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Universal Quantum Signatures of Chaos in Ballistic Transport.

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Abstract. – The conductance of a ballistic quantum dot (having chaotic classical dynamics and being coupled by ballistic point contacts to two electron reservoirs) is computed on the single assumption that its scattering matrix is a member of Dyson's circular ensemble. General formulae are obtained for the mean and variance of transport properties in the orthogonal ($\beta = 1$), unitary ($\beta = 2$), and symplectic ($\beta = 4$) symmetry class. Applications include universal conductance fluctuations, weak localization, sub-Poissonian shot noise, and normal-metal-superconductor junctions. The complete distribution $P(g)$ of the conductance g is computed for the case that the coupling to the reservoirs occurs via two quantum point contacts with a single transmitted channel. The result $P(g) \propto g^{-1+\beta/2}$ is qualitatively different in the three symmetry classes.

The search for signatures of chaotic behaviour in quantum-mechanical systems [1] has recently been extended to semiconductor nanostructures known as «quantum dots» [2, 3]. A quantum dot is essentially a mesoscopic electron billiard, consisting of a ballistic cavity connected by two small holes to two electron reservoirs. An electron which is injected through one of the holes will either return through the same hole, with probability R, or be transmitted through the other hole, with probability T. Classically, the uniform (ergodic) exploration of the boundaries yields $T = R$, if the two holes are of the same size and sufficiently small that direct transmission (without boundary reflections) can be ignored.

For a *closed* quantum dot (without holes), it is well known that one of the quantum signatures of its classically chaotic character consists in the Wigner-Dyson distribution of the energy levels [4, 5]. The Wigner-Dyson distribution was originally derived by random matrix

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theory (RMT), and is characterized by a repulsion of nearby levels which depends only on the symmetry of the Hamiltonian. The quantum analogue of the ergodic exploration of the dot boundaries by the classical trajectories consists in the Porter-Thomas distribution of the eigenfunctions, as confirmed by numerical studies of their amplitude distribution at the boundaries [6]. The quantum dot with holes is an *open*, rather than a closed system. Just as the Wigner-Dyson distribution describes the Hamiltonian H of the closed system, Dyson's circular ensemble [7] provides the statistical properties of the scattering matrix S of the open system. For ballistic dots (without impurities) the statistical ensemble can be generated by a change of the Fermi energy, the magnetic field or the shape of the dot. To have spectral or scattering properties given by the universal RMT description for H or S can be actually regarded as a precise definition of the somewhat vague concept of «quantum chaos». To what extent a real ballistic cavity is close to this precise universal limit is the subject of the theory of quantum billiards [5, 8].

Assuming this definition of a quantum chaotic system—and this is our only assumption—we will calculate the statistics of the transmission and reflection eigenvalues of the quantum dot, and hence its transport properties. This allows us to determine the universal quantum signatures of chaos in ballistic transport. Our investigation was motivated by a remarkable calculation by Mello of the variance of the conductance in the circular unitary ensemble [9]. The approach presented below recovers his result as a special case, and puts the quantum transport theory for a ballistic chaotic billiard on the same footing as the established theory for a disordered wire.

Dyson's circular ensemble characterizes a system where all scattering processes are equally probable, subject to the constraints of current conservation and time-reversal and spin rotation symmetry. There exist three symmetry classes: if a magnetic field B is applied, S is only unitary ($\beta = 2$, unitary ensemble); when $B = 0$, S is a unitary symmetric matrix in the absence of spin-orbit scattering ($\beta = 1$, orthogonal ensemble) or otherwise a unitary self-dual quaternion matrix ($\beta = 4$, symplectic ensemble). The dimension of S is $2N \times 2N$, where N is the number of transverse modes at the Fermi level in each of the two leads connecting the dot to the reservoirs. The probability $P_\beta(\mathrm{d}S)$ to find S in a neighbourhood $\mathrm{d}S$ of some given S is

