53

FRONT PROPAGATION INTO UNSTABLE STATES: SOME RECENT DEVELOPMENTS AND SURPRISES

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ABSTRACT

I review the differences and similarities between the marginal stability approach to front propagation into unstable states and the "pinch point" analysis for the space-time evolution of perturbations developed in plasma physics. I then briefly discuss the following developments and surprises: (i) the resolution of a discrepancy between the theory and experiments on Taylor vortex fronts; *(ii)* some new results for the regime where front propagation is dominated by nonlinear effects (nonlinear marginal stability regime); *(iii)* ongoing work on fronts and pulses in the complex Ginzburg-Landau equation.

INTRODUCTION

In the research communities represented here at this workshop on "Nonlinear Evolution of Spatio-Temporal structures in Dissipative Continuous Systems" we have in recent years seen a growing interest in front propagation into unstable states (see e.g. Refs. 1,2 and references therein). At the same time, especially due to the shift in attention from the Rayleigh-Bénard instability in a one component fluid to the Rayleigh-Bénard instability in binary mixtures, the awareness of the importance of the distinction between a convective instability and an absolute instability has become appreciated.[3] In a convectively unstable state, a perturbation grows but at the same time is convected away; the effect of the convection is strong enough that the perturbations are found to decay when viewed at a fixed position. At an absolute instability, on the other hand, the instability is strong enough that perturbations do grow when viewed at a fixed position. When a system of finite size exhibits a convective instability, the long term dynamics depend on whether perturbations have a chance to grow sufficiently during the time over which they are convected away from one side of the system to the other, and on the boundary conditions. As a result, the final state often depends on the size of the system. A nice example of these effects within the context of the Rayleigh-Bénard instability is given by the linear behavior as well as the nonlinear "sloshing" states in binary fluid mixtures.[3–5]

The term front propagation is often used to refer to describe situations in which the dynamical evolution of a spatially extended system that starts out in an unstable state, is governed mainly by the propagation of well defined spatially separated fronts or interfaces connecting the unstable region with a region in which the system is in some *nonlinear* (stable) state. Clearly, the dynamical properties of such fronts – e.g. their speed of propagation – determine to a large extent the qualitative behavior of a convectively unstable finite system like those found in binary fluid mixtures.[4]

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Convective instabilities abound in plasma physics[6] and in fluid dynamics[7] (essentially in all cases where a fluid flow past a fixed body becomes unstable). So it should come as no surprise that the *linear* spatio-temporal behavior at convective instabilities has been studied for over thirty years in these fields, and that the essential results are summarized in a standard book like the "Physical Kinetics" volume in the Landau-Lifshitz Course on Theoretical Physics.[8] Nevertheless, the relation between this work and that on front propagation has often been overlooked. Presumably, this is due to the fact that the theoretical work in plasmas and fluid dynamics is normally limited from the start to the linear behavior of perturbations, whereas originally most of the work on moving fronts concentrated on the development of fully nonlinear front solutions right away. That nevertheless some of the essential results of these rigorous approaches could be reformulated in terms of concepts related to the linear stability properties of the equations was recognized, to my knowledge, for the first time by Dee and Langer[9] and by Ben-Jacob et al.[10] However, the the similarity of some of their ideas with the tools already developed to determine the convective or absolute nature of an instability from an analysis of the linear evolution of perturbations, were not immediately recognized.

Most recently, I have obtained a number of new results for the rate of approach to the asymptotic front speed and for the mechanism of velocity selection in the regime dominated by the nonlinearities.[2] As explained above, in many theoretical discussions the effects of nonlinearities on the convective versus absolute nature of instabilities are not considered, even though our analysis indicates that they can be extremely important. Since these results have just appeared in a detailed paper, I will confine myself here to bringing some of the differences and similarities of the two approaches to the reader's attention. Moreover, I will highlight the recent resolution[11] of the longstanding discrepancy between theory and experiment on front propagation in Taylor-Couette vortex flow.[12] Finally, I will briefly draw attention to some of the implications of and surprises from my recent work as well as ongoing work in collaboration with P. C. Hohenberg.[13]

