PRIMALITY TESTING WITH FROBENIUS SYMBOLS

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In this lecture we discuss several primality testing algorithms that are based on the following trivial theorem.

Theorem. Let n be a positive integer. Then n is prime if and only if every divisor of n is a power of n.

In the actual primality tests one does not check that any r dividing n is a power of n, but that this is true for the images of r and n in certain groups: in Galois groups, in $(\mathbb{Z}/s\mathbb{Z})^*$ for certain auxiliary numbers s, or in the group of values of a Dirichlet character. We remark that it suffices to consider prime divisors r of n.

We begin with a few considerations from algebraic number theory. Let K be a finite abelian extension of the rational number field $\mathbb{Q}$, and suppose that the discriminant of K is relatively prime to n. By the Kronecker-Weber theorem, we have $K \subset \mathbb{Q}(\zeta_s)$ for some integer s with $\gcd(s, n) = 1$; here ζ_s denotes a primitive s-th root of unity. For any integer r that is coprime to s let σ_r be the restriction to K of the automorphism of $\mathbb{Q}(\zeta_s)$ sending ζ_s to ζ_s^r. Then σ_r belongs to the Galois group G of K over $\mathbb{Q}$. If r is prime, then σ_r is the Frobenius symbol of r for the extension $K/\mathbb{Q}$, and the field $K^{\sigma_r} = \{x \in K: \sigma_r(x) = x\}$ is the largest subfield of K in which r splits completely. Let now A be the ring of integers of K^{σ_n}. If n is actually prime, then it is a prime that splits completely in K^{σ_n}, so there is a ring homomorphism $A \to \mathbb{Z}/n\mathbb{Z}$ (mapping 1 to 1). Also, this ring homomorphism is usually not difficult to find. Suppose, for example, that $\alpha \in A$ is such that the index of $\mathbb{Z}[\alpha]$ in A is finite and relatively prime to n, and let f be the irreducible polynomial of α over $\mathbb{Z}$. Then finding a ring homomorphism $A \to \mathbb{Z}/n\mathbb{Z}$ is equivalent to finding a zero of (f mod n) in $\mathbb{Z}/n\mathbb{Z}$. There are good algorithms to find such a zero if n is prime. If conversely a zero is found, it does not follow that n is prime. But it does follow, by composing the map $A \to \mathbb{Z}/n\mathbb{Z}$ with the natural map $\mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/r\mathbb{Z}$, that for every prime divisor r of n there is a ring homomorphism $A \to \mathbb{Z}/r\mathbb{Z}$. This implies that r splits completely in K^{σ_n}, so $K^{\sigma_n} \subset K^{\sigma_r}$, and therefore σ_r <u>is a power of</u> σ_n <u>in the group</u> G, for every divisor r of n. If $K = \mathbb{Q}(\zeta_s)$ this just means that r is congruent to a power of n modulo s. We shall see below how such information can be used to decide whether n is prime or not.

If n is composite then the zero-finding routine that is used may not converge. Therefore it is advisable to apply the primality tests discussed in this lecture only if one is morally certain that n is prime. This certainty can be obtained by subjecting n to several pseudo-prime tests. The question is how to <u>prove</u> that n is prime.

We consider a special case of the test described above. Let s be the largest divisor of $n-1$ that one is able to factor completely, and let $K = \mathbb{Q}(\zeta_s)$. Then σ_n is the identity on K, and $A = \mathbb{Z}[\zeta_s]$. The irreducible polynomial of ζ_s over $\mathbb{Z}$ is the s-th cyclotomic polynomial Φ_s. If $a \in \mathbb{Z}$ satisfies

$$a^s \equiv 1 \bmod n,$$

$$\gcd(a^{s/q}-1, n) = 1 \quad \text{for every prime } q \text{ dividing } s,$$

then $(a \bmod n)$ is a zero of $(\Phi_s \bmod n)$ in $\mathbb{Z}/n\mathbb{Z}$. If n is actually prime, then such an a is usually not difficult to find, by manipulating with elements of the form $(b^{(n-1)/s} \bmod n)$. Conversely, if an a as above has been found then by the result proved above we know that any divisor r of n is congruent to a power of n modulo s, i.e. is congruent to 1 mod s. If we have $s > n^{1/2}$ then it follows immediately from this that n is prime. If the weaker inequality $s > n^{1/3}$ is satisfied we can also easily finish the primality test. Namely, if n is not prime then

$$n = (xs + 1)(ys + 1), \quad x > 0, \quad y > 0, \quad xy < s$$

for certain integers x, y. From $(x-1)(y-1) \geq 0$ we obtain $0 < x+y \leq s$, and since $x+y \equiv (n-1)/s \bmod s$ this means that we know the value of $x+y$. We also know that $n = (xs + 1)(ys + 1)$, so x and y can now be solved from a quadratic equation. The result tells us immediately whether n is prime or not.

The test just described is a classical one, and its correctness can easily be proved without Frobenius symbols. There are several refinements and extensions that we do not go into here.

