TOTAL
-.1599E-04	-.2642E-07	-.1601E-04
-.1079E-02	-.1782E-05	-.1080E-02

STRESS*STRAIN MOMENT*CURV.

NO(S)*EO(S)	NO(S)*EC(S)	NOC(S)*EOC(S)	TOTAL
.3025E-02	.3553E-02	.1874E-02	.8452E-02
.2041E+00	.2397E+00	.1264E+00	.5702E+00

NO(F,S)*EO(F,S)	NO(F,S)*EC(F,S)	NOC(F,S)*EOC(F,S)	TOTAL
-.2115E-02	.9741E-04	-.3993E-04	-.2058E-02
-.1427E+00	.6572E-02	-.2694E-02	-.1386E+00

NO(F)*EO(F,S)	NO(F,S)*EO(F)	NO(F)*EC(F,S)	NO(F,S)*EC(F)	TOTAL
-.4207E-02	-.3721E-02	-.1723E-02	-.2209E-02	-.1186E-01
-.2838E+00	-.2510E+00	-.1162E+00	-.1490E+00	-.8001E+00

SUB TOTAL	NO*EO,(F,S)	NC*EC,(F,S)	NOC*ECC,(F,S)	TOTAL
	-.1004E-01	-.3835E-02	-.3993E-04	-.1392E-01
	-.6776E+00	-.2587E+00	-.2694E-02	-.9389E+00

MO(S)*XO(S)	MC(S)*XC(S)	MOC(S)*XOO(S)	TOTAL
.4075E-02	.7572E-03	.8232E-03	.5655E-02
.2749E+00	.5108E-01	.5553E-01	.3815E+00

E1*E1	E1*E2	E1*B	B*B	E2*B	V1*V1
-.432017E-03	.348126E-04	-.701280E+00	-.171565E+01	.442458E-01	-.378309E-03
-.128262E-03	.103355E-04	-.208204E+00	-.509360E+00	.131362E-01	-.112316E-03

E4*E4	E5*E5	V1*E4	V1*E5	E4*E5
-.154637E-02	-.775296E+00	.131584E-02	-.112462E-01	-.577029E-02
-.459103E-03	-.230178E+00	.390660E-03	-.333888E-02	-.171315E-02

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- - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - -
                                                                        ENERGY ERROR IS   1.169 PER CENT
            CRITICAL LOAD=  -.51298651E-02   PN/E =    .11274429E-03   PN/PCL =   .93142067E+00
- - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - -
                                                 T.T.=  32.09  T. ENERGY=  .084
                                                 T.T.=  32.09  T. CONTRL=  .085

                      ----------------------------
                      ----    AI= 2.0    ----
                      ----------------------------
                                                              END OF SET UP, TOTAL TIME=   32.104
                                                 T.T.=  32.11  T. REDUCE=  .006
CONVERGENCE ON E-VALUE (ERR<  .100E-02) AFTER   5 ITERATIONS    .10532E+01
        CONVERGENCE ON E-VECTOR (ERR<    .100E-04) AFTER   3 ITERATIONS
        COMPLETE CONVERGENCE, E-VAL/VECT (ERR<   .100E-04) AFTER   6 ITERATIONS    .10487E+01
        CONVERGENCE ON E-VECTOR (ERR<   .100E-07) AFTER   2 ITERATIONS
        COMPLETE CONVERGENCE, E-VAL/VECT (ERR<   .100E-07) AFTER   3 ITERATIONS    .10487E+01
   Y OUTSIDE OF FUNVEC RANGE, 0- .1807E+01   Y= .2344E+01
        DIRECT ITERATION HAS FAILED, INVERSE ITERATION WILL BE TRIED
   Y OUTSIDE OF FUNVEC RANGE, 0- .1807E+01   Y= .1352E+02
        INVERSE ITERATION HAS FAILED WITH   1 E-VALUES FOUND
                                                 T.T.=  34.65  T. SLVITR= 2.543
   SLVITR HAS FAILED FOR L=  3   IFAIL=  2
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     CIRCUMFERENTIAL MODE NO.=   2   EIGEN-VALUE POSITION =  1    NO. OF TURNING POINTS ON HALF SHELL =   4
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  T.F.E. V0    UMO          UMC          UBO          UBO          L.P.
                .1068E-05    .1047E-05    .1430E-06    .1980E-08   -.2260E-05
                .4726E+00    .4633E+00    .6330E-01    .8761E-03   -.1000E+01
  T.F.E. V1    UMO          UMC          UBO          UBO          L.P.
                .1026E-05    .1451E-05    .4781E-06    .1119E-07   -.2982E-05     ENERGY ERROR=  -.531 PER CENT
                .3448E+00    .4880E+00    .1608E+00    .3763E-02   -.1003E+01

