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Random-matrix theory of parametric correlations in the spectra of disordered metals and chaotic billiards

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A random-matrix theory is developed for the adiabatic response to an external perturbation of the energy spectrum of a mesoscopic system The basic assumption is that spectral correlations are governed by level repulsion Following Dyson, the dependence of the energy levels on the perturbation parameter is modeled by a Brownian-motion process in a fictitious viscous fluid A Fokker–Planck equation for the evolution of the distribution function is solved to yield the correlation of level densities at different energies and different parameter values An approximate solution is obtained by asymptotic expansion and an exact solution by mapping onto a free-fermion model A generalization to multiple parameters is also considered, corresponding to Brownian motion in a fictitious world with multiple temporal dimensions Complete agreement is obtained with microscopic theory

1. Introduction

This is a theoretical investigation of the adiabatic response to an external perturbation of the energy spectrum of a complex quantum mechanical system. We consider a Hamiltonian $\mathcal{H}(X)$ which depends on a parameter X. The X-dependence of the energy levels $E_n(X)$, shown in fig. 1 by way of example, is taken from a calculation of the hydrogen atom in a magnetic field [1]. Only levels with the same cylindrical symmetry are shown. A weak X-dependence of the mean density of states is removed by a rescaling of the energy. What remains is an irregular oscillation of E_n as a function of X. Two levels which approach each other are repelled as X is increased further, leading to a sequence of avoided crossings at which the derivative $\dot{E}_n \equiv dE_n/dX$ changes sign. The average $\dot{E}_n$ is zero, averaged either over a range of X or over a range of n. The correlator of $E_n(X)$ and $E_m(X')$ is non-zero for nearby levels n, m and for nearby parameters X, X', and serves as a quantitative characterization of how the system responds to an external perturbation.
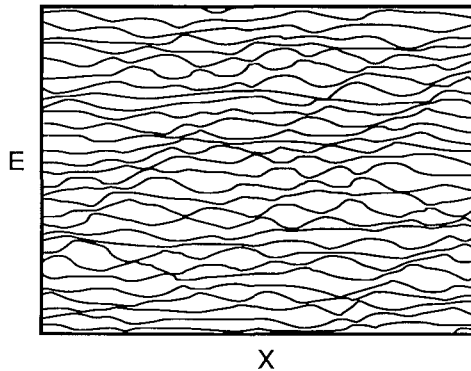
Fig 1 Typical parameter dependence of the energy levels, illustrating the phenomenon of level repulsion (parameter X and energy E in arbitrary units). This plot is based on a calculation of the spectrum of the hydrogen atom in a strong magnetic field by Goldberg et al [1]

Our investigation was motivated by a remarkable *universality* of the parametric correlations discovered by Szafer and Altshuler [2]. They considered a disordered metallic particle with the topology of a ring, enclosing a magnetic flux ϕ (measured in units of h/e). The energy levels $E_i(\phi)$ depend parametrically on ϕ. The dispersion is characterized by the "current density"

$$j(E, \phi) = \sum_i \delta(E - E_i(\phi)) \frac{\mathrm{d}}{\mathrm{d}\phi} E_i(\phi) .$$ (1.1)

Szafer and Altshuler applied diagrammatic perturbation theory [3] to compute the correlation function

$$C(\delta E, \delta\phi) = \overline{j(E, \phi)\, j(E + \delta E, \phi + \delta\phi)} ,$$ (1.2)

where the overline indicates an average over an ensemble of particles with different impurity configurations. The result was that the correlator $C(\delta E, \delta\phi)$ becomes universal for $\delta E = 0$,

$$C(0, X) = -\frac{2}{\pi^2 \beta X^2} ,$$ (1.3)

with $\beta = 2$ and $X = \delta\phi$. Eq. (1.3) is universal in the sense that it contains no microscopic parameters which characterize the particle, such as the diameter L, the mean level spacing Δ, the Fermi velocity v_F, or the mean free path l. It holds for $(\Delta/E_c)^{1/2} \ll \delta\phi \ll 1$, where $E_c \simeq \hbar v_F l / L^2$ is the Thouless energy.

Eq. (1.3) was proven for the case that the randomness in the energy spectrum is due to scattering by randomly located impurities. Numerical

simulations indicated that it applies generically to chaotic systems, even if there is no disorder and all randomness comes from scattering at irregularly shaped boundaries [2]. (The average in that case is taken over E and ϕ.) Further work on disordered systems by Simons and Altshuler [4] based on a non-perturbative "supersymmetry" formalism [5] has shown that eq. (1.3) with $\beta = 1$ and $X = \delta U$ applies if the external perturbation is a spatially fluctuating electrostatic potential $Us(r)$. (The function $s(r)$ should vary smoothly on the scale of the electron wavelength, with vanishing spatial average.) These analytical investigations assumed non-interacting electrons. Recent numerical simulations of a Hubbard model [6] have shown that eq. (1.3) remains valid in the presence of electron–electron interactions. The correlator (1.3) thus provides a universal quantum mechanical characterization of the response of a chaotic system to an external magnetic or electric field. Such universality calls for a *random-matrix theory* of parametric correlations. It is the purpose of this paper to present such a theory.

The basic principle of random-matrix theory (RMT) is that the spectral correlations are dominated by level repulsion [7]. Level repulsion is a direct consequence of the Jacobian $\Pi_{i<j} |E_i - E_j|^\beta$ associated with the transformation from the space of $N \times N$ Hermitian matrices $\mathcal{H}$ to the smaller space of N eigenvalues E_i. Level repulsion is universal in the sense that it is fully determined by the symmetry class of the Hamiltonian ensemble. There exist just three symmetry classes [8], characterized by the number $\beta = 1, 2, 4$ of independent components of the matrix elements of $\mathcal{H}$: $\beta = 1$ in zero magnetic field (real $\mathcal{H}$), $\beta = 2$ in non-zero field (complex $\mathcal{H}$), and $\beta = 4$ for strong spin-orbit scattering in zero magnetic field (quaternion $\mathcal{H}$). The three ensembles are called orthogonal ($\beta = 1$), unitary ($\beta = 2$), and symplectic ($\beta = 4$).

The Wigner–Dyson theory of random matrices yields a level-density correlation function $K(\delta E)$ which is *universal* for level separations δE greater than the mean level spacing Δ [9]. The function $K(\delta E)$ measures correlations between the level density

$$n(E, X) = \sum_{i=1}^{N} \delta(E - E_i(X)) \tag{1.4}$$

at different energies E and $E + \delta E$, but at the *same* value of the external parameter X:

$$K(\delta E) = \bar{n}(E, X)\, \bar{n}(E + \delta E, X) - \overline{n(E, X)\, n(E + \delta E, X)}. \tag{1.5}$$

The universal limiting form of K in the Wigner–Dyson theory is

$$K(\delta E) = \frac{1}{\pi^2 \beta \delta E^2} \qquad (1\ 6)$$

The universal correlator (1 6) was first obtained from RMT in the context of nuclear physics, and then applied to small metallic particles by Gorkov and Eliashberg [10] Much later, it was derived from a microscopic Hamiltonian by Efetov [5] and by Altshuler and Shklovskiĭ [3] The microscopic theory shows that eq (1 6) holds for a disordered metal in the energy range $\Delta \ll \delta E \ll E_c$ Numerical simulations have established that the Wigner–Dyson theory applies generically to systems with chaotic classical orbits [11], and also that it remains valid in the presence of electron–electron interactions [12]

The level-density correlation function (1 6) is thus universal in the same sense as the parametric correlation function (1 3) This suggests that it should be possible to derive eq (1 3) by some extension of the Wigner–Dyson theory to parameter-dependent Hamiltonians $\mathcal{H}(X)$ We will show that the *Brownian-motion model* used by Dyson [13] to construct a parameter-dependent ensemble of random matrices, yields parametric correlations in agreement with the microscopic theory of Altshuler, Simons and Szafer [2,4]

The outline of this paper is as follows In section 2 we formulate the problem of a random-matrix theory of parametric correlations and define the mapping onto Dyson's Brownian-motion model The correlation functions which we will calculate are summarized in section 3 In section 4 we present an asymptotic analysis which yields the correlation functions in the limit that the dimension N of the Hamiltonian matrix goes to infinity An exact result for the correlation functions in the Brownian-motion model for a special ensemble is given in section 5, and compared with the large-N result of the previous section In section 6 we extend the theory to parametric correlations involving multiple parameters We conclude in section 7, by comparing the results of random-matrix theory with the microscopic theory

The results of the asymptotic analysis were briefly announced in a recent letter [14]

2. Brownian-motion model

Starting point of our analysis is Dyson's Brownian-motion model [13] for the evolution of an ensemble of $N \times N$ Hermitian matrices as a function of an external parameter τ Dyson's idea was to regard τ as a fictitious "time", and to model the τ-dependence of the distribution of eigenvalues $P(\{E_n\}, \tau)$ by the one-dimensional Brownian motion of N classical particles at positions

$E_1(\tau), E_2(\tau), \ldots, E_N(\tau)$, in a fictitious viscous fluid with friction coefficient γ and temperature β^{-1}. Level repulsion is accounted for by the interaction potential $-\ln|E - E'|$ between particles at E and E'. The particles move in a confining potential $V(E)$, which is determined by the density of states.

