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Conductance distribution of a quantum dot with nonideal single-channel leads

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The entire distribution is computed of the conductance of a quantum dot connected to two electron reservoirs by leads with a single propagating mode, for arbitrary transmission probability Γ of the mode. The theory bridges the gap between previous work on ballistic leads ($\Gamma=1$) and on tunneling point contacts ($\Gamma\ll 1$).

An ensemble of mesoscopic systems has large sample-to-sample fluctuations in its transport properties, so that the average is not sufficient to characterize a single sample. To determine the complete distribution of the conductance is therefore a fundamental problem in this field. Early work focused on an ensemble of disordered wires. (See Ref. 1 for a review.) The distribution of the conductance in that case is either normal or log-normal, depending on whether the wires are in the metallic or insulating regime. Recently, it was found that a "quantum dot" has a qualitatively different conductance distribution.[2–4] A quantum dot is a small confined region, having a large level spacing compared to the thermal energy, which is weakly coupled by point contacts to two electron reservoirs. The classical motion within the dot is assumed to be ballistic and chaotic. An ensemble consists of dots with small variations in shape or in Fermi energy. The capacitance of a dot is assumed to be sufficiently large that the Coulomb blockade can be ignored, i.e., the electrons are assumed to be noninteracting. Two altogether different approaches have been taken to this problem.

Baranger and Mello,[3] and Jalabert, Pichard, and one of the authors[4] started from random-matrix theory.[5] The scattering matrix S of the quantum dot was assumed to be a member of the circular ensemble of $N\times N$ unitary matrices, as is appropriate for a chaotic billiard.[6,7] In the single-channel case ($N=2$), the distribution $P(T)$ of the transmission probability T [and hence of the conductance $G=(2e^2/h)T$] was found to be

$$P(T)=\tfrac{1}{2}\beta T^{-1+\beta/2}, \tag{1}$$

where $\beta\in\{1,2,4\}$ is the symmetry index of the ensemble ($\beta=1$ or 2 in the absence or presence of a time-reversal symmetry-breaking magnetic field; $\beta=4$ in zero magnetic field with strong spin-orbit interaction). Equation (1) was found to be in good agreement with numerical simulations of transmission through a chaotic billiard connected to ideal leads having a single propagating mode.[3] (The case $\beta=4$ was not considered in Ref. 3.)

Previously, Prigodin, Efetov, and Iida[2] had applied the method of supersymmetry to the same problem, but with a different model for the point contacts. They considered the case of broken time-reversal symmetry ($\beta=2$), for which Eq. (1) would predict a uniform conductance distribution. Instead, the distribution of Ref. 2 is strongly peaked near zero conductance. The tail of the distribution (towards unit transmission) is governed by resonant tunneling, and is consistent with earlier work by Jalabert, Stone, and Alhassid[8] on resonant tunneling in the Coulomb-blockade regime.

It is the purpose of the present paper to bridge the gap between these two theories, by considering a more general model for the coupling of the quantum dot to the reservoirs. Instead of assuming ideal leads, as in Refs. 3 and 4, we allow for an arbitrary transmission probability Γ of the propagating mode in the lead, as a model for coupling via a quantum point contact with conductance below $2e^2/h$. Equation (1) corresponds to $\Gamma=1$ (ballistic point contact). In the limit $\Gamma\ll 1$ (tunneling point contact) we recover, for $\beta=2$, the result of Ref. 2. We consider also $\beta=1$ and 4 and show that — in contrast to Eq. (1) — the limit $\Gamma\ll 1$ depends only weakly on the symmetry index β. In the crossover region from ballistic to tunneling conduction we find a remarkable Γ dependence of the conductance fluctuations: The variance is monotonically decreasing for $\beta=1$ and 2, but it has a *maximum* for $\beta=4$ at $\Gamma=0.74$.