BRIEF SKETCH OF HISTORICAL DEVELOPMENTS

The first investigations of how a class of initial conditions develops into a well-defined front solution propagating into an unstable state goes back to the work of Kolmogorov et al.[14] and Fisher.[15] These authors studied partial differential equation of the type

$$\frac{\partial \phi}{\partial t} = \frac{\partial^2 \phi}{\partial x^2} + F(\phi) , \qquad \textit{with} \; F(0) = 0, \quad F'(0) = 1 . \tag{1}$$

The simplest example of such an equation is

$$\frac{\partial \phi}{\partial t} = \frac{\partial^2 \phi}{\partial x^2} + \phi - \phi^3 . \tag{2}$$

In the latter equation, the homogeneous state $\phi = 0$ is unstable, whereas the homogeneous states $\phi = \pm 1$ are the absolutely stable states. The above authors were able to show that for a large class of functions like the one corresponding to (2), initial conditions $\phi(x, t=0)$ for which ϕ decays sufficiently rapidly for $x \to \infty$ develop under the dynamics (1) into a moving front solution $\phi(x - vt)$ with a particular value of the velocity. Subsequent investigations concentrated on equations of type (1), and culminated in the seminal work by Aronson and Weinberger.[16] The results of their rigorous mathematical analysis of (1) were popularized and reinterpreted for the physics community by Dee and Langer[9] and Ben-Jacob et al.[10] They noted that for a large class of functions $F(\phi)$ like the one in (2), the mathematical analysis showed that the asymptotic front velocity v_{as} was in fact *independent* of the details of the nonlinear form of $F(\phi)$ and equal in value to a particular velocity v^* that is singled out by the linear dispersion relation corresponding to Eqs. (1) and (2). That is, if one linearizes these equations and substitutes a Fourier mode of the form $e^{ikx-i\omega t}$ to determine the dispersion relation $\omega(k)$, one observes that the rigorous results for a large class of functions $F(\phi)$ amount to the statement that $v_{as}=v^*$, with v^* given by the solution ω^*, k^* of

$$v^* = \frac{\mathrm{Im}\omega}{\mathrm{Im}k} = \frac{\partial \mathrm{Im}\omega}{\partial \mathrm{Im}k}, \qquad \frac{\partial \mathrm{Im}\omega}{\partial \mathrm{Re}k} = 0 . \tag{3}$$

Stability considerations showed[9,10] that for equations of the type (1), v^* defined by (3) can interpreted as the velocity of the front at which front profiles are *marginally stable*. Since v^* is completely determined by the linear dynamics of small perturbations around the unstable state $\phi=0$, I refer to v^*, ω^* and k^* as the linear marginal stability values.

From the work of Aronson and Weinberger,[16] it is known that only for a certain class of functions $F(\phi)$ the front speed approaches v^* asymptotically. It is also possible to have $v_{as}=v^\dagger > v^*$ in (1). In such cases, Langer and coworkers[9,10] also showed that the point where $v=v^\dagger$ is the point where the stability of the uniformly moving profiles $\phi(x-vt)$ changes, in that front profiles with velocity $v>v^\dagger$ are stable while those with $v<v^\dagger$ are unstable. However, since in this case the stability can not be inferred from the linearized dynamical equation, but depends strongly on the whole nonlinear behavior of the profiles $\phi(x-vt)$, we refer to this case as nonlinear marginal stability.

Numerical studies demonstrated that the asymptotic speed of nonlinear fronts in several more complicated equations that do not admit uniformly translating front solutions, is in fact correctly predicted by the *linear* marginal stability value (3). This led me to reformulate some of the marginal stability ideas.[1,2] This reformulation, based on an analysis of the dynamics in the "leading edge" where perturbation are small enough that their dynamics is essentially governed by the linearized equations, not only leads to a better understanding of the physical mechanism that drives the front velocity either to the linear marginal stability value v^* or to the nonlinear marginal stability value $v^\dagger$, but also leads to explicit predictions for the rate of approach of the front velocity to v^*. We will give examples of this in the next sections.

