Three "Universal" Mesoscopic Josephson Effects

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Abstract. A recent theory is reviewed for the sample-to-sample fluctuations in the critical current of a Josephson junction consisting of a disordered point contact or microbridge. The theory is based on a relation between the supercurrent and the scattering matrix in the normal state. The root-mean-square amplitude rms I_c of the critical current I_c at zero temperature is given by rms $I_c \simeq e\Delta_0/\hbar$, up to a numerical coefficient of order unity (Δ_0 is the energy gap). This is the superconducting analogue of "Universal Conductance Fluctuations" in the normal state. The theory can also be applied to a ballistic point contact, where it yields the analogue of the quantized conductance, and to a quantum dot, where it describes supercurrent resonances. All three phenomena provide a measurement of the supercurrent unit $e\Delta_0/\hbar$, and are "universal" through the absence of a dependence on junction parameters.

1 Introduction

Nanostructures combining semiconducting and superconducting elements form a new class of systems in which to search for mesoscopic phenomena. The correspondence between transport of normal electrons and transport of Bogoliubov quasiparticles (being the elementary excitations of the superconductor) serves as a useful guide in the search. The appearance of a new length scale — the superconducting coherence length ξ — complicates the correspondence in an interesting way, by introducing two qualitatively different regimes: The *short-junction* regime (junction length $L \ll \xi$) and the *long-junction* regime ($L \gg \xi$). These two regimes appear over and above the transport regimes which depend on the relative magnitude of junction length and mean free path l: The *ballistic* regime ($l \gg L$) and the *diffusive* regime ($l \ll L$). A third transport regime in the normal state, that of *resonant tunneling*, is also subdivided into two new regimes, distinguished by the relative magnitude of ξ and the characteristic length $v_F \tau_{res}$, with τ_{res} the lifetime of the resonant state in the junction and v_F the Fermi velocity.

Our own interest in this field has focused on a set of three phenomena which provide, through the Josephson effect, a measurement of the supercurrent unit $e\Delta_0/\hbar$ (Δ_0 being the superconducting energy gap). This unit of current plays the role of the conductance quantum e^2/h in the normal state. The three phenomena are:

1. Discretization of the critical current of a ballistic point contact [1].

2. Resonant Josephson current through a quantum dot [2, 3].

3. Mesoscopic supercurrent fluctuations in a diffusive point contact [4].

These effects belong, respectively, to the ballistic, resonant-tunneling, and diffusive transport regimes, and within each regime to the short-junction limit.

Effect number 1 is the analogue of the quantized conductance of a quantum point contact [5]–[7]: The critical current I_c of a short and narrow constriction in a superconductor increases stepwise as a function of the constriction width, with step height $e\Delta_0/\hbar$ independent of the properties of the junction [1],

$$I_c = N\frac{e\Delta_0}{\hbar}. \tag{1}$$

The integer N is the number of transverse modes at the Fermi level which can propagate through the constriction. The short-junction limit $L \ll \xi_0$ (where $\xi_0 = \hbar v_F/\pi\Delta_0$ is the ballistic coherence length) is essential: Furusaki et al. [8] studied quantum size effects on the critical current in the long-junction limit and found a geometry-dependent behavior instead of Eq. (1).

Effect number 2 is described by the formula [3]

$$I_c^{res} = \frac{e}{\hbar}\frac{\Delta_0\Gamma_0}{\Delta_0 + \Gamma_0} \tag{2}$$

for the critical current through a bound state at the Fermi level which is coupled with equal tunnel rate $\Gamma_0/\hbar$ to two bulk superconductors. Since $\tau_{res} = \hbar/2\Gamma_0$, the "short-junction" criterion $\xi_0 \gg v_F\tau_{res}$ is equivalent to $\Gamma_0 \gg \Delta_0$. If $\Gamma_0 \gg \Delta_0$, the critical current on resonance becomes [2, 3]

$$I_c^{res} = \frac{e\Delta_0}{\hbar}, \tag{3}$$

which no longer depends on the tunnel rate.

Effect number 3 is the analogue of "Universal Conductance Fluctuations" in disordered normal metals [9]–[11]. The sample-to-sample fluctuations in the critical current of a disordered point contact have root-mean-square value [4]

$$\text{rms}\,I_c \simeq \frac{e\Delta_0}{\hbar}, \tag{4}$$

up to a numerical coefficient of order unity. These mesoscopic fluctuations are *universal* in the sense that they do not depend on the size of the junction or on the degree of disorder, as long as the criteria $l \ll L \ll Nl$ and $L \ll \xi$ for the diffusive, short-junction regime are satisfied. (Here $\xi = (\xi_0 l)^{1/2}$ is the diffusive coherence length.) The short-junction limit $L \ll \xi$ is again essential for universality. The opposite long-junction limit was considered previously by Al'tshuler and Spivak [12], who calculated the supercurrent fluctuations of an SNS junction in the limit $L \gg \xi$ (S=superconductor, N=normal metal, L=separation of the SN interfaces). Their result $\text{rms}\,I_c \simeq ev_F l/L^2$ depends on both l and L and is therefore *not* universal in the sense of UCF.

Table 1: Summary of the mesoscopic Josephson effects described in the text, in the ballistic, resonant-tunneling, and diffusive transport regimes. Universal (junction-independent) behavior is obtained in the short-junction limit, which in terms of the correlation energy E_c is given by $\Delta_0 \ll E_c$.

Mesoscopic Josephson Effects			
	short-junction	long-junction	E_c
ballistic	$I_c = Ne\Delta_0/\hbar$	geometry dep.	$\hbar v_F/L$
res.tunneling	$I_c^{res} = e\Delta_0/\hbar$	$I_c^{res} = e\Gamma_0/\hbar$	Γ_0
diffusive	$\text{rms}\,I_c \simeq e\Delta_0/\hbar$	$\text{rms}\,I_c \simeq eD/L^2$	$\hbar D/L^2$

We have summarized the three universal short-junction Josephson effects in Table 1, together with their non-universal long-junction counterparts. We have introduced the diffusion constant $D \simeq v_F l$ and the correlation energy (or Thouless energy) $E_c \simeq \hbar/\tau_{\text{dwell}}$, where τ_{dwell} is the dwell time in the junction. The distinction "short-junction" versus "long-junction" may alternatively be given in terms of the relative magnitude of E_c and Δ_0: In the short-junction regime $\Delta_0 \ll E_c$, while in the long-junction regime $\Delta_0 \gg E_c$. The universality is lost in the long-junction regime because E_c (which depends on junction parameters) becomes the characteristic energy for the Josephson effect instead of Δ_0. All these results hold in the zero-temperature limit, or more precisely for $k_B T \ll \min(\Delta_0, E_c)$.

