Scaling Theory of Conduction through a Normal-Superconductor Microbridge

C W J Beenakker, B Rejaei, and J A Melsen

Instituut-Lorentz, University of Leiden, P O Box 9506, 2300 RA Leiden, The Netherlands

(Received 23 December 1993)

The length dependence of the resistance of a disordered normal-metal wire attached to a superconductor is computed. The scaling of the transmission eigenvalue distribution with length is obtained exactly in the metallic limit by a transformation onto the isobaric flow of a two-dimensional ideal fluid. The resistance has a *minimum* for lengths near l/Γ, with l the mean free path and Γ the transmittance of the superconductor interface.

The resistance of a disordered wire increases with its length. In the metallic regime (length L much less than the localization length) the increase is linear, up to relatively small quantum corrections. The linear scaling seems obviously true regardless of the nature of the contacts to the wire, which would just contribute an additive, L-independent contact resistance according to Ohm's law. This is correct if the contacts are in the normal state. The purpose of this paper is to demonstrate and explain the complete breakdown of the linear scaling if one of the contacts is in the superconducting state. The resistance $R_{NS}(L)$ of a disordered normal-metal (N) wire (with mean free path l) connected to a superconductor (S) by a tunnel barrier (with transmission probability Γ) has a *minimum* when $L \simeq l/\Gamma$.

The existence of a resistance minimum, and its destruction by a voltage or magnetic field, was first proposed by van Wees *et al* [1], to explain the sharp dip in the differential resistance discovered by Kastalsky *et al* [2]. A similar anomaly has since been observed in a variety of semiconductor-superconductor junctions [3], and also in computer simulations [4,5]. A nonmonotonous L dependence of R_{NS} is implicit in the work of Volkov, Zaitsev, and Klapwijk [6], while Hekking and Nazarov [7] obtained a monotonous $R_{NS}(L)$ for a high tunnel barrier. Very recently, Nazarov [8], using a Green's function technique, has confirmed the results of Ref [6], in a highly interesting paper which has some overlap with the present work.

Our analysis builds on two theoretical results. The first result we use is a Landauer-type formula [9] for the conductance $G_{NS} \equiv 1/R_{NS}$,

$$ G_{NS} = \frac{4e^2}{h} \sum_{n=1}^{N} \left(\frac{T_n}{2 - T_n} \right)^2 \qquad (1) $$

The numbers $T_n \in [0, 1]$ are the eigenvalues of the matrix product $tt^\dagger$, with t the $N \times N$ transmission matrix of the disordered normal wire plus tunnel barrier (N being the number of transverse modes at the Fermi energy E_F). Equation (1) holds in the zero-temperature, zero-voltage, and zero-magnetic-field limit. Terms of order $(\Delta/E_F)^2$ are neglected (with Δ the superconducting energy gap), as well as contributions from disorder in the

superconductor (Disorder in the superconductor will increase the effective length L of the disordered region by an amount of the order of the superconducting coherence length)

The second result we use is a scaling equation [10],

$$ \frac{\partial}{\partial s}\rho(\lambda, s) = -\frac{2}{N}\frac{\partial}{\partial \lambda}\lambda(1 + \lambda)\rho(\lambda, s)\frac{\partial}{\partial \lambda}\int_0^\infty d\lambda'\, \rho(\lambda', s) \times \ln|\lambda - \lambda'|, \qquad (2) $$

where $s \equiv L/l$. We have made the conventional change of variables from T_n to $\lambda_n \equiv (1 - T_n)/T_n$, with $\lambda_n \in [0, \infty)$. The density ρ of the λ's is defined by $\rho(\lambda, L) = \langle \sum_n \delta(\lambda - \lambda_n) \rangle$, with $\langle \ \rangle$ the ensemble average. Equation (2) describes the scaling of the eigenvalue density with the length L of the disordered normal region. In this formulation the tunnel barrier at the NS interface appears as an initial condition,

$$ \rho(\lambda, 0) = N\delta\big(\lambda - (1 - \Gamma)/\Gamma\big), \qquad (3) $$

where Γ is the transmission probability of the barrier (For simplicity we assume a mode-independent transmission probability, i e , all T_n's equal to Γ when $L = 0$). Once ρ is known, one can compute the ensemble averaged conductance $\langle G_{NS} \rangle$ from Eq (1),

$$ \langle G_{NS} \rangle = \frac{4e^2}{h} \int_0^\infty d\lambda\, \rho(\lambda, s)(1 + 2\lambda)^{-2} \qquad (4) $$