With these definitions, $P(\{E_n\}, \tau)$ evolves according to the Fokker–Planck equation [13]

$$\gamma \frac{\partial P}{\partial \tau} = \sum_{i=1}^{N} \frac{\partial}{\partial E_i} \left(P \frac{\partial W}{\partial E_i} + \beta^{-1} \frac{\partial P}{\partial E_i} \right) , \tag{2.1}$$

$$W(\{E_n\}) = -\sum_{i<j} \ln|E_i - E_j| + \sum_i V(E_i) . \tag{2.2}$$

Eq. (2.1) has the $\tau \to \infty$ ("equilibrium") solution

$$P_{\text{eq}}(\{E_n\}) = Z^{-1} e^{-\beta W} , \tag{2.3}$$

where Z is such that P_{eq} is normalized to unity. Eq. (2.3), for $\beta = 1, 2$ and 4, is the eigenvalue distribution in the orthogonal, unitary, and symplectic ensemble [9]. It has the form of a Gibbs distribution, with the symmetry index β playing the role of inverse temperature. The fictitious energy W contains a logarithmic repulsive interaction plus a confining potential V. The function $V(E)$ is chosen such that P_{eq} yields the required average eigenvalue density (which depends on microscopic parameters, but is assumed to be independent of τ). The logarithmic interaction has a fundamental geometric origin: The factor $\exp(\beta \sum_{i<j} \ln|E_i - E_j|) = \prod_{i<j} |E_i - E_j|^\beta$ is the Jacobian associated with the transformation from the space of Hermitian matrices $\mathcal{H}$ to the smaller space of eigenvalues E_n.

The N-dimensional Fokker–Planck equation (2.1) is equivalent to N coupled Langevin equations,

$$\gamma \frac{dE_i}{d\tau} = -\frac{\partial W}{\partial E_i} + \mathcal{F}_i(\tau) , \qquad i = 1, 2, \ldots, N . \tag{2.4}$$

The random force $\mathcal{F}$ is a Gaussian white noise of zero mean, $\overline{\mathcal{F}_i(\tau)} = 0$, and variance

$$\overline{\mathcal{F}_i(\tau) \mathcal{F}_j(\tau')} = \frac{2\gamma}{\beta} \delta_{ij} \delta(\tau - \tau') . \tag{2.5}$$

The Fokker–Planck equation (2.1) and the Langevin equations (2.4) are equivalent levels of description of the Brownian motion [15].

The fictitious time τ needs still to be related to the perturbation parameter X in the Hamiltonian $\mathcal{H}(X)$ of the physical system one is modeling. Furthermore, we need a microscopic interpretation of the coefficient γ. These issues were not addressed in ref. [13], but are crucial for our purpose. Let $\tau = 0$ coincide with $X = 0$, so that

$$P(\{E_n\}, 0) = \prod_{i=1}^{N} \delta(E_i - E_i^0) , \tag{2.6}$$

with E_i^0 the eigenvalues of $\mathcal{H}(0)$. For $\tau > 0$ we then identify

$$\tau = X^2 . \tag{2.7}$$

This is the simplest relation between τ and X which is consistent with the average initial rate of change of the energy levels: On the one hand,

$$\overline{[E_i(X) - E_i^0]^2} = X^2 \overline{\left(\frac{\mathrm{d}E_i}{\mathrm{d}X}\right)^2} + \mathcal{O}(X^3) \tag{2.8}$$

is of order X^2 for small X, while on the other hand the ensemble average

$$\overline{[E_i(\tau) - E_i^0]^2} = \frac{2\tau}{\beta\gamma} + \mathcal{O}(\tau^2) \tag{2.9}$$

is of order τ for small τ, according to eqs. (2.4) and (2.5). The identification (2.7) also implies the relation

$$\frac{2}{\beta\gamma} = \overline{\left(\frac{\mathrm{d}E_i}{\mathrm{d}X}\right)^2} \tag{2.10}$$

between the friction coefficient and the mean-square rate of change of the energy levels.

Eq. (2.1) or (2.4) is the simplest description of the Brownian motion of the energy levels which is consistent with the equilibrium distribution (2.3). It is not the most general description: (i) One could include the "velocities" $\mathrm{d}E_n/\mathrm{d}\tau$ as independent stochastic variables, and work with a $2N$-dimensional evolution equation. In the case of Brownian motion in a physical fluid, the appropriate evolution equation is Kramer's equation [15]. It describes the dynamics of a Brownian particle on the time scale of the collisions with the fluid molecules. Since the viscous fluid in Dyson's Brownian-motion model is fictitious, it is not clear what the appropriate $2N$-dimensional evolution equation should be in this case. (ii) One could let γ be a matrix function $\gamma_{ij}(\{E_n\})$ of the configuration of energy levels. Such a configuration dependence (known in fluids as hydro-

dynamic interaction) would be an additional source of correlations, which is ignored. That is the basic assumption of Dyson's Brownian-motion model, that the spectral correlations are dominated by the fundamental geometric effect of level repulsion. The Brownian-motion model is known to provide a rigorous description of the transition between random-matrix ensembles of different symmetry [16]. However, there exists no derivation of eq. (2.1) or (2.4) from a microscopic Hamiltonian. Here we apply the Brownian-motion model to fluctuations around equilibrium in the random-matrix ensembles (2.3), and show that there is a complete agreement with the microscopic theory for disordered metals [2,4].

3. Correlation functions

We consider observables $A(X)$ of the form

$$A(X) = \sum_{i=1}^{N} a(E_i(X)) . \tag{3.1}$$

A quantity of the form (3.1) is called a *linear statistic* on the eigenvalues of $\mathscr{H}(X)$. The word "linear" indicates that A does not contain products of different eigenvalues, but the function $a(E)$ may well depend non-linearly on E. We assume that a varies smoothly on the scale of the mean level spacing Δ. (In particular, this excludes the case of a step function $a(E)$.) The correlator of A at two parameter values X and X' is $\overline{\delta A(X)\, \delta A(X')}$, where $\delta A \equiv A - \bar{A}$. The overline denotes an average over a range of X at constant $\delta X \equiv X' - X$ (or, alternatively, over an ensemble of statistically equivalent systems). Of particular interest is the integrated correlator

$$\chi_A = \int_0^\infty \mathrm{d}\delta X\, \overline{\delta A(X)\, \delta A(X + \delta X)} . \tag{3.2}$$

To compute the correlator of an arbitrary linear statistic we need the density correlation function

$$S(E, X, E', X') = \sum_{i,j} \overline{\delta(E - E_i(X))\, \delta(E' - E_j(X'))}$$

$$- \left(\sum_i \overline{\delta(E - E_i(X))} \right) \left(\sum_j \overline{\delta(E' - E_j(X'))} \right) . \tag{3.3}$$

The correlator of A at X and X' then follows from a double integration,

$$\overline{\delta A(X)\, \delta A(X')} = \int_{-\infty}^{\infty} \mathrm{d}E \int_{-\infty}^{\infty} \mathrm{d}E'\, a(E)\, a(E')\, S(E, X, E', X')\,. \tag{3.4}$$

We will also consider the correlator $\overline{\dot{A}(X)\, \dot{A}(X')}$ of the derivative $\dot{A} \equiv \mathrm{d}A/\mathrm{d}X$ of the linear statistic (3.1). This correlator follows from the density correlation function S by

$$\overline{\dot{A}(X)\, \dot{A}(X')} = \int_{-\infty}^{\infty} \mathrm{d}E \int_{-\infty}^{\infty} \mathrm{d}E'\, a(E)\, a(E')\, \frac{\partial^2}{\partial X\, \partial X'}\, S(E, X, E', X')\,. \tag{3.5}$$

Alternatively, we can compute the correlator of $\dot{A}$ from the current correlation function

$$C(E, X, E', X') = \sum_{i,j} \overline{\dot{E}_i(X)\, \dot{E}_j(X')\, \delta(E - E_i(X))\, \delta(E' - E_j(X'))}\,. \tag{3.6}$$

Since

$$\dot{A}(X) = \sum_{i=1}^{N} \dot{E}_i(X)\, \frac{\mathrm{d}}{\mathrm{d}E_i}\, a(E_i(X))\,, \tag{3.7}$$

one has, upon partial integration,

$$\overline{\dot{A}(X)\, \dot{A}(X')} = \int_{-\infty}^{\infty} \mathrm{d}E \int_{-\infty}^{\infty} \mathrm{d}E'\, a(E)\, a(E')\, \frac{\partial^2}{\partial E\, \partial E'}\, C(E, X, E', X')\,. \tag{3.8}$$

Comparison of eqs. (3.5) and (3.8) shows that the density and current correlation functions S and C are related by

$$\frac{\partial^2}{\partial E\, \partial E'}\, C(E, X, E', X') = \frac{\partial^2}{\partial X\, \partial X'}\, S(E, X, E', X')\,. \tag{3.9}$$

We assume that $S(E, X, E', X + \delta X) \equiv S(E, E', \delta X)$ depends only on the parameter increment δX. When considering a particular physical system, such as the hydrogen atom in a magnetic field, this may require a rescaling of energy levels, to eliminate a systematic drift in E_i versus X (cf. fig. 1). Since, by definition, $S(E, E', \delta X) = S(E', E, -\delta X)$, the correlators $\overline{\delta A(X)\, \delta A(X + \delta X)}$ and $\overline{\dot{A}(X)\, \dot{A}(X + \delta X)}$ are even functions of δX. Furthermore, we have the sum rule

$$\int\limits_0^\infty \mathrm{d}\delta X \; \overline{\dot{A}(X)\,\dot{A}(X+\delta X)} = 0 \,, \tag{3.10}$$

for any linear statistic.

