CONVERGENCE PROPERTIES OF LINEAR RECURRENCE SEQUENCES

by

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This paper provides a survey of the dissertation of the first named author [6]. The thesis deals with recurrence sequences $\{u_n\}_{n=0}^{\infty}$ of complex numbers satisfying

$$(1) \qquad a_k(n)u_{n+k} + a_{k-1}(n)u_{n+k-1} + \ldots + a_0(n)u_n = 0 \quad \text{for } n = 0, 1, 2, \ldots,$$

where the sequences $\{a_k(n)\}, \ldots, \{a_0(n)\}$ satisfy certain regularity conditions as $n \to \infty$. This kind of sequences plays an important role in analysis (the theory of orthogonal polynomials) and in combinatorics. Important applications in number theory can be found in Apéry's proof of the irrationality of $\zeta(3) = \sum_{n=1}^{\infty} n^{-3}$ and in other derivations of irrationality measures (cf. G.V. Chudnovsky [5] p. 344.) In most applications $k = 2$ and the coefficients a_2, a_1, a_0 are polynomials. We shall deal with the asymptotic behaviour of sequences $\{u_n\}$ as $n \to \infty$, in particular the existence of $\lim_{n\to\infty} u_{n+1}/u_n$. At the end we shall give some applications, one of which concerns the solution of a problem posed by Perron. It will appear that there are obvious similarities with the theory of linear differential equations, but also notable differences. The second author thanks several participants of the conference for their helpful comments.

1. Linear recurrences with constant coefficients.

For a better understanding we first recall some results on linear recurrences with constant coefficients. Let $a_0, a_1, \ldots, a_k$ be complex numbers. Suppose that $\{u_n\}_{n=0}^{\infty}$ is a sequence of complex numbers such that

(2) $$a_k u_{n+k} + a_{k-1} u_{n+k-1} + \ldots + a_0 u_n = 0 \quad \text{for } n = 0, 1, 2, \ldots.$$

Without loss of generality we may assume $a_0 a_k \neq 0$. Then the sequence is uniquely determined by any k consecutive values $u_r, u_{r+1}, \ldots, u_{r+k-1}$, and an explicit expression for u_n is given by the following result.

THEOREM 1. *Suppose $\{u_n\}_{n=0}^{\infty}$ satisfies (2). Consider the factorization of its characteristic polynomial*

(3) $$a_k z^k + a_{k-1} z^{k-1} + \ldots + a_0 = a_k \prod_{j=1}^{s} (z - \alpha_j)^{e_j}$$

where $\alpha_1, \ldots, \alpha_s$ are distinct complex numbers. Then

(4) $$u_n = \sum_{j=1}^{s} P_j(n) \alpha_j^n \qquad (n = 0, 1, 2, \ldots)$$

where P_j is a polynomial of degree at most $e_j - 1$ for $j = 1, \ldots, s$. The coefficients of the P_j's are determined by $a_0, a_1, \ldots, a_k$ and any k subsequent values $u_r, u_{r+1}, \ldots, u_{r+k-1}$.

On the other hand, every sequence of the form (4) satisfies the recurrence (2). Thus (2) *has k linearly independent solutions*

(5) $$\{n^{\rho-1} \alpha_j^n\}_{n=0}^{\infty} \qquad (\rho = 1, \ldots, e_j \; ; \; j = 1, \ldots, s)$$

and every solution of (2) is a linear combination of these solutions. We state some further corollaries of Theorem 1. Here and in the sequel we neglect the trivial solution which is constant zero.

(a) *If for some solution $\{u_n\}_{n=0}^{\infty}$ of (2) the limit $\alpha = \lim_{n\to\infty} u_{n+1}/u_n$ exists, then α is a root of the characteristic polynomial (3).* This is clear from (2) and (3).

(b) *If $s = k$, hence $e_1 = e_2 = \ldots = e_k = 1$, and the α_j have distinct absolute values, then $\lim_{n\to\infty} u_{n+1}/u_n$ exists for every solution $\{u_n\}_{n=0}^{\infty}$ of (2).* This is clear from (4).

(c) *For each root* α_j *of the characteristic polynomial (3) there are* e_j *linearly independent solutions of (2) such that* $\lim_{n\to\infty} u_{n+1}/u_n = \alpha_j$. These are given by (5).

In the case of linear recurrences with non-constant coefficients it is, in general, not possible to give an explicit formula for the solutions as in Theorem 1, but under suitable conditions Properties (a) - (c) can still be proved. There are also such recurrences for which (b) and (c) are false. We call k the order, s the rank and $\alpha_1, \ldots, \alpha_s$ the eigenvalues of recurrence (2). By (3) they depend on the coefficients of the characteristic polynomial only.