The nonlinear diffusion equation (2) was derived by Mello and Pichard [10] from a Fokker-Planck equation [11,12] for the joint distribution function of all N eigenvalues, by integrating out $N - 1$ eigenvalues and taking the large-N limit. This limit restricts its validity to the metallic regime ($N \gg L/l$), and is sufficient to determine the leading order contribution to the average conductance, which is $\mathcal{O}(N)$. The weak-localization correction, which is $\mathcal{O}(1)$, is neglected [13]. These $\mathcal{O}(1)$ corrections depend on the magnetic field B, whereas the $\mathcal{O}(N)$ contributions to ρ described by Eq (2) are insensitive to B. However, the relationship (4) between G_{NS} and ρ holds only for $B = 0$. (For $B \neq 0$, G_{NS} depends not only on the eigenvalues of $tt^\dagger$, but also on the eigenvectors [9].) A priori, Eq (2) holds only for a wire geometry (length L

much greater than width W), because the Fokker-Planck equation from which it is derived [12] requires $L \gg W$. Numerical simulations [14] indicate that the geometry dependence only appears in the $\mathcal{O}(1)$ corrections, and that the $\mathcal{O}(N)$ contributions are essentially the same for a wire, square, or cube.

Our method of solution is a variation on Carleman's method [15]. We introduce an auxiliary function,

$$F(z,s) = \int_0^\infty d\lambda' \frac{\rho(\lambda', s)}{z - \lambda'}, \qquad (5)$$

which is analytic in the complex z plane cut by the positive real axis. Furthermore,

$$\lim_{|z| \to \infty} F(z,s) = N/z. \qquad (6)$$

The function F has a discontinuity for $z = \lambda \pm i\epsilon$ (with $\lambda > 0$ and ϵ a positive infinitesimal). The limiting values $F_\pm(\lambda, s) \equiv F(\lambda \pm i\epsilon, s)$ are

$$F_\pm = \pm \frac{\pi}{i} \rho(\lambda, s) + \frac{\partial}{\partial \lambda} \int_0^\infty d\lambda' \rho(\lambda', s) \ln|\lambda - \lambda'|. \qquad (7)$$

Combination of Eqs. (2) and (7) gives

$$N \frac{\partial}{\partial s}(F_+ - F_-) = -\frac{\partial}{\partial \lambda} \lambda(1 + \lambda)(F_+^2 - F_-^2), \qquad (8)$$

which implies that the function

$$\mathcal{F}(z,s) = N \frac{\partial}{\partial s} F(z,s) + \frac{\partial}{\partial z} z(1 + z) F^2(z,s) \qquad (9)$$

is analytic in the whole complex plane, including the real axis. Moreover, $\mathcal{F} \to 0$ for $|z| \to \infty$, in view of Eq. (6). We conclude that $\mathcal{F} \equiv 0$, since the only analytic function which vanishes at infinity is identically zero.

It is convenient to make the mapping $z = \sinh^2 \zeta$ of the z plane onto the strip $\mathcal{S}$ in the ζ plane between the lines $y = 0$ and $y = -\pi/2$, where $\zeta = x + iy$. The mapping is conformal if we cut the z plane by the two half lines $\lambda > 0$ and $\lambda < -1$ on the real axis. On $\mathcal{S}$ we define the auxiliary function $U = U_x + iU_y$ by

$$U(\zeta, s) \equiv \frac{F}{2N} \frac{dz}{d\zeta} = \frac{\sinh 2\zeta}{2N} \int_0^\infty d\lambda' \frac{\rho(\lambda', s)}{\sinh^2 \zeta - \lambda'}. \qquad (10)$$

The equation $\mathcal{F} \equiv 0$ now takes the form

$$\frac{\partial}{\partial s} U(\zeta, s) + U(\zeta, s) \frac{\partial}{\partial \zeta} U(\zeta, s) = 0, \qquad (11)$$

which we recognize as *Euler's equation* of hydrodynamics: (U_x, U_y) is the velocity field in the (x, y) plane of a two-dimensional ideal fluid at constant pressure.

