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Convergence of solutions of $PSL(2, \mathbf{R})$-recurrences with parabolic limit*

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ABSTRACT

The aim of this paper is the investigation of the convergence of the solutions $\{z_n\}$ of a sequence of Möbius-transformations with parabolic limit. It is shown that either $\lim_{n\to\infty} z_n$ exists for all solutions or it exists for none of them. In the first part of the paper a description for the behaviour of solutions in the boundary region (between converging and non-converging type) is given with the aid of a certain class of renormalizations. A generalization of this idea is used in the second part to derive a necessary and sufficient condition for convergence in terms of the coefficients of the Möbius-transformations. Lastly, an application to second-order linear recurrences is given.

§1. INTRODUCTION

A $PSL(2,\mathbf{R})$-recurrence is a sequence $\{F_n\}_{n\geq 0}$ of Möbius-transformations $F_n : z \to (a_n z + b_n)/(c_n z + d_n)$ with $a_n, b_n, c_n, d_n \in \mathbf{R}$ and $a_n d_n \neq b_n c_n$, or alternatively, a sequence of matrix classes $M_n = \left(\begin{smallmatrix} a_n & b_n \\ c_n & d_n \end{smallmatrix}\right) \in PSL(2,\mathbf{R})$. The two concepts are related by: $M_n(z_1 e_1 + z_2 e_2) = z_1' e_1 + z_2' e_2$ if and only if $F_n(z_1/z_2) = z_1'/z_2' \in \mathbf{P}^1(\mathbf{R})$ where e_1, e_2 form a basis of unit vectors in $\mathbf{R}^2$. We study the asymptotic behaviour of the solutions of the recurrence, i.e. the sequences $\{z_n\}_{n\geq 0}$, $z_n \in \mathbf{P}^1(\mathbf{R})$, for which

$$z_{n+1} = F_n(z_n) \quad (n \in \mathbf{N}). \tag{1.1}$$

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From now on, we leave out the index set when considering sequences of maps, matrices and solutions, thus identifying two sequences whose terms are equal from a certain index on (we shall write $\{F_n\}$, $\{z_n\}$, etc). If the coefficients $(a_n : b_n : c_n : d_n)$ converge in $\mathbf{P}^3(\mathbf{R})$, then the following situations may occur: 1. The limit map F is constant (hence, degenerate); 2. F^2 is the identity map (i.e. $F^2 : z \to z$); 3. F is hyperbolic; 4. F is elliptic; 5. F is parabolic.

In case 3, F has two distinct fixpoints in $\mathbf{P}^1(\mathbf{R})$ and the stable fixpoints of F_n converge to the stable fixpoint s of F. In this case, all solutions of (1.1) up to one converge to the stable fixpoint s of F whereas the one remaining solution converges to the unstable fixpoint u of F. For let s_n, u_n be the stable and unstable fixpoints of F_n. Since $s \neq u$, there exists some $N \in \mathbf{N}$ and some open connected set $E \subset \mathbf{P}^1(\mathbf{R})$ such that $s_n \in E$, $u_n \notin E$ for $n \geq N$. Then $F_n(E) \subset E$ and if $z_n \in E$, then $\{z_n\}$ converges to s (if $n \geq N$). Define for $k \geq 0$, $E_k = (F_{n+k} \circ \cdots \circ F_N)^{-1}(E)$. Then $E \subset E_0 \subset E_1 \ldots$ and E_i is a connected open subset of $\mathbf{P}^1(\mathbf{R})$. On the other hand, the complements $E_i^c = \mathbf{P}^1(\mathbf{R}) \setminus E_i$ cannot be empty for all i, so E_i^c converges to some non-empty closed set D. Then $z_N \in D$ precisely if $\{z_n\}$ does not converge to s. Hence D does not depend on the particular choice of E and we see that for all points $z_N \in D$, $\{z_n\}$ converges to u. (Otherwise we could have included z_N in E for N large enough). Since the harmonic double ratio $(z_n^{(1)} : z_n^{(2)} : z_n^{(3)} : z_n^{(4)})$ is constant for all n (where $\{z_n^{(i)}\}$ $(i = 1, \ldots, 4)$ are solutions of (1.1)), we find, by letting $\{z_n^{(i)}\}$ converge to s for $i = 1, 2$ and to u for $i = 3$, so that $\{z_n^{(4)}\}$ cannot but converge to s unless $\{z_n^{(4)}\} = \{z_n^{(3)}\}$, that D consists of exactly one point. Of course, the same reasoning holds if we let F_n have coefficients in $\mathbf{C}$ instead of $\mathbf{R}$. In case 4, F has no real fixpoints (the real axis being invariant under $\{F_n\}$), so there cannot be convergence of the solutions. But here the complex case is considerably more difficult than the real case. See e.g. [5]. Finally, in case 5, F has one fixpoint (or rather, two coinciding fixpoints) in $\mathbf{P}^1(\mathbf{R})$, as follows by symmetry, if we consider the F_n as maps in $PSL(2, \mathbf{C})$. It is this case that will be the subject of our paper.

We can reduce the study of $PSL(2, \mathbf{R})$ (or $PSL(2, \mathbf{C})$) recurrences to the study of linear second-order recurrences and conversely. In order to see this, let $\{F_n\}$ be a $PSL(2, \mathbf{R})$-recurrence. Putting $z_n = (x_n : y_n) \in \mathbf{P}^1(\mathbf{R})$ we find:

$$(1.2) \qquad \begin{cases} x_{n+1} = a_n x_n + b_n y_n \\ y_{n+1} = c_n x_n + d_n y_n \end{cases}$$

which is equivalent to

$$(1.3a) \qquad y_{n+2} = (d_{n+1} + c_{n+1} a_n c_n^{-1}) y_{n+1} + (c_{n+1} b_n - a_n d_n c_{n+1} c_n^{-1}) y_n$$

$$(1.3b) \qquad y_{n+1}/y_n = d_n + c_n(x_n/y_n)$$

if $c_n \neq 0$. So, if $(a_n : b_n : c_n : d_n)$ converges, we see that a solution $\{z_n\}$ of (1.1) converges in $\mathbf{P}^1(\mathbf{R})$ if and only if $\lim_{n\to\infty} y_{n+1}/y_n$ exists for $\{y_n\}$ the solution of the second-order recurrence in (1.3a) corresponding to $\{z_n\}$ by way of (1.2). Now let $\{F_n\}$ be a $PSL(2, \mathbf{R})$-recurrence converging to a parabolic map F. For a suitable $G \in PSL(2, \mathbf{R})$, the sequence $\{G^{-1}F_nG\}$ converges to $G^{-1}FG : z \to$

$z/(z+1)$, which has a (double) fixpoint $z=0$. The corresponding linear recurrence (1.3a) has characteristic polynomial $\chi(X)=(X-1)^2$. Furthermore, since

$$U_{n+2}+P(n)\,U_{n+1}+Q(n)\,U_n=0$$

is equivalent to

$$V_{n+2}+2V_{n+1}+(4Q(n)/P(n)\,P(n+1))\,V_n=0$$

for $V_n=(\prod_{k=0}^{n-1}P(k)/2)\cdot U_n$ $(n\in\mathbf{N})$, we may suppose that (1.3a) has the form

$$(1.4)\qquad y_{n+2}-2y_{n+1}+(1-c(n))\,y_n=0$$

where $\lim_{n\to\infty}c(n)=0$, $c(n)\in\mathbf{R}$. Putting $y_{n+1}/y_n-1=w_n$, we find

$$(1.5)\qquad w_{n+1}=F_n(w_n)=\frac{w_n+c(n)}{w_n+1}\qquad(n\in\mathbf{N}).$$

Thus, when studying the convergence of the solutions of (1.1) with parabolic limit $F=\lim_{n\to\infty}F_n$, we may assume that $F_n(z)$ is of the form given in (1.5), with $\lim_{n\to\infty}c(n)=0$. It is a generalization of this type of recurrence that we shall study in the following sections.

We finish this section by discussing two notations that we shall use in the sequel. Firstly, if $\{a_n\}_{n\in\mathbf{N}}$ and $\{b_n\}_{n\in\mathbf{N}}$ are two sequences of real or complex numbers, $a_n\sim b_n$ will have the same meaning as $a_n-b_n=o(|a_n|)$ for $n\to\infty$. Secondly, $a_n\ll b_n$ means that $a_n<c\cdot b_n$ for all $n\in\mathbf{N}$ and for some positive constant c if a_n and b_n are non-negative real numbers.

§2. THE SETS $\mathcal{F}(\{d_n\},\{f_n\})$; ELEMENTARY PROPERTIES

We start by giving a few definitions.