The system under consideration is illustrated in the inset of Fig. 1(b). It consists of a quantum dot with two single-channel leads containing a tunnel barrier (transmission probability Γ). We assume identical leads for simplicity. The transmission properties of this system are studied in a transfer matrix formulation. The transfer matrix M_d of the quantum dot can be parametrized as[9,10]

$$M_d=\begin{pmatrix} u_1 & 0\\ 0 & v_1\end{pmatrix}\begin{pmatrix}\sqrt{1+\lambda_d} & \sqrt{\lambda_d}\\ \sqrt{\lambda_d} & \sqrt{1+\lambda_d}\end{pmatrix}\begin{pmatrix} u_2 & 0\\ 0 & v_2\end{pmatrix}, \tag{2}$$

where the parameter λ_d is related to the transmission probability T_d of the dot by

$$T_d=(1+\lambda_d)^{-1}. \tag{3}$$

The numbers u_j and v_j satisfy constraints that depend on the symmetry of the Hamiltonian of the quantum dot:

$$u_j=e^{i\phi_j}a_j,\quad v_j=e^{-i\phi_j}a_j, \tag{4}$$

with a_j a real ($\beta=1$), complex ($\beta=2$), or real quaternion ($\beta=4$) number of modulus one. In general the choice for u_j and v_j and their parametrization (4) is not unique. Uniqueness can be achieved by requiring that

$$a_1=1,\qquad 0\leq\phi_j<\pi\quad (j=1,2). \tag{5}$$

As in Refs. 3 and 4, we assume that the scattering matrix S_d of the quantum dot is a member of the circular ensemble,

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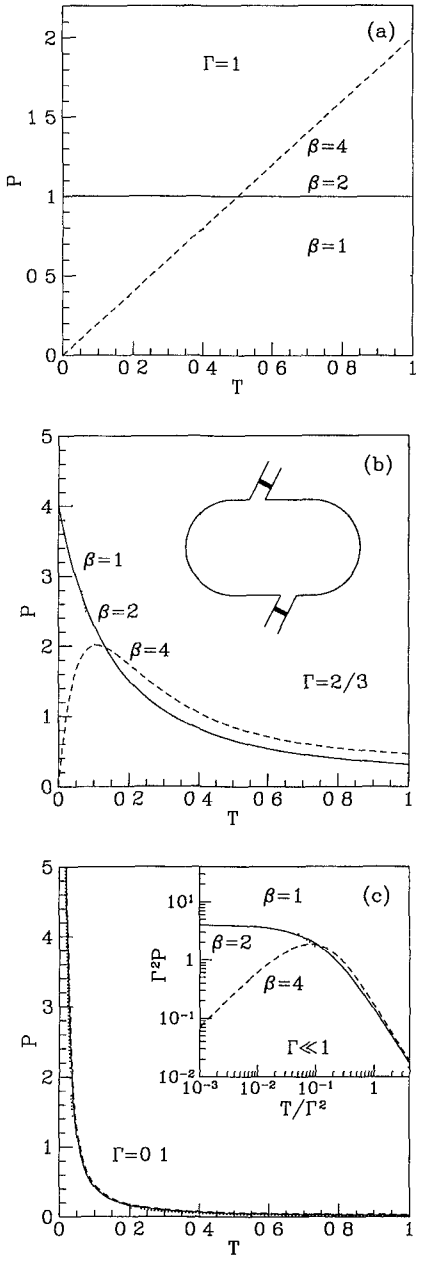

FIG. 1. Distribution of the transmission probability T through a quantum dot with nonideal single-channel leads, for three values of the transmission probability Γ of the leads. The curves are computed from Eq. (11) for each symmetry class ($\beta=1,2,4$). The inset of (b) shows the quantum dot, the inset of (c) shows the asymptotic behavior of $P(T)$ for $\Gamma \ll 1$ on a log-log scale.

which means that S_d is uniformly distributed in the unitary group (or the subgroup required by time reversal and/or spin rotation symmetry). The corresponding probability distribution of the transfer matrix M_d is

$$P_d(M_d)\, dM_d = \tfrac{1}{2}\beta(1+\lambda_d)^{-1-\beta/2} d\lambda_d d\phi_1 d\phi_2 da_2 . \quad (6)$$