Euler's equation is easily solved. For initial condition $U(\zeta, 0) = U_0(\zeta)$ the solution to Eq. (11) is

$$U(\zeta, s) = U_0\big(\zeta - sU(\zeta, s)\big). \qquad (12)$$

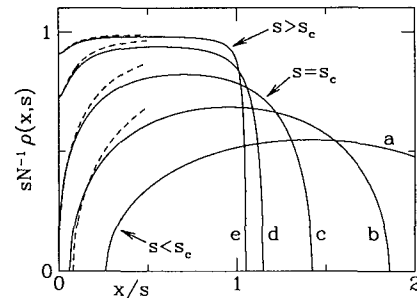To restrict the flow to $\mathcal{S}$ we demand that both ζ and

FIG. 1. Eigenvalue density $\rho(x, s)$ as a function of x (in units of $s = L/l$) for $\Gamma = 0.1$. Curves a,b,c,d,e are for $s = 2, 4, 9, 30, 100$, respectively. The solid curves are from Eq. (12), the dashed curves from Eq. (16). The resistance minimum is associated with the collision of the density profile with the boundary at $x = 0$, for $s = s_c = (1 - \Gamma)/\Gamma$.

$\zeta - sU(\zeta, s)$ lie in $\mathcal{S}$. From U we obtain the eigenvalue density, first in the x variables

$$\rho(x, s) = (2N/\pi)U_y(x - i\epsilon, s), \qquad (13)$$

and then in the λ variables ($\lambda = \sinh^2 x$)

$$\rho(\lambda, s) \equiv \rho(x, s)|dx/d\lambda| = \rho(x, s)|\sinh 2x|^{-1}. \qquad (14)$$

Equations (12)–(14) represent the *exact* solution of the nonlinear diffusion equation (2), for *arbitrary* initial conditions.

The initial condition (3) corresponds to

$$U_0(\zeta) = \tfrac{1}{2} \sinh 2\zeta \, (\cosh^2 \zeta - \Gamma^{-1})^{-1}. \qquad (15)$$

The solution of the implicit equation (12) is plotted in Fig. 1 (solid curves), for $\Gamma = 0.1$ and several values of $s = L/l$. For $s \gg 1$ and $x \ll s$ it simplifies to

$$x = \tfrac{1}{2} \text{arccosh} \, \tau - \tfrac{1}{2} \Gamma s(\tau^2 - 1)^{1/2} \cos \sigma, \qquad (16a)$$

$$\sigma \equiv \pi s N^{-1} \rho(x, s), \quad \tau \equiv \sigma(\Gamma s \sin \sigma)^{-1}, \qquad (16b)$$

shown dashed in Fig. 1. Equation (16) has been obtained independently by Nazarov [8]. For $s = 0$ (no disorder), ρ is a delta function at x_0, where $\Gamma \equiv 1/\cosh^2 x_0$. On adding disorder the eigenvalue density rapidly spreads along the x axis (curve a), such that $\rho \le N/s$ for $s > 0$. The sharp edges of the density profile, so uncharacteristic for a diffusion profile, reveal the hydrodynamic nature of the scaling equation (2). The upper edge is at

$$x_{\max} = s + \tfrac{1}{2} \ln(s/\Gamma) + \mathcal{O}(1). \qquad (17)$$

Since L/x has the physical significance of a localization length [12], this upper edge corresponds to a minimum localization length $\xi_{\min} = L/x_{\max}$ of order l. The lower edge at $x_{\min}$ propagates from x_0 to 0 in a "time" $s_c = (1 - \Gamma)/\Gamma$. For $1 \ll s \le s_c$ one has

$$x_{\min} = \tfrac{1}{2} \text{arccosh} \, (s_c/s) - \tfrac{1}{2}[1 - (s/s_c)^2]^{1/2}. \qquad (18)$$