In the following sections we will compute the density correlation function $S(E, E', X)$ from the Brownian-motion model described in section 2. In view of eq. (3.3) and the identification (2.7), we have the relation

$$
\begin{aligned}
S(E, E', X) = {} & \int\limits_{-\infty}^{\infty} \mathrm{d}E_1^0 \cdots \int\limits_{-\infty}^{\infty} E_N^0 \int\limits_{-\infty}^{\infty} \mathrm{d}E_1 \cdots \int\limits_{-\infty}^{\infty} \mathrm{d}E_N \\
& \times \left(\sum_{i,j} \delta(E - E_i^0)\,\delta(E' - E_j) \right) \\
& \times P_{\mathrm{eq}}(\{E_n^0\})\,[P(\{E_n\}, X^2) - P_{\mathrm{eq}}(\{E_n\})]
\end{aligned}
\tag{3.11}
$$

between the density correlation function and the solution $P(\{E_n\}, \tau)$ of the Fokker–Planck equation (2.1) with initial condition (2.6). Once we have S, the current correlation function C and the correlators of A and $\dot{A}$ follow from eqs. (3.4), (3.5) and (3.9).

4. Asymptotic solution

In this section we compute the large-N asymptotic limits of the density and current correlation functions $S(E, E', X)$ and $C(E, E', X)$. By "asymptotic" we mean that the expressions obtained hold in the limit $N \to \infty$ in the energy range $|E - E'| \gg \Delta$ for all X and in the parameter range $X \gg \Delta\sqrt{\gamma}$ for all E, E'. A justification of our asymptotic analysis will be given in section 5, when we compare with an exact result for $\beta = 2$. We assume that the $N \to \infty$ limit is accompanied by a rescaling of the confining potential $V(E)$ in eq. (2.2), such that the mean density of states remains the same. An explicit example of such a rescaling is given in section 5.

The first step in the analysis is to reduce the Fokker–Planck equation (2.1) to an evolution equation for the average density of eigenvalues

$$\rho(E, \tau) = \int\limits_{-\infty}^{\infty} \mathrm{d}E_1 \cdots \int\limits_{-\infty}^{\infty} \mathrm{d}E_N\, P(\{E_n\}, \tau) \sum_{i=1}^{N} \delta(E - E_i) \,. \tag{4.1}$$

This problem was solved by Dyson [13] in the limit $N \to \infty$, with the result

$$\gamma \frac{\partial}{\partial \tau} \rho(E, \tau) = \frac{\partial}{\partial E} \left[\rho(E, \tau) \frac{\partial}{\partial E} \left(V(E) - \int_{-\infty}^{\infty} dE' \, \rho(E', \tau) \ln|E - E'| \right) \right].$$

(4.2)

Corrections to eq. (4.2) are smaller by an order $N^{-1} \ln N$. To the same order, the equilibrium density $\rho_{eq}(E)$ (defined as in eq. (4.1) with P replaced by P_{eq}) satisfies [13]

$$\frac{\partial}{\partial E} \left(V(E) - \int_{-\infty}^{\infty} dE' \, \rho_{eq}(E') \ln|E - E'| \right) = 0.$$

(4.3)

To make this paper selfcontained, we present Dyson's derivation of eq. (4.2) in the appendix.

The next step is to reduce eq. (4.2) to a diffusion equation by linearizing ρ around ρ_{eq}. This is consistent with the large-N limit, since ρ is of order N while fluctuations in the density are of order one [13]. We write $\rho(E, \tau) = \rho_{eq}(E) + \delta\rho(E, \tau)$ and find, to first order in $\delta\rho$,

$$\frac{\partial}{\partial \tau} \delta\rho(E, \tau) = \frac{\partial}{\partial E} \int_{-\infty}^{\infty} dE' \, D(E, E') \frac{\partial}{\partial E'} \delta\rho(E', \tau),$$

(4.4)

$$D(E, E') = -\gamma^{-1} \rho_{eq}(E) \ln|E - E'|.$$

(4.5)

Eq. (4.4) has the form of a non-local diffusion equation, with diffusion kernel (4.5).

To proceed we assume a constant density of states $\rho_{eq}(E) \equiv \rho_0 \equiv 1/\Delta$ over the energy range of interest (which is the energy range where the function $a(E)$ in the linear statistic (3.1) differs appreciably from zero). The diffusion kernel can then be taken to be translationally invariant, $D(E, E') \equiv D(E' - E)$, with Fourier transform

$$D(k) \equiv \int_{-\infty}^{\infty} dE \, e^{ikE} D(E) = \frac{\rho_0 \pi}{\gamma |k|}.$$

(4.6)

Eq. (4.4) becomes an ordinary differential equation in k-space, with solution

$$\delta\rho(k, \tau) = \delta\rho(k, 0) \exp[-k^2 D(k)\tau].$$

(4.7)

In view of eq. (2.6), the initial condition on the eigenvalue density is

$$\rho(E, 0) = \sum_{i=1}^{N} \delta(E - E_i^0) \,. \tag{4.8}$$

We define the equilibrium average $\langle f \rangle_{\rm eq}$ of an arbitrary function $f(\{E_n^0\})$ of the initial configuration by

$$\langle f \rangle_{\rm eq} = \int_{-\infty}^{\infty} dE_1^0 \cdots \int_{-\infty}^{\infty} dE_N^0 \, P_{\rm eq}(\{E_n^0\}) \, f(\{E_n^0\}) \,. \tag{4.9}$$

Using also definition (4.1), eq. (3.11) for the density correlation function $S(E, E', X)$ can be written as

$$\begin{aligned}
S(E, E', X) &= \langle \rho(E, 0) \, \rho(E', X^2) \rangle_{\rm eq} - \rho_{\rm eq}(E) \, \rho_{\rm eq}(E') \\
&= \langle \delta\rho(E, 0) \, \delta\rho(E', X^2) \rangle_{\rm eq} \,.
\end{aligned} \tag{4.10}$$

In the second equality we have used that $\langle \rho(E, \tau) \rangle_{\rm eq} = \rho_{\rm eq}(E)$. The correlation function $K(E, E')$ is defined by (cf. eq. (1.5))

$$K(E, E') = -\langle \delta\rho(E, 0) \, \delta\rho(E', 0) \rangle_{\rm eq} = -S(E, E', 0) \,. \tag{4.11}$$

Over the energy range of a constant density of states, the correlation functions $S(E, E', X) \equiv S(E' - E, X)$ and $K(E, E') \equiv K(E' - E)$ are translationally invariant, with Fourier transforms $S(k, X)$ and $K(k)$. According to eqs. (4.7), (4.10) and (4.11), we have

$$S(k, X) = -K(k) \exp[-k^2 D(k) X^2] \,. \tag{4.12}$$

The function $K(k)$ is known [9]. In the limit $N \to \infty$, one has asymptotically

$$K(k) = -\frac{|k|}{\pi\beta} \,, \tag{4.13}$$

independent of $V(E)$ [17]. Eq. (4.13) is the Fourier transform of eq. (1.6), and holds for energy scales $k^{-1} \gg \Delta$ large compared to the mean level spacing. (This is the relevant regime, since the function $a(E)$ in the linear statistic (3.1) is assumed to be smooth on the scale of the level spacing.)