$$P_\beta(\mathrm{d}S) = \frac{1}{V_\beta}\,\mu_\beta(\mathrm{d}S), \tag{1}$$

where $V_\beta = \int \mu_\beta(\mathrm{d}S)$ is the total volume of the S-matrix space and $\mu_\beta(\mathrm{d}S)$ is the β-dependent measure of the neighbourhood $\mathrm{d}S$ of S. In the original work of Dyson [7], these measures are expressed in eigenvalue-eigenvector coordinates. This is a suitable representation to obtain the distribution of the scattering phase shifts, but is not very convenient for a study of conduction through the quantum dot. A transport property A can generally be expressed as a linear statistic $A = \sum_{n=1}^{N} f(T_n)$ on the transmission eigenvalues T_n. The T_n's are *not* eigenvalues of S and are not in any simple way related to the scattering phase shifts. Instead, T_n is an eigenvalue of the matrix product $tt^\dagger$, where the transmission matrix t is an $N \times N$ submatrix of S. The measures $\mu_\beta(\mathrm{d}S)$ have recently been calculated in the transmission-eigenvalue representation [10]. Since this technical advance is at the basis of our analysis, we briefly sketch the derivation for the orthogonal ensemble ($\beta = 1$).

For $\beta = 1$ the scattering matrix is unitary symmetric, so that it can be represented in the form $S = YY^{\mathrm{T}}$, where Y is unitary. Note that this decomposition is not unique. An infinitesimal neighbourhood $\mathrm{d}S$ of S is given by $\mathrm{d}S = iY\,\mathrm{d}Q\,Y^{\mathrm{T}}$, with $\mathrm{d}Q$ a real symmetric matrix. It has been shown by Dyson [7] that if the matrix elements $\mathrm{d}Q_{ij}$ vary through some small intervals of lengths $\mathrm{d}\mu_{ij}$, the measure μ_1 equals $\mu_1(\mathrm{d}S) = \prod_{i \leq j} \mathrm{d}\mu_{ij}$, independent of Y. We use this freedom to choose Y in the form

$$Y = \mathcal{U}\,\mathcal{O}\,\mathcal{S} = \begin{pmatrix} u & 0 \\ 0 & u' \end{pmatrix} \begin{pmatrix} ((1-\sqrt{\mathcal{R}})/2)^{1/2} & -((1+\sqrt{\mathcal{R}})/2)^{1/2} \\ ((1+\sqrt{\mathcal{R}})/2)^{1/2} & ((1-\sqrt{\mathcal{R}})/2)^{1/2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & i1 \end{pmatrix}, \tag{2}$$

where $\mathcal{R}$ is an $N \times N$ diagonal matrix with elements $R_n = 1 - T_n$. Since $i\,\mathrm{d}Q = \mathrm{d}Y^{\mathrm{T}} Y^* + Y^\dagger \mathrm{d}Y$, and since Y and $\mathrm{d}Y$ can be expressed in terms of the matrices $\mathcal{U}$, $\mathcal{O}$, $\mathcal{S}$ and their neighbourhoods $\mathrm{d}\mathcal{U}$ and $\mathrm{d}\mathcal{O}$, one can easily get $\mathrm{d}Q$ in this parametrization. The result is an expression for $\mu_1(\mathrm{d}S)$ in terms of the measures $\mu(\mathrm{d}\mathcal{U})$ and $\mu(\mathrm{d}\mathcal{R})$ associated with the matrices $\mathcal{U}$ and $\mathcal{R}$, times a Jacobian:

$$\mu_1(\mathrm{d}S) = \prod_{i<j} |R_i - R_j| \prod_i (1 - R_i)^{-1/2} \mu(\mathrm{d}\mathcal{R})\,\mu(\mathrm{d}\mathcal{U}). \tag{3}$$

Integration over the unitary matrix $\mathcal{U}$ gives the reflection eigenvalue distribution $P(R_1, R_2, \ldots, R_N)$ in the circular orthogonal ensemble. The calculations in the unitary [10] and symplectic [11] ensembles proceed similarly.