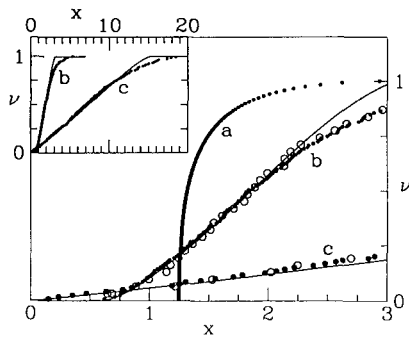It follows that the maximum localization length $\xi_{\max} = $

FIG. 2. Comparison between theory and simulation of the integrated eigenvalue density $\nu(x,s) \equiv N^{-1} \int_0^x dx' \, \rho(x',s)$, for $\Gamma = 0.18$. The labels a,b,c indicate, respectively, $s = 0, 0.7, 11.7$. Solid curves are from Eq. (12); data points are the x_n's from the simulation plotted in ascending order versus $n/N \equiv \nu$ (filled data points are for a square geometry, open points for an aspect ratio $L/W = 3.8$). The theoretical curve for $s = 0$ is a step function at $x_0 = 1.5$ (not shown). The inset shows the full range of x; the main plot shows only the small-x region, to demonstrate the disorder-induced opening of channels for tunneling through a barrier. (Note that, since $T \equiv 1/\cosh^2 x$, x near zero corresponds to near-unit transmission.)

$L/x_{\min}$ *increases* if disorder is added to a tunnel junction. This paradoxical result, that disorder enhances transmission, becomes intuitively obvious from the hydrodynamic correspondence, which implies that $\rho(x,s)$ spreads both to larger *and* smaller x as the fictitious time s progresses. When $s = s_c$ the diffusion profile hits the boundary at $x = 0$ (curve c), so that $x_{\min} = 0$. This implies that for $s > s_c$ there exist scattering states (eigenfunctions of $tt^{\dagger}$) which tunnel through the barrier with near-unit transmission probability, even if $\Gamma \ll 1$. The number N_{open} of transmission eigenvalues close to 1 (so-called "open channels" [16]) is of the order of the number of x_n's in the range 0 to 1 (since $T_n \equiv 1/\cosh^2 x_n$ vanishes exponentially if $x_n > 1$). For $s \gg s_c$ (curve e) we estimate

$$N_{\text{open}} \simeq \rho(0,s) = N(s + \Gamma^{-1})^{-1}, \tag{19}$$

where we have used Eq. (16).

To test these analytical results we have carried out numerical simulations similar to those reported in Ref. [5]. The disordered normal region was modeled by a tight-binding Hamiltonian on a square lattice (lattice constant a), with a random impurity potential at each site (uniformly distributed between $\pm \frac{1}{2} U_D$). The tunnel barrier was introduced by assigning a nonrandom potential energy $U_B = 2.3 E_F$ to a single row of sites at one end of the lattice, corresponding to a mode-averaged transmission probability $\Gamma = 0.18$. The Fermi energy was chosen at $E_F = 1.5 u_0$ from the band bottom (with $u_0 \equiv \hbar^2/2ma^2$). We chose U_D between 0 and $1.5 u_0$, corresponding to s between 0 and 11.7 [17]. Two geometries were considered: $L = W = 285a$ (corresponding to $N = 119$), and $L = 285a$, $W = 75a$ (corresponding to $N = 31$).
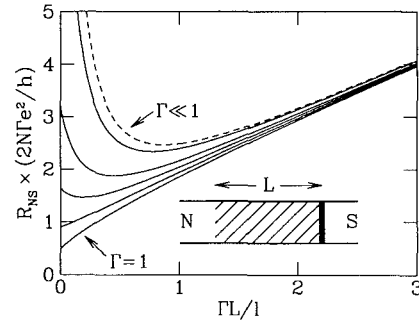
FIG. 3. Dependence of the resistance R_{NS} on the length L of the disordered normal region (shaded in the inset), for different values of the transmittance Γ of the NS interface. Solid curves are computed from Eq. (20), for $\Gamma = 1, 0.8, 0.6, 0.4, 0.1$ from bottom to top. For $\Gamma \ll 1$ the dashed curve is approached, in agreement with Ref. [6].