Combining eqs. (4.6), (4.12) and (4.13), we conclude that the density correlation function is given by

$$S(k, X) = \frac{|k|}{\pi\beta} \exp(-\xi^2 |k|) \,, \tag{4.14}$$

$$\xi \equiv X(\pi\rho_0/\gamma)^{1/2} \,. \tag{4.15}$$

The E-space correlation function becomes, upon inverse Fourier transformation,

$$S(E, X) = \frac{1}{2\pi^2\beta} \frac{\partial^2}{\partial E^2} \ln(\xi^4 + E^2) \,. \tag{4.16}$$

The current correlation function $C(E, E', X, X') \equiv C(E' - E, X' - X)$ is obtained from S by means of relation (3.9), which in k-space takes the form

$$C(k, X) = -\frac{1}{k^2} \frac{\partial^2}{\partial X^2} S(k, X) \,. \tag{4.17}$$

We find from eqs. (4.14) and (4.17):

$$C(k, X) = \frac{2\rho_0}{\beta\gamma} (1 - 2\xi^2|k|) \exp(-\xi^2|k|) \,, \tag{4.18}$$

$$C(E, X) = \frac{1}{2\pi^2\beta} \frac{\partial^2}{\partial X^2} \ln(\xi^4 + E^2) \,. \tag{4.19}$$

The asymptotic results for the correlation functions given above can be used to compute the $N \to \infty$ limit of the integrated correlator χ_A, defined by eqs. (3.2) and (3.4). The k-space expression for χ_A is

$$\chi_A = \frac{1}{2\pi} \int\limits_0^\infty dX \int\limits_{-\infty}^\infty dk \, |a(k)|^2 S(k, X) \,, \tag{4.20}$$

$$a(k) \equiv \int\limits_{-\infty}^\infty dE \, e^{ikE} a(E) \,. \tag{4.21}$$

Substituting the asymptotic formula (4.14), and carrying out the integral over X, we obtain the result

$$\chi_A = \frac{1}{2\pi^2\beta} (\gamma/\rho_0)^{1/2} \int\limits_0^\infty dk \, |a(k)|^2 k^{1/2} \,. \tag{4.22}$$

5. Exact solution

The Fokker–Planck equation (2.1) can be solved exactly for the *Gaussian ensemble*, which is the case of a parabolic potential $V(E) = cE^2$ (c is an arbitrary positive constant). The eigenfrequencies and eigenfunctions of the Fokker–Planck equation were constructed by Sutherland [18], by mapping it onto a Schrödinger equation. Here we use the same method to compute the correlation functions for $\beta = 2$, and compare with the asymptotic $N \to \infty$ results of section 4.

5.1. Sutherland's method

To map the Fokker–Planck equation (2.1) onto a Schrödinger equation we substitute

$$P(\{E_n\}, \tau) = e^{-\frac{1}{2}\beta W(\{E_n\})} \Psi(\{E_n\}, \tau) , \tag{5.1}$$

where W is given by eq. (2.2) with $V(E) = cE^2$. Sutherland [18] used a different mapping (with β instead of $\frac{1}{2}\beta$ in the exponent), but this one is more suitable for our purpose. Substitution of eq. (5.1) into eq. (2.1) yields for Ψ the equation

$$-\frac{\partial \Psi}{\partial \tau} = -\frac{1}{\beta\gamma} \sum_{i=1}^{N} \frac{\partial^2 \Psi}{\partial E_i^2} + \frac{1}{2\gamma} \Psi \sum_{i=1}^{N} \left[\frac{1}{2}\beta \left(\frac{\partial W}{\partial E_i}\right)^2 - \frac{\partial^2 W}{\partial E_i^2} \right] . \tag{5.2}$$

The expression between square brackets is evaluated as follows:

$$\sum_{i=1}^{N} \frac{\partial^2 W}{\partial E_i^2} = \sum_i \sum_{j(\neq i)} \frac{1}{(E_i - E_j)^2} + 2cN , \tag{5.3}$$

$$\sum_{i=1}^{N} \left(\frac{\partial W}{\partial E_i}\right)^2 = \sum_i \sum_{j(\neq i)} \sum_{k(\neq i)} \frac{1}{E_i - E_j} \frac{1}{E_i - E_k} + 4c^2 \sum_i E_i^2 - 4c \sum_i \sum_{j(\neq i)} \frac{E_i}{E_i - E_j}$$

$$= \sum_i \sum_{j(\neq i)} \frac{1}{(E_i - E_j)^2} + 4c^2 \sum_i E_i^2 - 2cN(N - 1) . \tag{5.4}$$

In the final equality we have used that for any three distinct indices i, j, k

$$\frac{1}{E_i - E_j} \frac{1}{E_i - E_k} + \frac{1}{E_j - E_i} \frac{1}{E_j - E_k} + \frac{1}{E_k - E_i} \frac{1}{E_k - E_j} \equiv 0 , \tag{5.5}$$

so that the triple sum over $k \neq i \neq j$ collapses to a double sum over $i \neq j$.

Collecting results, we find that Ψ satisfies a Schrödinger equation in imaginary time ($\tau \equiv it$),

$$-\frac{\partial \Psi}{\partial \tau} = (\mathcal{H}_S - U_0)\Psi \,, \tag{5.6}$$

$$\mathcal{H}_S = -\frac{1}{\beta\gamma}\sum_i \frac{\partial^2}{\partial E_i^2} + \frac{\beta-2}{4\gamma}\sum_i \sum_{j(\neq i)} \frac{1}{(E_i - E_j)^2} + \frac{\beta c^2}{\gamma}\sum_i E_i^2 \,, \tag{5.7}$$

$$U_0 = N\frac{c}{\gamma} + N(N-1)\frac{\beta c}{2\gamma} \,. \tag{5.8}$$

The Sutherland Hamiltonian $\mathcal{H}_S$ has an inverse-square interaction and a parabolic confining potential. The interaction is attractive for $\beta = 1$ and repulsive for $\beta = 4$. For $\beta = 2$ the interaction vanishes. Since $\exp(-\beta W)$ is a time-independent solution of the Fokker–Planck equation (2.1), $\exp(-\frac{1}{2}\beta W)$ is a time-independent solution of the Schrödinger equation (5.6) (in view of eq. (5.1)). Hence once has the eigenvalue equation

$$\mathcal{H}_S \, e^{-\frac{1}{2}\beta W} = U_0 \, e^{-\frac{1}{2}\beta W} \,. \tag{5.9}$$

For a particular ordering of the "coordinates" $E_1, E_2, \ldots, E_N$, the function $\Psi_0 \propto \exp(-\frac{1}{2}\beta W)$ is an eigenfunction of the N-fermion Hamiltonian $\mathcal{H}_S$. Since it is nodeless, it is the ground state at energy U_0. Anti-symmetrization yields the fermion ground-state wavefunction [18,19]

$$\Psi_0(\{E_n\}) = C \, e^{-\frac{1}{2}\beta W(\{E_n\})} \prod_{i<j} \frac{E_i - E_j}{|E_i - E_j|} \,, \tag{5.10}$$

with C a normalization constant. (Alternatively, we could work with the symmetric wavefunction $\exp(-\frac{1}{2}\beta W)$, which is the ground state for hard-core bosons.)

We obtain the N-particle Green's function $G(\{E_n\}, \tau)$ of the Schrödinger equation (5.6) from $P(\{E_n\}, \tau)$ by the similarity transformation

$$G(\{E_n\}, \tau) = \exp[\tfrac{1}{2}\beta W(\{E_n\})] \, P(\{E_n\}, \tau) \exp[-\tfrac{1}{2}\beta W(\{E_n^0\})] \,. \tag{5.11}$$

For $\tau > 0$, the function G satisfies

$$-\frac{\partial G}{\partial \tau} = (\mathcal{H}_S - U_0)G \,, \tag{5.12}$$

in view of eqs. (5.1) and (5.6). The initial condition is

$$G(\{E_n\}, 0) = \prod_{i=1}^{N} \delta(E_i - E_i^0) \,, \tag{5.13}$$

in view of eq. (2.6). Hence G is indeed a Green's function. In operator notation,

$$G(\tau) = e^{-(\mathcal{H}_S - U_0)\tau} \,. \tag{5.14}$$

We note that since the Fokker–Planck equation conserves the ordering of the levels $E_1, E_2, \ldots, E_N$ for $\tau \geqslant 0$, we can write eq. (5.11) equivalently in terms of the anti-symmetrized wavefunction (5.10),

$$G(\{E_n\}, \tau) = \Psi_0^{-1}(\{E_n\}) \, P(\{E_n\}, \tau) \, \Psi_0(\{E_n^0\}) \,. \tag{5.15}$$

We are now ready to relate the equilibrium density-correlation function in Dyson's classical Brownian-motion model to the ground-state density-correlation function in Sutherland's quantum many-body problem in imaginary time. In fact, we will see that the two correlation functions are *identical*. We define the ground-state expectation value $\langle A \rangle_0$ of an operator A,

$$\langle A \rangle_0 = \int_{-\infty}^{\infty} dE_1 \cdots \int_{-\infty}^{\infty} dE_N \, \Psi_0^+ A \Psi_0 \,. \tag{5.16}$$