As the above sketch indicates, the above line of research started with a fully nonlinear analysis of a particular class of relatively simple partial differential equations. From the very first investigations on, it was clear that the propagation of fronts into unstable states can depend critically on the nonlinearities in the dynamical equation. However, the recent realization that the linear marginal stability prediction for v^* in terms of the linear dispersion relation $\omega(k)$ appears to be correct for a large class of equations, has focussed attention on the fact that to a large extent the dynamical selection takes place in the "leading edge" governed by the linearized equations.

The line of research that developed independently since the late fifties in plasma physics[6,8] and fluid dynamics started almost at the other end: here the attention was confined right away to the linear dynamical evolution of perturbations, with the advantage that the arguments were general enough that no specific assumption concerning the specific form of the equations needed to be made. On the other hand, even to date the effect of nonlinearities is hardly ever discussed explicitly in these approaches, even though some of the most interesting applications are to cases in which the perturbations grow sufficiently large that nonlinear dynamical patterns, e.g. vortices,[17] are generated. Moreover, the argument is commonly formulated in such a way that it only applies in the limit $t\to\infty$, x fixed. When perturbations around an absolutely unstable state are investigated, this is not the proper limit: to make contact with the work on front propagation, we should then analyze the equation in a frame moving with speed v'. This speed should be chosen self-consistently so that it agrees with the front speed predicted by the analysis, since only then we stay within the front region with an analysis based on keeping the co-moving coordinate $x'=x-vt$ fixed. In other words, if the analysis predicts the selection of a mode $e^{-i\omega t+ikx}$ in the lab frame, we need to choose

$$v' = \frac{\text{Im}\omega}{\text{Im}k} \,. \tag{4}$$

Consider now a system whose reference state is spatially homogeneous in the x-direction. If we linearize the dynamical equation about this state then we get upon Fourier transformation in space-time an expression for the Green's function $G(x', t)$ in the comoving frame $x' = x - v't$ of the form[6]

$$G(x', t) = \int_{C_\omega} d\omega' \int_{C_k} dk' \, \frac{e^{-i\omega' t + ik'x'}}{D(\omega', k')} \,. \tag{5}$$

Here the contour C_ω in the ω' plane is dictated by causality considerations, while the k' integration is along the real k' axis. The zeroes of $D(\omega', k')$ determine the dispersion relation $\omega'(k')$ which is related to the dispersion relation $\omega(k')$ in the fixed frame according to

$$\omega'(k') = \omega(k') - v'k' \,. \tag{6}$$

To extract the long-time behavior of the Green's function (5), we use the fact that the integration contours in (5) can be deformed continuously in an suitable manner. For an arbitrary fixed complex value of ω', $D(\omega', k')$ will have a number of zeroes in the complex k' plane. This situation is sketched in the rightmost column of Fig. 1. When the contour C_ω is deformed, the poles in the k' plane move, but we can deform the contour C_k continuously around the poles without changing the value of the integral *until* two poles "pinch off" the k' contour – see Fig. 1. As a result, the asymptotic $t \to \infty$ behavior of (5) is determined by points in the complex k' plane where two poles arrive from opposite sides of the real axis and pinch off the C_k contour.

The location of the pinch point where two roots coincide is given by[6,8]

$$\frac{d\omega'(k')}{dk'} = 0 \,. \tag{7}$$

Upon using (6) and the consistency requirement (4), this can be written as

$$\frac{d\omega(k')}{dk'} = v' = \frac{\text{Im}\omega(k')}{\text{Im}k'} \,. \tag{8}$$

From the real and imaginary part of this equation one recovers the linear marginal stability equations (3). Hence, *provided* one analyzes the evolution of perturbations in a comoving frame, the pinch point coincides with the linear marginal stability point, and $v' = v^*$.