In Fig. 2 we compare the integrated eigenvalue density $\nu(x,s) \equiv N^{-1} \int_0^x dx' \, \rho(x',s)$, which is the quantity following directly from the simulation. The points are raw data from a single sample. (Sample-to-sample fluctuations are small, because the x_n's are self-averaging quantities [12].) The simulation unambiguously demonstrates the appearance of open channels (x_n's near 0) on adding disorder to a tunnel barrier, and is in good agreement with our analytical result (12), *without any adjustable parameters*. No significant geometry dependence was found, as anticipated.

The average resistance of the NS junction is obtained directly from the complex velocity field U on the imaginary axis. From Eqs. (4) and (10) we find

$$R_{NS} = (h/2Ne^2) \lim_{\zeta \to -i\pi/4} (\partial U/\partial \zeta)^{-1}$$

$$= (h/2Ne^2)(s + Q^{-1}), \tag{20a}$$

$$Q = \frac{\phi}{s \cos\phi} \left(\frac{\phi}{\Gamma s \cos\phi}(1 + \sin\phi) - 1 \right), \tag{20b}$$

where $\phi \in (0, \pi/2)$ is determined by

$$\phi[1 - \tfrac{1}{2}\Gamma(1 - \sin\phi)] = \Gamma s \cos\phi. \tag{20c}$$

For $\Gamma \ll 1$ (or $s \gg 1$) Eq. (20c) simplifies to $\phi = \Gamma s \cos\phi$, hence $Q = \Gamma \sin\phi$, in precise agreement with Ref. [6]. The scaling of the resistance with length is plotted in Fig. 3. For $\Gamma = 1$ the resistance increases monotonically with L. The ballistic limit $L \to 0$ equals $h/4Ne^2$, half the Sharvin resistance of a normal junction because of Andreev reflection [18]. For $\Gamma \lesssim 0.5$ a resistance minimum develops, somewhat below $L = l/\Gamma$. The resistance minimum is associated with the crossover from a quadratic to a linear dependence of R_{NS} on $1/\Gamma$. The two asymptotic dependences are (for $\Gamma \ll 1$ and $s \gg 1$)

$$R_{NS} = (h/2Ne^2)s^{-1}\Gamma^{-2}, \quad \text{if } \Gamma s \ll 1, \tag{21a}$$

$$R_{NS} = (h/2Ne^2)(s + \Gamma^{-1}), \quad \text{if } \Gamma s \gg 1. \tag{21b}$$

to be contrasted with the classical series resistance [5],

$$R_{NS}^{\text{class}} = (h/2Ne^2)(s + 2\Gamma^{-2}),\qquad(22)$$

which holds if phase coherence is destroyed by a voltage or magnetic field Equation (22) would follow from a naive application of Ohm's law to the NS junction, with the tunnel barrier contributing an additive, disorder-independent amount to the total resistance The quadratic dependence on $1/\Gamma$ in Eqs. (21a) and (22) is as expected for tunneling into a superconductor, being a two-particle process [7]. The linear dependence on $1/\Gamma$ in Eq. (21b) was first noted in numerical simulations [5], as a manifestation of "reflectionless tunneling": It is as if one of the two quasiparticles can tunnel into the superconductor without reflection. Comparison of Figs. 1 and 3 provides the explanation. The resistance minimum occurs when the lower edge of the density profile reaches $x = 0$ (curve c in Fig. 1), and signals the appearance of scattering states which can tunnel through the barrier with probability close to 1. For $\Gamma s \gg 1$ (curve e), R_{NS} is dominated by the N_{open} transmission eigenvalues close to 1. From Eqs. (1) and (19) we estimate $R_{NS} \simeq h/e^2 N_{\text{open}} = (h/Ne^2)(s + \Gamma^{-1})$, up to a numerical prefactor, consistent with the asymptotic result (21b).