The transfer matrix M_b of the tunnel barrier in the lead is given by

$$M_b = \begin{pmatrix} \sqrt{1+\mu} & \sqrt{\mu} \\ \sqrt{\mu} & \sqrt{1+\mu} \end{pmatrix}, \quad (7)$$

with $\Gamma=(1+\mu)^{-1}$. The transfer matrix M of the total system follows from the matrix product

$$M = M_b M_d M_b . \quad (8)$$

From Eqs. (2)–(8) we straightforwardly compute the transmission probability T of the total system and its probability distribution $P(T)$. The result for T is

$$T = [1+\lambda_d + m\lambda_d \cos^2\psi_- + m(\lambda_d+1)\cos^2\psi_+ + 2\sqrt{\lambda_d(\lambda_d+1)m(m+1)}\cos\psi_- \cos\psi_+]^{-1}, \quad (9)$$

where we have abbreviated

$$m = 4(1-\Gamma)\Gamma^{-2}, \quad \psi_\pm = \phi_1 \pm \phi_2 . \quad (10)$$

The variables a_j, and with them all β dependence, drop out of this expression. Equation (9) can be inverted[11] to yield λ_d in terms of ϕ_1 and ϕ_2 for given T and Γ. The probability distribution $P(T)$ then follows from

$$P(T) = \frac{\beta}{2\pi^2} \int_0^\pi d\phi_1 \int_0^\pi d\phi_2 (1+\lambda_d)^{-1-\beta/2} \left| \frac{\partial \lambda_d}{\partial T} \right|, \quad (11)$$

where the integration is over all $\phi_i \in (0,\pi)$ for which λ_d is real and positive.

For $\Gamma=1$ the function $P(T)$ is given by Eq. (1), as found in Refs. 3 and 4. In Fig. 1 the crossover from a ballistic to a tunneling point contact is shown. For $\Gamma \ll 1$ and $T \ll 1$, $\Gamma^2 P(T)$ becomes a Γ-independent function of T/Γ^2, which is shown in the inset of Fig. 1(c). Several asymptotic expressions for $P(T)$ can be obtained from Eq. (11) for $\Gamma \ll 1$,

$$\beta=1: \quad P(T) = \begin{cases} \dfrac{8}{\pi^2 \Gamma} T^{-1/2} & (T \ll \Gamma^2) \\ \dfrac{\Gamma}{\pi^2} T^{-3/2} & (\Gamma^2 \ll T \ll 1), \end{cases} \quad (12a)$$

$$\beta=2: \quad P(T) = 4\Gamma \frac{\Gamma^2+T}{(\Gamma^2+4T)^{5/2}} \quad (T \ll 1), \quad (12b)$$

$$\beta=4: \quad P(T) = 24T\Gamma \frac{3\Gamma^4+4T\Gamma^2+3T^2}{(\Gamma^2+4T)^{9/2}} \quad (T \ll 1). \quad (12c)$$

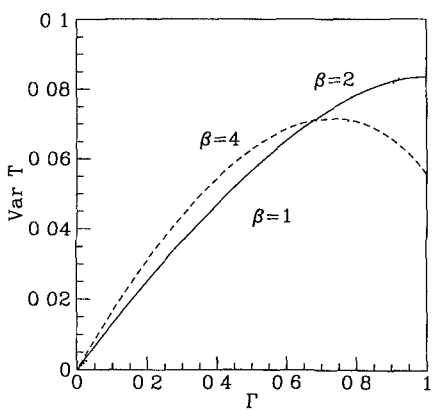

FIG. 2. Variance of the transmission probability T as a function of the transmission probability of the leads Γ.

The $\beta=2$ expression (12b) for $P(T)$ in the tunneling regime agrees precisely with the supersymmetry calculation of Prigodin, Efetov, and Iida.[2,12] Equations (12) do not cover the range near unit transmission. As $T\to 1$ (and $\Gamma\ll 1$), $P(T)\to c_\beta\Gamma$, with $c_1=1/2\pi$, $c_2=\frac{1}{4}$, and $c_4=\frac{3}{8}$.

A quite remarkable feature of the quantum dot with ideal leads is the strong β dependence of $P(T)$ [cf. Fig. 1(a)]. For $\Gamma\ll 1$, the β dependence is much less pronounced. For $T\gg\Gamma^2$ the leads dominate the transmission properties of the total system, thereby suppressing the β dependence of $P(T)$ (although not completely). For very small transmission coefficients ($T\ll\Gamma^2$) the nonideality of the leads is of less importance, and the characteristic β dependence of Eq. (1) is recovered [see inset of Fig. 1(c)].