The final result is conveniently written in terms of a new set of variables $\lambda_n \in [0, \infty]$, related to the reflection and transmission eigenvalues by $R_n \equiv \lambda_n/(1+\lambda_n)$, $T_n \equiv 1/(1+\lambda_n)$. The distribution $P(\lambda_1, \lambda_2, \ldots, \lambda_N)$ of the λ-variables takes the form of a Gibbs distribution,

$$P(\{\lambda_n\}) = Z^{-1} \exp[-\beta \mathcal{H}(\{\lambda_n\})], \tag{4a}$$

$$\mathcal{H}(\{\lambda_n\}) = -\sum_{i<j} \ln|\lambda_i - \lambda_j| + \sum_i V_\beta(\lambda_i), \tag{4b}$$

$$V_\beta(\lambda) = \left(N + \frac{1}{2}\beta^{-1}(2-\beta)\right) \ln(1+\lambda), \tag{4c}$$

where Z is a normalization constant. The symmetry parameter $\beta \in \{1, 2, 4\}$ plays the role of an inverse temperature. The fictitious «Hamiltonian» $\mathcal{H}$ consists of a logarithmic pairwise interaction plus a one-body potential $V_\beta(\lambda)$. This potential is symmetry independent to order N, while the term of order N^0 depends on β.

Remarkably, the distribution (4) is identical to the global maximum-entropy ansatz for the transfer matrix of a diffusive conductor [12]—except for the one-body potential, which is different: the potential $V_{\mathrm{d}}(\lambda)$ for a disordered wire of length L and mean free path l is $V_{\mathrm{d}}(\lambda) = (Nl/L) \ln^2(\sqrt{\lambda} + \sqrt{1+\lambda}) + \mathcal{O}(N^0)$. The potential (4c), in contrast, contains no microscopic parameters and increases more slowly with λ. In the case of a disordered wire, it is known [13] that the logarithmic repulsion $-\ln|\lambda_i - \lambda_j|$ is only rigorously valid for the weakly reflected scattering channels ($\lambda_i, \lambda_j \ll 1$). In the ballistic chaotic dot, the logarithmic repulsion which we have found is a direct consequence of the basic assumption that the scattering matrix belongs to the circular ensemble.

We consider transport properties of the form $A = \sum_{n=1}^{N} a(\lambda_n)$. To calculate the expectation value $\langle A \rangle = \int_0^\infty a(\lambda)\rho(\lambda)\,\mathrm{d}\lambda$ for a ballistic chaotic system, we need the density $\rho(\lambda)$ of the λ's in the circular ensemble. For this purpose, we use Dyson's large-N expansion [7]

$$\int_0^\infty \frac{\rho(\lambda')}{\lambda - \lambda'}\,\mathrm{d}\lambda' + \frac{\beta - 2}{2\beta}\frac{\mathrm{d}}{\mathrm{d}\lambda} \ln \rho(\lambda) = \frac{\mathrm{d}}{\mathrm{d}\lambda} V_\beta(\lambda), \tag{5}$$

where $⨍$ denotes the principal value. We decompose $\rho = \rho_N + \delta\rho$ into a contribution ρ_N of order N (giving the «Boltzmann conductance») and a symmetry-dependent correction $\delta\rho$ of order N^0 (responsible for the «weak-localization effect»). From eqs. (4) and (5) we find, order by order [1],

$$⨍_0^\infty \frac{\rho_N(\lambda')}{\lambda - \lambda'} \mathrm{d}\lambda' = \frac{N}{1+\lambda} \Rightarrow \rho_N(\lambda) = \frac{N}{\pi(1+\lambda)\sqrt{\lambda}}, \tag{6}$$

$$⨍_0^\infty \frac{\delta\rho(\lambda')}{\lambda - \lambda'} \mathrm{d}\lambda' = \frac{\beta-2}{4\beta}\frac{1}{\lambda} \Rightarrow \delta\rho(\lambda) = \frac{\beta-2}{4\beta}\delta_+(\lambda), \tag{7}$$

where the one-sided delta-function satisfies $\int_0^\infty \delta_+(\lambda)\,\mathrm{d}\lambda = 1$. The transmission eigenvalue density $\rho_N(T) = \rho_N(\lambda)|\mathrm{d}\lambda/\mathrm{d}T|$ (with $T = (1+\lambda)^{-1}$) has a *bimodal* distribution with peaks near-unit and near-zero transmission. This is a familiar result for a diffusive conductor, but was not previously established for a chaotic dot. Both peaks have the same strength for the latter, while the relationship between the strengths of the peaks for a diffusive wire is governed by l/L. This difference can be understood by comparing eq. (6) and the eigenvalue density $\rho^{(\mathrm{d})}$ for a disordered wire. To order N and for $L \gg l$ one has [14]

$$\rho_N^{(\mathrm{d})}(\lambda) = \frac{Nl}{2L}\frac{1}{\sqrt{\lambda(1+\lambda)}}, \qquad \text{for } \lambda < \lambda_c \simeq \frac{1}{4}\exp[2L/l]. \tag{8}$$