2. **Linear recurrences with almost-constant coefficients**

Using the notation of Section 1 we now allow a perturbation sequence $\{\varepsilon_\ell(n)\}$ in the coefficient a_ℓ. Let $\{u_n\}_{n=0}^{\infty}$ satisfy

$$(6) \qquad (a_k + \varepsilon_k(n))u_{n+k} + (a_{k-1} + \varepsilon_{k-1}(n))u_{n+k-1} + \ldots + (a_0 + \varepsilon_0(n))u_n = 0,$$

where $\varepsilon_\ell(n) \to 0$ as $n \to \infty$ for $\ell = 0, 1, \ldots, k$. Since $a_0 a_k \neq 0$, we can define N as an integer such that $(a_k + \varepsilon_k(n))(a_0 + \varepsilon_0(n)) \neq 0$ for $n \geq N$. We restrict our attention to $\{u_n\}_{n=N}^{\infty}$. These sequences are well defined. We define characteristic polynomial, eigenvalue α_j, order k and rank s as before. They depend only on the constants $a_k, a_{k-1}, \ldots, a_0$ and not on the perturbation sequences. The following generalizations of Properties $(a) - (c)$ are known. (cf. [12] Ch. 10)

(a1) *If for some solution* $\{u_n\}_{n=0}^{\infty}$ *of (6) the limit* $\alpha = \lim_{n\to\infty} u_{n+1}/u_n$ *exists, then* α *is an eigenvalue of the recurrence (6).* Clear from (6) and (3).

(b1) *If* $e_1 = e_2 = \ldots = e_k = 1$ *and the* α_j *have distinct absolute values, then* $\lim_{n\to\infty} u_{n+1}/u_n$ *exists for every solution* $\{u_n\}_{n=N}^{\infty}$ *of (6).* This was proved by Poincaré [15] in 1885.

(c1) *If* $e_1 = e_2 = \ldots = e_k = 1$ *and the* α_j *have distinct absolute values, then for each eigenvalue* α_j *there is a solution* $\{u_n^{(j)}\}_{n=N}^{\infty}$ *such that* $u_{n+1}^{(j)}/u_n^{(j)} \to \alpha_j$. This was proved by Perron [13] in 1909.

In spite of several results in this direction due to Maté, Nevai and others, [4], [7], [8], [9], [10], [11], a general result analogous to (c) in the case of equal absolute values was not available up to now. The following theorem provides such a result if the absolute values of the perturbations are sufficiently small.

THEOREM 2. (Kooman [6]). *In the above notation put* $E = \max_{j=1,\ldots,s} e_j$.

If

$$\sum_{n=N}^{\infty} n^{E-1}|\varepsilon_\ell(n)| < \infty \quad \text{for } \ell = 0, 1, \ldots, k, \tag{7}$$

then (6) has k linearly independent solutions $\{u_n^{(\rho,j)}\}$ *such that*

$$\lim_{n\to\infty} \frac{u_n^{(\rho,j)}}{n^{\rho-1}\alpha_j^n} = 1 \quad \text{for } \rho = 1, \ldots, e_j; j = 1, \ldots, s. \tag{8}$$

If the coefficients $a_j + \varepsilon_j(n)$ of (6) are all real, then the solutions $\{u_n^{(\rho,j)}\}_{n=N}^{\infty}$ can be chosen to be real for each j with $\alpha_j \in \mathbf{R}$. If $e_1 = e_2 = \ldots = e_k = 1$, then $E = 1$ and Theorem 2 implies Property (c) also in the case of roots of equal absolute values provided that $\sum_{n=N}^{\infty} |\varepsilon_\ell(n)| < \infty$ for $\ell = 0, \ldots, k$. A refinement of Theorem 2 which will be mentioned in the next section implies the following generalization of Property (c).

(c2) *For each eigenvalue of (6) there are* e_j *linearly independent solutions of (6) such that* $\lim_{n\to\infty} u_{n+1}/u_n = \alpha_j$ *provided that* $\sum_{n=N}^{\infty} n^{E_j-1}|\varepsilon_\ell(n)| < \infty \quad (\ell = 0, \ldots, k)$ *where* $E_j = \max_{|\alpha_h|=|\alpha_j|} e_h$.

These e_j solutions can be chosen in such a way that they behave asymptotically as in (8).

The following examples suggest that (7) is a natural condition for Theorem 2.

1) (cf. [6] Proposition 5.3) Consider the recurrence

$$u_{n+2} - 2u_{n+1} + (1 + \frac{1}{n^2})u_n = 0.$$

The characteristic equation is $(z-1)^2$ and $E = 2$. Condition (7) is not fulfilled. The recurrence has no real solution $\{u_n\}_{n=N}^{\infty}$ such that $u_{n+1}/u_n \to 1$ as $n \to \infty$ (or converges to some other limit). As we shall see in Section 4 the recurrence has two linearly independent solutions $\{u_n^{(1)}\}_{n=N}^{\infty}$ and $\{u_n^{(2)}\}_{n=N}^{\infty}$ with

$$\lim_{n\to\infty}\frac{u_n^{(1)}}{n^{\alpha}}=1,\qquad \lim_{n\to\infty}\frac{u_n^{(2)}}{n^{\beta}}=1\quad\text{where }\alpha,\beta=\frac{1}{2}\pm\frac{1}{2}i\sqrt{3}$$

and of course every solution is a linear combination of these solutions.