The density operator is

$$n(E) = \sum_{i=1}^{N} \delta(E - E_i) \,, \tag{5.17a}$$

$$n(E, \tau) = e^{\mathcal{H}_S \tau} n(E) \, e^{-\mathcal{H}_S \tau} \,, \tag{5.17b}$$

in the Schrödinger and Heisenberg picture, respectively. Combining eqs. (4.1), (4.8), (4.9) and (5.14)–(5.17), one then finds

$$\langle \rho(E', \tau) \, \rho(E, 0) \rangle_{eq} = \langle n(E') \, e^{-(\mathcal{H}_S - U_0)\tau} n(E) \rangle_0$$
$$= \langle n(E', \tau) \, n(E, 0) \rangle_0 \,. \tag{5.18}$$

Hence the density correlation function $S(E, E', X)$, defined in eq. (4.10), is identical to

$$S(E, E', X) = \langle n(E', X^2) \, n(E, 0) \rangle_0 - \langle n(E) \rangle_0 \langle n(E') \rangle_0 \,. \tag{5.19}$$

5 2   Gaussian unitary ensemble

The significance of the formal relationship (5 19) is that the quantum mechanical correlator on the right-hand side can be computed exactly using the known excitation spectrum of the Sutherland Hamiltonian [18] The problem of computing the time-dependent correlation functions of $\mathcal{H}_S$ was previously considered by Simons, Lee and Altshuler, in connection with a microscopic theory of parametric correlations [20] We will return to their work in section 7 The case $\beta = 2$ is particularly simple, since $\mathcal{H}_S$ is then the Hamiltonian of *non-interacting* fermions This is the case of the Gaussian unitary ensemble

The single-particle eigenfunctions $\phi_p(E)$ and eigenvalues ε_p of $\mathcal{H}_S$ are (cf eq (5 7) with $\beta = 2$)

$$\phi_p(E) = (2c/\pi)^{1/4}(2^p p!)^{-1/2} \, e^{-cE^2} H_p(E\sqrt{2c}) \, , \tag{5 20}$$

$$\varepsilon_p = (p + \tfrac{1}{2})\omega \, , \qquad p = 0, 1, 2, \quad , \tag{5 21}$$

$$\omega \equiv 2c/\gamma \tag{5 22}$$

The functions $H_p(x)$ are the Hermite polynomials The density operator (5 17) becomes, in second quantization,

$$n(E) = \sum_{p\,q=0}^{\infty} \phi_p(E) \, \phi_q(E) \, c_p^\dagger c_q \, , \tag{5 23a}$$

$$n(E, \tau) = \sum_{p\,q=0}^{\infty} \phi_p(E) \, \phi_q(E) \, e^{(\varepsilon_p - \varepsilon_q)\tau} c_p^\dagger c_q \, , \tag{5 23b}$$

where $c_p^\dagger$ and c_p are fermion creation and annihilation operators in state p The average density in the N-fermion ground state is

$$\langle n(E) \rangle_0 \equiv \rho_{\mathrm{eq}}(E) = \sum_{p=0}^{N-1} \phi_p^2(E) \tag{5 24}$$

To compute the density fluctuations, we need the ground-state expectation value

$$\langle n(E', \tau) \, n(E, 0) \rangle_0$$

$$= \sum_{p\,q\,p\,q=0}^{\infty} \phi_p(E') \, \phi_q(E') \, \phi_p(E) \, \phi_q(E) \, e^{(\varepsilon_p - \varepsilon_q)\tau} \langle c_p^\dagger c_q \, c_p^\dagger c_q \rangle_0 \tag{5 25}$$

The average of the product of four c's evaluates to

$$\langle c_p^\dagger{}_{'} c_{q'} c_p^\dagger c_q \rangle_0 = \delta_{pq} \delta_{p'q'} \theta(N - 1 - p') \, \theta(N - 1 - p)$$
$$+ \delta_{pq'} \delta_{p'q} \theta(N - 1 - q) \, \theta(p - N) , \tag{5.26}$$

where δ_{pq} is the Kronecker delta and the function $\theta(x)$ equals 1 if $x \geq 0$ and 0 if $x < 0$. Collecting results, we find for the density correlation function (5.19) the formula

$$S(E, E', X) = \sum_{p=N}^{\infty} \sum_{q=0}^{N-1} \phi_p(E) \, \phi_p(E') \, \phi_q(E) \, \phi_q(E') \, e^{(\varepsilon_q - \varepsilon_p) X^2} . \tag{5.27}$$

The infinite series over p in eq. (5.27) can be reduced to a finite sum by using an addition theorem for Hermite polynomials:

$$G_0(E, E', \tau) \equiv \sum_{p=0}^{\infty} \phi_p(E) \, \phi_p(E') \, e^{-\varepsilon_p \tau}$$
$$= \left(\frac{c}{\pi \sinh \omega\tau} \right)^{1/2} \exp\left(\frac{c}{\sinh \omega\tau} [2EE' - (E^2 + E'^2) \cosh \omega\tau] \right) . \tag{5.28}$$

This is the familiar result for the (imaginary time) Green's function of a one-dimensional harmonic oscillator (with coordinate E, mass γ, and oscillator frequency ω). Substitution into eq. (5.27) yields

$$S(E, E', X) = G_0(E, E', X^2) \sum_{q=0}^{N-1} \phi_q(E) \, \phi_q(E') \, e^{\varepsilon_q X^2}$$
$$- \sum_{p=0}^{N-1} \sum_{q=0}^{N-1} \phi_p(E) \, \phi_p(E') \, \phi_q(E) \, \phi_q(E') \, e^{(\varepsilon_q - \varepsilon_p) X^2} . \tag{5.29}$$

For $X = 0$ the function G_0 becomes a delta function, so that eq. (5.29) reduces to

$$S(E, E', 0) = \delta(E - E') \sum_{p=0}^{N-1} \phi_p^2(E) - \left(\sum_{p=0}^{N-1} \phi_p(E) \, \phi_p(E') \right)^2 . \tag{5.30}$$

Eq. (5.30) was obtained by Mehta [9] using an approach known as the "method of orthogonal polynomials". Our eq. (5.29) extends this exact result to parametric correlations.

5.3. Large-N limit

It is instructive to see how the result (4.14) of the asymptotic analysis in section 4 follows (for $\beta = 2$) from the large-N limit of the exact result (5.27) for the Gaussian unitary ensemble.

We wish to evaluate the density correlation function (5.27) in the limit $N \to \infty$, $c \to 0$, while the product cN remains constant (to ensure a constant density of states, see below). Using the asymptotic form of the Hermite polynomials H_p for $p \gg 1$, one has for the eigenfunctions (5.20) the large-p expressions

$$\phi_{2p}(E) = (-1)^p (2c/p\pi^2)^{1/4} \, e^{-cE^2} \cos(E\sqrt{8cp}) \,, \tag{5.31a}$$

$$\phi_{2p+1}(E) = (-1)^p (2c/p\pi^2)^{1/4} \, e^{-cE^2} \sin(E\sqrt{8cp}) \,. \tag{5.31b}$$

We need to compute the series

$$\sum_{p=0}^{N-1} \phi_p(E) \, \phi_p(E') \, e^{\epsilon_p X^2} = \sum_{p=0}^{\frac{1}{2}(N-1)} \phi_{2p}(E) \, \phi_{2p}(E') \exp[(2p + \tfrac{1}{2})\omega X^2]$$
$$+ \sum_{p=0}^{\frac{1}{2}(N-2)} \phi_{2p+1}(E) \, \phi_{2p+1}(E') \exp[(2p + \tfrac{3}{2})\omega X^2] \,, \tag{5.32}$$

in the limit $N \to \infty$, $c \to 0$ at constant cN. Note that $c \to 0$ implies $\omega \to 0$, in view of eq. (5.22). Combining eqs. (5.31) and (5.32), and replacing the sum over p by an integral, we find in this limit

$$\sum_{p=0}^{N-1} \phi_p(E) \, \phi_p(E') \exp(\epsilon_p X^2) = \rho_0 \int_0^1 ds \, \exp(\alpha X^2 s^2) \cos[\pi\rho_0(E - E')s] \,, \tag{5.33}$$

with the definitions

$$\rho_0 \equiv \frac{2}{\pi} (cN)^{1/2} \,, \tag{5.34}$$

$$\alpha \equiv N\omega \equiv \frac{\pi^2 \rho_0^2}{2\gamma} \,. \tag{5.35}$$

Similarly,

$$\sum_{p=N}^{\infty} \phi_p(E)\, \phi_p(E') \exp(-\varepsilon_p X^2) = \rho_0 \int_1^{\infty} ds \, \exp(-\alpha X^2 s^2) \cos[\pi\rho_0(E - E')s] \, .$$

$$(5.36)$$

Substitution of eq. (5.33) into eq. (5.24) gives

$$\rho_{\mathrm{eq}}(E) = \rho_0 \, , \tag{5.37}$$

justifying the identification (5.34). The limit $N \to \infty$ yields a uniform density of states in any *fixed* energy range. At finite N, the density $\rho_{\mathrm{eq}}(E)$ vanishes for $\rho_0|E| \gtrsim 2N/\pi$, as follows from a more accurate evaluation of eq. (5.24) [9].

