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EUCLID'S ALGORITHM IN LARGE DEDEKIND DOMAINS

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Abstract. It is proved that any Dedekind domain with many more elements than prime ideals is Euclidean

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Let A be a Dedekind domain, and denote by Z the set of its non-zero prime ideals. It is well known that A is a principal ideal domain if Z is finite. An infinite analogue of this result was obtained by Claborn [1; 2, chapter III, section 13]. He proved that A is a principal ideal domain if

(1) $\quad\quad \#A > (\#Z)^{\mathfrak{a}}$,

where $\mathfrak{a}$ is the least infinite cardinal and $\#S$ denotes the cardinality of S.

If Z is finite then A is not only a principal ideal domain but even a *Euclidean* domain [4, Proposition 5]. The latter statement means that there exists a map ϕ from $A-\{0\}$ to a well-ordered set W such that for all $a, b \in A$ with $b \neq 0$, $a \notin Ab$, there exists $r \in a+Ab$ with $\phi(r) < \phi(b)$. For finite Z one can take for W the set of non-negative integers.

It is a natural question whether Claborn's result can be extended in a similar way, *i.e.* whether A is Euclidean if (1) holds. In the present paper we show that this is indeed the case. For W we take a well-ordered set of order type ω^2, where ω is the least infinite ordinal. The elements of W can be written in a unique way as $\omega a + b$, where a, b are non-negative integers; and $\omega a + b < \omega a' + b'$ if and only if either $a < a'$ or $a = a'$, $b < b'$.

We shall see that the other results that Claborn obtained in [1] can be extended in an analogous way.

We let K denote the field of fractions of A, and $v_{\mathfrak{p}}$, for $\mathfrak{p} \in Z$, the normalized exponential valuation of K corresponding to $\mathfrak{p}$. The group of units of A is denoted by A^*.

Claborn's first result [1, Proposition; 2, Proposition 13.7] states that A is a principal ideal domain if A contains a field k satisfying $\#A = \#k > \#Z$. A sharper result is as follows.

(2) Proposition. *Let A be a Dedekind domain, and suppose that A contains a subset k with the properties*

(3) $\quad\quad \#k > \#Z$,

(4) $\quad\quad \lambda - \mu \in A^* \cup \{0\}$ *for all* $\lambda, \mu \in k$.

Then A is Euclidean.

Proof. For $x \in A - \{0\}$, let $\phi(x) = \sum_{\mathfrak{p} \in Z} v_{\mathfrak{p}}(x)$. We prove that A is Euclidean with respect to ϕ.

Let $a, b \in A$, $b \neq 0$, $a \notin Ab$. First suppose that for some $\lambda \in k$ we have $A \cdot (a + \lambda b) = Aa + Ab$. Then

$$v_{\mathfrak{p}}(a+\lambda b)=\min\{v_{\mathfrak{p}}(a),v_{\mathfrak{p}}(b)\}\leqslant v_{\mathfrak{p}}(b)$$

for all $\mathfrak{p}\in Z$, with strict inequality for at least one $\mathfrak{p}$. Hence the element $r=a+\lambda b$ of $a+Ab$ satisfies $\phi(r)<\phi(b)$, as required.

Next suppose that no such λ exists. Then for every $\lambda\in k$ there exists $\mathfrak{p}_\lambda\in Z$ such that $a+\lambda b\in\mathfrak{p}_\lambda\cdot(Aa+Ab)$. The map $k\to Z$ sending λ to $\mathfrak{p}_\lambda$ is not injective, by (3), so there are λ, $\mu\in k$, $\lambda\neq\mu$, with $\mathfrak{p}_\lambda=\mathfrak{p}_\mu$. Then $(\lambda-\mu)b=(a+\lambda b)-(a+\mu b)\in\mathfrak{p}_\lambda\cdot(Aa+Ab)$, so $b\in\mathfrak{p}_\lambda\cdot(Aa+Ab)$, by (4). We conclude that $Aa+Ab=A\cdot(a+\lambda b)+Ab$ is contained in $\mathfrak{p}_\lambda\cdot(Aa+Ab)$, which is a contradiction. This proves (2).