Definition. Let $\{d_n\},\{f_n\}$ be sequences of real numbers. We define $\mathcal{F}(\{d_n\},\{f_n\})$ as the recurrence (1.1) defined by the sequence of Möbius-transformations $\{F_n\}$ where

$$(2.1)\qquad F_n(z)=\frac{z-d_n}{f_n z+1}\qquad(n\in\mathbf{N}).$$

In the sequel, we shall consider only a special type of recurrences $\mathcal{F}(\{d_n\},\{f_n\})$, which will appear to be a sort of natural generalization of the case that $\lim_{n\to\infty}F_n$ exists and is parabolic. We let $\mathcal{S}$ be the set of sequences $\{f_n\}$ with $f_n\in\mathbf{R}$, $\lim_{n\to\infty}f_{n+1}/f_n=1$ and $\sum_{n=0}^{\infty}f_n=+\infty$. We shall, when studying $\mathcal{F}(\{d_n\},\{f_n\})$, assume $\{f_n\}\in\mathcal{S}$ and, moreover, $d_n\in\mathbf{R}$ and $\lim_{n\to\infty}d_nf_n=0$. In this case, we can always choose a representant with $d_nf_n>-1$, so that $\mathcal{F}(\{d_n\},\{f_n\})$ becomes a $PSL(2,\mathbf{R})$-recurrence and, moreover, F_n preserves the orientation of $\mathbf{P}^1(\mathbf{R})$ (as a subset of $\mathbf{P}^1(\mathbf{C})$, i.e. F_n maps the upper half-plane in $\mathbf{C}$ onto itself). We have the following results:

Lemma 2.1. *Let F_n be as in (2.1) with $d_nf_n>-1$ and $f_n>0$. Then*

(a) *F_n preserves the orientation of any three points in $\mathbf{P}^1(\mathbf{R})$.*

(b) *If $|z|^2>c\,|d_n|\,f_n^{-1}$ $(c\geq1)$, then $F_n(z)^{-1}-z^{-1}>c'f_n$ where $c'>0$ if $c>1$ and $c'=0$ if $c=1$.*

(c) *If $|z|^2\leq|d_n|\,f_n^{-1}$, then $F_n(z)\leq(|d_n|\,f_n^{-1})^{1/2}$.*

Proof. The straightforward proof is left to the reader. □

The preservation of orientation implies the following

Proposition 2.2. *If* $\{f_n\} \in \mathcal{S}$, $d_n f_n > -1$, *then either* $\{f_n z_n\}$ *converges for all solutions* $\{z_n\}$ *of* $\mathcal{F}(\{d_n\}, \{f_n\})$, *or it diverges for all solutions.*

Proof. Suppose there is some solution $\{z_n^{(0)}\}$ such that $\{f_n z_n^{(0)}\}$ converges. By Lemma 2.1, we have for any solution $\{z_n\}$ of $\mathcal{F}(\{d_n\}, \{f_n\})$ that for n large enough, and some $c > 1$, that $z_n^{(0)} \leq z_n$ and either $z_{n+1}^{-1} - z_n^{-1} > c' f_n$ or $f_n |z_n| \leq \sqrt{c}(|d_n| f_n)^{1/2}$, where the right-hand part becomes arbitrarily small. Fix some number $0 < \varepsilon < 2/c'$ and let N be so large that $c \cdot |d_n| f_n < \varepsilon^2, f_{n+1}/f_n < 1 + (c'\varepsilon/2) < 2$ and $f_n |z_n^{(0)}| < \varepsilon$ for $n \geq N$. By $\sum_{n \in \mathbf{N}} f_n = \infty$ we conclude that $-\varepsilon < -f_m |z_m^{(0)}| \leq f_m z_m < \varepsilon$ for some number $m \geq N$. For $n \geq N$ we have: If $f_n z_n \leq 0$, then $f_{n+1} z_{n+1} \leq \sqrt{c}(f_n |d_n|)^{1/2} < \varepsilon$, if $0 < f_n z_n < \varepsilon/2$, then $f_{n+1} z_{n+1} < f_n z_n$, and if $f_n z_n \geq \varepsilon/2$, then $f_{n+1} z_{n+1} < \varepsilon$. Hence $f_n |z_n| < \varepsilon$ for all $n \geq m$. Since ε was arbitrary, we conclude that $\lim_{n \to \infty} f_n z_n = 0$. □

From Proposition 2.2 it follows that we can distinguish two distinct types of recurrences $\mathcal{F}(\{d_n\}, \{f_n\})$: converging-type recurrences for all of whose solutions $\{z_n\}$ the sequence $\{z_n f_n\}$ converges (in which case it must converge to zero), and diverging-type recurrences, where $\{z_n f_n\}$ does not converge for any of its solutions $\{z_n\}$. We can now introduce an ordering on the set of solutions of a converging type recurrence: For two solutions $\{z_n\}, \{z_n'\}$ we define $\{z_n\} \leq \{z_n'\}$ if $z_n \leq z_n'$ for all n large enough. This yields a total ordering on the set of solutions. For $\{z_n\}$ fixed, the set U consisting of the numbers $\alpha \in \mathbf{P}^1(\mathbf{R})$ such that $\{z_n'\} > \{z_n\}$ and $z_0' = \alpha$ is open and connected. The complement of U in $\mathbf{P}^1(\mathbf{R})$ is closed, connected and non-empty, since it contains z_0. Hence it contains the number ζ_0 such that $\{\zeta_n\} = \inf(\{z_n'\} : \{z_n'\} \leq \{z_n\})$, where the infimum is taken over the solutions of $\mathcal{F}(\{d_n\}, \{f_n\})$. $\{\zeta_n\}$ is the solution for which $\{\zeta_n\} \leq \{z_n'\}$ for all solutions $\{z_n'\}$, so it des not depend on $\{z_n\}$. We call it the subdominant solution of $\mathcal{F}(\{d_n\}, \{f_n\})$. The following proposition gives, for $\{f_n\} \in \mathcal{S}$ fixed, a characterisation for sequences that are subdominant solutions of some recurrence $\mathcal{F}(\{d_n\}, \{f_n\})$.

Proposition 2.3. *Let* $\{z_n\}$ *be a sequence of real numbers such that* $\lim_{n \to \infty} f_n z_n = 0$ *for* $\{f_n\} \in \mathcal{S}$. *Let N be such that* $f_n |z_n| < 1$ *and* $f_n |z_{n+1}| < 1$ *for* $n \geq N$ *and define*

$$(2.2) \qquad G_n = \prod_{k=N}^{n-1} \frac{1 + f_k z_k}{1 - f_k z_{k+1}} \qquad (n \in N,\ n \geq N).$$

Then $\{z_n\}$ *is the subdominant solution of* $\mathcal{F}(\{z_n - z_{n+1} - f_n z_n z_{n+1}\}, \{f_n\})$ *if and only if* $\sum_{n=N}^{\infty} f_n G_n^{-1} = \infty$.

Proof. First note that $G_n > 0$ for all $n \geq N$ and that

$$\mathcal{F}(\{z_n - z_{n+1} - f_n z_n z_{n+1}\}, \{f_n\})$$

is of converging type. Put $d_n = z_n - z_{n+1} - f_n z_n z_{n+1}$ for $n \in \mathbf{N}$. Then

$$\begin{pmatrix} 1 & -z_{n+1} \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & -d_n \\ f_n & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & z_n \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 - f_n z_{n+1} & 0 \\ 0 & 1 + f_n z_n \end{pmatrix}$$

so that, for any solution $\{z'_n\}$ of $\mathcal{F}(\{d_n\}, \{f_n\})$,

$$(z'_{n+1} - z_{n+1})^{-1} = \frac{1 + f_n z_n}{1 - f_n z_{n+1}} \cdot (z'_n - z_n)^{-1} + \frac{f_n}{1 - f_n z_{n+1}}$$

whence, for $n \geq m \geq N$,

$$[G_n(z'_n - z_n)]^{-1} = [G_m(z'_m - z_m)]^{-1} + \sum_{k=m}^{n-1} f_k G_k^{-1}(1 + f_k z_k)^{-1}. \tag{2.3}$$

Take $m = N$. Then, if $\{z_n\}$ is subdominant, we see that the right-hand side becomes positive for all $\{z'_n\} \neq \{z_n\}$ as soon as n is sufficiently large. Hence, taking into account that $\lim_{k \to \infty} f_k z_k = 0$, we find that indeed $\sum_{k=N}^{\infty} f_k G_k^{-1} = \infty$. Conversely, if $\sum_{k=N}^{\infty} f_k G_k^{-1} = \infty$, then the left-hand side becomes positive for all $\{z'_n\} \neq \{z_n\}$ if n is sufficiently large, so that $\{z'_n\} \geq \{z_n\}$ for all solutions $\{z'_n\}$. □

§3. SOME CONDITIONS FOR CONVERGENCE OR DIVERGENCE

In this section, we state a number of sufficient conditions for $\mathcal{F}(\{d_n\}, \{f_n\})$ to be of converging or diverging type. As above, we suppose $\{f_n\} \in \mathcal{S}$, $\lim_{n \to \infty} f_n d_n = 0$, so that the results of §2 hold. We write $\{a_n\} \geq$ (or $>$) $\{a'_n\}$ for two real-valued sequences if $a_n \geq$ (or $>$) a'_n for n sufficiently large. We start with a few lemmas:

Lemma 3.1. *If $\mathcal{F}(\{d_n\}, \{f_n\})$ is of converging type and $\{d'_n\} \leq \{d_n\}$, $\lim_{n \to \infty} d'_n f_n = 0$, then $\mathcal{F}(\{d'_n\}, \{f_n\})$ is of converging type. Moreover, if $\{\zeta'_n\}, \{\zeta_n\}$ are the subdominant solutions of $\mathcal{F}(\{d'_n\}, \{f_n\})$ and $\mathcal{F}(\{d_n\}, \{f_n\})$ respectively, then $\{\zeta'_n\} \leq \{\zeta_n\}$.*

Proof. Let $\{z_n\}$ be some solution of $\mathcal{F}(\{d_n\}, \{f_n\})$ such that $f_n |z_n| < 1$ for all $n \geq N$. Let $\{z'_n\}$ be the solution of $\mathcal{F}(\{d'_n\}, \{f_n\})$ such that $z'_N = z_N$. Then $z_n \leq z'_n$ for all $n \geq N$. The reasoning that $\lim_{n \to \infty} f_n |z'_n| = 0$ goes exactly as in the proof of Proposition 2.2. Suppose $\zeta'_N > \zeta_N$ for infinitely many $N \in \mathbf{N}$. Then $\{\zeta'_n\} \geq \{\zeta_n\}$. But since $\{d'_n\} \leq \{d_n\}$, for the solution $\{\zeta''_n\}$ of $\mathcal{F}(\{d'_n\}, \{f_n\})$ with $\zeta''_N = \zeta_N$ we would have $\{\zeta''_N\} < \{\zeta'_n\}$ so $\{\zeta'_n\}$ cannot be subdominant. □

Lemma 3.2. *If $\{f_n\} \in \mathcal{S}$, then for N fixed, N large enough,* $\lim_{n \to \infty} (f_N + f_n)/(f_N + \cdots + f_n) = 0$.