In the present paper we review a scattering theory for the Josephson effect, which provides a unified treatment of the three universal (short-junction) phenomena. In Sec. 2 we show, following Ref. [13], how the supercurrent can be obtained directly from the quasiparticle excitation spectrum of the Josephson junction. This approach is equivalent to the usual method which starts from the finite-temperature Greens function, but is more convenient in the short-junction regime, where — as we shall see — the excitation spectrum has a particularly simple form. In Sec. 3 we follow Ref. [4] in relating the excitation spectrum to the normal-state scattering matrix of the junction. In Sec. 4 we take the short-junction limit and obtain a simple relation between the supercurrent and the eigenvalues of the transmission matrix product $tt^\dagger$. This relation plays the role for the Josephson effect of the Landauer formula for the conductance [7]. We emphasize that the transmission matrix t refers to *normal* electrons. Other scattering approaches to the Josephson effect [14, 15] relate the supercurrent to the scattering matrix for Bogoliubov quasiparticles. The usefulness of the present method is that one can use directly the many properties of t known from normal-state transport theory. Indeed, as we shall see in Sec. 5, the three universal Josephson effects follow as special cases of the general scattering formula of Sec. 4.

2 Supercurrent from Excitation Spectrum

The quasiparticle excitation spectrum of the Josephson junction consists of the positive eigenvalues of the Bogoliubov-de Gennes (BdG) equation [16]. The BdG equation has the form of two Schrödinger equations for electron and hole wavefunctions u(r) and v(r), coupled by the pair potential $\Delta(\mathbf{r})$:

$$\begin{pmatrix} \mathcal{H}_0 & \Delta \\ \Delta^* & -\mathcal{H}_0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \varepsilon \begin{pmatrix} u \\ v \end{pmatrix}. \tag{5}$$

Here $\mathcal{H}_0 = \mathbf{p}^2/2m + V(\mathbf{r}) - E_F$ is the single-electron Hamiltonian, containing an electrostatic potential $V(\mathbf{r})$ (zero vector potential is assumed). The excitation energy ε is measured relative to the Fermi energy E_F.

In a uniform system with $\Delta(\mathbf{r}) \equiv \Delta_0 e^{i\phi}$, $V(\mathbf{r}) \equiv 0$, the solution of the BdG equation is

$$\begin{aligned} \varepsilon &= \left((\hbar^2 k^2/2m - E_F)^2 + \Delta_0^2 \right)^{1/2}, \\ u(\mathbf{r}) &= \mathcal{V}^{-1/2}(2\varepsilon)^{-1/2} e^{i\phi/2} \left(\varepsilon + \hbar^2 k^2/2m - E_F \right)^{1/2} e^{i\mathbf{k}\cdot\mathbf{r}}, \\ v(\mathbf{r}) &= \mathcal{V}^{-1/2}(2\varepsilon)^{-1/2} e^{-i\phi/2} \left(\varepsilon - \hbar^2 k^2/2m + E_F \right)^{1/2} e^{i\mathbf{k}\cdot\mathbf{r}}. \end{aligned} \tag{6}$$

The eigenfunction is normalized to unit probability in a volume $\mathcal{V}$,

$$\int_{\mathcal{V}} d\mathbf{r} \left(|u|^2 + |v|^2 \right) = 1. \tag{7}$$

The excitation spectrum is continuous, with excitation gap Δ_0. The eigenfunctions (u, v) are plane waves characterized by a wavevector $\mathbf{k}$. The coefficients of the plane waves are the two coherence factors of the BCS theory. This simple excitation spectrum is modified by the presence of the Josephson junction. The spectrum acquires a discrete part due to the non-uniformities in $\Delta(\mathbf{r})$ near the junction. The discrete spectrum corresponds to bound states in the gap $(0 < \varepsilon < \Delta_0)$, localized within a coherence length from the junction. In addition, the continuous spectrum is modified. As we shall show in Sec. 3, in the short-junction limit the supercurrent is entirely determined by the *discrete* part of the excitation spectrum.

Consider an SNS junction with normal region at $|x| < L/2$. Let $\Delta(\mathbf{r}) \rightarrow \Delta_0 e^{\mp i\phi/2}$ for $x \rightarrow \pm\infty$. To determine $\Delta(\mathbf{r})$ near the junction one has to solve the self-consistency equation [16]

$$\Delta(\mathbf{r}) = |g(\mathbf{r})| \sum_{\varepsilon > 0} v^*(\mathbf{r})u(\mathbf{r})[1 - 2f(\varepsilon)], \tag{8}$$

where the sum is over all states with positive eigenvalue, and $f(\varepsilon) = [1 + \exp(\varepsilon/k_B T)]^{-1}$ is the Fermi function. The coefficient g is the interaction constant of the BCS theory of superconductivity. At an SN interface, g drops abruptly (over atomic distances) to zero. (We assume non-interacting electrons in the normal region.) Therefore, $\Delta(\mathbf{r}) \equiv 0$ for $|x| < L/2$ regardless of Eq. (8).

At the superconducting side of the SN interface, $\Delta(\mathbf{r})$ recovers its bulk value Δ_0 only at some distance from the interface (proximity effect). There exists a class of Josephson junctions where the suppression of $\Delta(\mathbf{r})$ on approaching the SN interface can be neglected, and one can use the step-function model

$$\Delta(\mathbf{r}) = \begin{cases} \Delta_0 e^{i\phi/2} & \text{if } x < -L/2, \\ 0 & \text{if } |x| < L/2, \\ \Delta_0 e^{-i\phi/2} & \text{if } x > L/2. \end{cases} \tag{9}$$

The step-function pair potential is also referred to in the literature as a "rigid boundary-condition". Likharev [17] discusses in detail the conditions under which this model is valid:

1. If the width W of the junction is small compared to the coherence length, the non-uniformities in $\Delta(\mathbf{r})$ extend only over a distance of order W from the junction (because of "geometrical dilution" of the influence of the narrow junction in the wide superconductor). Since non-uniformities on length scales $\ll \xi$ do not affect the dynamics of the quasiparticles, these can be neglected and the step-function model holds. Constrictions in the short-junction limit belong in general to this class of junctions. Note that, if the length and width of the junction are $\ll \xi$, it is irrelevant whether it is made from a normal metal or from a superconductor. In the literature, such a junction is referred to as an ScS junction, without specifying whether the constriction (c) is N or S.

2. Alternatively, the step-function model holds if the resistivity of the junction region is much bigger than the resistivity of the bulk superconductor. This condition is formulated more precisely by Kupriyanov et al. [18], who have also studied deviations from the ideal step-function model due to the proximity effect and due to current-induced suppression of the pair potential. A superconductor — semiconductor — superconductor junction is typically in this second category.

In equilibrium, a phase difference ϕ in the pair potential induces a stationary current I through the junction (DC Josephson effect). The current–phase relationship $I(\phi)$ is 2π-periodic and antisymmetric $[I(\phi + 2\pi) = I(\phi), I(-\phi) = -I(\phi)]$. The maximum $I_c \equiv \max I(\phi)$ is known as the critical current. There is a thermodynamic relation

$$I = \frac{2e}{\hbar} \frac{dF}{d\phi} \tag{10}$$

between the equilibrium current and the derivative of the free energy F with respect to the phase difference. To apply this relation we need to know how to obtain F from the BdG equation. The required formula was derived by Bardeen et al. [19] from the Greens function expression for F. An alternative derivation, directly from the BdG equation, has been given in Ref. [13]. The result is

$$F = -2k_B T \sum_{\varepsilon > 0} \ln \left[2 \cosh(\varepsilon/2k_B T) \right] + \int d\mathbf{r} \, |\Delta|^2/|g| + \text{Tr} \, \mathcal{H}_0. \tag{11}$$

The first term in Eq. (11) (the sum over ε) can be formally interpreted as the free energy of non-interacting electrons, all of one single spin, in a "semiconductor" with Fermi level halfway between the "conduction band" (positive ε) and the "valence band" (negative ε). This semiconductor model of a superconductor appeals to intuition, but does not give the free energy correctly. The second term in Eq. (11) corrects for a double-counting of the interaction energy in the semiconductor model. The third term $\mathrm{Tr}\,\mathcal{H}_0$ (i.e. the sum of the single-electron eigenenergies) cancels a divergence at large ε of the series in the first term.