2) (cf. [6] p.88) Consider the recurrence

$$u_{n+2}-(1+\frac{(-1)^n}{n})u_n=0.$$

The characteristic equation is z^2-1 and $E=1$. We have

$u_{n+2}=(1+1/n)u_n$ for n even, hence $|u_{2n}|\to\infty$ as $n\to\infty$, but

$u_{n+2}=(1-1/n)u_n$ for n odd, hence $u_{2n+1}\to 0$ as $n\to\infty$.

Thus $\{u_{n+1}/u_n\}$ does not converge. Again $\sum_{n=N}^{\infty} n^{E-1}|\varepsilon_2(n)|=\sum_{n=N}^{\infty}\frac{1}{n}$.

It can also occur that all $\{u_{n+1}/u_n\}$ converge to one eigenvalue and none to the other.

3) (cf. [6] p.88) Consider the recurrence

$$p_n u_{n+2}+(p_{n+1}-p_n)u_{n+1}-p_{n+1}u_n=0\quad\text{with } p_n=1+\frac{(-1)^n}{n}. \tag{9}$$

The characteristic polynomial is z^2-1, hence $\alpha_1=1,\alpha_2=-1$ and $E=1$. It is easy to check that every solution of (9) is of the form

$$u_n=\lambda\sum_{\nu=1}^{n-1}(-1)^{\nu}p_{\nu}+\mu\qquad(\lambda,\mu\in\mathbf{C}).$$

Since $\sum_{\nu=0}^{n-1}(-1)^{\nu}p_{\nu}\to\infty$ as $n\to\infty$ we obtain

$$\frac{u_{n+1}}{u_n}=1+\frac{(-1)^n p_n\lambda}{\lambda\sum_{\nu=1}^{n-1}(-1)^{\nu}p_{\nu}+\mu}\to 1\quad\text{for all }\lambda\text{ and }\mu\text{ with }|\lambda|+|\mu|\neq 0.$$

Again condition (7) is not satisfied since $\sum_{n=N}^{\infty}\frac{1}{n}$ diverges.

After the examples 1) - 3) it will be obvious that if (7) is not satisfied, the signs (or in the complex case the arguments) of the $\varepsilon_{\ell}(n)$ will have to be taken into account. We return to this question in Section 4.

3. **On the proof of Theorem 2.**

As we have seen Theorem 2 seems to give a rather natural condition and is anyway not far from the best possible. The proof consists of two parts. First the roots of the characteristic polynomial are separated according to their absolute values, using the following theorem.

<u>THEOREM 3.</u> *Consider the recurrence relation*

$$(a_k + \varepsilon_k(n))u_{n+k} + \ldots + (a_0 + \varepsilon_0(n))u_n = 0 \tag{10}$$

with $a_k \neq 0, (a_k + \varepsilon_k(n))(a_0 + \varepsilon_0(n)) \neq 0$ for $n \geq N$ *and* $\varepsilon_\ell(n) \to 0$ *as* $n \to \infty$ *for* $\ell = 0, 1, \ldots, k$. *Suppose (10) has eigenvalues* α_i *with multiplicities* e_i. *Put for some* j

$$m = \sum_{|\alpha_i|=|\alpha_j|} e_i.$$

Then there exist m *linearly independent solutions* $\{u_n^{(1)}\}_{n\geq N}, \ldots, \{u_n^{(m)}\}_{n\geq N}$ *of (10) and a linear recurrence of order* m

$$u_{n+m} + (b_{m-1} + \delta_{m-1}(n))u_{n+m-1} + \ldots + (b_0 + \delta_0(n))u_n = 0 \quad (n \geq N) \tag{11}$$

with $b_0 + \delta_0(n) \neq 0$ *for* $n \geq N$ *and* $\delta_\ell(n) \to 0$ *as* $n \to \infty$ *for* $\ell = 0, \ldots, m-1$ *such that* $\{u_n^{(1)}\}_{n\geq N}, \ldots, \{u_n^{(m)}\}_{n\geq N}$ *constitute a basis of solutions of (11) and that (11) has characteristic polynomial*

$$Q(z) = z^m + b_{m-1}z^{m-1} + \ldots + b_0 = \prod_{|\alpha_i|=|\alpha_j|} (z - \alpha_i)^{e_i}.$$

Moreover, if the coefficients of (10) are all real, then the coefficients of (11) can all be taken real as well.