Proof. The simple proof will be left to the reader. □

Lemma 3.3. *If $\{f_n\} \in \mathcal{S}$ and $f_n = F_{n+1} - F_n$ then $\{f_n/F_n\} \in \mathcal{S}$.*

Proof. $\lim_{n\to\infty} F_n = \infty$ and by $\lim_{n\to\infty} f_{n+1}/f_n = 1$, hence by Lemma 3.2, $\lim_{n\to\infty} f_n/F_n = 0$, so $\lim_{n\to\infty} F_{n+1}/F_n = 1$. Moreover, for numbers $M > N$ large enough,

$$\sum_{k=N}^{M-1} f_k/F_k \geq F_N \sum_{k=N}^{M-1} f_k/F_k F_{k+1} = 1 - F_N/F_M \geq \tfrac{1}{2}$$

so that $\sum_{k=N}^{\infty} f_k/F_k = +\infty$ (where N is so large that $F_n > 0$ for $n \geq N$). □

Lemma 3.4. *Let $\{D_n\}$ be a sequence of matrices in $\mathcal{M}_2(K)$ and $\{\alpha(n)\}, \{\beta(n)\}$ be sequences of complex numbers such that $\{|\alpha(n)|\} \geq \{|\beta(n)|\} > \{0\}$. If $\sum_{n=N}^{\infty} \|D_n\| \cdot |\beta(n)|^{-1} < \infty$ then there are matrices $J_n \in GL(2, K)$ such that $\lim_{n\to\infty} J_n = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and $J_{n+1}(\operatorname{diag}(\alpha(n), \beta(n)) + D_n) J_n^{-1} = \operatorname{diag}(\alpha(n), \beta(n))$ ($K = \mathbf{R}$ or $\mathbf{C}, n \in \mathbf{N}$).*

Proof. See [6], Lemma 4.1. □

Proposition 3.5. *Let $\{f_n\}$ and $\{d_n\}$ be sequences such that $\{f_n\} \in \mathcal{S}$ and $\lim_{n\to\infty} d_n f_n = 0$. Then*

(a) *If $\{d_n\} \leq \{0\}$ then $\mathcal{F}(\{d_n\}, \{f_n\})$ is of converging type.*

(b) *If $\mathcal{F}(\{d_n\}, \{f_n\})$ is of converging type and $\{d_n\} \geq \{0\}$, then $\sum_{k=N}^{\infty} d_k < \infty$.*

(c) *If $d_n = (\varepsilon + \mathrm{o}(1))\,\Delta(F_n^{-1}) + d_n'$ where $\Delta F_n = F_{n+1} - F_n = f_n$ ($n \in \mathbf{N}$) and $\sum_{n=N}^{\infty} |d_n'| F_{n+1} < \infty$, then, if $\varepsilon > -\frac{1}{4}$, $\mathcal{F}(\{d_n\}, \{f_n\})$ is of converging type and has a subdominant solution $\{\zeta_n\}$ with $\lim_{n\to\infty} \zeta_n F_n = \delta_1$, whereas $\lim_{n\to\infty} z_n F_n = \delta_2$ for all other solutions $\{z_n\}$, where $\delta_1 < \delta_2$ are the roots of $X^2 - X - \varepsilon = P(X)$. If $\varepsilon < -\frac{1}{4}$, then $\mathcal{F}(\{d_n\}, \{f_n\})$ is of diverging type.*

Proof. (a) This follows immediately from Lemma 3.1 and the fact that $\mathcal{F}(\{0\}, \{f_n\})$ is of converging type (it has $\{0\}$ as a solution).

(b) Let $\{z_n\}$ be a solution of $\mathcal{F}(\{d_n\}, \{f_n\})$. By $z_n - z_{n+1} = f_n z_n z_{n+1} + d_n$ we have for $N \in \mathbf{N}$ so large that $|f_n z_n| < 1$ for $n \geq N$, that $z_{n+1} < z_n/(1 + f_n z_n) < z_n$ for $n \geq N$, so that $\{z_n\} \geq \{0\}$ (compare with Lemma 3.1). But then, for n large enough $\sum_{k=n}^{\infty} d_k \leq \sum_{k=n}^{\infty} (z_k - z_{k+1}) = z_n - \lim_{k\to\infty} z_k$.

(c) Set

$$G_n = \begin{pmatrix} 1 & -d_n \\ f_n & 1 \end{pmatrix} \quad (n \in \mathbf{N})$$

and let $d_n = \varepsilon\, \Delta F_n^{-1} + d_n'$. Put

$$H_n = \frac{1}{F_n} \cdot \begin{pmatrix} \delta_1 & \delta_2 \\ 1 & 1 \end{pmatrix}.$$

Then

$$(3.1) \qquad H_{n+1}^{-1} \cdot G_n \cdot H_n = \operatorname{diag}(1 + \delta_1 f_n/F_n,\ 1 + \delta_2 f_n/F_n) + D_n$$

where $\|D_n\| = O(F_{n+1}\,|d_n'|)$. If $\varepsilon > -\frac{1}{4}$, then $\delta_1, \delta_2 \in \mathbf{R}$ and

$$(3.2) \qquad \sum_{n=N}^{\infty} \left| \left| \frac{1+\delta_1 f_n/F_n}{1+\delta_2 f_n/F_n} \right| - 1 \right| = \infty.$$

If $\varepsilon < -\frac{1}{4}$, then $\delta_2 = \overline{\delta_1} \notin \mathbf{R}$ and

$$(3.3) \qquad \left| \frac{1+\delta_1 f_n/F_n}{1+\delta_2 f_n/F_n} \right| = 1 \quad (n \in \mathbf{N}).$$

Lemma 3.4 now yields that there exist $J_n \in GL(2,\mathbf{R})$ in case $\varepsilon > -\frac{1}{4}$, $J_n \in GL(2,\mathbf{C})$ if $\varepsilon < -\frac{1}{4}$ such that $\{J_n\}$ converges to the identity matrix and $J_{n+1} H_{n+1}^{-1} G_n H_n J_n^{-1} = \operatorname{diag}(1+\delta_1 f_n/F_n,\ 1+\delta_2 f_n/F_n)$. So $\mathcal{F}(\{d_n\},\{f_n\})$ has solutions $\{z_n^{(i)}\} = \{(x_n^{(i)} : y_n^{(i)})\}$ with $(x_n^{(i)} : y_n^{(i)})^t = \lambda_n^{(i)} H_n J_n^{-1} e_i$. So $z_n^{(i)} = (\delta_i + o(1))\,F_n^{-1}$ $(i = 1,2,\ n \in \mathbf{N})$. If $\varepsilon > -\frac{1}{4}$, then $z_n^{(i)} \in \mathbf{R}$ and since $\lim_{n\to\infty} z_n^{(i)} f_n = 0$, $\mathcal{F}(\{d_n\},\{f_n\})$ is indeed of converging type. For a solution $\{z_n\} \neq \{z_n^{(1)}\}$ it follows from (3.2) that $\lim_{n\to\infty} z_n F_n = \delta_2$, since $z_n = ((\lambda x_n^{(1)} + \mu x_n^{(2)}) : (\lambda y_n^{(1)} + \mu y_n^{(2)}))$ with $\mu \neq 0$ $(\lambda_n^{(1)}/\lambda_n^{(2)} \to 0$ as $n \to \infty)$. So, $\{z_n^{(1)}\}$ is the subdominant solution. Moreover, if $d_n - d_n' = (\varepsilon + o(1))\,\Delta(F_n^{-1})$ instead of $\varepsilon \cdot \Delta F_n^{-1}$, Lemma 3.1 yields that, for the subdominant solution $\{z_n^{(1)}\}$, for any $\delta > 0$

$$\limsup z_n^{(1)} F_n \leq \delta_1 + \delta, \qquad \liminf z_n^{(1)} F_n \geq \delta_1 - \delta.$$