 TPE DELTA V0  UMO          UMC          UBO          UBO          L.P.
                .5600E-07    .7628E-07    .2259E-07    .4669E-09   -.1553E-06
                .3605E+00    .4911E+00    .1454E+00    .3006E-02   -.1000E+01
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NODE	E1*E1	E1*E2	E1*B	B*B	E2*B	V1*V1	E4*E4	E5*E5	V1*E4	V1*E5	E4*E5	NODE
1	-.223E-05	.649E-06	-.794E-02	-.364E-01	.231E-02	-.932E-03	-.925E-03	-.369E-01	.217E-02	-.278E-02	-.277E-02	1
4	-.204E-04	.293E-05	-.889E-01	-.499E+00	.127E-01	-.559E-03	-.411E-03	-.719E+00	.111E-02	-.294E-01	-.251E-01	4
8	-.778E-08	-.997E-09	-.121E-02	-.250E+00	-.155E-03	-.299E-04	-.374E-05	-.119E+01	-.248E-04	-.271E-03	.960E-04	8
12	-.184E-04	-.652E-05	-.264E-01	-.485E-01	-.933E-02	-.194E-03	-.422E-03	-.666E+00	.652E-03	-.340E-01	-.493E-01	12
16	-.812E-04	-.110E-04	-.175E+00	-.146E+00	-.238E-01	-.349E-03	-.486E-03	-.125E+00	.899E-03	-.340E-01	-.385E-01	16
20	-.864E-05	.109E-05	-.482E-01	-.326E+00	.608E-02	-.159E-03	-.186E-03	-.205E-02	.395E-03	-.171E-02	-.183E-02	20
24	-.873E-06	.891E-06	.518E-02	-.358E-01	-.527E-02	-.373E-04	-.470E-04	-.536E-02	.958E-04	-.135E-02	-.149E-02	24
28	-.831E-05	.607E-05	-.739E-02	-.616E-02	.539E-02	-.244E-05	-.342E-05	-.648E-02	.587E-05	-.855E-03	-.942E-03	28
32	-.389E-05	.134E-05	-.100E-01	-.263E-01	.346E-02	-.210E-05	-.308E-05	-.158E-02	.540E-05	.286E-03	.330E-03	32
36	-.376E-07	.970E-08	-.690E-03	-.160E-01	.178E-03	-.419E-05	-.394E-05	-.322E-04	.952E-05	.828E-05	.805E-05	36
40	-.337E-05	.183E-05	-.333E-03	-.246E-04	.180E-03	-.489E-06	-.345E-06	-.325E-03	.731E-06	.119E-03	.874E-04	40
44	-.386E-05	-.128E-05	-.348E-02	-.146E-02	-.115E-02	-.127E-07	-.159E-08	-.382E-04	.620E-08	-.144E-04	-.399E-05	44
48	-.946E-10	.905E-10	.571E-05	-.405E-03	-.544E-05	-.204E-07	-.270E-10	-.591E-07	-.165E-08	-.272E-06	.956E-08	48