Note that the case $m = 1$ implies the Poincaré-Perron Theorem (b1) - (c1) and even more. Whereas the latter theorem requires that all zeros of the characteristic polynomial have distinct moduli, Theorem 3 ensures that for each zero α_j with $e_j = 1$ and $|\alpha_i| \neq |\alpha_j|$ for $i \neq j$ there exists a solution $\{u_n\}_{n \geq N}$ of *(10)* such that $u_{n+1}/u_n \to \alpha_j$ as $n \to \infty$.

It follows from the proof of Theorem 3 that the order of growth of the δ_ℓ 's in *(11)* is not larger than that of the ε_ℓ 's in *(10)*.

In the second part of the proof of Theorem 2 an iteration method is used in order to construct a solution of (6) that is very close to a solution of the corresponding unperturbed recurrence (2).
Thus, let $\{u_n^{(0)}\}_{n \geq N}$ with $u_n^{(0)} = n^{\rho-1}\alpha^n$ be such that

$$(12) \qquad a_k u_{n+k}^{(0)} + \ldots + a_0 u_n^{(0)} = 0 \qquad (n \geq N).$$

For $i = 1, 2, 3, \ldots$ we construct sequences $\{u_n^{(i)}\}_{n \geq N}$ such that

$$(13) \qquad a_k u_{n+k}^{(i)} + \ldots + a_0 u_n^{(i)} = -(\varepsilon_k(n) u_{n+k}^{(i-1)} + \ldots + \varepsilon_0(n) u_n^{(i-1)})$$

and such that the numbers $|u_n^{(i)}|$ are very small. Condition (7) ensures that $u_n^{(i)}$ can be chosen in such a way that $\sum_{i=1}^{\infty} |u_n^{(i)}|/u_n^{(0)} \to 0$ as $n \to \infty$. Hence, if we define $v_n = \sum_{i=0}^{\infty} u_n^{(i)} (n \geq N)$, then $\{v_n\}_{n \geq N}$ is a solution of (6) and

$$\lim_{n \to \infty} \frac{v_n}{u_n^{(0)}} = 1.$$

Using the estimation for the numbers $u_n^{(i)}$ it is also possible to indicate the speed of convergence of the solutions. Roughly speaking, the loss is at most n^E. Using the separation of eigenvalues as given in Theorem 3 we obtain the following refinement of Theorem 2.

<u>THEOREM 4.</u> *Consider the recurrence relation (10) with* $a_0 a_k \neq 0$, $(a_k + \varepsilon_k(n))(a_0 + \varepsilon_0(n)) \neq 0$ *for* $n \geq N$ *and* $\varepsilon_\ell(n) \to 0$ *as* $n \to \infty$ *for* $\ell = 0, 1, \ldots, k$. *Let* α_j *be a zero of the characteristic polynomial with multiplicity* e_j. *Put* $E_j = \max_{|\alpha_i| = |\alpha_j|} e_i$.

Let $h_j \geq E_j$ be such that

$$\sum_{n=N}^{\infty} n^{h_j-1}|\varepsilon_\ell(n)| < \infty \qquad \text{for } \ell = 0, 1, \ldots, k.$$

Then there exist e_j linearly independent solutions $\{u_n^{(\rho,j)}\}_{n=N}^{\infty}$ such that

$$u_n^{(\rho,j)} = n^{\rho-1}\alpha_j^n(1 + o(n^{E_j-h_j})) \quad \text{for } \rho = 0, 1, \ldots, e_j - 1.$$

4. **Second-order recurrences with rational functions as coefficients.**

The asymptotic behaviour of arbitrary recurrence sequences $\{u_n\}_{n=0}^{\infty}$ satisfying (1) is quite complicated. In Chapters 5 and 6 of his thesis Kooman [6] analyzed the case $k = 2$ which often occurs in applications. Note that this is the first non-trivial case, since the solutions of first order recurrences can be expressed as sums of products of the coefficients. On the other hand, the second-order recurrences provide the essential difficulties in a nutshell. We distinguish between

I the eigenvalues have distinct absolute values,

II the eigenvalues are equal,

III the eigenvalues have the same absolute values, but are not equal.

The method of treatment in each case is to reduce the second-order recurrence in u_n to a first-order recurrence in some expression of u_{n+1} and u_n, for example $(u_{n+1} - u_n)/u_n$. In contrast to Kooman's thesis we shall restrict our attention here to the case that the coefficients are elements of $\mathbf{R}(X)$. We define the degree $\deg r$ of a rational function $r(X)$ to be d if $\lim_{x\to\infty} r(x)x^{-d}$ has a finite, non-zero limit. Furthermore we put $d = -\infty$ if $r = 0$.