Hence, $\lim_{n\to\infty} z_n^{(1)} F_n = \delta_1$, and similarly, $\lim_{n\to\infty} z_n F_n = \delta_2$ for any other solution $\{z_n\}$. Now for the case $\varepsilon < -\frac{1}{4}$. We may assume $\{z_n^{(2)}\} = \{\bar{z}_n^{(1)}\}$. Then we see that $\mathcal{F}(\{d_n\},\{f_n\})$ has a real solution $\{z_n\}$ with

$$z_n = \frac{z_n^{(1)}\lambda(n) + z_n^{(2)}\bar{\lambda}(n)}{\lambda(n) + \bar{\lambda}(n)}$$

where $|\lambda(n)| = 1$, $\lambda(n+1)^2 = \lambda(n)^2 \cdot (1+\delta_1 f_n/F_n)/(1+\delta_2 f_n/F_n)$. By

$$\sum_{n=N}^{\infty} f_n/F_n = \infty,$$

we see that $\arg \lambda(n)$ does not converge, and for infinitely many n,

$$\left| \arg \lambda(n+1) - \frac{\pi}{2} \right| < \arg(1+\delta_1 f_n/F_n),$$

so that for $\varepsilon > 0$ fixed,

$$|\Re \lambda(n+1)| < (1+\varepsilon)\,|\Im \delta_1|\, f_n/F_n$$

for infinitely many n. For such an $n \in \mathbf{N}$,

$$|f_{n+1} z_{n+1}| \geq \frac{\Re(\lambda(n+1)\, z_{n+1}^{(1)}) \cdot F_{n+1}}{(1+\varepsilon)\,|\Im \delta_1|} \cdot \frac{f_{n+1}}{f_n} \cdot \frac{F_n}{F_{n+1}}$$

and the expression on the right tends to $(1+\varepsilon)^{-1}$ as $n \to \infty$. Hence, $\{f_n z_n\}$ does not converge. Finally, if $d_n - d_n' = (\varepsilon + o(1))\,\Delta(F_n^{-1})$, $\varepsilon < -\frac{1}{4}$, then $\{d_n\} \geq \{(\varepsilon/2 + \frac{1}{8})\,\Delta F_n^{-1} + d_n''\}$ and $\sum_{n=N}^{\infty} |d_n''|\, F_{n+1} < \infty$ so that $\mathcal{F}(\{d_n\}, \{f_n\})$ is indeed of diverging type (by Lemma 3.1 again). □

Example. Consider the linear recurrence

$$(3.4) \qquad u_{n+2} - 2u_{n+1} + (1 - c(n))\, u_n = 0.$$

For $\{u_n\} \neq \{0\}$ a solution of (3.4), $\{(u_{n+1}/u_n) - 1\}$ is a solution of $\mathcal{F}(\{-c(n)\}, \{1\})$ and conversely. Suppose $c(n) \in \mathbf{R}$ $(n \in \mathbf{N})$. We take $\{F_n\} = \{n\}$. By Proposition 3.5(c) the following facts hold: If $n^2 c(n) \leq -\frac{1}{4} - \varepsilon$ $(\varepsilon > 0)$ for n large enough, then $\lim_{n\to\infty} (u_{n+1}/u_n)$ does not exist for any real solution of (3.4). If $n^2 c(n) \geq -\frac{1}{4} + \varepsilon$ $(\varepsilon > 0)$ for n large enough, then $\lim_{n\to\infty} (u_{n+1}/u_n) = 1$ for all non-trivial solutions $\{u_n\}$ of (3.4). Moreover, if $\lim_{n\to\infty} n^2 c(n) = \gamma > -\frac{1}{4}$, then $\lim_{n\to\infty} n((u_{n+1}^{(i)}/u_n^{(i)}) - 1) = \delta_i$ for solutions $\{u_n^{(i)}\}$ $(i = 1, 2)$ of (3.4), where δ_1, δ_2 are the zeros of $X^2 - X - \gamma$. The last result holds also if $c(n)$ complex, and not $\gamma \leq -\frac{1}{4}$. Finally, if $c(n) \in \mathbf{C}$, $\lim_{n\to\infty} n^2 c(n) = \gamma \neq -\frac{1}{4}$ and $\sum_{n=N}^{\infty} |nc(n) - \gamma/n| < \infty$ then (3.4) has solutions $\{u_n^{(i)}\}$ $(i = 1, 2)$ such that $\lim_{n\to\infty} n((u_{n+1}^{(i)}/u_n^{(i)}) - 1) = \delta_i$. This follows from the proof of Proposition 3.5(c), but in fact, this case (and its n-th order analogon) has been treated extensively in [5].

§4. AFFINE RENORMALIZATIONS

Under the conditions of Proposition 3.5(c) we see that if ε approaches $-\frac{1}{4}$ from above, then $\lim_{n\to\infty} z_n^{(1)}/z_n$ (where $\{z_n^{(1)}\}$ is the subdominant solution of $\mathcal{F}(\{d_n\}, \{f_n\})$, and $\{z_n\}$ any other solution) tends to 1. If $\varepsilon < -\frac{1}{4}$, then $\mathcal{F}(\{d_n\}, \{f_n\})$ is of diverging type. So $\varepsilon = -\frac{1}{4}$ constitutes in a way a boundary case. We shall investigate such cases more closely with the aid of what will be called 'affine renormalizations'.

Let $\{f_n\} \in \mathcal{S}$ be a fixed sequence and $\{\hat{d}_n\}$ some sequence of real numbers such that $\lim_{n\to\infty} \hat{d}_n f_n = 0$ and $\mathcal{F}(\{\hat{d}_n\}, \{f_n\})$ is of converging type. Let $\{\zeta(n)\}$ be its subdominant solution and define

$$(4.1) \qquad G_n = G_n(\{\zeta(n)\}, \{f_n\}) = \prod_{k=0}^{n-1} \frac{1 + f_k\, \zeta(k)}{1 - f_k\, \zeta(k+1)} \qquad (n \in \mathbf{N}).$$

Note that $(1 + f_k\, \zeta(k))/(1 - f_k(k+1)) = (1 + f_k\, \zeta(k))^2/(1 + \hat{d}_k f_k) > 0$ if $\hat{d}_k f_k > -1$ unless $f_k\, \zeta(k) = -1$ or $\zeta(k) = \infty$. If $f_k\, \zeta(k) = -1$, then $\zeta(k+1) = \infty$ and $f_{k+1}\zeta(k+2) = 1$, and

$$\frac{1 + f_k\, \zeta(k)}{1 - f_k \zeta(k+1)} \cdot \frac{1 + f_{k+1}\zeta(k+1)}{1 - f_{k+1}\zeta(k+2)} = \frac{1 + f_k\, \hat{d}_k}{1 + f_{k+1}\hat{d}_{k+1}} \cdot \frac{(1 + f_k\, \zeta(k+1))^2}{(1 - f_k\, \zeta(k+1))^2} > 0.$$

So, assuming (as is legitimate) $f_k\, \hat{d}_k > -1$ for all k, we have $G_n > 0$ for almost all n. By Proposition 2.3, $\sum_{n=N}^{\infty} f_n G_n^{-1} = \infty$ (N so large that $G_n > 0$ for $n \geq N$). The recurrence $\mathcal{F}(\{\hat{d}_n\}, \{f_n\})$ defines an affine transformation as follows:

Proposition 4.1. *Every solution $\{z_n\}$ of a non-degenerate recurrence $\mathcal{F}(\{d_n\}, \{f_n\})$ corresponds uniquely to a solution $\{z_n^\nu\} = \{G_n(z_n - \zeta(n))\}$ of $\mathcal{F}(\{d_n^\nu\}, \{f_n^\nu\})$ where*

$$d_n^\nu = G_{n+1}(d_n - \hat{d}_n)/(1 + f_n\,\zeta(n)), \qquad f_n^\nu = f_n\,G_n^{-1}/(1 + f_n\,\zeta(n)),$$

and $\{f_n^\nu\} \in \mathcal{S}$. In particular, $\lim_{n\to\infty} d_n^\nu f_n^\nu = 0$ if and only if $\lim_{n\to\infty} d_n f_n = 0$. Furthermore, $\mathcal{F}(\{d_n\}, \{f_n\})$ is of converging type if and only if $\mathcal{F}(\{d_n^\nu\}, \{f_n^\nu\})$ is.

Proof. Putting

$$M_n = \begin{pmatrix} 1 & -d_n \\ f_n & 1 \end{pmatrix} \quad \text{and} \quad P_n = \begin{pmatrix} G_n & -G_n\,\zeta(n) \\ 0 & 1 \end{pmatrix}$$

we find

$$P_{n+1} M_n P_n^{-1} = \begin{pmatrix} 1 & -d_n^\nu \\ f_n^\nu & 1 \end{pmatrix} \quad (n \in \mathbf{N})$$

(where we take representants for elements of $PSL(2, \mathbf{R})$). Since $\lim_{n\to\infty} f_n\,\zeta(n) = 0$ we have $\lim_{n\to\infty} G_{n+1}/G_n = 1$, so that $\lim_{n\to\infty} f_{n+1}^\nu/f_n^\nu = 1$. From $\sum_{n=N}^\infty f_n G_n^{-1} = \infty$ it then follows that $\{f_n^\nu\} \in \mathcal{S}$. Finally, $\lim_{n\to\infty} d_n f_n = 0$, $\lim_{n\to\infty} z_n f_n = 0$ imply $\lim_{n\to\infty} d_n^\nu f_n^\nu = 0$, $\lim_{n\to\infty} z_n^\nu f_n^\nu = 0$ and conversely. □

Remark 4.1. Transformations of the type described in the above propostion will be called 'affine renormalizations' and will be denoted by $\nu(\{\zeta(n)\}, \{f_n\})$. Note that an affine renormalization defines the sequence (germ) $\{G_n\}$ up to a multiplicative constant $\lambda \in \mathbf{R}$, $\lambda \neq 0$, so the same holds for $\{d_n^\nu\}, \{z_n^\nu\}, \{(f_n^\nu)^{-1}\}$.