Substitution of eqs. (5.33) and (5.36) into eqs. (5.27) gives an integral expression for the density correlation function $S(E, E', X) \equiv S(E' - E, X)$,

$$S(E, X) = \rho_0^2 \int_0^1 ds \int_1^{\infty} ds' \, \exp[\alpha X^2(s^2 - s'^2)] \cos(\pi\rho_0 Es) \cos(\pi\rho_0 Es') \, .$$

$$(5.38)$$

The Fourier transform $S(k, X) \equiv \int dE \, S(E, X) \exp(ikE)$ with respect to the energy increment becomes

$$S(k, X) = \frac{\rho_0}{\xi^2|k|} \exp(-\xi^2|k| q_{\mathrm{max}}) \sinh(\xi^2|k| q_{\mathrm{min}}) \, , \tag{5.39}$$

$$q_{\mathrm{min}} \equiv \min\left(1, \frac{|k|}{2\pi\rho_0}\right), \qquad q_{\mathrm{max}} \equiv \max\left(1, \frac{|k|}{2\pi\rho_0}\right). \tag{5.40}$$

The variable ξ was defined in eq. (4.15).

The result (5.39) holds in the large-N limit (at constant density of states) in any fixed k-range. We now further restrict ourselves to energy scales bigger than the mean level spacing $\Delta \equiv \rho_0^{-1}$, i.e. to the range $k \ll \rho_0$. Eq. (5.39) then simplifies to

$$S(k, X) = \frac{|k|}{2\pi} \exp(-\xi^2|k|) \, , \tag{5.41}$$

in agreement with eq. (4.14) for $\beta = 2$. The correlation function $S(k, X)$ is only appreciably different from zero if $\xi^2|k| \lesssim 1$. Hence the restriction $k \ll \rho_0$ on eq. (5.41) becomes irrelevant if $\xi^2\rho_0 \gg 1$. This implies that the asymptotic expressions for the correlation functions $S(E, X)$ (and $C(E, X)$) of section 4 hold for $N \to \infty$ in the energy range $E \gg \Delta$ for all X and in the parameter range

$X \gg \Delta \sqrt{\gamma}$ for all E. If *both* $E \lesssim \Delta$ *and* $X \lesssim \Delta\sqrt{\gamma}$ one cannot use eq. (5.41), but should use instead the full expressions (5.38) or (5.39).

Once we have the density correlation function $S(k, X)$, the current correlation function $C(k, X)$ follows directly in view of the relation (4.17). From eq. (5.39) one thus finds

$$
\begin{aligned}
C(k, X) = \frac{2\pi\rho_0^2}{\gamma \xi^4 |k|^3} \exp(-\xi^2 |k| q_{\max}) \\
\times \{ \xi^2 |k| q_{\min} \cosh(\xi^2 |k| q_{\min}) \, (3 + 4\xi^2 |k| q_{\max}) \\
- \sinh(\xi^2 |k| q_{\min}) \, [3 + 3\xi^2 |k| q_{\max} + 2\xi^4 k^2 (q_{\max}^2 + q_{\min}^2)] \} .
\end{aligned}
\tag{5.42}
$$

Eq. (5.42) holds for $N \to \infty$ and any k. For $k \ll \rho_0$ it reduces to the asymptotic expression (4.18) of section 4 (with $\beta = 2$).

6. Extension of multiple parameters

In this section we show how the Brownian-motion model of section 2 can be extended to a parameter *vector* $X = X_1, X_2, \ldots, X_d$, relevant for a statistical description of the dispersion relation of a d-dimensional crystalline lattice [4]. The Brownian motion of the energy levels $E_n(X)$ then takes place in a fictitious world with multiple temporal dimensions $\tau = \tau_1, \tau_2, \ldots, \tau_d$.

We assume that any systematic drift in the energy levels is eliminated by a rescaling, so that

$$
\overline{\partial_\mu E_n(X)} = 0 .
\tag{6.1}
$$

(We abbreviate $\partial_\mu \equiv \partial/\partial X_\mu$.) We also assume that the different parameters X_μ are independent, that is to say

$$
\overline{\partial_\mu E_n(X) \, \partial_\nu E_n(X)} = 0 , \qquad \text{if } \mu \ne \nu .
\tag{6.2}
$$

Let $\tau = 0$ coincide with $X = 0$. The initial condition on the distribution function $P(\{E_n\}, \tau)$ is

$$
P(\{E_n\}, \tau = 0) = \prod_{i=1}^{N} \delta(E_i - E_i^0) ,
\tag{6.3}
$$

with E_i^0 the eigenvalues of $\mathcal{H}(X = 0)$. For $\tau_\mu > 0$ the distribution function

evolves according to the multiple-time-dimensional generalization of the Fokker–Planck equation (2.1),

$$\frac{1}{d} \sum_{\mu=1}^{d} \gamma_\mu \frac{\partial P}{\partial \tau_\mu} = \sum_{i=1}^{N} \frac{\partial}{\partial E_i} \left(P \frac{\partial W}{\partial E_i} + \beta^{-1} \frac{\partial P}{\partial E_i} \right) . \tag{6.4}$$

By comparing the initial average rate of change of the energy levels as a function of X and τ,

$$\overline{(E_i(X) - E_i^0)^2} = \sum_{\mu=1}^{d} X_\mu^2 \overline{(\partial_\mu E_i)^2} + \mathcal{O}(X^3) , \tag{6.5}$$

$$\overline{(E_i(\tau) - E_i^0)^2} = \frac{2}{\beta} \sum_{\mu=1}^{d} \frac{\tau_\mu}{\gamma_\mu} + \mathcal{O}(\tau^2) , \tag{6.6}$$

we arrive as in section 2 at the identifications

$$\tau_\mu = X_\mu^2 , \tag{6.7}$$

$$2/\beta\gamma_\mu = \overline{(\partial_\mu E_i)^2} . \tag{6.8}$$

The Fokker–Planck equation (6.4) can now be reduced to a non-local diffusion equation as in section 4,

$$\frac{1}{d} \sum_{\mu=1}^{d} \gamma_\mu \frac{\partial}{\partial \tau_\mu} \delta\rho(E, \tau) = -\frac{\partial}{\partial E} \int_{-\infty}^{\infty} dE' \, \rho_{\text{eq}}(E) \ln|E - E'| \frac{\partial}{\partial E'} \delta\rho(E', \tau) , \tag{6.9}$$

valid asymptotically for $N \to \infty$. For a constant density of states ρ_0 the diffusion kernel becomes translationally invariant. Eq. (6.9) then has the k-space solution

$$\delta\rho(k, \tau) = \delta\rho(k, 0) \exp\left(-\pi\rho_0 |k| \sum_{\mu=1}^{d} \tau_\mu/\gamma_\mu \right) , \tag{6.10}$$

which implies for the density correlation function (cf. eq. (4.14))

$$S(k, X) = \frac{|k|}{\pi\beta} \exp\left(-\pi\rho_0 |k| \sum_{\mu=1}^{d} X_\mu^2/\gamma_\mu \right) . \tag{6.11}$$

Here we have also used the identification (6.7). The E-space correlation function becomes, upon inverse Fourier transformation,

$$S(E, X) = \frac{1}{2\pi^2\beta} \frac{\partial^2}{\partial E^2} \ln\left[E^2 + \left(\pi\rho_0 |k| \sum_{\mu=1}^{d} X_\mu^2/\gamma_\mu\right)^2 \right] .$$ (6.12)

The current correlation function

$$C_{\mu\nu}(E, X, E', X') = \sum_{\iota\,\jmath} \overline{[\partial_\mu E_\iota(X)][\partial_\nu E_\jmath(X')]\delta(E - E_\iota(X))\,\delta(E' - E_\jmath(X'))}$$ (6.13)

is related to the density correlation function $S(E, X, E', X')$ by

$$\frac{\partial^2}{\partial E\,\partial E'} C_{\mu\nu}(E, X, E', X') = \frac{\partial^2}{\partial X_\mu\,\partial X_\nu'} S(E, X, E', X') .$$ (6.14)

Because of translational invariance, $C_{\mu\nu}(E, X, E', X') = C_{\mu\nu}(E' - E, X' - X)$, $S(E, X, E', X') = S(E' - E, X' - X)$. In k-space eq. (6.14) then takes the form

$$C_{\mu\nu}(k, X) = -\frac{1}{k^2} \frac{\partial^2}{\partial X_\mu\,\partial X_\nu} S(k, X) .$$ (6.15)

From eqs. (6.11) and (6.15) we find for the current correlation function the k- and E-space expressions

$$C_{\mu\nu}(k, X) = \frac{2\rho_0}{\beta}\left(\frac{\delta_{\mu\nu}}{\gamma_\mu} - 2\pi\rho_0|k|\frac{X_\mu X_\nu}{\gamma_\mu\gamma_\nu}\right) \exp\left(-\pi\rho_0|k| \sum_{\lambda=1}^{d} X_\lambda^2/\gamma_\lambda\right) ,$$ (6.16)

$$C_{\mu\nu}(E, X) = \frac{1}{2\pi^2\beta} \frac{\partial^2}{\partial X_\mu\,\partial X_\nu} \ln\left[E^2 + \left(\pi\rho_0|k| \sum_{\mu=1}^{d} X_\mu^2/\gamma_\mu\right)^2 \right] .$$ (6.17)

For $d = 1$ the correlation functions (6.12) and (6.17) reduce to the results (4.16) and (4.19) of section 4.