Let $p(X), q(X)$ be rational functions in X with real coefficients. Consider the recurrence relation

$$(14) \qquad u_{n+2} - p(n)u_{n+1} - q(n)u_n = 0 \quad (n = N, N+1, \ldots).$$

If $p(n) = 0$ for all n, then it is easy to calculate the solutions $\{u_n\}$ by separating terms with even and with odd index. Hence we may assume that $p(n)$ and $q(n)$ exist and $p(n)q(n) \neq 0$ for $n \geq N$. We shall transform (14) into a recurrence with only one free coefficient. Put

$$v_n = u_n \prod_{k=N}^{n-2} \frac{2}{p(k)} \qquad (n = N+1, N+2, \ldots).$$

Then $\{v_n\}_{n=N+2}^{\infty}$ satisfies the recurrence

$$v_{n+2} - 2v_{n+1} - \frac{4q(n)}{p(n)p(n-1)} v_n = 0 \quad (n = N+1, N+2, \ldots). \tag{15}$$

Put

$$b = \lim_{n\to\infty} -\frac{4q(n)}{p(n)p(n-1)}.$$

If $b = -\infty$, then we are in case III with real eigenvalues of opposite signs,

if $-\infty < b < 1$, then we are in case I with two real eigenvalues,

if $b = 1$, then we are in case II,

if $1 < b \leq \infty$, then we are in case III with a pair of conjugate non-real eigenvalues.

We treat each case separately:

Case I. $(-\infty < b < 1)$ (cf. [6] Theorem 5.1)

Put $z^2 - 2z + b = (z-\alpha)(z-\beta)$. Then $\alpha, \beta \in \mathbf{R}$ with $|\alpha| \neq |\beta|$. Without loss of generality we may assume $\alpha > |\beta|$. By the theorem of Poincaré-Perron (properties (b1) - (c1)), the recurrence (15) has linearly independent solutions $\{v_n^{(1)}\}$ and $\{v_n^{(2)}\}$ such that

$$\lim_{n\to\infty} \frac{v_{n+1}^{(1)}}{v_n^{(1)}} = \alpha, \quad \lim_{n\to\infty} \frac{v_{n+1}^{(2)}}{v_n^{(2)}} = \beta. \tag{16}$$

If

$$\frac{4q(n)}{p(n)p(n-1)} + b = 0(\frac{1}{n^2}) \tag{17}$$

we even obtain from Theorem 2 that

$$\lim_{n\to\infty}\frac{v_n^{(1)}}{\alpha^n}=1 \text{ and } \lim_{n\to\infty}\frac{v_n^{(2)}}{\beta^n}=1 \quad \text{provided that } \beta\neq 0.$$

It can be shown that even if (17) is not satisfied, there exist real numbers γ, δ such that

$$\lim_{n\to\infty}\frac{v_n^{(1)}}{n^\gamma\alpha^n}=1 \text{ and } \lim_{n\to\infty}\frac{v_n^{(2)}}{n^\delta\beta^n}=1 \quad \text{provided that } \beta\neq 0.$$

If $\beta=0$, then $\{v_n^{(2)}\}$ can be chosen such that

$$\frac{v_{n+1}^{(2)}}{v_n^{(2)}}=0(\frac{1}{n}).$$

It follows from (16) and $|\beta|<\alpha$ that for the corresponding solutions $\{u_n^{(1)}\}$ and $\{u_n^{(2)}\}$ of (14) we have

$$\lim_{n\to\infty}\frac{u_n^{(2)}}{u_n^{(1)}}=0.$$

The sequence $\{u_n^{(2)}\}$ is a solution of (14) with an exceptionally small rate of growth. Such exceptional solutions play a role in some irrationality proofs, such as for $\zeta(3)$. (cf. Application 5.2)

Case II. $(b=1)$

Put

$$C(n)=1+\frac{4q(n)}{p(n)p(n-1)}.$$

Then $\lim_{n\to\infty}C_n=0$. We first consider the case that $\deg C\leq -2$. Then we distinguish between two cases IIa and IIb. Subsequently we distinguish between two cases IIc and IId when $\deg C=-1$.

Subcase IIa. Suppose $b=1, \lim_{n\to\infty}n^2C(n)=\gamma>-\frac{1}{4}$ with $\gamma\in\mathbf{R}$ (cf. [6] Thm. 5.4). Let ξ be the root of $x^2-x-\gamma$ with Re $\xi>\frac{1}{2}$. Using that

$$w_n=n(\frac{v_{n+1}}{v_n}-1)-\xi$$

satisfies

$$w_{n+1} = \frac{(1+(1-\xi)/n)w_n + (n+1)C(n) - \gamma/n}{w_n/n + 1 + \xi/n} \tag{18}$$

it can be shown that $w_n \to 0$ as $n \to \infty$ for some solution $\{w_n\}$ of (18). It follows that (15) has real solutions $\{v_n^{(1)}\}$ and $\{v_n^{(2)}\}$ such that

$$\lim_{n\to\infty} \frac{v_n^{(1)}}{n^{\xi}} = \lim_{n\to\infty} \frac{v_n^{(2)}}{n^{1-\xi}} = 1.$$

Hence for every non-trivial solution $\{u_n\}$ of (14)

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = 1. \tag{19}$$

<u>Subcase IIb.</u> Suppose $b = 1, \lim_{n\to\infty} n^2 C(n) = \gamma \le -\frac{1}{4}$ with $\gamma \in \mathbf{R}$ (cf. [6] Thm. 5.8).