The following proposition asserts that a sort of converse of Proposition 4.1 holds:

Proposition 4.2. *Let $\{f_n\} \in \mathcal{S}$ be fixed. Any (affine) transformation that transforms solutions $\{z_n\}$ of recurrences $\mathcal{F}(\{d_n\}, \{f_n\})$ into solutions $\{z_n^\nu\} = \{G_n(z_n - \zeta(n))\}$ of some recurrence $\mathcal{F}(\{\hat{d}_n\}, \{\hat{f}_n\})$ with $\{\hat{f}_n\} \in \mathcal{S}$, $G_n \neq 0$, is an affine renormalization $\nu(\{\zeta(n)\}, \{f_n\})$.*

Proof. From the fact that $\{z_n^\nu\}$ is a solution of $\mathcal{F}(\{\hat{d}_n\}, \{\hat{f}_n\})$ it follows that $G_{n+1}/G_n = (1 + f_n\,\zeta(n))/(1 - f_n\,\zeta(n+1))$ $(n \in \mathbf{N})$ as in Proposition 4.1, and

$$\hat{f}_n = f_n G_n^{-1}/(1 + f_n\,\zeta(n)), \qquad \hat{d}_n = G_{n+1}(d_n - \tilde{d}_n)/(1 + f_n(n))$$

with

$$\tilde{d}_n = \zeta(n) - \zeta(n+1) - f_n\,\zeta(n) \cdot \zeta(n+1) \quad (n \in \mathbf{N}).$$

So $\hat{f}_{n+1}/\hat{f}_n = f_{n+1}/f_n \cdot (1 - f_n\,\zeta(n+1))/(1 + f_{n+1}\,\zeta(n+1))$ $(n \in \mathbf{N})$ and

$$\lim_{n\to\infty} \hat{f}_{n+1}/\hat{f}_n = 1$$

imply $\lim_{n\to\infty} f_n\,\zeta(n) = 0$, so that $\{\zeta(n)\}$ is the solution of a recurrence $\mathcal{F}(\{\tilde{d}_n\}, \{f_n\})$ with $\lim_{n\to\infty} \tilde{d}_n f_n = 0$. This in its turn implies $\lim_{n\to\infty} G_{n+1}/G_n = 1$ and from $\sum_{n=N}^\infty \hat{f}_n = \infty$ it then follows that $\sum_{n=N}^\infty f_n G_n^{-1} = \infty$, so that $\{\zeta(n)\}$ is indeed the subdominant solution of $\mathcal{F}(\{\tilde{d}_n\}, \{f_n\})$. □

Corollary 4.3. *The composition of two affine renormalizations* $\nu = \nu(\{\zeta(n)\}, \{f_n\})$ *and* $\nu' = \nu(\{\vartheta(n)\}, \{f_n^\nu\})$ *is an affine renormalization* $\nu(\{\eta(n)\}, \{f_n\})$, *where* $\eta(n)^\nu = \vartheta(n)$ *for all n. Moreover,*

$$G_n(\{\eta(n)\}, \{f_n\}) = G_n(\{\eta(n)^\nu\}, \{f_n^\nu\}) \cdot G_n(\{\zeta(n)\}, \{f_n\}).$$

Proof. Putting $G_n = G_n(\{\zeta(n)\}, \{f_n\})$, $G_n' = G_n(\{\eta(n)^\nu\}, \{f_n^\nu\})$ for $n \in \mathbf{N}$, we have

$$(z_n^\nu)^{\nu'} = G_n'(G_n(z_n - \zeta(n)) - G_n(\eta(n) - \zeta(n))) = G_n' G_n(z_n - \eta(n)).$$

From Proposition 4.2 it now follows that $G_n' G_n = G_n(\{\eta(n)\}, \{f_n\})$ and $\nu' \circ \nu = \nu(\{\eta(n)\}, \{f_n\})$. □

Application. Let $\{f_n\} \in \mathcal{S}$, $d_n = -\frac{1}{4} \Delta F_n^{-1} + d_n'$, where $F_{n+1} - F_n = f_n$ $(n \in \mathbf{N})$. If $\{d_n'\} = \{0\}$, then $\{\zeta(n)\} = \{\frac{1}{2} F_n^{-1}\}$ is a solution of $\mathcal{F}(\{d_n\}, \{f_n\})$. Moreover, it is the subdominant solution, by $(G_n(\{\zeta(n)\}, \{f_n\}))/(G_N(\{\zeta(n)\}, \{f_n\})) = F_n/F_N$ $(n \geq N)$ and $\sum_{j=N}^\infty f_j G_j^{-1} = F_N G_N^{-1} \sum_{j=N}^\infty f_j/F_j = \infty$ (see Lemma 3.3), where $G_n = G_n(\{\zeta(n)\}, \{f_n\})$. So $\{\zeta(n)\}$ defines an affine renormalization $\nu = \nu(\{\zeta(n)\}, \{f_n\})$ with

$$z_n^\nu = F_n z_n - \tfrac{1}{2}, \quad f_n^\nu = f_n(F_n + \tfrac{1}{2} f_n)^{-1}$$

and

$$d_n^\nu = F_{n+1} d_n'(1 + \tfrac{1}{2} f_n/F_n)^{-1}.$$

Repeating the argument and using the fact that a (finite) succession of affine renormalizations gives an affine renormalization, we obtain

Theorem 4.4. *Define sequences* $\{f_n^{(i)}\}$ *as follows:* $\{f_n^{(0)}\} = \{f_n\} \in \mathcal{S}$, $\{f_n^{(i+1)}\} = \{f_n^{(i)}(F_n^{(i)} + \frac{1}{2} f_n^{(i)})^{-1}\}$, *where* $F_0^{(i)} = \lambda_i \in \mathbf{R}$ *and* $\Delta F_n^{(i)} = f_n^{(i)}$ $(n, i \in \mathbf{N})$. *Then* $\{\zeta_j(n)\} = \{\frac{1}{2}(1/(F_n^{(0)}) + 1/(F_n^{(0)} F_n^{(1)}) + \cdots + 1/(F_n^{(0)} \cdot \cdots \cdot F_n^{(j)}))\}$ *is the subdominant solution of* $\mathcal{F}(\{d_j(n)\}, \{f_n\})$ *and defines an affine renormalization* $\nu(\{\zeta_j(n)\}, \{f_n\})$ *with* $z_n^\nu = F_n^{(0)} \cdot \cdots \cdot F_n^{(j)}(z_n - \zeta_j(n))$, $f_n^\nu = f_n^{(j+1)}$ *and*

$$d_j(n) = \frac{1}{4} \sum_{i=0}^{j} \frac{-1}{F_{n+1}^{(0)} \cdot \cdots \cdot F_{n+1}^{(i-1)}} \cdot \Delta(F_n^{(i)})^{-1} \cdot (1 + f_n \zeta_{i-1}(n)) \quad (n \in \mathbf{N}).$$

Proof. Putting $\nu_j = \nu(\{1/(2F_n^{(j)})\}, \{f_n^{(j)}\})$, $(j \in \mathbf{N})$, we have by Corollary 4.3, $\nu_{j-1} \circ \cdots \circ \nu_0 = \nu(\{\zeta_{j-1}(n)\}, \{f_n\})$ and if we set $z_n^{(j)} = (z_n^{(j-1)})^{\nu_j}$, then $z_n^{(j)} = F_n^{(j-1)} z_n^{(j-1)} - \frac{1}{2}$ by the above remarks. So $z_n^{(j)} = F_n^{(j-1)} \cdot \cdots \cdot F_n^{(0)}(z_n - \zeta_{j-1}(n))$, so that $\{\zeta_{j-1}(n)\}$ is indeed of the form mentioned in the statement of the theorem. The fact that the $\{\zeta_j(n)\}$ are subdominant solutions of recurrences $\mathcal{F}(\{d_j(n)\}, \{f_n\})$ follows from Proposition 4.2, so it suffices to determine the numbers $d_j(n)$. Writing $d_n^{(j+1)} = (d_n^{(j)})^{\nu_{j+1}}$ we have

$$d_n^{(j+1)} = F_{n+1}^{(j)}(d_n^{(j)} + \tfrac{1}{4} \Delta(F_n^{(j)})^{-1})(1 + \tfrac{1}{2} f_n^{(j)}/F_n^{(j)})^{-1}$$

and, by Proposition 4.1,

$$d_n^{(j)} = F_{n+1}^{(j-1)} \cdot \dots \cdot F_{n+1}^{(0)}(d_n^{(0)} - d_{j-1}(n)) \cdot (1 + \zeta_{j-1}(n) f_n)^{-1}.$$