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NODE	U	V	W	EQ(S)	EC(S)	EQF(S)	EQ(F)	EC(F)	NODE
0	-.823637E-02	.827693E-02	.176120E-01						U
1	0.	0.	0.						1
2	.823524E-02	-.827986E-02	.176122E-01	.630659E+00	-.612985E+00	-.621740E+00	-.398816E-02	-.399937E-02	2
3	.160432E-01	-.163999E-01	.692119E-01	.633397E+00	-.565077E+00	-.599755E+00	-.398931E-02	-.403260E-02	3
4	.229834E-01	-.242073E-01	.151179E+00	.632652E+00	-.491528E+00	-.562765E+00	-.399171E-02	-.408345E-02	4
5	.286062E-01	-.315395E-01	.257757E+00	.622474E+00	-.396072E+00	-.510042E+00	-.399602E-02	-.414560E-02	5
6	.324727E-01	-.382140E-01	.381449E+00	.597504E+00	-.285974E+00	-.441103E+00	-.400303E-02	-.421101E-02	6
7	.341975E-01	-.410328E-01	.513532E+00	.555280E+00	-.168561E+00	-.356415E+00	-.401357E-02	-.427072E-02	7
8	.335108E-01	-.487958E-01	.644674E+00	.497668E+00	-.512498E-01	-.257873E+00	-.402835E-02	-.431564E-02	8
9	.303210E-01	-.523221E-01	.765600E+00	.431108E+00	.590003E-01	-.149058E+00	-.404786E-02	-.433740E-02	9
10	.247650E-01	-.544747E-01	.867769E+00	.365445E+00	.156136E+00	-.351330E-01	-.407228E-02	-.432907E-02	10
11	.172238E-01	-.551826E-01	.944013E+00	.311581E+00	.235335E+00	.776086E-01	-.410138E-02	-.428574E-02	11
12	.829565E-02	-.541572E-01	.989075E+00	.278547E+00	.293194E+00	.182495E+00	-.413446E-02	-.420499E-02	12
13	-.127457E-02	-.523963E-01	.100000E+01	.270865E+00	.327801E+00	.273370E+00	-.417040E-02	-.408699E-02	13
14	-.106986E-01	-.491777E-01	.976322E+00	.287041E+00	.338717E+00	.345440E+00	-.420765E-02	-.393460E-02	14
15	-.192579E-01	-.450406E-01	.920035E+00	.319729E+00	.326686E+00	.395898E+00	-.424433E-02	-.375321E-02	15
16	-.264014E-01	-.402597E-01	.835328E+00	.357588E+00	.294503E+00	.424176E+00	-.427827E-02	-.355046E-02	16
17	-.318053E-01	-.351163E-01	.728119E+00	.388275E+00	.244835E+00	.431806E+00	-.430712E-02	-.333591E-02	17
18	-.353837E-01	-.298727E-01	.605435E+00	.401640E+00	.181996E+00	.421936E+00	-.432852E-02	-.312064E-02	18
19	-.372524E-01	-.247512E-01	.471741E+00	.392187E+00	.110668E+00	.398643E+00	-.434015E-02	-.291673E-02	19
20	-.376612E-01	-.199234E-01	.343280E+00	.359958E+00	.357787E-01	.366224E+00	-.433991E-02	-.273674E-02	20
21	-.369157E-01	-.155069E-01	.217521E+00	.310023E+00	-.378509E-01	.328594E+00	-.432661E-02	-.259309E-02	21
22	-.353090E-01	-.115697E-01	.102767E+00	.250654E+00	-.105645E+00	.288906E+00	-.429706E-02	-.249744E-02	22
23	-.330774E-01	-.813913E-02	.292991E-02	.191059E+00	-.164571E+00	.249405E+00	-.425221E-02	-.245996E-02	23
24	-.303834E-01	-.521234E-02	-.795256E-01	.139275E+00	-.211353E+00	.211470E+00	-.419122E-02	-.248873E-02	24
25	-.273228E-01	-.276708E-02	-.143497E+00	.100719E+00	-.244584E+00	.175794E+00	-.411457E-02	-.258907E-02	25
26	-.239477E-01	-.770352E-03	-.189070E+00	.775594E-01	-.263725E+00	.142606E+00	-.402355E-02	-.276301E-02	26
27	-.202950E-01	.815096E-03	-.217276E+00	.688590E-01	-.269215E+00	.111894E+00	-.392024E-02	-.300875E-02	27
28	-.164122E-01	.202678E-02	-.229841E+00	.712663E-01	-.262249E+00	.835802E-01	-.380778E-02	-.332039E-02	28
29	-.123757E-01	.290061E-02	-.228936E+00	.799997E-01	-.244614E+00	.576401E-01	-.368984E-02	-.368767E-02	29
30	-.829852E-02	.347065E-02	-.216954E+00	.898822E-01	-.218978E+00	.341592E-01	-.357123E-02	-.409604E-02	30
31	-.432734E-02	.377005E-02	-.196336E+00	.962531E-01	-.187140E+00	.133424E-01	-.345711E-02	-.452692E-02	31
32	-.631259E-03	.383243E-02	-.169442E+00	.956485E-01	-.151636E+00	-.451611E-02	-.335337E-02	-.495827E-02	32
33	.261529E-02	.369339E-02	-.138164E+00	.862149E-01	-.114631E+00	-.190888E-01	-.326613E-02	-.536557E-02	33
34	.525271E-02	.339154E-02	-.105392E+00	.676666E-01	-.780359E-01	-.300858E-01	-.320154E-02	-.572310E-02	34
35	.715659E-02	.296862E-02	-.720019E-01	.422401E-01	-.434628E-01	-.373303E-01	-.316530E-02	-.600554E-02	35
36	.825626E-02	.246866E-02	-.398639E-01	.124356E-01	-.122120E-01	-.408281E-01	-.316222E-02	-.618972E-02	36
37	.854902E-02	.193593E-02	-.103522E-01	-.173813E-01	.147208E-01	-.408187E-01	-.319566E-02	-.625647E-02	37
38	.810765E-02	.141212E-02	.153592E-01	-.426127E-01	.366329E-01	-.377965E-01	-.326716E-02	-.619226E-02	38
39	.707811E-02	.933066E-03	.363212E-01	-.590530E-01	.531060E-01	-.324905E-01	-.337620E-02	-.599057E-02	39
40	.566488E-02	.525740E-03	.518535E-01	-.638367E-01	.640008E-01	-.257972E-01	-.352010E-02	-.565275E-02	40
41	.410342E-02	.206147E-03	.616016E-01	-.563084E-01	.694538E-01	-.186708E-01	-.369394E-02	-.518840E-02	41
42	.262245E-02	-.212775E-04	.655907E-01	-.385123E-01	.698702E-01	-.119870E-01	-.389054E-02	-.461514E-02	42
43	.140351E-02	-.162567E-03	.642590E-01	-.149817E-01	.659077E-01	-.641072E-02	-.410057E-02	-.395794E-02	43
44	.548890E-03	-.231546E-03	.584456E-01	.833602E-02	.584451E-01	-.230357E-02	-.431303E-02	-.324804E-02	44
45	.689168E-04	-.246190E-03	.493224E-01	.258643E-01	.485302E-01	.299073E-03	-.451609E-02	-.252153E-02	45
46	-.105608E-03	-.224882E-03	.382718E-01	.338360E-01	.373120E-01	.163054E-02	-.469818E-02	-.181773E-02	46
47	-.904068E-04	-.183459E-03	.267389E-01	.319025E-01	.259656E-01	.207707E-02	-.484881E-02	-.117730E-02	47
48	-.657880E-05	-.133628E-03	.160920E-01	.224506E-01	.156232E-01	.205333E-02	-.495888E-02	-.640356E-03	48
49	.547434E-04	-.827915E-04	.751650E-02	.993366E-02	.731136E-02	.180086E-02	-.502056E-02	-.244363E-03	49
50	.528172E-04	-.344449E-04	.194280E-02	-.595309E-03	.189843E-02	.176884E-02	-.502730E-02	-.221601E-04	50
51	0.	0.	0.						51
52	-.500132E-04	-.126687E-03	.198310E-02						52