From Eqs. (10) and (11) one obtains [13]

$$
\begin{aligned}
I = & -\frac{2e}{\hbar}\sum_p \tanh(\varepsilon_p/2k_{\mathrm B}T)\frac{d\varepsilon_p}{d\phi}\\
& -\frac{2e}{\hbar}2k_{\mathrm B}T\int_{\Delta_0}^{\infty}d\varepsilon \ln\left[2\cosh(\varepsilon/2k_{\mathrm B}T)\right]\frac{d\rho}{d\phi}+\frac{2e}{\hbar}\frac{d}{d\phi}\int d\mathbf{r}\,|\Delta|^2/|g|,\ (12)
\end{aligned}
$$

where we have rewritten $\sum_{\varepsilon>0}$ as a sum over the discrete positive eigenvalues ε_p ($p=1,2,\dots$), and an integration over the continuous spectrum with density of states $\rho(\varepsilon)$. The term $\mathrm{Tr}\,\mathcal{H}_0$ in Eq. (11) does not depend on ϕ, and therefore does not contribute to I. The spatial integral of $|\Delta|^2/|g|$ does contribute in general. If the step-function model for $\Delta(\mathbf r)$ holds, however, $|\Delta|$ is independent of ϕ so that this contribution can be disregarded. A calculation of the Josephson current from Eq. (12) then requires only knowledge of the eigenvalues, not of the eigenfunctions.

3 Excitation spectrum from scattering matrix

Following Ref. [4], we now show how to relate the excitation spectrum of Bogoliubov quasiparticles to the scattering matrix of normal electrons. The supercurrent then follows from Eq. (12). The model considered is illustrated in Fig. 1. It consists of a disordered normal region between two superconducting regions S_1 and S_2. The disordered region may or may not contain a geometrical constriction. To obtain a well-defined scattering problem we insert ideal

$c_e^+\rightarrow$	$c_e^+\rightarrow$			$\leftarrow c_e^-$	$\leftarrow c_e^-$
$c_h^-\rightarrow$	$c_h^-\rightarrow$			$\leftarrow c_h^+$	$\leftarrow c_h^+$
S_1	N_1	disordered region	N_2		S_2
$c_e^-\leftarrow$	$c_e^-\leftarrow$			$\rightarrow c_e^+$	$\rightarrow c_e^+$
$c_h^+\leftarrow$	$c_h^+\leftarrow$			$\rightarrow c_h^-$	$\rightarrow c_h^-$

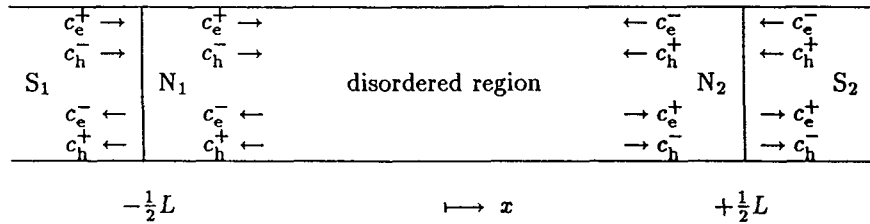$-\tfrac12 L$ $\qquad\longmapsto x\qquad$ $+\tfrac12 L$

Figure 1: Top: Superconductor — normal-metal — superconductor Josephson junction containing a disordered normal region. Scattering states in the normal and superconducting leads are indicated schematically. Normal scattering (disordered region) and Andreev reflection (SN interface) are spatially separated.

(impurity-free) normal leads N_1 and N_2 to the left and right of the disordered region. The leads should be long compared to the Fermi wavelength $\lambda_{\mathrm F}$, but short compared to the coherence length ξ_0. The SN interfaces are located at $x=\pm L/2$. We assume that the only scattering in the superconductors consists of Andreev reflection at the SN interfaces, i.e. we consider the case that the disorder is contained entirely within the normal region. The spatial separation of Andreev and normal scattering is the key simplification which allows us to relate the supercurrent directly to the normal-state scattering matrix.

We first construct a basis for the scattering matrix (s-matrix). In the normal lead N_1 the eigenfunctions of the BdG equation (5) can be written in the form

$$
\begin{aligned}
\Psi^{\pm}_{n,e}(N_1) &= \begin{pmatrix}1\\0\end{pmatrix}(k_n^e)^{-1/2}\Phi_n(y,z)\exp[\pm ik_n^e(x+\tfrac12 L)],\\
\Psi^{\pm}_{n,h}(N_1) &= \begin{pmatrix}0\\1\end{pmatrix}(k_n^h)^{-1/2}\Phi_n(y,z)\exp[\pm ik_n^h(x+\tfrac12 L)],\quad (13)
\end{aligned}
$$

where the wavenumbers k_n^e and k_n^h are given by

$$
k_n^{e,h}\equiv(2m/\hbar^2)^{1/2}(E_{\mathrm F}-E_n+\sigma^{e,h}\varepsilon)^{1/2},\qquad (14)
$$

and we have defined $\sigma^e\equiv 1$, $\sigma^h\equiv -1$. The labels e and h indicate the electron or hole character of the wavefunction. The index n labels the modes, $\Phi_n(y,z)$ is the transverse wavefunction of the n-th mode, and E_n its threshold energy:

$$
[(p_y^2+p_z^2)/2m+V(y,z)]\Phi_n(y,z)=E_n\Phi_n(y,z).\qquad (15)
$$

The eigenfunction Φ_n is normalized to unity, $\int dy\int dz\,|\Phi_n|^2=1$. The eigenfunctions in lead N_2 are chosen similarly, but with $x+\tfrac12 L$ replaced by $x-\tfrac12 L$.