Hence,

$$d_j(n)^{(j)} = F_{n+1}^{(j-1)} \cdot \dots \cdot F_{n+1}^{(0)}(d_j(n) - d_{j-1}(n)) \cdot (1 + \zeta_{j-1}(n) f_n)^{-1}$$

and, on the other hand, $d_j(n)^{(j+1)} = 0$ since $\zeta_j(n)^{(j+1)} = 0$, so that $d_j(n)^{(j)} = -\frac{1}{4}\Delta(F_n^{(j)})^{-1}$ and

$$d_j(n) - d_{j-1}(n)$$
$$= \frac{1}{4} \cdot \frac{1 + f_n \zeta_{j-1}(n)}{F_{n+1}^{(j-1)} \cdot \dots \cdot F_{n+1}^{(0)}} \cdot -\Delta(F_n^{(j)})^{-1} \quad (j \geq 1,\ n \in \mathbf{N}). \qquad \square$$

Corollary 4.5. *Define $f_n^{(i)}$ and $F_n^{(i)}$ $(i, n \in \mathbf{N})$ as in Theorem 4.4. Then if for some $j \in \mathbf{N}$ and $\varepsilon > 0$, $d_n = d_n' + d_n''$ with*

$$d_n' \leq d_j(n) - \varepsilon(d_j(n) - d_{j-1}(n))$$

and

$$\sum_{n=N}^{\infty} |d_n''| F_{n+1}^{(0)} \cdot \dots \cdot F_{n+1}^{(j)} < \infty,$$

then $\mathcal{F}(\{d_n\}, \{f_n\})$ is of converging type, and if

$$d_n' \geq d_j(n) + \varepsilon(d_j(n) - d_{j-1}(n))$$

and

$$\sum_{n=N}^{\infty} |d_n''| F_{n+1}^{(0)} \cdot \dots \cdot F_{n+1}^{(j)} < \infty,$$

then $\mathcal{F}(\{d_n\}, \{f_n\})$ is of diverging type.

Proof. By Theorem 4.4, application of the affine renormalization $\nu = \nu(\{\zeta_{j-1}(n)\}, \{f_n\})$ yields

$$d_n^{\nu} = F_{n+1}^{(j-1)} \cdot \dots \cdot F_{n+1}^{(0)}(d_n - d_{j-1}(n))(1 + f_n \zeta_{j-1}(n))^{-1}$$
$$\leq F_{n+1}^{(j-1)} \cdot \dots \cdot F_{n+1}^{(0)} d_n''(1 + f_n \zeta_{j-1}(n))^{-1} + \frac{1-\varepsilon}{4} |\Delta(F_n^{(j)})^{-1}|$$

for large n. Now apply Lemma 3.1, Proposition 3.5(c) and Proposition 4.1. The second case is completely analogous to the first one. $\square$

Remark 4.2. If we choose $F_0^{(0)}, F_0^{(1)}, \dots$ in such a way that for a certain number $N \in \mathbf{N}$, $\lambda_i = F_N^{(i)} \geq 1$ for all i and $s := \sum_{j=0}^{\infty} (F_N^{(0)} \cdot \dots \cdot F_N^{(j)})^{-1} < \infty$, then $F_n^{(i)} \geq \lambda_i$ for $n \geq N$ and $\zeta(n) = \lim_{j\to\infty} \zeta_j(n) = \frac{1}{2}\sum_{j=0}^{\infty} (F_n^{(0)} \cdot \dots \cdot F_n^{(j)})^{-1}$ exists for $n \geq N$, and $0 < \zeta(n) < s \cdot (F_N^{(0)}/F_n^{(0)})$, so that $\lim_{n\to\infty} \zeta(n) = 0$. Moreover, since $\{\zeta(n)\}$ is the uniform limit (in n) of $\{\zeta_j(n)\}$, it is the subdominant solution of $\mathcal{F}(\{\hat{d}_n\}, \{f_n\})$, where $\hat{d}_n = \lim_{j\to\infty} d_j(n)$. So it defines an affine renormalization $\{z_n\} \to \{z_n^{\nu}\} = \{G_n(\{\zeta(n)\}, \{f_n\}) \cdot (z_n - \zeta(n))\}$. We determine $G_n(\{\zeta(n)\}, \{f_n\})$.

Lemma 4.6. $\{(\lambda_0 \cdot \cdots \cdot \lambda_i)^{-1} F_n^{(0)} \cdot \cdots \cdot F_n^{(i)}\}$ *converges as* $i \to \infty$ *for all* $n \geq N$.

Proof. Since $f_n^{(i+1)} = f_n^{(i)}/(F_n^{(i)} + \frac{1}{2} f_n^{(i)}) < \log F_{n+1}^{(i)}/F_n^{(i)}$ $(i \in \mathbf{N},\ n \geq N)$ we have

$$F_n^{(i+1)} - F_N^{(i+1)} < \log F_n^{(i)} - \log F_N^{(i)},$$

so

$$0 \leq F_n^{(i+1)}/\lambda_{i+1} - 1 < \lambda_{i+1}^{-1} \log(F_n^{(i)}/\lambda_i) \leq \log F_n^{(i)}/\lambda_i$$

so that $\lim_{i \to \infty} F_n^{(i)}/\lambda_i = 1$ for $n \geq N$. Furthermore, if we put $F_n^{(i)}/\lambda_i - 1 = \varepsilon_i(n)$, we have

$$\varepsilon_{i+1}(n) < \lambda_{i+1}^{-1} \cdot \log(1 + \varepsilon_i(n)) < \lambda_{i+1}^{-1} \cdot \varepsilon_i(n)$$

so that

$$0 \leq \varepsilon_i(n) < (\lambda_1 \cdot \cdots \cdot \lambda_i)^{-1} \varepsilon_0(n)$$

and $\sum_{i=0}^{\infty} \varepsilon_i(n) < \infty$ for $n \geq N$. □

We claim: $G_n = \lim_{i \to \infty} F_n^{(0)} \cdot \cdots \cdot F_n^{(i)}(\lambda_0 \cdot \cdots \cdot \lambda_i)^{-1}$. We write H_n for $\lim_{i \to \infty} F_n^{(0)} \cdot \cdots \cdot F_n^{(i)}(\lambda_0 \cdot \cdots \cdot \lambda_i)^{-1}$. By Proposition 4.1 and Theorem 4.4, we have that $\{z_n^{(i)}\} = \{F_n^{(0)} \cdot \cdots \cdot F_n^{(i-1)}(z_n - \zeta_{i-1}(n))\}$ is a solution of $\mathcal{F}(\{d_n^{(i)}\}, \{f_n^{(i)}\})$ where

$$d_n^{(i)} = F_{n+1}^{(0)} \cdot \cdots \cdot F_{n+1}^{(i-1)}(d_n - d_{i-1}(n))(1 + f_n \zeta_{i-1}(n))^{-1}$$

and

$$f_n^{(i)} = (f_n/F_n^{(0)} \cdot \cdots \cdot F_n^{(i-1)})(1 + f_n \zeta_{i-1}(n))^{-1} \quad (i \in \mathbf{N}).$$

So, $\{(\lambda_0 \cdot \cdots \cdot \lambda_{i-1})^{-1} z_n^{(i)}\}$ is a solution of

$$\mathcal{F}(\{d_n^{(i)}(\lambda_0 \cdot \cdots \cdot \lambda_{i-1})^{-1}\}, \{f_n^{(i)}(\lambda_0 \cdot \cdots \cdot \lambda_{i-1})\}).$$

Taking limits, we find that $\{\lim_{i \to \infty} z_n^{(i)}(\lambda_0 \cdot \cdots \cdot \lambda_{i-1})^{-1}\} = \{H_n(z_n - \zeta(n))\}$ is a solution of $\mathcal{F}(\{\tilde{d}_n\}, \{\tilde{f}_n\})$ with

$$\tilde{d}_n = H_{n+1}(d_n - \hat{d}_n)(1 + f_n \zeta(n))^{-1}, \qquad \tilde{f}_n = f_n H_n^{-1}(1 + f_n \zeta(n))^{-1}.$$

In particular, $\{G_n\} = \{H_n\}$.

The affine renormalization $\nu(\{\zeta(n)\}, \{f_n\})$ is in a way the limit of $\nu_{j-1} \circ \cdots \circ \nu_0 = \nu(\{\zeta_{j-1}(n)\}, \{f_n\})$.

§5. APPLICATION TO LINEAR SECOND-ORDER RECURRENCES

We apply the theory derived above to linear recurrences, although we shall use slightly different affine renormalizations than those defined in §4, in order to obtain explicit formulae for the coefficients of the renormalized recurrences.

Let $f_n^{(0)} = 1$ for all $n \in \mathbf{N}$ and let $f_n^{(i)}$ and $F_n^{(i)}$ be as in §4, so

$$f_n^{(i+1)} = f_n^{(i)}/(F_n^{(i)} + \tfrac{1}{2} f_n^{(i)}) \quad \text{and} \quad \Delta F_n^{(i)} = f_n^{(i)},$$

where $F_0^{(i)} = \lambda_i \in \mathbf{R}$ $(i = 0, \ldots, J)$. We define sequences $\{h_n^{(i)}\}, \{H_n^{(i)}\}$ $(i = 0, \ldots, J)$ inductively: $h_n^{(0)} = f_n^{(0)}$, $H_n^{(0)} = F_n^{(0)}$ and, for $i \geq 0$, $h_n^{(i+1)} = \log H_{n+1}^{(i)} - \log H_n^{(i)}$, $\Delta H_n^{(i)} = h_n^{(i)}$ with $H_0^{(i)}$ chosen such that

$$\lim_{n \to \infty} (F_n^{(i)} - H_n^{(i)}) = 0.$$

Lemma 5.1. *The above construction is well-defined. Moreover,* $f_n^{(i)} - h_n^{(i)} = O(n^{-3})$, $F_n^{(i)} - H_n^{(i)} = O(n^{-2})$ $(n \to \infty)$.