Let ξ and $\overline{\xi}$ be the roots of $X^2 - X - \gamma$. By considering the recurrence

$$w_{n+1} = \frac{w_n - d_n}{1 + w_n/n} \qquad \text{where } d_n = (n+1)C(n) - \gamma/n,$$

it can be shown that there exist solutions $\{v_n^{(1)}\}$ and $\{v_n^{(2)}\}$ of (15) with

$$\lim_{n\to\infty} \frac{v_n^{(1)}}{n^{\xi}} = \lim_{n\to\infty} \frac{v_n^{(2)}}{n^{\overline{\xi}}} = 1 \qquad \text{if } \gamma < -\frac{1}{4} \quad \text{and}$$

$$\lim_{n\to\infty} \frac{v_n^{(1)}}{n^{\frac{1}{2}} \log n} = \lim_{n\to\infty} \frac{v_n^{(2)}}{n^{\frac{1}{2}}} = 1 \qquad \text{if } \gamma = -\frac{1}{4}.$$

<u>Subcase IIc.</u> Suppose $b = 1$ and $\lim_{n\to\infty} nC(n) > 0$ (cf. [6] Thm. 5.10).

By proving that the recurrence

$$w_{n+1} = \frac{w_n(1 - \sqrt{C(n+1)}) + (1 + \sqrt{C(n)})(1 - \sqrt{C(n+1)/C(n)})}{(1 + (1+w_n)\sqrt{C(n)})\sqrt{C(n+1)/C(n)}}$$

has some solution $\{w_n^{(0)}\}$ such that $\lim_{n\to\infty} w_n^{(0)} = 0$, it can be shown that (15) has solutions $\{v_n^{(1)}\}$ and $\{v_n^{(2)}\}$ such that

$$\lim_{n\to\infty} \sqrt{C(n)^{-1}}\left(\frac{v_{n+1}^{(1)}}{v_n^{(1)}} - 1\right) = 1, \quad \lim_{n\to\infty} \sqrt{C(n)^{-1}}\left(\frac{v_{n+1}^{(2)}}{v_n^{(2)}} - 1\right) = -1.$$

It follows that (19) holds for all non-trivial solutions of (14). Note that for the solutions $\{u_n^{(1)}\}, \{u_n^{(2)}\}$ corresponding to $\{v_n^{(1)}\}, \{v_n^{(2)}\}$ we have

$$\lim_{n\to\infty} \frac{u_n^{(2)}}{u_n^{(1)}} = 0.$$

<u>Subcase IId.</u> Suppose $b = 1$ and $\lim_{n\to\infty} nC(n) < 0$ (cf. [6] Thm. 5.12).

By proving that the recurrence relation

$$w_{n+1} = \frac{w_n + e_n r_n}{w_n r_n + e_n} \quad \text{where } r_n = \frac{\sqrt{-C(n)} - \sqrt{-C(n+1)}}{\sqrt{-C(n)} + \sqrt{-C(n+1)}}, e_n = \frac{i - \sqrt{-C(n)}}{i + \sqrt{-C(n)}}$$

has some solution which tends to 0 as $n \to \infty$, it can be shown that (15) has solutions $\{v_n^{(1)}\}$ and $\{v_n^{(2)}\}$ such that

$$\lim_{n\to\infty} \frac{1}{\sqrt{-C(n)}}\left(\frac{v_{n+1}^{(1)}}{v_n^{(1)}} - 1\right) = i \quad \text{and} \quad \lim_{n\to\infty} \frac{1}{\sqrt{-C(n)}}\left(\frac{v_{n+1}^{(2)}}{v_n^{(2)}} - 1\right) = -i.$$

<u>Case III</u> ($b = -\infty$ or $1 < b \le \infty$).

By multiplying u_n by a suitable function of n we can transform (14) into a recurrence relation with almost-constant coefficients and characteristic polynomial $(z - \alpha)(z - \beta)$, where

$$\alpha = 1, \beta = -1 \qquad \text{if } b = -\infty \tag{20}$$

and

$$|\alpha| = |\beta| = 1, \qquad \alpha = \overline{\beta}, \alpha \ne \beta \qquad \text{if } 1 < b \le \infty. \tag{21}$$

The subcases IIIa and IIIb correspond to $b = -\infty$ and subcase IIIc to $1 < b \le \infty$.