The density goes to zero abruptly near a cut-off $\lambda_c \gg 1$, in such a way that $\int_0^\infty \rho_N^{(\mathrm{d})}(\lambda)\,\mathrm{d}\lambda = N$.

The term of order N^0, which yields the weak-localization (antilocalization) corrections for a disordered wire when $\beta \neq 2$, is given by [15]

$$\delta\rho^{(\mathrm{d})}(\lambda) = \frac{\beta-2}{2\beta}[\delta_+(\lambda) + (\lambda+\lambda^2)^{-1/2}(4\ln^2[\sqrt{\lambda}+\sqrt{1+\lambda}] + \pi^2)^{-1}], \tag{9}$$

which is not as strongly peaked near $\lambda = 0$ as the delta-function result (7) for a chaotic dot.

We now use eqs. (6) and (7) to calculate the expectation value $\langle T \rangle$ of the total transmission probability $T = \mathrm{Tr}\, tt^\dagger = \sum_n (1+\lambda_n)^{-1}$. According to the Landauer formula, T equals the conductance g (measured in units of $2e^2/h$, the factor 2 comes from spin degeneracy). The result is

$$\langle T \rangle = \frac{1}{2}N + \delta T, \qquad \delta T = \frac{1}{4}\beta^{-1}(\beta-2). \tag{10}$$

For $\beta = 2$, one finds $\langle T \rangle = (1/2)N = \langle R \rangle$ (where $\langle R \rangle = N - \langle T \rangle$ is the local reflection probability). This is the quantum analogue of what we expect from the «ergodic» exploration

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[1] The result (7) for the $\mathcal{O}(N^0)$ correction $\delta\rho$ holds only for $\lambda \ll N^{2/3}$, because for larger λ's the $\mathcal{O}(N)$ contribution $\rho_N \approx N\lambda^{-3/2}$ no longer dominates the density and the large-N expansion fails. The large-λ tail ensures that $\int_0^\infty \delta\rho(\lambda)\,\mathrm{d}\lambda = 0$, but is irrelevant for the conductance of the quantum dot. (The range $\lambda \gtrsim N^{2/3} \gg 1$ gives a contribution to g of order $N^{-2/3}$, which can be neglected relative to the contribution of order 1 which is retained.)

of the dot boundaries by the classical trajectories. Quantum interference then breaks the equality $\langle T \rangle = \langle R \rangle$ by an amount δT, due to weak localization ($\beta = 1$) or anti-localization ($\beta = 4$). The value $\delta T = -1/4$ for $\beta = 1$ is in agreement with the result of Iida, Weidenmüller and Zuk for a similar model [16], and demonstrates the point raised in ref. [17]: weak localization is not only given by coherent backscattering, but has an off-diagonal (in mode index and classical trajectory labels) component. In the same way one can compute the average of any other linear statistics. We give two examples to illustrate the generality of our approach.

The first example is the shot-noise power P which is given by [18] $P = P_0 \operatorname{Tr} tt^\dagger(1 - tt^\dagger)$, with $P_0 = (2e^2/h) 2eU$ (U is the applied voltage). In this case $a(\lambda) = P_0 \lambda(1 + \lambda)^{-2}$. Since $\delta\rho(\lambda)\, a(\lambda) \equiv 0$ for any β, there is *no* weak-localization correction for the shot noise of a chaotic dot—in contrast to a diffusive conductor, where a weak-localization effect does exist [19]. Integration of $\rho_N(\lambda)\, a(\lambda)$ gives the average shot noise power $\langle P \rangle = \frac{1}{8} NP_0 = \frac{1}{4} P_{\text{Poisson}}$, which is four times smaller than the Poisson noise $P_{\text{Poisson}} = gP_0 = 2eI$ associated with a current I of uncorrelated electrons. The 1/4 reduction in a chaotic dot is to be compared with the 1/3 reduction of shot noise in a diffusive conductor [20].