In the superconducting lead S_1, where $\Delta=\Delta_0\exp(i\phi/2)$, the eigenfunctions are

$$
\begin{aligned}
\Psi^{\pm}_{n,e}(S_1) &= \begin{pmatrix}e^{i\eta^e/2}\\e^{-i\eta^e/2}\end{pmatrix}(2q_n^e)^{-1/2}(\varepsilon^2/\Delta_0^2-1)^{-1/4}\Phi_n\exp[\pm iq_n^e(x+\tfrac12 L)],\\
\Psi^{\pm}_{n,h}(S_1) &= \begin{pmatrix}e^{i\eta^h/2}\\e^{-i\eta^h/2}\end{pmatrix}(2q_n^h)^{-1/2}(\varepsilon^2/\Delta_0^2-1)^{-1/4}\Phi_n\exp[\pm iq_n^h(x+\tfrac12 L)].\ (16)
\end{aligned}
$$

We have defined

$$
\begin{aligned}
q_n^{e,h} &\equiv (2m/\hbar^2)^{1/2}[E_{\mathrm F}-E_n+\sigma^{e,h}(\varepsilon^2-\Delta_0^2)^{1/2}]^{1/2},\qquad (17)\\
\eta^{e,h} &\equiv \tfrac12\phi+\sigma^{e,h}\arccos(\varepsilon/\Delta_0).\qquad (18)
\end{aligned}
$$

The square roots are to be taken such that $\mathrm{Re}\,q^{e,h}\geq 0$, $\mathrm{Im}\,q^e\geq 0$, $\mathrm{Im}\,q^h\leq 0$. The function $\arccos t\in(0,\pi/2)$ for $0<t<1$, while $\arccos t\equiv -i\ln[t+(t^2-1)^{1/2}]$ for $t>1$. The eigenfunctions in lead S_2, where $\Delta=\Delta_0\exp(-i\phi/2)$, are obtained by replacing ϕ by $-\phi$ and L by $-L$.

The wavefunctions (13) and (16) have been normalized to carry the same amount of quasiparticle current, because we want to use them as the basis for a unitary s-matrix. Unitarity of the s-matrix is a consequence of the fact that the BdG equation conserves the quasiparticle current density

$$\mathbf{j}_{\text{qp}}(\mathbf{r}) = \frac{1}{m} \text{Re} \begin{pmatrix} u^* \\ v^* \end{pmatrix} \begin{pmatrix} p & \emptyset \\ \emptyset & -p \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}. \tag{19}$$

The quasiparticle current I_{qp} through a lead is given by

$$I_{\text{qp}} = \int dy \int dz \, (\mathbf{j}_{\text{qp}})_x = \frac{\hbar}{m} \text{Re} \int dy \int dz \left(u^* \frac{\partial}{i \partial x} u - v^* \frac{\partial}{i \partial x} v \right). \tag{20}$$

One easily verifies that, provided the wavenumbers $k_n^{\text{e,h}}$ and $q_n^{\text{e,h}}$ are real (i.e. for propagating modes), each of the wavefunctions (13,16) carries the same amount of quasiparticle current $|I_{\text{qp}}|$. (The direction of the current is opposite for Ψ_e^+, Ψ_h^- and Ψ_e^-, Ψ_h^+.) We note that the wavefunctions (13) and (16) carry *different* amounts of *charge* current (i.e. electrical current), which is given by

$$I_{\text{charge}} = -e \frac{\hbar}{m} \text{Re} \int dy \int dz \left(u^* \frac{\partial}{i \partial x} u + v^* \frac{\partial}{i \partial x} v \right). \tag{21}$$

This does not contradict the unitarity of the s-matrix constructed from these wavefunctions, because the BdG equation does not conserve the charge of the quasiparticles. (Andreev reflection [20], i.e. the reflection of an electron into a hole at an SN interface, is an example of a non-charge-conserving scattering process.)

A wave incident on the disordered normal region is described in the basis (13) by a vector of coefficients

$$c_{\text{N}}^{\text{in}} \equiv \left(c_e^+(N_1), c_e^-(N_2), c_h^-(N_1), c_h^+(N_2) \right), \tag{22}$$

as shown schematically in Fig. 1. (The mode-index n has been suppressed for simplicity of notation.) The reflected and transmitted wave has vector of coefficients

$$c_{\text{N}}^{\text{out}} \equiv \left(c_e^-(N_1), c_e^+(N_2), c_h^+(N_1), c_h^-(N_2) \right). \tag{23}$$

The s-matrix s_{N} of the normal region relates these two vectors, $c_{\text{N}}^{\text{out}} = s_{\text{N}} c_{\text{N}}^{\text{in}}$. Because the normal region does not couple electrons and holes, this matrix has the block-diagonal form

$$s_{\text{N}}(\varepsilon) = \begin{pmatrix} s_0(\varepsilon) & \emptyset \\ \emptyset & s_0(-\varepsilon)^* \end{pmatrix}, \ s_0 \equiv \begin{pmatrix} r_{11} & t_{12} \\ t_{21} & r_{22} \end{pmatrix}. \tag{24}$$

Here s_0 is the unitary and symmetric s-matrix associated with the single-electron Hamiltonian $\mathcal{H}_0$. The reflection and transmission matrices $r(\varepsilon)$ and $t(\varepsilon)$ are $N \times N$ matrices, $N(\varepsilon)$ being the number of propagating modes at energy ε. (We assume for simplicity that the number of modes in leads N_1 and N_2 is the same.) The dimension of $s_{\text{N}}(\varepsilon)$ is $2N(\varepsilon) + 2N(-\varepsilon)$.

We will make use of two more s-matrices. For energies $0 < \varepsilon < \Delta_0$ there are no propagating modes in the superconducting leads S_1 and S_2. We can then define an s-matrix s_{A} for Andreev reflection at the SN interfaces by $c_{\text{N}}^{\text{in}} = s_{\text{A}} c_{\text{N}}^{\text{out}}$. The elements of s_{A} can be obtained by matching the wavefunctions

(13) at $|x| = L/2$ to the decaying wavefunctions (16). Since $\Delta_0 \ll E_{\text{F}}$ one may ignore normal reflections at the SN interface and neglect the difference between $N(\varepsilon)$ and $N(-\varepsilon)$. This is known as the Andreev approximation [20]. The result is

$$s_{\text{A}} = \alpha \begin{pmatrix} \emptyset & r_{\text{A}} \\ r_{\text{A}}^* & \emptyset \end{pmatrix}, r_{\text{A}} \equiv \begin{pmatrix} e^{i\phi/2} \mathbf{1} & \emptyset \\ \emptyset & e^{-i\phi/2} \mathbf{1} \end{pmatrix}, \tag{25}$$

where $\alpha \equiv \exp[-i \arccos(\varepsilon/\Delta_0)]$. The matrices $\mathbf{1}$ and $\emptyset$ are the unit and null matrices, respectively. Andreev reflection transforms an electron mode into a hole mode, without change of mode index. The transformation is accompanied by a phase shift, which consists of two parts: 1. A phase shift $-\arccos(\varepsilon/\Delta_0)$ due to the penetration of the wavefunction into the superconductor; 2. A phase shift equal to plus or minus the phase of the pair potential in the superconductor (*plus* for reflection from hole to electron, *minus* for the reverse process).