Proof. For $i = 0$ the lemma is trivial. Suppose the assertion holds for $i \leq j-1 \leq J-1$. Since for $n \to \infty$, $f_n^{(j)} \sim f_n^{(j-1)}/F_n^{(j-1)}$, we have $f_n^{(j-1)}/F_n^{(j-1)} = O(f_n^{(1)}) = O(n^{-1})$. Furthermore,

$$\begin{aligned} f_n^{(j)} &= \log F_{n+1}^{(j-1)} - \log F_n^{(j-1)} + O((f_n^{(j-1)}/F_n^{(j-1)})^3) \\ &= \log F_{n+1}^{(j-1)} - \log F_n^{(j-1)} + O(n^{-3}) \quad (n \to \infty). \end{aligned}$$

Hence,

$$\begin{aligned} f_n^{(j)} &= h_n^{(j)} + O(n^{-3}) + O\left(\frac{f_n^{(j-1)}}{F_n^{(j-1)}} - \frac{f_n^{(j-1)} + O(n^{-3})}{F_n^{(j-1)}(1 + O(n^{-2}))} \right) \\ &= h_n^{(j)} + O(n^{-3}) \quad (n \to \infty). \end{aligned}$$

and

$$\begin{aligned} F_n^{(j)} &= \lambda_j + \sum_{k=0}^{n-1} \{\log(F_{k+1}^{(j-1)}/F_k^{(j-1)}) + O(k^{-3})\} \\ &= \lambda_j' + \log F_n^{(j-1)} + O(n^{-2}) \\ &= \lambda_j' + \log H_n^{(j-1)} + O(n^{-2}) = H_n^{(j)} + O(n^{-2}) \quad (n \to \infty) \end{aligned}$$

if we choose $H_0^{(j)} = \lambda_j'$. □

Lemma 5.2. *Let* $h_n^{(i)}, H_n^{(i)}$ *be as above* $(i = 0, 1, \ldots, J, n \in \mathbf{N})$. *Put*

$$\tilde{d}_{-1}(n) = 0, \qquad \tilde{d}_i(n) - \tilde{d}_{i-1}(n) = \frac{1}{4} \cdot \frac{h_n^{(i)}}{H_{n+1}^{(0)} \cdot \cdots \cdot H_{n+1}^{(i)} H_n^{(i)}}$$

for $n \in \mathbf{N}$, $i = 0, \ldots, J$. *Then* $\mathcal{F}(\{d(n)\}, \{1\})$ *is of converging type if, for some* $\varepsilon > 0$ *and* $d'(n)$ *such that* $\sum_{n \in \mathbf{N}} |d'(n)| H_{n+1}^{(0)} \cdot \cdots \cdot H_{n+1}^{(J)} < \infty$,

$$\{d(n)\} - d'(n)\} \leq \{\tilde{d}_J(n)(1 - \varepsilon) + \tilde{d}_{J-1}(n) \cdot \varepsilon\},$$

and $\mathcal{F}(\{d(n)\}, \{1\})$ *is of diverging type if for such* ε *and* $\{d'(n)\}$ *as above,*

$$\{d(n) - d'(n)\} \geq \{\tilde{d}_J(n)(1 + \varepsilon) - \tilde{d}_{J-1}(n) \cdot \varepsilon\}.$$

Proof. Considering the fact that $H_0^{(0)}, \ldots, H_0^{(J)}$ can be chosen freely, it suffices to show that $\sum_{n=N}^{\infty} |\tilde{d}_j(n) - d_j(n)| \cdot F_{n+1}^{(0)} \cdot \cdots \cdot F_{n+1}^{(J)} < \infty$ for $0 \leq j \leq J$ with $d_j(n)$ as in Corollary 4.5, since $\lim_{n \to \infty} H_n^{(j)}/F_n^{(j)} = 1$ $(0 \leq j \leq J)$. By Lemma 5.1,

$$\frac{h_n^{(j)}}{H_{n+1}^{(0)} \cdot \cdots \cdot H_{n+1}^{(j)} H_n^{(j)}} = \frac{f_n^{(j)} + O(n^{-3})}{F_{n+1}^{(0)} \cdot \cdots \cdot F_{n+1}^{(j)} F_n^{(j)} (1 + O(n^{-2} F_n^{(j)-1}))}$$

so that $d_0(n) = \tilde{d}_0(n)$ and

$$\begin{aligned}&\tilde{d}_j(n) - \tilde{d}_{j-1}(n)\\&\quad = (d_j(n) - d_{j-1}(n))\cdot(1+\mathrm{O}(1/n)) + \mathrm{O}(n^{-3})\cdot\frac{1}{F_{n+1}^{(0)}\cdot\dots\cdot F_{n+1}^{(j)}F_n^{(j)}},\end{aligned}$$

where we use that $\zeta_{j-1}(n) = \mathrm{O}(1/n)$. But

$$\begin{aligned}&\sum_{n=N}^{\infty}\left(\frac{1}{n^3} + \frac{f_n^{(j)}}{n}\right)\cdot\frac{F_{n+1}^{(j+1)}\cdot\dots\cdot F_{n+1}^{(J)}}{F_n^{(j)}}\\&\quad\ll \sum_{n=N}^{\infty}\frac{1}{n^2}\cdot\frac{F_{n+1}^{(j+1)}\cdot\dots\cdot F_{n+1}^{(J)}}{F_n^{(j)}} < \infty\end{aligned}$$

for $j = 1, \dots, J$, since $F_n^{(i+1)} = \mathrm{O}(\log F_n^{(i)})$ for $i \in \mathbf{N}$, $n \to \infty$. □

For $J = 0$ we have $\tilde{d}_0(n) = \frac{1}{4}/((n+\lambda_0)(n+\lambda_0+1)) = 1/(4n^2) + \mathrm{O}(n^{-3})$. In general

$$\begin{aligned}H_n^{(J)} &= \lambda'_J + \log(\lambda'_{J-1} + \dots + \log(\lambda_0 + n)\dots)\\&= \lambda'_J + \log_J n + \mathrm{O}\left(\frac{1}{\log_{J-1} n}\right)\end{aligned}$$

where $\log_0 n = n$, $\log_j n = \log(\log_{j-1} n)$ $(j \geq 1)$, so that

$$\tilde{d}_j(n) - \tilde{d}_{j-1}(n) = \frac{\Delta \log_j n}{4n(\log n)\cdot\dots\cdot(\log_j n)^2}\left(1 + \mathrm{O}\left(\frac{1}{\log_j n}\right)\right).$$

Since

$$\begin{aligned}&\sum_{n=N}^{\infty}\frac{1}{\log_j n}\cdot\frac{1}{(2n\log n\dots\log_j n)^2}\cdot H_{n+1}^{(0)}\cdot\dots\cdot H_{n+1}^{(J)}\\&\quad\ll \sum_{n=N}^{\infty}\frac{\log_{j+1} n\dots\log_J n}{n\log n\dots(\log_j n)^2} < \infty \quad (N\text{ large enough}),\end{aligned}$$

we can reformulate Lemma 5.2 as follows:

Corollary 5.3. *$\mathcal{F}(\{d(n)\},\{1\})$ is of converging type if for some $\varepsilon > 0$ and $\{d'(n)\}$ such that for N large enough $\sum_{n=N}^{\infty} |d'(n)|\cdot(n\log n\dots\log_J n) < \infty$,*

$$\{d(n) - d'(n)\} \leq \left\{\frac{1}{4}\sum_{j=0}^{J}(n\log n\dots\log_j n)^{-2} - \varepsilon(n\log n\dots\log_J n)^{-2}\right\}$$

and of diverging type if, for such an ε and $\{d'(n)\}$

$$\{d(n) - d'(n)\} \geq \left\{\frac{1}{4}\sum_{j=0}^{J}(n\log n\dots\log_j n)^{-2} + \varepsilon(n\log n\dots\log_J n)^{-2}\right\}.$$

This result can now be applied directly to recurrences of type (1.4), using the correspondences shown in §1.

The following theorem gives a characterisation in terms of affine renormalizations for a recurrence of type $\mathcal{F}(\{d_n\}, \{f_n\})$ to be of converging type.

Theorem 6.1. *Let* $\{f_n\} \in \mathcal{S}$, $\{d_n\}$ *a real-valued sequence such that* $\lim_{n\to\infty} d_n f_n = 0$. *Then* $\mathcal{F}(\{d_n\}, \{f_n\})$ *is of converging type if and only if for all affine renormalizations* $\nu = \nu(\{\zeta(n)\}, \{f_n\})$ *such that* $\{d_n^\nu\} \geq \{0\}$ *we have* $\sum_{n=N}^{\infty} d_n^\nu < \infty$.