If A is the ring of integers in an algebraic number field then condition (3) can be substantially weakened, see [3, Theorem (1.4)].

For a subset $Y\subset Z$, we define the subring $A_Y\subset K$ by

$$A_Y=\{x\in K:\ v_{\mathfrak{p}}(x)\geqslant 0 \text{ for all } \mathfrak{p}\in Y\}.$$

Notice that $A_Z=A$. Claborn [1, Theorem; 2, Theorem 13.8] proved that every ideal of A_Y is generated by an element of A if the inequality $\#A>(\#Y)^{\mathfrak{a}}$ is satisfied. To formulate our stronger result we need a definition. Let the pair (A,Y) be called *Euclidean* if there exist a well-ordered set W and a map $\phi:A-\{0\}\to W$ such that for all a, $b\in A$, $b\neq 0$, $a\notin A_Yb$, there exists $r\in a+Ab$ with $\phi(r)<\phi(b)$. We have $A_Z=A$, and (A,Z) is Euclidean if and only if A is.

Let (A,Y) be Euclidean and $\mathfrak{b}$ a non-zero A_Y-ideal. Then $\mathfrak{b}$ is generated by $\mathfrak{b}\cap A$, and if $b\in\mathfrak{b}\cap A$ has minimal ϕ-value then it follows easily that $A_Yb=\mathfrak{b}$. Hence, if (A,Y) is a Euclidean pair, then every ideal of A_Y is generated by an element of A. This shows that the following theorem is indeed sharper than Claborn's result.

(5) Theorem. *Let A be a Dedekind domain, and Y a set of non-zero prime ideals of A such that $\#A>(\#Y)^{\mathfrak{a}}$, where $\mathfrak{a}$ denotes the least infinite cardinal. Then (A,Y) is a Euclidean pair.*

The proof uses the following lemma. Let W be the well-ordered set of order type ω^2 defined above.

(6) Lemma. *Let A be Dedekind, $Y\subset Z$ a subset, and suppose that there exists a finite subset $X\subset Y$ with the property that for every $x\in A_X-A_Y$ there exists $q\in A$ such that $(x+q)^{-1}\in A_Y$. Then (A,Y) is a Euclidean pair with respect to the map $\phi:A-\{0\}\to W$ defined by*

$$\phi(x)=\omega\cdot\sum_{\mathfrak{p}\in X}v_{\mathfrak{p}}(x)+\sum_{\mathfrak{p}\in Y-X}v_{\mathfrak{p}}(x).$$

Proof of (6). Let a, $b\in A$, $b\neq 0$, $a\notin A_Y\cdot b$. We have to find $r\in a+Ab$ such that $\phi(r)<\phi(b)$.

First suppose that $v_{\mathfrak{p}}(a)\geqslant v_{\mathfrak{p}}(b)$ for all $\mathfrak{p}\in X$. Then $x=a/b$ belongs to A_X, but not to A_Y, so by the hypothesis of the lemma there exists $q\in A$ such that $(x+q)^{-1}=b/(a+qb)$ belongs to A_Y. Then $b\in A_Y\cdot(a+qb)$, and therefore $A_Y\cdot(a+qb)=A_Ya+A_Yb$. Hence $r=a+qb\in a+Ab$ satisfies

$$v_{\mathfrak{p}}(a+qb)=\min\{v_{\mathfrak{p}}(a),v_{\mathfrak{p}}(b)\}\leqslant v_{\mathfrak{p}}(b)$$

for all $\mathfrak{p}\in Y$, with strict inequality for at least one $\mathfrak{p}$ because $a\notin A_Yb$. It follows that $\phi(r)<\phi(b)$.