For $\varepsilon > \Delta_0$ we can define the s-matrix s_{SNS} of the whole junction, by $c_{\text{S}}^{\text{out}} = s_{\text{SNS}} c_{\text{S}}^{\text{in}}$. The vectors

$$c_{\text{S}}^{\text{in}} \equiv \left(c_e^+(S_1), c_e^-(S_2), c_h^-(S_1), c_h^+(S_2) \right), \tag{26}$$

$$c_{\text{S}}^{\text{out}} \equiv \left(c_e^-(S_1), c_e^+(S_2), c_h^+(S_1), c_h^-(S_2) \right). \tag{27}$$

are the coefficients in the expansion of the incoming and outgoing wave in leads S_1 and S_2 in terms of the wavefunctions (16) (cf. Fig. 1). By matching the wavefunctions (13) and (16) at $|x| = L/2$, we arrive after some algebra (using again $\Delta_0 \ll E_{\text{F}}$) at the matrix-product expression

$$s_{\text{SNS}} = U^{-1} (1 - M)^{-1} (1 - M^\dagger) s_{\text{N}} U, \tag{28}$$

$$U \equiv \begin{pmatrix} r_{\text{A}} & \emptyset \\ \emptyset & r_{\text{A}}^* \end{pmatrix}^{\frac{1}{2}}, M \equiv \alpha s_{\text{N}} \begin{pmatrix} \emptyset & r_{\text{A}} \\ r_{\text{A}}^* & \emptyset \end{pmatrix}.$$

One can verify that the three s-matrices defined above (s_{N}; s_{A} for $0 < \varepsilon < \Delta_0$; s_{SNS} for $\varepsilon > \Delta_0$) are *unitary* ($s^\dagger s = s s^\dagger = 1$) and satisfy the *symmetry relation* $s(\varepsilon, \phi)_{ij} = s(\varepsilon, -\phi)_{ji}$, as required by quasiparticle-current conservation and by time-reversal invariance, respectively.

We are now ready to relate the excitation spectrum of the Josephson junction to the s-matrix of the normal region. First the discrete spectrum. The condition $c_{\text{in}} = s_{\text{A}} s_{\text{N}} c_{\text{in}}$ for a bound state implies $\text{Det}(1 - s_{\text{A}} s_{\text{N}}) = 0$. Using Eqs. (24), (25), and the *folding-identity*

$$\text{Det} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \text{Det} (ad - aca^{-1}b) \tag{29}$$

(which holds for arbitrary square matrices a, b, c, d of equal dimension, with $\text{Det } a \neq 0$), we find the equation

$$\text{Det} \left[1 - \alpha(\varepsilon_p)^2 r_{\text{A}}^* s_0(\varepsilon_p) r_{\text{A}} s_0(-\varepsilon_p)^* \right] = 0, \tag{30}$$

which determines the discrete spectrum. The density of states of the continuous spectrum is related to s_{SNS} by the general relation [21]

$$\rho = \frac{1}{2\pi i} \frac{\partial}{\partial \varepsilon} \ln \operatorname{Det} s_{\mathrm{SNS}} + \text{constant}, \qquad (31)$$

where "constant" indicates a ϕ-independent term. From Eqs. (28) and (29) we find

$$\frac{\partial \rho}{\partial \phi} = -\frac{1}{\pi} \frac{\partial^2}{\partial \phi \partial \varepsilon} \operatorname{Im} \ln \operatorname{Det} \left[1 - \alpha(\varepsilon)^2 r_A^* s_0(\varepsilon) r_A s_0(-\varepsilon)^* \right], \qquad (32)$$

which determines the ϕ-dependence of the continuous spectrum.

4 Short-Junction Limit

In the short-junction limit $L \ll \xi$, the determinantal equations (30) and (32) can be simplified further. As mentioned in the Introduction, the condition $L \ll \xi$ is equivalent to $\Delta_0 \ll E_c$, where the correlation energy $E_c \equiv \hbar/\tau_{\mathrm{dwell}}$ is defined in terms of the dwell time τ_{dwell} in the junction. The elements of $s_0(\varepsilon)$ change significantly if ε is changed by at least E_c [22, 23]. We are concerned with ε of order Δ_0 or smaller (since $\rho(\varepsilon, \phi)$ becomes independent of ϕ for $\varepsilon \gg \Delta_0$). For $\Delta_0 \ll E_c$ we may thus approximate $s_0(\varepsilon) \approx s_0(-\varepsilon) \approx s_0(0) \equiv s_0$. Eq. (30) may now be simplified by multiplying both sides by $\operatorname{Det} s_0$ and using $s_0^* s_0 = 1$ (unitarity plus symmetry of s_0), as well as the folding identity (29). The result can be written in the form

$$\operatorname{Det} \left[(1 - \varepsilon_p^2/\Delta_0^2) 1 - t_{12} t_{12}^\dagger \sin^2(\phi/2) \right] = 0. \qquad (33)$$

For $\varepsilon > \Delta_0$ one can see that $\operatorname{Det}[1 - \alpha(\varepsilon)^2 r_A^* s_0 r_A s_0^*]$ is a real number. (Use that α is real for $\varepsilon > \Delta_0$ and that the determinant $\operatorname{Det}(1 - aa^*)$ is real for arbitrary matrix a.) Eq. (32) then reduces to $\partial \rho/\partial \phi = 0$, from which we conclude that *the continuous spectrum does not contribute to $I(\phi)$ in the short-junction limit.*

Eq. (33) can be solved for ε_p in terms of the eigenvalues T_p ($p = 1, 2, \ldots N$) of the hermitian $N \times N$ matrix $t_{12} t_{12}^\dagger$,

$$\varepsilon_p = \Delta_0 \left[1 - T_p \sin^2(\phi/2) \right]^{1/2}. \qquad (34)$$

Since $t_{12} t_{12}^\dagger = r_{11} t_{21}^\dagger t_{21} r_{11}^{-1}$ (as follows from unitarity of s_0), the matrices $t_{12} t_{12}^\dagger$ and $t_{21} t_{21}^\dagger$ have the same set of eigenvalues. We can therefore omit the indices of $tt^\dagger$. Unitarity of s_0 also implies that $0 \leq T_p \leq 1$ for all p.

Substitution of Eq. (34) into Eq. (12) yields the Josephson current

$$I(\phi) = \frac{e\Delta_0}{2\hbar} \sum_{p=1}^{N} \frac{T_p \sin \phi}{[1 - T_p \sin^2(\phi/2)]^{1/2}} \tanh \left(\frac{\Delta_0}{2k_B T} [1 - T_p \sin^2(\phi/2)]^{1/2} \right). \qquad (35)$$

Eq. (35) holds for an arbitrary transmission matrix $tt^\dagger$, i.e. for arbitrary disorder potential. It is the *multi-channel* generalization of a formula first obtained by Haberkorn et al. [24] (and subsequently rederived by several authors [25]–[27]) for the *single-channel* case (appropriate for a geometry such as a planar tunnel

barrier, where the different scattering channels are uncoupled). A formula of similar generality for the conductance is the multi-channel Landauer formula [23, 28]

$$G = \frac{2e^2}{h} \operatorname{Tr} tt^\dagger \equiv \frac{2e^2}{h} \sum_{p=1}^{N} T_p. \qquad (36)$$

In contrast to the Landauer formula, Eq. (35) is a *non-linear* function of the transmission eigenvalues T_p. It follows that knowledge of the conductance (i.e. of the sum of the eigenvalues) is *not sufficient* to determine the supercurrent.