An advantage of the pinch point analysis is that it can also be used to study perturbations in the lab frame or in any other frame moving with a nonzero velocity. Such an analysis is, of course, only useful as long as perturbations are still so small that a nonlinear front has not developed yet. Here, we focus on the aspects most relevant for a comparison with front propagation.

Fig. 1 illustrates the various differences between the marginal stability analysis[1,2] and the pinch point analysis.[6-8] *(i)* The former is conveniently formulated in terms of an analysis of the local wavenumber $q \equiv -i\, \partial(ln\phi)/\partial x$. The dynamical equation for q supports not only the intuitive interpretation of the dynamical velocity selection, but can also be used to derive the rate of approach to the asymptotic velocity v^*. The pinch point analysis, on the other hand, is a straightforward asymptotic analysis of the Green's function with the aid of contour integra-

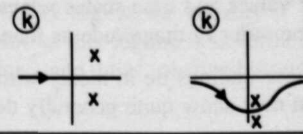
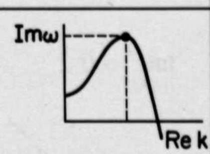
	MARGINAL STABILITY	PINCH POINT
type of analysis	leading edge dynamics $\phi = e^{iu(x,t)}$, $q \equiv u_x$ $q_t = (f' - f_q q') q_x + \mathcal{L} q$	contour integration 
related conditions	maximum growth rate 	integration contour has to be "pinched" off by two poles
initial conditions	can be important	often ignored
importance of nonlinearities	linear marginal stability = ⇕ nonlinear marginal stability	pinch point ? (nonlinearities not considered)

Figure 1. Illustration of the difference between the marginal stability approach and the pinch point analysis.

$$\left.\frac{\partial \mathrm{Im}\omega}{\partial \mathrm{Re}k}\right|_{k=k^\dagger} = 0 \,. \tag{11}$$

The above equation can be used to write e.g. Rek as a function of Imk, and according to our arguments, the solution relevant for the nonlinear marginal stability regime is the one corresponding to the root with the second smallest value of Imk. Together with the usual relation $v^\dagger = \mathrm{Im}\omega/\mathrm{Im}k^\dagger$, our arguments then imply that $v^\dagger$ and Imk should obey a particular relation in the nonlinear marginal stability regime.

We stress that while my explicit calculations of $v^\dagger$ for pattern forming equations that are not of the form (1) all rely on perturbative amplitude expansions, Eq. (11) is a nonperturbative result that can be tested even far into the nonlinear marginal stability regime. I have done so for the following extension of the Swift-Hohenberg equation,[18]

$$\frac{\partial \phi}{\partial t} = -2\,\frac{\partial^2 \phi}{\partial x^2} - \frac{\partial^4 \phi}{\partial x^4} + (\varepsilon - 1)\phi + b\,\phi^2 - \phi^3 \,. \tag{12}$$

Fig. 4 presents some of our data for the velocity of fronts propagating into the unstable state $\phi = 0$ for $\varepsilon = 1/4$. The full line in the figure denotes the branch of solutions of (11) corresponding to the smallest root Imk, and the dashed branch the next smallest one. The minimum of the curve corresponds to the point v^*, Imk^*. The open circles are the data points from our simulations for various values of b as indicated, and the inset shows the measured value of the velocity as a function of b. Clearly, to within our numerical accuracy, the data lie on the dashed branch, and our simulations therefore support the validity of our picture of nonlinear marginal stability for Eq. (12).

We remark that although this was not noted in Ref. 2, I expect my analysis of the nonlinear marginal stability regime to apply only to pattern forming equations that admit a two-parameter family of moving front solutions. Indeed, if this is the case, there is a one-parameter family of front solutions for every fixed value of the velocity v, and therefore one expects there