Secondly, suppose that $v_{\mathfrak{p}}(a)<v_{\mathfrak{p}}(b)$ for at least one $\mathfrak{p}\in X$. Since X is finite, the approximation theorem for Dedekind domains implies that there exists $r\in A$ with the following properties:

$$v_{\mathfrak{p}}(r-a)\geqslant v_{\mathfrak{p}}(b) \text{ for all } \mathfrak{p}\in Z \text{ with } v_{\mathfrak{p}}(a)<v_{\mathfrak{p}}(b),$$

1 [1, Section 2.4, Proposition 2]

$$v_{\mathfrak{p}}(r)=v_{\mathfrak{p}}(b) \text{ for all } \mathfrak{p}\in X \text{ with } v_{\mathfrak{p}}(a)\geqslant v_{\mathfrak{p}}(b),$$
$$v_{\mathfrak{p}}(r)=v_{\mathfrak{p}}(b) \text{ for all } \mathfrak{p}\in Z-X \text{ with } v_{\mathfrak{p}}(a)\geqslant v_{\mathfrak{p}}(b)>0.$$

Then we have $v_{\mathfrak{p}}(r-a)\geqslant v_{\mathfrak{p}}(b)$ for *all* $\mathfrak{p}\in Z$, so $r\in a+Ab$. Also, $v_{\mathfrak{p}}(r)\leqslant v_{\mathfrak{p}}(b)$ for all $\mathfrak{p}\in X$, with strict inequality if $v_{\mathfrak{p}}(a)<v_{\mathfrak{p}}(b)$, which occurs for at least one $\mathfrak{p}\in X$. Hence $\sum_{\mathfrak{p}\in X} v_{\mathfrak{p}}(r) < \sum_{\mathfrak{p}\in X} v_{\mathfrak{p}}(b)$, and it follows that $\phi(r)<\phi(b)$, as required. This proves (6).

Notice that the lemma implies that (A, Y) is a Euclidean pair if Y is *finite*.

Proof of the theorem. It suffices to show that some for finite subset $X\subset Y$ the condition of the lemma is satisfied. By the remark just made we may assume that Y is infinite. Let $\mathfrak{p}\in Z$, and let $\hat{A}_{\mathfrak{p}}$ be the $\mathfrak{p}$-adic completion of A. Then from

$$(\# Y)^{\mathfrak{a}}<\# A\leqslant \#\hat{A}_{\mathfrak{p}}=(\# A/\mathfrak{p})^{\mathfrak{a}}$$

we see that $\# Y<\# A/\mathfrak{p}$. So $A/\mathfrak{p}$ is infinite for every $\mathfrak{p}\in Z$.

Suppose that there does not exist a finite subset $X\subset Y$ satisfying the condition of (6), *i.e.*:

(7) for every finite $X\subset Y$ there exists $x\in A_X-A_Y$ such that $(x+q)^{-1}\notin A_Y$ for all $q\in A$.

We derive a contradiction.

Using (7) we construct a sequence $(x_m)_{m=0}^{\infty}$ of elements of $K-A_Y$ with the following two properties:

(8) $(x_n+q)^{-1}\notin A_Y$ for all $n\geqslant 0$ and all $q\in A$,

(9) if $X_n=\{\mathfrak{p}\in Y\colon v_{\mathfrak{p}}(x_n)<0\}$ then $X_i\cap X_j=\varnothing$ for all $i, j\geqslant 0$, $i\neq j$.

The construction is by induction on m. Let $m\geqslant 0$, and let x_n, for $0\leqslant n<m$, be such that (8), (9) hold when restricted to i, j, $n<m$. Applying (7) to $X=\bigcup_{n<m} X_n$ we find $x_m\in A_X-A_Y$ such that $(x_m+q)^{-1}\notin A_Y$ for all $q\in A$. For $n<m$ we then have $x_m\in A_X\subset A_{X_n}$, so $X_n\cap X_m=\varnothing$. Hence (8) and (9) hold for i, j, $n\leqslant m$. This concludes the induction step and the construction of the sequence $(x_m)_{m=0}^{\infty}$.