7. Comparison with microscopic theory

7.1. Diagrammatic perturbation theory

The asymptotic analysis of section 4 yields the density and current correlation functions in the limit $N \to \infty$ if $E \gg \Delta \equiv 1/\rho_0$ for all X and if $X \gg X_c \equiv \Delta\sqrt{\gamma}$ for all E. We will now show that the random-matrix theory (RMT) in this

regime agrees with the diagrammatic perturbation theory of Szafer, Altshuler, and Simons [2,4].

When $E \to 0$, our result (4.19) for the current correlation function $C(E, X)$ reduces to

$$C(0, X) = \frac{2}{\pi^2 \beta} \frac{\partial^2}{\partial X^2} \ln|X| = -\frac{2}{\pi^2 \beta X^2}, \qquad \text{if } X \neq 0, \tag{7.1}$$

independent of the microscopic parameters ρ_0 and γ. Eq. (7.1), obtained here from RMT, is precisely the universal correlator (1.3) which Szafer and Altshuler [2] derived from diagrammatic perturbation theory.

At $X = 0$, the function $C(0, X)$ according to eq. (7.1) has an integrable singularity consisting of a positive peak such that the integral over all X vanishes. This is a special case of the general sumrule

$$\int_0^\infty dX \, C(E, X) = 0, \tag{7.2}$$

which follows from eq. (4.19) (cf. also eq. (3.10)). The peak of positive correlation has infinitesimal width in the limit $E \to 0$. At non-zero E the peak has a finite width of order $X_c (E/\Delta)^{1/2}$, as illustrated in fig. 2, where we have plotted $C(E, X)$ from eq. (4.19) for $E = 0.1\Delta$ (dashed curve).

As discussed in section 5, the asymptotic formula (7.1) becomes exact only for $X \gg X_c$. (Compare with the solid curve in fig. 2, computed from the exact result (5.42).) Using the definition of the generalized Thouless energy [4]

$$\mathscr{E}_c \equiv \Delta^{-1} \overline{\dot{E}_i^2}, \tag{7.3}$$

and the relationship (2.10) between γ and $\overline{\dot{E}_i^2}$, one can write

$$X_c \equiv \Delta\sqrt{\gamma} \equiv \left(\frac{2\Delta}{\beta\mathscr{E}_c}\right)^{1/2}. \tag{7.4}$$

In ref. [2] the parameter X is the magnetic flux increment in units of h/e. Then $\mathscr{E}_c$ is the conventional Thouless energy [21] $E_c \simeq \hbar v_F l/L^2$, related to the conductance g (in units of e^2/h) by $g \simeq E_c/\Delta$. The Aharonov–Bohm periodicity implies in this case the additional restriction $X \ll 1$ to eq. (7.1) (which is compatible with the condition $X \gg X_c$ because $X_c \simeq g^{-1/2} \ll 1$ in the metallic regime).

We have shown that the $E \to 0$ limit of $C(E, X)$ obtained from RMT agrees with the microscopic theory. What about non-zero energy differences? This is
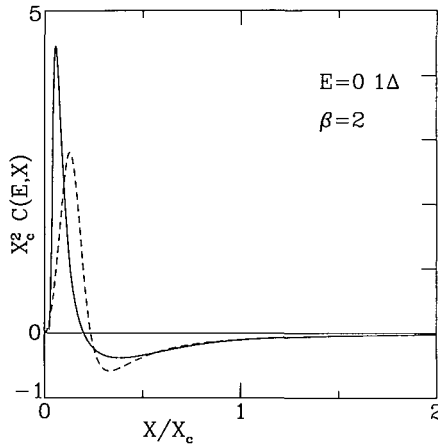
Fig 2   Current correlation function $C(E, X)$ at $E = 0.1\Delta$ as a function of the parameter X, scaled by $X_c \equiv \Delta\sqrt{\gamma}$ The solid curve is the $N \to \infty$ limit of the exact solution for the Gaussian unitary ensemble, obtained by numerical inversion of the Fourier transform (5 42) The dashed curve is the result (4 19) of the asymptotic analysis for $\beta = 2$, valid for $N \to \infty$ and $X \gg X_c$ As $E \to 0$, the peak of positive correlation becomes an integrable singularity at $X = 0$, such that $\int_0^\infty dX\, C(E, X) = 0$

most easily discussed in terms of the density correlation function $S(E, X)$, to which $C(E, X)$ is related via eq. (4.17). Using $\xi \equiv X(\tfrac{1}{2}\pi\beta\mathscr{E}_c)^{1/2}$, we find that the result (4.16) can be rewritten identically as

$$S(E, X) = \frac{1}{\pi^2\beta} \operatorname{Re}\left(iE + \tfrac{1}{2}\pi\beta\mathscr{E}_c X^2\right)^{-2}, \tag{7.5}$$

which agrees with the diagrammatic perturbation theory [2,4] provided $E \ll E_c$. The deviation between RMT and the microscopic theory on energy scales greater than the Thouless energy E_c is well known from the work by Altshuler and Shklovskiĭ on parameter-independent correlations [3].

One can similarly show that the correlation functions for *multiple* parameters X_μ ($\mu = 1, 2, \ldots, d$), obtained from RMT in section 6, agree with the results which Simons and Altshuler [4] obtained by microscopic theory. In particular, we find from eq. (6.17) that

$$\sum_{\mu=1}^{d} \gamma_\mu C_{\mu\mu}(0, X) = \frac{2d - 4}{\pi^2\beta}\left(\sum_{\mu=1}^{d} X_\mu^2/\gamma_\mu\right)^{-1}, \tag{7.6}$$

in agreement with ref. [4]. The correlator (7.6) is the multiple-parameter generalization of the universal correlator (1.3). Simons and Altshuler have

discussed the physical origin of the different sign of the correlator for $d < 2$ and $d > 2$.

7.2. Non-linear sigma model

The restriction on the asymptotic analysis that either $X \gg X_c$ or $E \gg \Delta$ is removed by the exact solution of section 5 for the Gaussian unitary ensemble ($\beta = 2$). The density correlation function in the limit $N \rightarrow \infty$ is given for this random-matrix ensemble by eq. (5.38) in E-space and by eq. (5.39) in k-space. In terms of the generalized Thouless energy (7.3), the E-space expression can be written as

$$S(E, X) = \Delta^{-2} \int_0^1 ds \int_1^\infty ds' \exp\left(\frac{\pi^2 \mathscr{E}_c}{2\Delta} X^2 (s^2 - s'^2)\right) \cos(\pi s E / \Delta) \cos(\pi s' E / \Delta) .$$

(7.7)

This is precisely the result of the microscopic theory of Simons and Altshuler [4]. If either $X \gg X_c$ or $E \gg \Delta$, eq. (7.7) reduces to eq. (4.16) with $\beta = 2$. Simons and Altshuler were able to extend the microscopic theory to the regime $X \lesssim X_c$, $E \lesssim \Delta$, which is not obtainable by perturbation theory, by using a supersymmetry formulation followed by a mapping onto a non-linear sigma model [5]. As emphasized by these authors, it is quite remarkable that the microscopic parameters enter only via the quantities Δ and $\mathscr{E}_c$, so that a rescaling of the E and X variables maps all density correlation functions onto a single universal function.

Simons and Altshuler have also computed the small E and X behavior of $S(E, X)$ from the microscopic theory in the presence of time-reversal symmetry, i.e. for $\beta = 1$. Again, they used a mapping onto a non-linear sigma model to go beyond perturbation theory. We have no RMT result for the small E and X behavior in this case, which would correspond to the orthogonal ensemble. The case $\beta = 4$ of strong spin-orbit scattering (symplectic ensemble in RMT) has not yet been treated by microscopic theory, and only in the asymptotic limit by RMT.