Let now s be a positive integer that is coprime to n. We assume that the complete prime factorization of s is known. Instead of assuming that s divides $n - 1$ we now require that the order t of $(n \bmod s)$ in the unit group $(\mathbb{Z}/s\mathbb{Z})^*$ is relatively small. If n is prime, then the residue class field of any prime ideal of $\mathbb{Z}[\zeta_s]$ containing n is the finite field $\mathbb{F}_{n^t}$. Also, if $a \in \mathbb{F}_{n^t}^*$ is the image of ζ_s then

$$a^s = 1,$$

$$a^{s/q} - 1 \in \mathbb{F}_{n^t}^* \quad \text{for each prime } q \text{ dividing } s,$$

$$\prod_{i=0}^{t-1} (X - a^{n^i}) \text{ has coefficients in } \mathbb{F}_n.$$

The latter property comes from the fact that the polynomial $\prod_{i=0}^{t-1} (X - \zeta_s^{n^i})$ has coefficients in the ring previously denoted by A (for $K = \mathbb{Q}(\zeta_s)$). There are, again, good methods to construct $\mathbb{F}_{n^t}$ and a as above, if n is prime. Suppose, conversely, that one has constructed a ring extension R of $\mathbb{Z}/n\mathbb{Z}$ and an element $a \in R$ having the above properties, with $\mathbb{F}_{n^t}$, $\mathbb{F}_n$ replaced by R, $\mathbb{Z}/n\mathbb{Z}$. Then there is a ring homomorphism $\mathbb{Z}[\zeta_s] \to R$ mapping ζ_s to a, and the subring generated by the coefficients of $g = \prod_{i=0}^{t-1} (X - \zeta_s^{n^i})$ is mapped to $\mathbb{Z}/n\mathbb{Z}$. But from the fact that g is the irreducible polynomial of ζ_s over A it is easy to derive that this subring is equal to A. That gives us the desired ring homomorphism $A \to \mathbb{Z}/n\mathbb{Z}$, which permits us to conclude that every divisor of n is congruent to a power of n modulo s. If $s > n^{1/2}$ then this conclusion immediately leads to the complete factorization of n, by trying the remainders of $1, n, \ldots, n^{t-1}$ modulo s as divisors. The weaker condition $s > n^{1/3}$ is also sufficient to finish the test, by a procedure that is somewhat more complicated than the one described before.

As an example we treat the Lucas-Lehmer test for Mersenne numbers $n = 2^m - 1$, with $m > 2$. Let $e_1 = 4$, $e_{i+1} = e_i^2 - 2$. Then it is asserted that n is prime if and only if $e_{m-1} \equiv 0 \bmod n$. The case that m is <u>even</u> is easy and uninteresting, by looking mod 3. So let m be odd, and define

$$R = (\mathbb{Z}/n\mathbb{Z})[T]/(T^2 - \sqrt{2}\cdot T - 1)$$

where $\sqrt{2} = (2^{(m+1)/2} \bmod n) \in \mathbb{Z}/n\mathbb{Z}$. Denote the image of T in R by a, and let $b = \sqrt{2} - a = -a^{-1}$ be "the" other zero of $X^2 - \sqrt{2}\cdot X - 1$ in R. Then $a^{2^i} + b^{2^i} = (e_i \bmod n)$. If n is prime then one easily checks that R is a field in which a and b are conjugate, so $a^n = b$ by the theory of finite fields. Multiplying by a one gets $a^{2^m} = -1$, so $(e_{m-1} \bmod n) = a^{2^{m-1}} + b^{2^{m-1}}$ $= a^{2^{m-1}} + a^{-2^{m-1}} = 0$. Conversely, assume that $(e_{m-1} \bmod n) = 0$. Then

$$a^{2^m} = -1, \qquad a^{2^{m+1}} = 1$$

and from $a^n = a^{2^m-1} = -a^{-1} = b$ we find

$$(X - a)(X - a^n) = (X - a)(X - b) = X^2 - \sqrt{2}\cdot X - 1,$$

a polynomial with coefficients in $\mathbb{Z}/n\mathbb{Z}$. Applying the preceding theory with $s = 2^{m+1}$, $t = 2$ we conclude that every divisor of n is congruent to 1 or n mod s. From $s > n$ it now follows that n is prime.

To prove that, in the general case, a suitable value for s can always be found we invoke a result of Pomerance and Odlyzko. They proved that for each $n > e^e$ there exists a positive integer t with

$$t < (\log n)^{c \log\log\log n},$$

where c is an absolute effectively computable constant, such that the number

$$s = \prod_{q \text{ prime},\ q-1 \text{ divides } t} q$$

exceeds $n^{1/2}$. If gcd (s, n) = 1 then Fermat's theorem implies that $n^t \equiv 1 \bmod s$, so the order of (n mod s) in $(\mathbb{Z}/s\mathbb{Z})^*$ is relatively small. This value for s can be used for all n of the same order of magnitude. Given n, one can often make better choices of s by employing known prime factors of $n^i - 1$ for various small values of i.

It is probably possible to treat Adleman's new primality test (see Séminaire Bourbaki, exp. 576) from the same point of view. Let s, t be as in the result of Pomerance and Odlyzko. The $\mathbb{Q}(\zeta_s)$ can be written as the compositum of a collection of cyclic fields, each of which has prime power degree p^k and prime conductor q, with p^k dividing t and q dividing s. These fields have much smaller degrees over $\mathbb{Q}$ than $\mathbb{Q}(\zeta_s)$, and are therefore more attractive from a computational point of view. Employing Gaussian sums as Lagrange resolvents for these fields one can design tests that, as before, permit one to conclude that every divisor of n is congruent to a power of n modulo s. It is, in fact, more efficient to do the actual calculations with Jacobi sums, in the rings $\mathbb{Z}[\zeta_{p^k}]/n\mathbb{Z}[\zeta_{p^k}]$. This version of Adleman's test is being programmed by H. Cohen on the minicomputer in Bordeaux.

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