LOAD POTENTIAL

P*DV(S)	P*DV(F,S)	TOTAL
.5120E-04	-.2020E-06	.5100E-04
.2622E-02	-.1035E-04	.2612E-02

STRESS*STRAIN MOMENT*CURV.

NO(S)*EO(S)	NO(S)*EC(S)	NOO(S)*EOC(S)	TOTAL
.3176E-02	.2365E-02	.4024E-02	.9566E-02
.1627E+00	.1211E+00	.2061E+00	.4899E+00

NO(F,S)*EO(F,S)	NO(F,S)*EO(F,S)	NOO(F,S)*EOC(F,S)	TOTAL
-.1407E-02	-.4039E-04	-.8381E-03	-.2286E-02
-.7208E-01	-.2069E-02	-.4292E-01	-.1171E+00

NO(F)*EO(F,S)	NO(F,S)*EO(F)	NO(F)*EC(F,S)	NO(F,S)*EO(F)	TOTAL
-.4128E-02	-.4378E-02	-.4483E-02	-.4233E-02	-.1722E-01
-.2114E+00	-.2242E+00	-.2296E+00	-.2168E+00	-.8820E+00

SUB TOTAL NO*EO,(F,S)	NO*EO,(F,S)	NOC*EOC,(F,S)	TOTAL
-.9913E-02	-.8756E-02	-.8381E-03	-.1951E-01
-.5077E+00	-.4485E+00	-.4292E-01	-.9991E+00

MO(S)*XO(S)	MO(S)*XC(S)	MOC(S)*XOC(S)	TOTAL
.4196E-02	.3462E-02	.2223E-02	.9881E-02
.2149E+00	.1773E+00	.1139E+00	.5061E+00

E1*E1	E1*E2	E1*B	B*B	E2*B	V1*V1
-.198429E-03	.146332E-05	-.423158E+00	-.176205E+01	-.183636E-01	-.133635E-02
-.447240E-04	.329818E-06	-.953758E-01	-.397150E+00	-.413699E-02	-.301201E-03

E4*E4	E5*E5	V1*E4	V1*E5	E4*E5
-.174330E-02	-.201381E+01	.334759E-02	-.985403E-01	-.118996E+00
-.392923E-03	-.453894E+00	.754514E-03	-.222100E-01	-.268205E-01

ENERGY ERROR IS -.052 PER CENT

CRITICAL LOAD= -.50520654E-02 PN/E = .11103440E-03 PN/PCL = .91729472E+00

T.T.= 34.74 T. ENERGY= .085
T.T.= 34.74 T. CONTRL= .085