5 Universal Josephson Effects

5.1 Quantum Point Contact

Consider the case that the weak link consists of a ballistic constriction ($l \gg L$) with a conductance quantized at $G = 2N_0 e^2/h$ (a *quantum point contact* [7]). The integer N_0 is the number of occupied one-dimensional subbands (per spin direction) in the constriction, or alternatively the number of transverse modes at the Fermi level which can propagate through the constriction. Note that $N_0 \ll N$. A quantum point contact is characterized by a special set of transmission eigenvalues, which are equal to either 0 or 1:

$$T_p = \begin{cases} 1 & \text{if } 1 \leq p \leq N_0, \\ 0 & \text{if } N_0 < p \leq N, \end{cases} \qquad (37)$$

where the eigenvalues have been ordered from large to small. We emphasize that Eq. (37) is valid whether the transport through the constriction is adiabatic or not. In the special case of adiabatic transport, the transmission eigenvalue T_p is equal to the transmission probability $\mathcal{T}_p$ of the p-th subband. In the absence of adiabaticity there is no simple relation between T_p and $\mathcal{T}_p$.

The discrete spectrum (34) in the short-junction limit contains an N_0-fold degenerate state at energy $\varepsilon = \Delta_0|\cos(\phi/2)|$. Eq. (35) for the supercurrent becomes

$$I(\phi) = N_0 \frac{e\Delta_0}{\hbar} \sin(\phi/2) \tanh \left(\frac{\Delta_0}{2k_B T} \cos(\phi/2) \right), \quad |\phi| < \pi. \qquad (38)$$

At $T = 0$ the current–phase relationship is given by $I(\phi) = N_0(e\Delta_0/\hbar)\sin(\phi/2)$ for $|\phi| < \pi$, and continued periodically for $|\phi| > \pi$. At $\phi = \pi$ the function $I(\phi)$ has a discontinuity which is smeared at finite temperatures. The ratio $I(\phi)/(\pi\Delta_0 G/e)$ at $T = 0$ is plotted in Fig. 2 (solid curve). The critical current $I_c = N_0 e\Delta_0/\hbar$ is *discretized* in units of $e\Delta_0/\hbar$. In the classical limit $N_0 \to \infty$ we recover the results of Kulik and Omel'yanchuk [29] for a *classical* ballistic point contact.

Eq. (38) was derived in Ref. [1] under the assumption of adiabatic transport. The present derivation (taken from Ref. [4]) does not assume adiabaticity. As discussed in Ref. [1], Eq. (38) breaks down if the Fermi level lies within Δ_0 of the threshold energy E_n of a subband. In the present con-
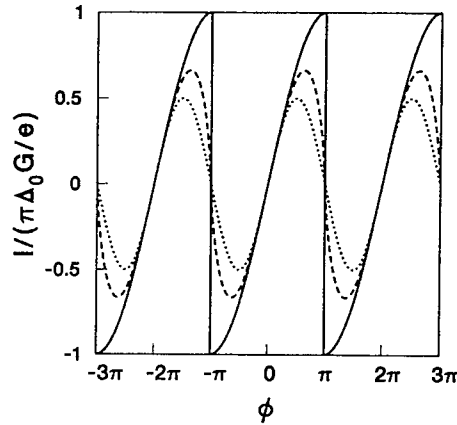
Figure 2: Current–phase relationship at $T = 0$ of a ballistic point contact [solid curve, Eq. (38)], a disordered point contact (ensemble averaged) [dashed curve, Eq. (49)], and a tunnel junction [dotted curve, Eq. (50)]. The current is normalized by the normal-state conductance of the junction.

text this breakdown can be understood from the fact that the dwell time $\tau_{\text{dwell}} = (L/v_{\text{F}})(1 - E_n/E_{\text{F}})^{-1/2}$ becomes infinitely long as E_{F} approaches E_n, so that the "short-junction" condition $\Delta_0 \ll E_{\text{c}} \equiv \hbar/\tau_{\text{dwell}}$ is violated.

5.2 Quantum Dot

Consider a small confined region (of dimensions comparable to the Fermi wavelength), which is weakly coupled by tunnel barriers to two electron reservoirs. We assume that transport through this *quantum dot* occurs via resonant tunneling through a single bound state. Let ε_{res} be the energy of the resonant level, relative to the Fermi energy E_{F} in the reservoirs, and let $\Gamma_1/\hbar$ and $\Gamma_2/\hbar$ be the tunnel rates through the left and right barriers. We denote $\Gamma \equiv \Gamma_1 + \Gamma_2$. If $\Gamma \ll \Delta E$ (with ΔE the level spacing in the quantum dot) and $T \ll \Gamma/k_{\text{B}}$, the conductance G in the case of non-interacting electrons has the form

$$G = \frac{2e^2}{h} \frac{\Gamma_1\Gamma_2}{\varepsilon_{\text{res}}^2 + \frac{1}{4}\Gamma^2} \equiv \frac{2e^2}{h} T_{\text{BW}}, \qquad (39)$$

where T_{BW} is the Breit-Wigner transmission probability at the Fermi level. The normal-state scattering matrix $s_0(\varepsilon)$ which yields this conductance has matrix elements [30]

$$(s_0(\varepsilon))_{nm} = \left(\delta_{nm} - \frac{i\sqrt{\Gamma_{1n}\Gamma_{2m}}}{\varepsilon - \varepsilon_{\text{res}} + i\Gamma/2} \right) e^{i(\delta_{1n} + \delta_{2m})}, \qquad (40)$$

where $\sum_{n=1}^{N} \Gamma_{1n} \equiv \Gamma_1$, $\sum_{n=1}^{N} \Gamma_{2n} \equiv \Gamma_2$. The phases δ_{1n}, δ_{2m}, as well as the basis with respect to which the matrix elements (40) are calculated, need not

be further specified. Büttiker [31] has shown how the conductance (39) follows, via the Landauer formula (36), from the Breit-Wigner scattering matrix (40).

In Ref. [3] Van Houten and the author have calculated the supercurrent through the quantum dot from the Breit-Wigner formula (40) in the special case of single-channel leads ($N = 1$). Just as for the conductance [31]–[33] one would expect that the results are not changed for multi-channel leads ($N > 1$). We will now demonstrate this explicitly in the limit $\Gamma \gg \Delta_0$ (which for resonant tunneling corresponds to the "short-junction" limit, cf. Sec. 1). The transmission matrix product $t_{12}t_{12}^{\dagger}$ (evaluated at the Fermi level $\varepsilon = 0$) following from the scattering matrix (40) is

$$\left(t_{12}t_{12}^{\dagger} \right)_{nm} = T_{\text{BW}}\gamma_n\gamma_m^*, \quad \gamma_n \equiv e^{i\delta_n}\sqrt{\Gamma_{1n}/\Gamma_1}. \qquad (41)$$

Its eigenvalues are

$$T_p = \begin{cases} T_{\text{BW}} & \text{if } p = 1, \\ 0 & \text{if } 2 \leq p \leq N. \end{cases} \qquad (42)$$

The single non-zero eigenvalue has eigenvector $u_n = \gamma_n$, the other eigenvectors span the plane orthogonal to the vector γ. Substitution into Eq. (35) yields the current–phase relationship for a wide resonance ($\Gamma \gg \Delta_0$),

$$I(\phi) = \frac{e\Delta_0}{2\hbar} \frac{T_{\text{BW}}\sin\phi}{[1 - T_{\text{BW}}\sin^2(\phi/2)]^{1/2}} \tanh\left(\frac{\Delta_0}{2k_{\text{B}}T}[1 - T_{\text{BW}}\sin^2(\phi/2)]^{1/2} \right). \qquad (43)$$

The critical current at zero temperature is

$$I_{\text{c}} = \frac{e\Delta_0}{\hbar}[1 - (1 - T_{\text{BW}})^{1/2}], \text{ if } \Gamma \gg \Delta_0, \qquad (44)$$

in agreement with Ref. [3]. Since $T_{\text{BW}} = 1$ on resonance ($\varepsilon_{\text{R}} = 0$) in the case of equal tunnel rates ($\Gamma_1 = \Gamma_2$), we obtain the result (3) discussed in the Introduction.