The moments of $P(T)$ can be computed in closed form for all Γ directly from Eq. (9). The first two moments are [recall that $m=4(1-\Gamma)\Gamma^{-2}$]

$$\langle T\rangle=\begin{cases}\frac{1}{2}m^{-1}\left[\sqrt{1+m}-\dfrac{1}{\sqrt{m}}\ln(\sqrt{1+m}+\sqrt{m})\right] & (\beta=1)\\ \frac{2}{3}m^{-2}[(m-2)\sqrt{1+m}+2] & (\beta=2)\\ \frac{4}{15}m^{-3}[(3m^2-4m+8)\sqrt{1+m}-32] & (\beta=4),\end{cases}\tag{13}$$

$$\langle T^2\rangle=\begin{cases}\frac{3}{64}m^{-2}\left[(4m-18)\sqrt{1+m}+\left(\dfrac{18}{\sqrt{m}}+8\sqrt{m}\right)\ln(\sqrt{1+m}+\sqrt{m})\right] & (\beta=1)\\ \frac{4}{15}m^{-3}[(m^2+2m+16)\sqrt{1+m}-10m-16] & (\beta=2)\\ \frac{4}{35}m^{-4}[(3m^3+2m^2-40m-144)\sqrt{1+m}+112m+144] & (\beta=4).\end{cases}\tag{14}$$

For $\Gamma\ll 1$ one has asymptotically

$$\langle T^n\rangle=\frac{\beta\Gamma}{2(\beta+1)}\prod_{j=1}^{n-1}\frac{(\beta+2j)(2j-1)}{2j(\beta+2j+1)}.\tag{15}$$

The Γ dependence of the variance $\mathrm{Var}T=\langle T^2\rangle-\langle T\rangle^2$ of the transmission probability is shown in Fig. 2. In the crossover regime between a ballistic point contact ($\Gamma=1$) and a tunneling point contact ($\Gamma\ll 1$), the three symmetry classes show striking differences. For $\beta=1$ and 2 the conductance fluctuations decrease monotonically upon decreasing Γ, whereas they show nonmonotonic behavior for $\beta=4$. Notice also that the transition $\beta=1\ \to\ \beta=2$, by application of a magnetic field, reduces fluctuations for $\Gamma>\Gamma_c$ but increases fluctuations for $\Gamma<\Gamma_c$, where $\Gamma_c=0.92$.

In summary, we have computed the transmission probability of a ballistic and chaotic cavity for all possible values of the symmetry index β and for arbitrary values of the transparency Γ of the single-channel leads. Our results describe the conductance of a quantum dot in the crossover regime from a coupling to the reservoirs by ballistic to tunneling point contacts. The theory unifies and extends known results.[2–4] The characteristic β dependence of the distribution function that was found for ideal leads [Eq. (1)] is strongly suppressed for transmission probabilities T larger than Γ^2. A closely related phenomenon is the nontrivial Γ dependence of the conductance fluctuations for the three symmetry classes. The theory is relevant for experiments on chaotic scattering in quantum dots with adjustable point contacts, which are of great current interest.[13–15]

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[11] Inversion of Eq. (9) requires some care. Since a shift $\psi_+\to\psi_++\pi$ changes the sign of the term containing the square root in Eq. (9), solving for λ_d with ψ_1 and $\psi_1+\pi$ yields exactly

two (complex) solutions in total. This allows one to construct a single-valued function $\lambda_d(\psi_+,\psi_-)$ such that these two solutions are given by $\lambda_d(\psi_+,\psi_-)$ and $\lambda_d(\psi_+ + \pi,\psi_-)$. This function λ_d is understood as the inverse of Eq. (9).

[12]To compare with Ref. 2 we identify $\alpha_1 = \alpha_2 = \frac{1}{2}\Gamma \ll 1$ and take the limit $\alpha \to 0$ of Eq. (7) in that paper. This yields our Eq. (12b). Here α_1, α_2, and α are, respectively, the level broadening (divided by the level spacing) due to coupling to lead 1 and 2, and due to inelastic scattering processes (which we have not included in our formulation, whence $\alpha \to 0$).

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