It is essential for the occurrence of a resistance minimum that $1/R_{NS}$ depends *nonlinearly* on the transmission eigenvalues. Indeed, if we compute the normal-state resistance $1/R_N = (2e^2/h)\sum_n T_n$ from the eigenvalue density (12), we find the linear scaling $R_N = (h/2Ne^2)(s + \Gamma^{-1})$ for all Γ and s. The crossover to a quadratic dependence on $1/\Gamma$ cannot occur in this case, because of the linear relation between $1/R_N$ and T_n in the normal state.

In summary, we have presented a scaling theory for the resistance of a normal-superconductor microbridge. The scaling of the density ρ of transmission eigenvalues with length L is governed by Euler's equation for the isobaric flow of a two-dimensional ideal fluid: L corresponds to time and ρ to the y component of the velocity field on the x axis, with L/x corresponding to the localization length This hydrodynamic correspondence provides an explanation for the resistance minimum which is both exact and intuitive

We thank Yu V Nazarov for valuable discussions and R A Jalabert for advice on the numerical simulations.

[1] B J van Wees, P de Vries, P Magnée, and T M Klapwijk, Phys Rev Lett **69**, 510 (1992)

[2] A Kastalsky, A W Kleinsasser, L H Greene, R Bhat, F P Milliken, and J P Harbison, Phys Rev Lett **67**, 3026 (1991)

[3] C Nguyen, H Kroemer, and E L Hu, Phys Rev Lett **69**, 2847 (1992), K -M H Lenssen *et al*, Surf Sci (to be published), S J M Bakker *et al* (to be published)

[4] Y Takane and H Ebisawa, J Phys Soc Jpn **62**, 1844 (1993)

[5] I K Marmorkos, C W J Beenakker, and R A Jalabert, Phys Rev B **48**, 2811 (1993)

[6] A F Volkov, A V Zaitsev, and T M Klapwijk, Physica (Amsterdam) **210C**, 21 (1993)

[7] F W J Hekking and Yu V Nazarov, Phys Rev Lett **71**, 1625 (1993)

[8] Yu V Nazarov (to be published)

[9] C W J Beenakker, Phys Rev B **46**, 12841 (1992)

[10] P A Mello and J -L Pichard, Phys Rev B **40**, 5276 (1989)

[11] O N Dorokhov, Pis'ma Zh Eksp Teor Fiz **36**, 259 (1982) [JETP Lett **36**, 318 (1982)], P A Mello, P Pereyra, and N Kumar, Ann Phys (N Y) **181**, 290 (1988)

[12] For a review, see A D Stone, P A Mello, K A Muttalib, and J -L Pichard, in *Mesoscopic Phenomena in Solids*, edited by B L Al'tshuler, P A Lee, and R A Webb (North-Holland, Amsterdam, 1991)

[13] It is possible to extend Eq (2) to include $\mathcal{O}(1)$ corrections to the density For a study of weak localization along these lines see C W J Beenakker, Phys Rev B **49**, 2205 (1994), A M S Macêdo and J T Chalker (to be published)

[14] K Slevin, J -L Pichard, and K A Muttalib, J Phys I (France) **3**, 1387 (1993)

[15] T Carleman, Ark Mat Astr Fys **16** (26), 1 (1922)

[16] Y Imry, Europhys Lett **1**, 249 (1986)

[17] The parameter s was computed numerically by equating $G_N = (2Ne^2/h)(1 + s)^{-1}$, where G_N is the conductance of the disordered normal region This is accurate for all s, see M J M de Jong, Phys Rev B **49**, 7778 (1994)

[18] G E Blonder, M Tinkham, and T M Klapwijk, Phys Rev B **25**, 4515 (1982)