For completeness, we also quote the formula for the current–phase relationship in the opposite regime of a narrow resonance (derived in Ref. [3]):

$$I_{\text{c}} = \frac{e}{\hbar}(\varepsilon_{\text{res}}^2 + \frac{1}{4}\Gamma^2)^{1/2}[1 - (1 - T_{\text{BW}})^{1/2}], \text{ if } \Gamma, \varepsilon_{\text{res}} \ll \Delta_0. \qquad (45)$$

As shown in Fig. 3, the lineshapes (44) and (45) of a resonance in the critical current (solid curves) differ substantially from the lorentzian lineshape (39) of a conductance resonance (dotted curve). For $\Gamma_1 = \Gamma_2$, I_{c} has a *cusp* at $\varepsilon_{\text{res}} = 0$ (which is rounded at finite temperatures). On resonance, the maximum critical current $I_{\text{c}}^{\text{res}}$ equals $(2e\Delta_0/\hbar\Gamma)\min(\Gamma_1, \Gamma_2)$ and $(e/\hbar)\min(\Gamma_1, \Gamma_2)$ for a wide and narrow resonance, respectively. An analytical formula, Eq. (2), for the crossover between these two regimes can be obtained for the case of equal tunnel rates [3]. Off-resonance, I_{c} has the lorentzian decay $\propto 1/\varepsilon_{\text{res}}^2$ in the case $\Gamma \gg \Delta_0$ of a wide resonance, but a slower decay $\propto 1/\varepsilon_{\text{res}}$ in the case $\Gamma, \varepsilon_{\text{res}} \ll \Delta_0$. Near $\varepsilon_{\text{res}} \simeq \Delta_0$ this linear decay of the narrow resonance crosses over to a quadratic decay (not shown in Fig. 3).
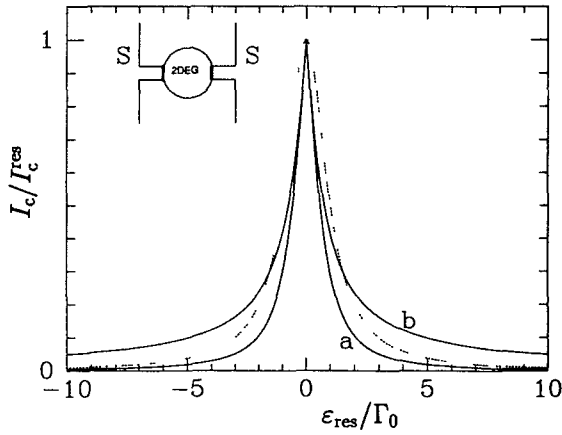
Figure 3: Normalized critical current versus energy of the resonant level, at zero temperature and for equal tunnel barriers ($\Gamma_1 = \Gamma_2 \equiv \Gamma_0$). The two solid curves are the results (44) and (45) for the two regimes $\Gamma_0 \gg \Delta_0$ (curve a) and $\Gamma_0, \varepsilon_{\text{res}} \ll \Delta_0$ (curve b). The dotted curve is the Breit-Wigner transmission probability (39). The inset shows schematically the quantum-dot Josephson junction. Taken from Ref. [3].

Since we have assumed non-interacting quasiparticles, the above results apply to a quantum dot with a small charging energy U for double occupancy of the resonant state. Glazman and Matveev have studied the influence of Coulomb repulsion on the resonant supercurrent [2]. The influence is most pronounced in the case of a narrow resonance, when the critical current is suppressed by a factor Γ/Δ_0 (for $U, \Delta_0 \gg \Gamma$). In the case of a wide resonance, the Coulomb repulsion does not suppress the supercurrent, but slightly broadens the resonance by a factor $\ln(\Gamma/\Delta_0)$ (for $U, \Gamma \gg \Delta_0$). The broadening is a consequence of the Kondo effect, and occurs only for $\varepsilon_{\text{res}} < 0$, so that the resonance peak becomes somewhat asymmetric [2].

5.3 Disordered Point Contact

We now turn to the regime of diffusive transport through a disordered point contact. We first consider the average supercurrent (averaged over an ensemble of impurity configurations) and then the fluctuations from the average. This section corrects Ref. [4].

5.3.1 Average Supercurrent

The transmission eigenvalue T_p is related to a channel-dependent localization length ζ_p by

$$T_p = \cosh^{-2}(L/\zeta_p). \tag{46}$$

The inverse localization length is uniformly distributed between 0 and $1/\zeta_{\text{min}} \simeq 1/l$ for $l \ll L \ll Nl$ [34]. One can therefore write

$$\frac{\left\langle \sum_{p=1}^{N} f(T_p) \right\rangle}{\left\langle \sum_{p=1}^{N} T_p \right\rangle} = \frac{\int_0^{L/\zeta_{\text{min}}} dx \, f(\cosh^{-2} x)}{\int_0^{L/\zeta_{\text{min}}} dx \, \cosh^{-2} x} = \int_0^{\infty} dx \, f(\cosh^{-2} x), \tag{47}$$

where $\langle \ldots \rangle$ indicates the ensemble average and $f(T)$ is an arbitrary function of the transmission eigenvalue such that $f(T) \to 0$ for $T \to 0$. In the second equality in Eq. (47) we have used that $L/\zeta_{\text{min}} \simeq L/l \gg 1$ to replace the upper integration limit by ∞. Note that, in view of the Landauer formula (36), the denominator $\left\langle \sum_{p=1}^{N} T_p \right\rangle$ is just $h/2e^2$ times the average conductance $\langle G \rangle$.

Combining Eqs. (47) and (35) we find for the average supercurrent the expression ($0 < \phi < \pi$)

$$
\begin{aligned}
\langle I(\phi) \rangle &= \frac{e\Delta_0}{2\hbar} \frac{h}{2e^2} \langle G \rangle \int_0^{\infty} dx \, \frac{(\cosh x)^{-2} \sin\phi}{[1 - (\cosh x)^{-2} \sin^2(\phi/2)]^{1/2}} \\
&\quad \times \tanh\left(\frac{\Delta_0}{2k_{\text{B}}T}[1 - (\cosh x)^{-2} \sin^2(\phi/2)]^{1/2} \right) \\
&= \frac{\pi\Delta_0}{e} \langle G \rangle \cos(\phi/2) \int_{\cos(\phi/2)}^{1} dz \, \frac{\tanh(\Delta_0 z/2k_{\text{B}}T)}{[z^2 - \cos^2(\phi/2)]^{1/2}}.
\end{aligned} \tag{48}
$$

At $T = 0$ this integral can be evaluated in closed form, with the result

$$\langle I(\phi) \rangle = \frac{\pi\Delta_0}{e} \langle G \rangle \cos(\phi/2) \operatorname{arctanh}[\sin(\phi/2)], \tag{49}$$

plotted in Fig. 2 (dashed curve). Note that the derivative $dI/d\phi$ diverges at $\phi = \pi$. The critical current is $I_c^{(0)} \equiv \max \langle I(\phi) \rangle = 1.32 \, (\pi\Delta_0/2e)\langle G \rangle$, reached at $\phi \equiv \phi_c^{(0)} = 1.97$.