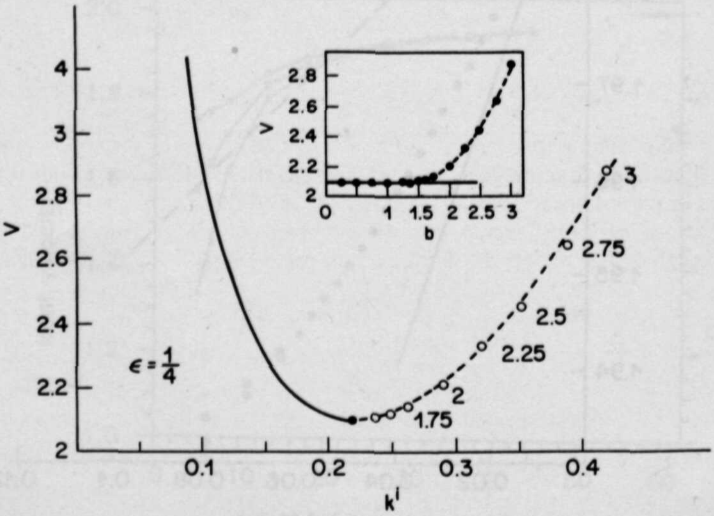


Figure 4. The measured front velocity v versus $k^i = \mathrm{Im}k$ for Eq. (12) with $\varepsilon = 1/4$. Open circles are data points for $b > 1.5$ with the value of b as indicated. The inset shows the velocity as a function of b. For $b \leq 1.5$, we have linear marginal stability, and for $b \geq 1.5$ nonlinear marginal stability.

to be a particular moving front solution that satisfies the condition (11) at $v=v^{\dagger}$. For the Swift-Hohenberg equation, it is indeed known[19] that there is a two-parameter family of moving front solutions, and our picture appears to be consistent. I do not know, however, whether the nonlinear marginal stability analysis can be extended to equations that admit only a one-parameter family of solutions. Further study of this question appears necessary.

COMPLEX GINZBURG-LANDAU EQUATION NEAR A SUBCRITICAL BIFURCATION

Recently, Hohenberg and I[13] have obtained a number of exciting new results for the existence and stability of fronts and pulses in the complex Ginzburg-Landau equation near a critical bifurcation,

$$\frac{\partial A}{\partial t} = (1+c_1)\frac{\partial^2 A}{\partial x^2} + \varepsilon A + (1+c_3)\,|A|^2 A + (-1+c_5)\,|A|^4 A \,. \tag{13}$$

In this investigation, both front propagation into an unstable state $\varepsilon>0$ and front propagation into a metastable state for $\varepsilon<0$ was analyzed. An extremely surprising finding of our work is that an exact analytic front solution of this equation can be found that allows one to compute $v^{\dagger}$ both for $\varepsilon>0$ and for $\varepsilon<0$ analytically. With the aid of this exact solution, a number of new and interesting additional results are obtained: *(i)* For fixed c_1, c_3 and c_5, there usually is a range of values of $\varepsilon<0$ where stable periodic nonlinear pulse solutions with stationary envelope are found. The upper edge of this range is determined by the condition $v^{\dagger}(\varepsilon)=0$. *(ii)* In a co-dimension one subspace of parameter space, we have obtained exact analytic nonlinear pulse solutions. The analytic form we find has subsequently been used by Niemela et al.[20] to successfully fit pulse shapes found in convection experiments. *(iii)* In large regions of the c_1, c_3, c_5 parameter space, nonzero initial conditions never dynamically develop into a front solution that propagates into the $A=0$ state for any $\varepsilon<0$. In this case, stable pulses continue to exist all the way up to $\varepsilon=0$. *(iv)* In this parameter range where pulses exist for all $\varepsilon\to 0^-$, all fronts propagating into the unstable state $A=0$ for $\varepsilon>0$ propagate with the linear marginal stability velocity v^*. This contradicts my earlier speculation that for arbitrarily small but positive ε, one would always expect fronts to propagate with $v^{\dagger}$ rather than v^* near a subcritical bifurcation.[2]

CONCLUSION

The results summarized in the last three section illustrate that important progress is being made concerning front propagation into unstable states. Moreover, as our recent work on the complex Ginzburg-Landau equation indicates, the distinction between front propagation into unstable states and metastable states may be less large than we originally believed. I expect these type of problems to remain a fruitful area for research in the next few years.

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