If $(a_m)_{m=0}^{\infty}$ is any sequence of elements of A, then plainly also $(y_m)_{m=0}^{\infty}=(x_m+a_m)_{m=0}^{\infty}$ satisfies (8) and (9), with $x_.$ replaced by $y_.$. We claim that for a suitable choice of $(a_m)_{m=0}^{\infty}$ the sequence $(y_m)_{m=0}^{\infty}$ has the following additional property:

(10) there is no $\mathfrak{p}\in Y$ such that there exist i, j, k with

$$v_{\mathfrak{p}}(y_i-y_j)>0,\ v_{\mathfrak{p}}(y_j-y_k)>0,\ i<j<k.$$

The proof is again by induction. Let $m\geqslant 0$, and let $a_n\in A$, for $n<m$, be such that (10) holds when restricted to $k<m$. The only $\mathfrak{p}\in Y$ which can possibly violate (10), with $k=m$, are those for which $v_{\mathfrak{p}}(y_i-y_j)>0$ for certain i, j with $i<j<m$. There are only finitely many such $\mathfrak{p}$, since $y_i=y_j$ would imply that $X_i=X_j$, so $X_i=\varnothing$ by (9), contradicting that $x_i\notin A_Y$. Notice that $v_{\mathfrak{p}}(y_i-y_j)>0$, with $i<j<m$, implies that $\mathfrak{p}\notin X_i$ and $\mathfrak{p}\notin X_j$. If $\mathfrak{p}\in X_m$, then regardless of the choice of a_m we have $v_{\mathfrak{p}}(y_j-y_m)<0$. If $\mathfrak{p}\notin X_m$, then we have $v_{\mathfrak{p}}(y_j-y_m)=0$ provided that

$$a_m\not\equiv y_j-x_m \bmod \mathfrak{p}$$

(in the local ring at $\mathfrak{p}$). Hence, for (10) to be valid with $k=m$, it suffices that a_m avoids a finite set of residue classes modulo each of a finite number of prime ideals of A. Since $A/\mathfrak{p}$ is infinite for all $\mathfrak{p}\in Z$, the approximation theorem guarantees the existence of an element $a_m\in A$ satisfying these conditions. This completes our inductive proof of (10). [1]

From (8), (9) (with y for x) and (10) we derive a contradiction. Fix $q \in A$. Then for each $n \geqslant 0$ there exists $\mathfrak{p}_n \in Y$ with $v_{\mathfrak{p}_n}(y_n + q) > 0$, by (8). If $\mathfrak{p}_i = \mathfrak{p}_j = \mathfrak{p}_k$ for $i < j < k$, then with $\mathfrak{p} = \mathfrak{p}_i$ we obtain a contradiction to (10). Hence each $\mathfrak{p} \in Y$ occurs at most twice as $\mathfrak{p}_n$, and the map f_q: $\{0, 1, 2, \ldots\} \to Y$ defined by $f_q(n) = \mathfrak{p}_n$ has infinite image.

The number of maps $\{0, 1, 2, \ldots\} \to Y$ is $(\# Y)^{\alpha}$, so from $\# A > (\# Y)^{\alpha}$ it follows that there exist $q \neq r$ in A with $f_q = f_r$. For $\mathfrak{p} = f_q(n)$ we then have $v_{\mathfrak{p}}(y_n + q) > 0$, $v_{\mathfrak{p}}(y_n + r) > 0$, and therefore

$$v_{\mathfrak{p}}(q - r) > 0 \text{ for all } \mathfrak{p} \text{ in the image of } f_q.$$

But f_q has infinite image, so it follows that $q - r = 0$, a contradiction.

This proves the theorem.

(11) Corollary. *Let A be a Dedekind domain, and suppose that the set Z of non-zero prime ideals of A satisfies $\# A > (\# Z)^{\alpha}$. Then A is Euclidean.*

This follows from (5), with $Y = Z$.

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