In ref. [20], Simons, Lee, and Altshuler have argued convincingly (although not completely proven) that the density correlation functions of the non-linear sigma model and the Sutherland Hamiltonian are equivalent for $\beta = 1, 2$ and 4. In the present paper, in section 5, we have proven the equivalence of the density correlation functions of Dyson's Brownian-motion model and the Sutherland Hamiltonian. Taken together this is evidence for the complete

equivalence of the non-linear sigma model and RMT, although the cases $\beta = 1$ and $\beta = 4$ still lack a complete proof.

7.3. Conclusion

We have studied the response to an external perturbation of the energy levels of a quantum mechanical system by means of the Brownian-motion model introduced by Dyson in the theory of random matrices. Our results for the energy and parameter-dependent level-density and current-density correlation functions $S(E, X)$ and $C(E, X)$ agree with the microscopic theory for a disordered metallic particle, for energy scales below the Thouless energy E_c. This establishes the validity of Dyson's basic assumption, that parametric correlations are dominated by level repulsion and therefore solely dependent on the symmetry of the Hamiltonian.

It is likely that the approach developed in this paper can also be used to describe parametric correlations in random transmission matrices. The analogue of level repulsion for the transmission eigenvalues is known [22], and leads to a pair correlation function $K(T, T')$ which differs from eq. (1.6) for $K(E, E')$ but has the same universal β-dependence [17]. This suggests that the analogue of the universal correlator (1.3) exists as well for the transmission eigenvalues, with obvious implications for the conductance of a mesoscopic system.

Appendix. Derivation of eq. (4.2)

For completeness, we present here Dyson's derivation [13] of the non-linear diffusion equation (4.2) from the Fokker–Planck equation (2.1), in the limit $N \to \infty$.

We multiply eq. (2.1) by $\delta(E - E_i)$, integrate over $E_1, E_2, \ldots, E_N$, and sum over i. The result is

$$\gamma \frac{\partial}{\partial \tau} \rho(E, \tau) = \frac{\partial}{\partial E} \left(\beta^{-1} \frac{\partial}{\partial E} \rho(E, \tau) \right.$$

$$\left. + \int_{-\infty}^{\infty} dE_1 \cdots \int_{-\infty}^{\infty} dE_N \, P(\{E_n\}, \tau) \sum_{i=1}^{N} \delta(E - E_i) \frac{\partial W}{\partial E_i} \right), \quad (A.1)$$

where $\rho(E, \tau)$ is defined in eq. (4.1). Substitution of the definition (2.2) of $W(\{E_n\})$ into eq. (A.1) leads to

$$\gamma \frac{\partial}{\partial \tau} \rho(E, \tau) = \frac{\partial}{\partial E} \left(\beta^{-1} \frac{\partial}{\partial E} \rho(E, \tau) + \rho(E, \tau) \frac{d}{dE} V(E) \right.$$

$$\left. - \mathcal{P} \int_{-\infty}^{\infty} dE' \frac{\rho_2(E, E', \tau)}{E - E'} \right), \quad (A.2)$$

where $\mathcal{P} \int$ indicates the principal value of the integral. The pair density $\rho_2(E, E', \tau)$ is defined by

$$\rho_2(E, E', \tau) = \int_{-\infty}^{\infty} dE_1 \cdots \int_{-\infty}^{\infty} dE_N \, P(\{E_n\}, \tau) \sum_{i \neq j} \delta(E - E_i) \, \delta(E' - E_j) .$$

$$(A.3)$$

The pair density is symmetric in the energy arguments, $\rho_2(E, E', \tau) = \rho_2(E', E, \tau)$, and satisfies the normalization

$$\int_{-\infty}^{\infty} dE' \, \rho_2(E, E', \tau) = (N - 1)\rho(E, \tau) . \quad (A.4)$$

Following ref. [13] we decompose the pair density into a correlated and an uncorrelated part,

$$\rho_2(E, E', \tau) = \rho(E, \tau) \, \rho(E', \tau) \, [1 - y(E, E', \tau)] . \quad (A.5)$$

The function $y(E, E', \tau) = y(E', E, \tau)$ is symmetric in E and E', and satisfies

$$\int_{-\infty}^{\infty} dE' \, y(E, E', \tau) \, \rho(E', \tau) = 1 , \quad (A.6)$$

in view of the normalization (A.4). Substitution of the definition (A.5) into eq. (A.2) leads to

$$\gamma \frac{\partial}{\partial \tau} \rho(E, \tau) = \frac{\partial}{\partial E} \left(\beta^{-1} \frac{\partial}{\partial E} \rho(E, \tau) + \rho(E, \tau) \frac{\partial}{\partial E} [V(E) + U(E, \tau)] \right. $$
$$\left. + \rho(E, \tau) I(E, \tau) \right), \tag{A.7}$$

with the definitions

$$I(E, \tau) = \mathcal{P} \int_{-\infty}^{\infty} dE' \, \rho(E', \tau) \frac{y(E, E', \tau)}{E - E'}, \tag{A.8}$$

$$U(E, \tau) = - \int_{-\infty}^{\infty} dE' \, \rho(E', \tau) \ln|E - E'|. \tag{A.9}$$

Eq. (A.7) is still exact. To introduce the approximation we need one further piece of notation. We re-express the function $y(E, E', \tau)$ in terms of the sum and difference variables $t = \frac{1}{2}(E + E')$ and $s = E' - E$:

$$y(E, E', \tau) = Y(\tfrac{1}{2}(E + E'), E' - E, \tau) \equiv Y(t, s, \tau). \tag{A.10}$$

The function $Y(t, s, \tau) = Y(t, -s, \tau)$ is even in s. The normalization (A.6) becomes

$$\int_{-\infty}^{\infty} ds \, Y(E + \tfrac{1}{2}s, s, \tau) \, \rho(E + s, \tau) = 1. \tag{A.11}$$

Similarly, the integral (A.8) takes the form

$$I(E, \tau) = -\mathcal{P} \int_{-\infty}^{\infty} ds \, Y(E + \tfrac{1}{2}s, s, \tau) \, \rho(E + s, \tau) \, s^{-1}. \tag{A.12}$$

By substituting the Taylor expansions

$$Y(E + \tfrac{1}{2}s, s, \tau) = Y(E, s, \tau) + \tfrac{1}{2}s \frac{\partial}{\partial E} Y(E, s, \tau) + \cdots, \tag{A.13}$$

$$\rho(E + s, \tau) = \rho(E, \tau) + s \frac{\partial}{\partial E} \rho(E, \tau) + \cdots, \tag{A.14}$$

into eq. (A.12), we obtain an expansion of $I(E, \tau)$ in higher and higher moments $Y_p(E, \tau)$ of $Y(E, s, \tau)$ with respect to s,

$$Y_p(E, \tau) = \int_{-\infty}^{\infty} ds\, Y(E, s, \tau)\, s^p \, . \tag{A.15}$$

Because of the symmetry $Y(t, s, \tau) = Y(t, -s, \tau)$ only even moments contribute ($Y_p(E, \tau) \equiv 0$ for p odd). Following Dyson [13], we *neglect the second and higher moments*. An order of magnitude estimate suggests that the error involved in neglecting Y_p for $p \geqslant 2$ is of order N^{-2}. Dyson argues that the error is actually of order $N^{-2} \ln N$, by comparison with exact results for the distribution of the spacing of eigenvalues.

Since Y_{-1} and Y_1 are identically zero, only Y_0 contributes to $I(E, \tau)$ to second order. Substitution of the Taylor expansions (A.13) and (A.14) into eq. (A.12) yields

$$I(E, \tau) = -\tfrac{1}{2} \rho(E, \tau) \frac{\partial}{\partial E} Y_0(E, \tau) - Y_0(E, \tau) \frac{\partial}{\partial E} \rho(E, \tau) \, . \tag{A.16}$$

Similarly, substitution of the Taylor expansions into eq. (A.11) yields

$$\rho(E, \tau)\, Y_0(E, \tau) = 1 \, . \tag{A.17}$$

Combining eqs. (A.16) and (A.17), we find

$$I(E, \tau) = -\tfrac{1}{2} \frac{\partial}{\partial E} \ln \rho(E, \tau) \, . \tag{A.18}$$

Hence eq. (A.7) takes the form

$$\gamma \frac{\partial}{\partial \tau} \rho(E, \tau) = \frac{\partial}{\partial E} \left[\rho(E, \tau) \frac{\partial}{\partial E} \left(V(E) - \int_{-\infty}^{\infty} dE'\, \rho(E', \tau) \ln|E - E'| \right. \right.$$
$$\left. \left. + \frac{2 - \beta}{2\beta} \ln \rho(E, \tau) \right) \right] \, . \tag{A.19}$$

This is eq. (4.2), except for the final term, proportional to $(2 - \beta)/2\beta$. As noted in ref. [13], this term is of order $\ln N$ and can be neglected relative to the other terms, which are of order N. Dropping that term, we obtain eq. (4.2).

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