<u>Subcase IIIa.</u> $\deg p \le -2, \deg(q(X) - 1) < 0$ (cf. [6] Cor. 6.1).

The transformed recurrence has real solutions $\{u_n^{(1)}\}_{n\ge N}, \{u_n^{(2)}\}_{n\ge N}$ such that

$$\lim_{n\to\infty} \frac{u_{n+1}^{(1)}}{u_n^{(1)}} = 1, \lim_{n\to\infty} \frac{u_{n+1}^{(2)}}{u_n^{(2)}} = -1 \quad \text{and} \quad \lim_{n\to\infty} \left|\frac{u_n^{(2)}}{u_n^{(1)}}\right| = 1.$$

Subcase IIIb. $\deg p = -1, \deg(q(X) - 1) < 0$ (cf. [6] Thm. 6.4).

The transformed recurrence has real solutions $\{u_n^{(1)}\}_{n \geq N}, \{u_n^{(2)}\}_{n \geq N}$ such that

$$\pm \lim_{n \to \infty} \frac{u_{n+1}^{(1)}}{u_n^{(1)}} = 1, \quad \pm \lim_{n \to \infty} \frac{u_{n+1}^{(2)}}{u_n^{(2)}} = -1 \text{ and } \lim_{n \to \infty} \frac{u_n^{(2)}}{u_n^{(1)}} = 0.$$

The chosen sign should be the sign of $p(n)$ as $n \to \infty$.

Subcase IIIc. Suppose $1 < b \leq \infty$ (cf. [6] Thm. 6.2). Let eigenvalues α, β be determined by (21). The transformed recurrence has solutions $\{u_n^{(1)}\}_{n \geq N}, \{u_n^{(2)}\}_{n \geq N}$ such that

$$\lim_{n \to \infty} \frac{u_{n+1}^{(1)}}{u_n^{(1)}} = \alpha, \quad \lim_{n \to \infty} \frac{u_{n+1}^{(2)}}{u_n^{(2)}} = \beta, \quad u_n^{(2)} = \overline{u_n^{(1)}} \quad \text{for all } n.$$

For real solutions $\{u_n\}$, $\lim_{n \to \infty} \frac{u_{n+1}}{u_n}$ does not exist (cf. (a) in §1).

From the above results we obtain the following extension of the Poincaré-Perron Theorem:

THEOREM 5. *Let $p, q \in \mathbf{R}(X), q \neq 0$. Suppose the recurrence relation*

$$(14) \qquad u_{n+2} - p(n)u_{n+1} - q(n)u_n = 0 \qquad (n = N, N+1, \ldots)$$

has characteristic polynomial $(z - \alpha)(z - \beta)$ with $\alpha, \beta \in \mathbf{C}$. Then there exist two linearly independent solutions $\{u_n^{(1)}\}_{n \geq N}$ and $\{u_n^{(2)}\}_{n \geq N}$ of (14) such that

$$\lim_{n \to \infty} \frac{u_{n+1}^{(1)}}{u_n^{(1)}} = \alpha, \quad \lim_{n \to \infty} \frac{u_{n+1}^{(2)}}{u_n^{(2)}} = \beta.$$

5. Applications.

5A. Continued fractions.

Let $p, q \in \mathbf{R}(X)$, $p, q \neq 0$. We consider the continued fraction

$$\frac{q(1)|}{|p(1)} + \frac{q(2)|}{|p(2)} + \frac{q(3)|}{|p(3)} + \ldots \quad . \tag{22}$$

A natural question is whether the limit exists. We say that the continued fraction converges in the broad sense if the limit

$$\lim_{n\to\infty} \frac{q(1)|}{|p(1)} + \frac{q(2)|}{|p(2)} + \ldots + \frac{q(n)|}{|p(n)}$$

exists or if

$$\lim_{n\to\infty} p(1) + \frac{q(2)|}{|p(2)} + \ldots + \frac{q(n)|}{|p(n)} = 0.$$

Perron [14] pp. 271-273 investigated for which p, q there is convergence in the broad sense. Kooman [6] Ch.7 gave a complete answer.

THEOREM 6. *Put* $r(n) = 1 + 4q(n)/p(n)p(n-1)$.

The continued fraction (22) converges in the broad sense if and only if

(i) $\deg r \leq -2$ *and* $\lim_{x\to\infty} x^2 r(x) \geq -1/4$,

(ii) $\deg r = -1$ *and* $\lim_{x\to\infty} x r(x) > 0$,

(iii) $\deg r = 0$ *and* $\lim_{x\to\infty} r(x) > 0$,

(iv) $\deg r = 1$ or 2 *and* $\lim_{x\to\infty} r(x) = \infty$.