The second example is the conductance G_{NS} of the dot if one of the two attached reservoirs is a superconductor. This case corresponds to [21] $a(\lambda) = (4e^2/h)(1 + 2\lambda)^{-2}$ if $\beta = 1, 4$ (G_{NS} is not a linear statistic for $\beta = 2$). Again, we find a noticeable difference between the diffusive disordered wire and the ballistic chaotic dot. In the disordered case, the conductance G_{N} in the normal state is unchanged if one of the reservoirs becomes superconducting ($\langle G_{\text{N}} \rangle = \langle G_{\text{NS}} \rangle$, up to a weak-localization correction of order N^0). In the ballistic chaotic case, eq. (6) yields (to order N): $\langle G_{\text{NS}} \rangle = (2e^2/h)(2 - \sqrt{2})N$, which differs to order N from the result $\langle G_{\text{N}} \rangle = (2e^2/h)(N/2)$ in the normal state.

So far we have focused on the expectation values in the circular ensemble. Fluctuations around the average in this ensemble can be computed using the general formulae of ref. [21], which hold for any ensemble with a logarithmic interaction (regardless of the form of the one-body potential). The variance in the large-N limit is given by

$$\operatorname{Var} A = -\frac{1}{\beta}\frac{1}{\pi^2}\int_0^\infty \mathrm{d}\lambda \int_0^\infty \mathrm{d}\lambda' \left(\frac{\mathrm{d}a(\lambda)}{\mathrm{d}\lambda}\right)\left(\frac{\mathrm{d}a(\lambda')}{\mathrm{d}\lambda'}\right) \ln\left|\frac{\sqrt{\lambda} - \sqrt{\lambda'}}{\sqrt{\lambda} + \sqrt{\lambda'}}\right| . \tag{11}$$

This can be used to compute the analogue of the «Universal Conductance Fluctuations» (UCF) in a ballistic chaotic cavity. One obtains, for example, $\operatorname{Var} g = 1/8\beta$, $\operatorname{Var} P/P_0 = 1/64\beta$ for the fluctuations in the conductance and shot noise, respectively. The $(1/\beta)$-dependence is the same as for a disordered wire, but the numerical coefficients are somewhat different due to the difference in interaction potential [13].

The results for mean and variance given above require $N \gg 1$. The opposite regime $N = 1$ is also of interest. This would apply to a semiconductor quantum dot which is coupled to the reservoirs by two quantum point contacts with a quantized conductance of $2e^2/h$. The probability distribution (4) reduces for $N = 1$ to $P(\lambda) = (1/2)\beta(1 + \lambda)^{-1-\beta/2}$. This implies for the (dimensionless) conductance $g = (1 + \lambda)^{-1}$ the distribution

$$P(g) = \frac{1}{2}\beta g^{-1+\beta/2}, \qquad 0 \leqslant g \leqslant 1 . \tag{12}$$

This is a remarkable result: in the presence of magnetic field ($\beta = 2$), any value of the conductance between 0 and $2e^2/h$ is equally probable. In non-zero field it is more probable to find a small than a large conductance, provided that the boundary scattering preserves spin rotation symmetry ($\beta = 1$). In the presence of spin-orbit scattering at the boundary ($\beta = 4$), however, a large conductance is more probable than a small one. To observe this qualitatively different behaviour presents a challenge for experimentalists.

In summary, we have calculated the distribution of the transmission and reflection eigenvalues characterizing Dyson's circular ensemble. Relying on a definition of «quantum chaos» based on the applicability of this ensemble to describe scattering in a ballistic chaotic cavity, we have extracted from the joint probability distribution (4) the expectation values and the fluctuations of arbitrary linear statistics related to quantum transport. We have mainly stressed the differences between ballistic chaotic dots and disordered wires. Another important point, which we underline in conclusion, consists in the qualitative differences existing between a fully chaotic dot and a dot where the classical dynamics is integrable and for which larger quantum fluctuations are generally expected [2, 3].

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Additional remark.

Upon completion of this manuscript we received a preprint by Baranger and Mello, in which some of our results are obtained by a different method.

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