Eq. (49) for the average supercurrent in a disordered point contact agrees with the result of Kulik and Omel'yanchuk [35]. These authors pointed out that a disordered weak link and a tunnel junction with the *same* conductance exhibit nevertheless a quite *different* Josephson effect. For a tunnel junction with $T_p \ll 1$ for all p one may linearize Eq. (35) in T_p, with the result (at zero temperature)

$$I = \frac{e\Delta_0}{2\hbar} \sin\phi \sum_{p=1}^{N} T_p = \frac{\pi\Delta_0}{2e} G \sin\phi, \tag{50}$$

which is also plotted for comparison in Fig. 2 (dotted curve). The corresponding critical current is $(\pi\Delta_0/2e)G$. The well-known result (50) for a tunnel junction [36] differs from the result (49) for a disordered weak link [35]. Zaïtsev [25] has explained the difference in terms of different boundary conditions for the Greens functions at the interface between the junction and the bulk superconductor. The present work yields an alternative perspective on this issue: Although the tunnel junction and the disordered weak link may have the same conductance (i.e. the same value of $\operatorname{Tr} tt^{\dagger}$), *the supercurrents differ because the distribution of transmission eigenvalues is different.* In a tunnel junction all N eigenvalues

are $\ll 1$, while in the disordered system a fraction l/L of the eigenvalues is of order unity, the remainder being exponentially small [34].

5.3.2 Supercurrent Fluctuations

The analysis of Kulik and Omel'yanchuk is based on a diffusion equation for the *ensemble-averaged* Greens function, and can not therefore describe the mesoscopic fluctuations of $I(\phi)$ from the average. In contrast, Eq. (35) holds for a *specific* member of the ensemble of impurity configurations. The statistical properties of the transmission eigenvalues T_n in this ensemble are known from the theory of conductance fluctuations [11]. A general result [34, 37] is that a *linear statistic* $\mathcal{L}(f) \equiv \sum_{n=1}^{N} f(T_n)$ on the eigenvalues (with $f(T) \to 0$ for $T \to 0$ and $f(T) = \mathcal{O}(1)$ for $T \lesssim 1$) fluctuates with root-mean-square value of order unity

$$\operatorname{rms} \mathcal{L} \equiv \left(\langle \mathcal{L}^2 \rangle - \langle \mathcal{L} \rangle^2\right)^{1/2} = \mathcal{O}(1), \tag{51}$$

independent of N, L, or l as long as $l \ll L \ll Nl$. An additional requirement is that the junction is not much wider than long: $W_y, W_z \lesssim L$ where W_y and W_z are the widths in the y- and z-direction. If the junction is much wider than long in one direction, then $\operatorname{rms} \mathcal{L}$ is of order $(W_y W_z / L^2)^{1/2}$. It is not necessary for these results that $f(T)$ is itself a linear function of T. In the special case that $f(T) = T$, the coefficient on the right-hand-side of Eq. (51) is known [10]:

$$\operatorname{rms} \sum_{n=1}^{N} T_n \equiv C_{\text{UCF}} = 0.365, \text{ if } W_y, W_z \ll L. \tag{52}$$

According to Eq. (35) the supercurrent $I(\phi)$ is a linear statistic on T_n, so that we conclude that (at zero temperature) $\operatorname{rms} I(\phi) \simeq e\Delta_0/\hbar$ times a function of ϕ which is independent of the length of the junction or the degree of disorder. At small ϕ we may linearize in T and use Eq. (52), with the result

$$\operatorname{rms} I(\phi) = \tfrac{1}{2} C_{\text{UCF}}(e\Delta_0/\hbar)\phi \tanh(\Delta_0/2k_B T) + \mathcal{O}(\phi^2). \tag{53}$$

The critical current $I_c \equiv \max I(\phi) \equiv I(\phi_c)$ is not by definition a linear statistic on T_n, since the phase ϕ_c at which the maximum supercurrent is reached depends itself on all the transmission eigenvalues. It is therefore not possible in general to write I_c in the form $\sum_n f(T_n)$, as required for a linear statistic. However, I_c does become a linear statistic in the limit $L/Nl \ll 1$ (or equivalently $\langle G \rangle \gg e^2/h$) of a good conductor (which is the appropriate limit for mesoscopic fluctuations). To see this, we write

$$I(\phi) = \langle I(\phi) \rangle [1 + \epsilon X(\phi)], \ \epsilon \equiv L/Nl, \tag{54}$$

where the function $X(\phi)$ accounts for the sample-to-sample fluctuations of $I(\phi)$ around the ensemble average $\langle I(\phi) \rangle$. One has $\langle X \rangle = 0$, $\operatorname{rms} X = \mathcal{O}(1)$. We now expand I_c to lowest order in ϵ. Defining $\phi_c \equiv \phi_c^{(0)} + \epsilon \phi_c^{(1)}$, where $\max \langle I(\phi) \rangle \equiv \langle I(\phi_c^{(0)}) \rangle$, one can write

$$
\begin{aligned}
I_c &\equiv \langle I(\phi_c) \rangle [1 + \epsilon X(\phi_c)] \\
&= \left\langle I(\phi_c^{(0)} + \epsilon \phi_c^{(1)}) \right\rangle \left[1 + \epsilon X(\phi_c^{(0)} + \epsilon \phi_c^{(1)})\right] \\
&= \left\langle I(\phi_c^{(0)}) \right\rangle \left[1 + \epsilon X(\phi_c^{(0)}) + \mathcal{O}(\epsilon^2)\right] = I(\phi_c^{(0)}). \tag{55}
\end{aligned}
$$

In the third equality we have used that, by definition, $d\langle I \rangle/d\phi = 0$ at $\phi = \phi_c^{(0)}$. Since $I(\phi_c^{(0)})$ is a linear statistic on T_n (note that $\phi_c^{(0)}$ is by definition independent of T_n), we conclude that the critical current I_c is a linear statistic on the transmission eigenvalues in the limit $\epsilon \equiv L/Nl \to 0$, and hence that $\operatorname{rms} I_c = \operatorname{rms} I(\phi_c^{(0)}) \simeq e\Delta_0/\hbar$. The value of the numerical coefficient remains to be calculated.

Experimentally, sample-to-sample fluctuations are not easily studied. Instead, fluctuations in a given sample as a function of Fermi energy E_F are more accessible. Josephson junctions consisting of a two-dimensional electron gas (2DEG) with superconducting contacts allow for variation of E_F in the 2DEG by means of a gate voltage [38]. Point-contact junctions can be defined in the 2DEG either lithographically or electrostatically (using split gates) [7]. For such a system one would expect that if E_F is varied on the scale of E_c, the low-temperature critical current will fluctuate by an amount of order $e\Delta_0/\hbar$, independent of the properties of the junction.

Acknowledgements

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