The underlying equation is $y_n y_{n+1} + p(n) y_n + q(n) = 0$ which can be transformed to $u_{n+2} - p(n) u_{n+1} - q(n) u_n = 0$ where $u_n = (-1)^{n+1} y_{n+1} y_n \ldots y_1$.

5B. Irrationality measures.

In 1978 Apéry [1] proved the irrationality of $\zeta(3) = \sum_{n=1}^{\infty} n^{-3}$ (cf. Reyssat [16], Beukers [2], van der Poorten [17].) Actually it follows that for all positive integers $p, q > 0$ sufficiently large relative to $\varepsilon > 0$:

$$|\zeta(3) - \frac{p}{q}| > q^{-(\theta+\epsilon)} \text{ where } \theta = 1 + \frac{4\log(1+\sqrt{2})+3}{4\log(1+\sqrt{2})-3} = 13.417820\ldots.$$

See [17] p.199. A similar irrationality measure can be derived for $\zeta(2)$. Apéry's proof is based on the recurrence relation

$$n^3 u_n - (34n^3 - 51n^2 + 27n - 5)u_{n-1} + (n-1)^3 u_{n-2} = 0. \tag{23}$$

Let $\{a_n\}_{n=0}^{\infty}$ be the solution of (23) with $a_0 = 0, a_1 = 6$ and $\{b_n\}_{n=0}^{\infty}$ the solution of (23) with $b_0 = 1, b_1 = 5$. Then

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \zeta(3).$$

Kooman [6] Ch.2 studied the set of numbers which can be obtained in this way, that is, which are the limit of the quotient of the n-th terms of solutions of the same recurrence relation with elements from $\mathbf{Z}[n]$ as coefficients and integer initial values. He showed that this is a countable set which forms a field. This field contains all real algebraic numbers, but also $e^k (k \in \mathbf{Q}), \log k (k \in \mathbf{Q}_{>1}), \arctan k (k \in \mathbf{Q}, |k| \le 1), \zeta(k)(k \in \mathbf{Z}_{>1})$ and various other sets of well known numbers. For example $\zeta(k) = \lim_{n\to\infty} a_n^{(k)}/b_n^{(k)}$ where $\{a_n^{(k)}\}, \{b_n^{(k)}\}$ satisfy the recurrence relation

$$(n+2)^k u_{n+2} - ((n+2)^k + (n+1)^k)u_{n+1} + (n+1)^k u_n = 0$$

and $a_0^{(k)} = 0, a_1^{(k)} = 1, b_0^{(k)} = b_1^{(k)} = 1$. However, this recurrence relation is of no use for an irrationality proof, since such a proof requires a recurrence with an eigenvalue which is very small in absolute value. There is a theory of transforming recurrences into recurrences with accelerated convergence (see Brezinski [3]), but nobody has found a suitable recurrence to prove the irrationality of $\zeta(5)$.

5C. **Convergence of the sequence** $\{\sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^k}{k!}\}_{n=0}^{\infty}$.

The functional analyst C.B. Huijsmans asked us whether the sequence $\{s_n\}_{n=0}^{\infty}$ defined by

$$s_n \;=\; \sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^k}{k!}$$

satisfies $|s_n| < 1$ for all n. This would be very surprising, since the terms composing s_n can be quite large,

$$\binom{n}{k}\frac{(-1)^k}{k!} \approx e^{2k} \approx e^{2\sqrt{n}} \qquad \text{if } k \approx \sqrt{n}.$$

However, computations showed that $|s_n| < 1$ for $n < 100$. It can be shown that $\{s_n\}_{n=0}^{\infty}$ satisfies the recurrence relation

$$\text{(24)} \qquad u_{n+2} - (2 - \frac{2}{n})u_{n+1} + (1 - \frac{1}{n})u_n = 0 \qquad (n = 0, 1, 2, \ldots).$$

According to section IId of Section 4 (24) has solutions $\{u_n^{(1)}\}$ and $\{u_n^{(2)}\}$ such that

$$\lim_{n\to\infty} \sqrt{n}(\frac{u_{n+1}^{(1)}}{u_n^{(1)}} - 1) = i \quad \text{and} \quad \lim_{n\to\infty} \sqrt{n}(\frac{u_{n+1}^{(2)}}{u_n^{(2)}} - 1) = -i.$$

An analysis of the proof yielded that

$$s_n = \frac{c}{n^{1/4}}\sin(2\sqrt{n} + \varphi) + o(\frac{1}{n^{1/4}}) \quad (n \to \infty)$$

for some real constants $c \neq 0$ and φ. This shows that $s_n \to 0$ as $n \to \infty$ and that there are infinitely many sign changes where the distance between consecutive sign changes increases almost linearly.

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