We shall need the following results:

Theorem 6.2. *Let* $\{z_n\}, \{z_n'\}$ *be solutions of recurrences* $\mathcal{F}(\{d_n\}, \{f_n\})$, $\mathcal{F}(\{d_n'\}, \{f_n\})$ *respectively, with* $\{f_n\} \in \mathcal{S}$ *and* $\{1\} > \{d_n f_n\} \geq \{0\}$ *as well as* $\{1\} > \{d_n' f_n\} \geq \{0\}$ *such that for some* $p, q \in \mathbf{N}$, $z_p = z_p'$, $z_q = z_q'$ *and* $0 \leq z_n, z_n'$ *(or* $0 \geq z_n, z_n'$*) for* $p \leq n \leq q$: *If* $d_n' = 0$ *for* $p \leq n < q - 1$ *(*$d_n' = 0$ *for* $q - 1 \geq n > p$*, respectively), then* $\sum_{n=p}^{q-1} d_n' \leq \sum_{n=p}^{q-1} d_n$.

Proof. It suffices to show the result for $q = p + 2$, the general case then follows by induction. Thus

$$(6.1) \qquad d_p + d_{p+1} = \frac{c + d_p(z_p f_p + z_{p+2} f_{p+1})}{1 + z_p f_p}$$

with c independent of d_p, d_{p+1}. The result holds for all values of z_p, z_{p+1}, z_{p+2}. So if $z_n \geq 0$ for $n = p, p+1, p+2$, then $d_p + d_{p+1}$ is minimal if $d_p = 0$. The other case goes analogously. □

Lemma 6.3. *Let* $\{z_n\}$ *be a solution of* $\mathcal{F}(\{d_n\}, \{f_n\})$ *such that* $\{f_n^{-1}\} > \{d_n\} \geq \{0\}$ *and such that for some numbers* $N, L \in \mathbf{N}$, $z_{N+l} > 0$ $(l = 0, \ldots, L)$ *and* $z_{N+L+1} \leq 0$. *Then*

$$\sum_{n=N}^{N+L} d_n \geq (z_N^{-1} + f_N + \cdots + f_{N+L-1})^{-1}.$$

Proof. By Lemma 6.2 we may assume that $d_n = 0$ for $n = N, \ldots, N+L-2$. Moreover, by (6.1), $d_{N+L-1} + d_{N+L} \geq d_{N+L-1}' + d_{N+L}'$ where either d_{N+L-1}' or $d_{N+L}' = 0$. In the first case, $d_{N+L}' \geq z_{N+L}' = z_N/(1 + (f_N + \cdots + f_{N+L-1})z_N)$. In the second case, $d_{N+L-1}' \geq z_{N+L-1}' = z_N/(1 + (f_N + \cdots + f_{N+L-2})z_N)$. □

Proof of Theorem 6.1. The necessity of the condition follows from Proposition 3.5(b). Conversely, let $\mathcal{F}(\{d_n\}, \{f_n\})$ be of diverging type. We first assume that $\{d_n\} \geq \{0\}$. Let $\{z_n\}$ be a solution of $\mathcal{F}(\{d_n\}, \{f_n\})$ and let $\{m(j)\}, \{n(j)\}$ be sequences of indices (still to be fixed) such that $-\infty \leq z_{m(j)} < 0 < z_{m(j)+1}$ and $z_{n(j)+1} < 0 \leq z_{n(j)}$ and $m(j-1) < n(j) < m(j)$ for $j \in \mathbf{N}$. We construct an affine renormalization $\nu = \nu(\{\zeta(n)\}, \{f_n\})$ such that $\{\zeta(n)\} \geq \{0\}$ and $\{d_n^\nu\} \geq \{0\}$, $\sum_{k=n(j-1)+1}^{n(j)} d_k^\nu \geq 1$ for all j. Suppose that $\{\zeta(n)\}_{n \leq n(j-1)}$ has been constructed. Set $G_n = G_n(\{\zeta(n)\}, \{f_n\})$ (so $\{G_n\}_{n \leq n(j-1)}$ is known), and

$$\Gamma_i = G_{n(i)}(1 + f_{n(i)}\,\zeta(n(i))) \quad (n, i \in \mathbf{N}).$$

Moreover, we assume that for some $0 < \varepsilon_{j-1} = \varepsilon < \frac{1}{2}$,

$$f_{n(j-1)}/G_{n(j-1)} < \varepsilon, \qquad 1 - \varepsilon < f_n/f_{n-1} < 1 + \varepsilon \tag{6.2}$$

for $n \geq n(j-1)$. Let now n_1 be such that $n_1 > n(j-1)$, $f_{n(j-1)+1} + \cdots + f_{n_1 - 1} > \Gamma_{j-1}$ and let $p(j), m(j-1)$ be such that $p(j) \geq m(j-1) \geq n_1$ and

$$0 < z_\nu^{-1} \leq (\Gamma_{j-1}^{-1} - (f_{n(j-1)+1} + \cdots + f_{\nu-1})^{-1})^{-1} + \sum_{l=n(j-1)+1}^{\nu-1} f_l$$

for $m(j-1) < \nu \leq p(j)$ and either

$$z_\nu^{-1} > (\Gamma_{j-1}^{-1} - (f_{n(j-1)+1} + \cdots + f_{\nu-2})^{-1})^{-1} + \sum_{l=n(j-1)+1}^{\nu-1} f_l$$

for $\nu = p(j) + 1$ or $z_{p(j)+1} \leq 0$. This is possible since $\{z_n\}$ is a diverging solution. We now define $l_j^{-1} := \zeta(n(j-1)+1) := \Gamma_{j-1}^{-1} - \Phi_j^{-1}$ where $\Phi_j = f_{n(j-1)+1} + \cdots + f_{p(j)-1}$. Then $l_j > \Gamma_{j-1} > 0$. Further, let $n(j)$ be such that $z_{n(j)+1} < 0$ and $z_n \geq 0$ for $m(j-1) < n \leq n(j)$. Clearly, $n(j) \geq p(j)$. For $0 \leq K \leq p(j) - n(j)$ we define sequences $\{\zeta(n, K)\}$ which are solutions of $\mathcal{F}(\{d_n(K)\}, \{f_n\})$ such that $\zeta(n(j-1)+1, K) = l_j^{-1}$, $d_n(K) = 0$ for $n = n(j-1)+1, \ldots, p(j)-1$, $\zeta(n, K) = z_n$ for $n = p(j)+1, \ldots, p(j)+[K]$, $d_{p(j)+[K]}(K) = (K - [K]) \cdot d_{p(j)+[K]}$ and $d_n(K) = 0$ for $n = p(j) + [K] + 1, \ldots,$ $n(j) - 1$. It is then clear that $0 \leq d_n(K) \leq d_n$ for $n = n(j-1)+1, \ldots, n(j)-1$. Moreover, by Lemma 6.3 and for $\nu = \nu(\{\zeta(n, K)\}, \{f_n\})$,

$$\sum_{k=m(j-1)+1}^{p(j)} d_k^\nu \geq G_{p(j)}(\tilde{z}_{p(j)} - \zeta(p(j))) \tag{6.3}$$

where $\{\zeta(n)\}$ is the solution of $\mathcal{F}(\{\hat{d}_n\}, \{f_n\})$ with $\zeta(n(j-1)+1) = l_j^{-1}$ and $\hat{d}_k = 0$ for $k = n(j-1)+1, \ldots, p(j)-1$, and $\{\tilde{z}_n\}$ is the solution of $\mathcal{F}(\{\hat{d}_n\}, \{f_n\})$ with $\tilde{z}_{n(j-1)+1} = \infty$. Then $\tilde{z}_{p(j)} = \Phi_j^{-1}$, $\zeta(p(j)) = (l_j + \Phi_j)^{-1}$ and

$$\sum_{k=n(j-1)+1}^{p(j)} d_k^\nu \geq G_{p(j)} \cdot \frac{l_j}{\Phi_j(l_j + \Phi_j)} \geq \Gamma_{j-1} \cdot \frac{l_j + \Phi_j}{l_j \Phi_j} = 1. \tag{6.4}$$

We show that we can choose K and $m(j-1)$ such that $\hat{d}_{n(j)} \geq 0$ for all j and $\lim_{n \to \infty} f_n \zeta(n) = 0$. In any case, $m(j-1) \geq n_1$, so that $l_j > \Gamma_{j-1}$. Then, by (6.2), $f_{n(j-1)+1} \zeta(n(j-1)+1) < (f_{n(j-1)+1}/G_{n(j-1)}) < (1+\varepsilon)\varepsilon$. Writing $\{\hat{\zeta}(n)\}$ for the solution of $\mathcal{F}(\{0\}, \{f_n\})$ with $\hat{\zeta}(p(j)) = \zeta(p(j))$, we have, for $n(j-1) < n \leq n(j)$,

$$\begin{aligned} 0 \leq f_n \zeta(n) \leq f_n \hat{\zeta}(n) &= \frac{f_n}{l_j + f_{n(j-1)+1} + \cdots + f_{n-1}} \\ &\leq \left(\frac{1}{\varepsilon(1+\varepsilon)} (1-\varepsilon)^{n-n(j-1)-1} + (1-\varepsilon)^{n-n(j-1)-1} + \cdots + (1-\varepsilon) \right)^{-1} \\ &\leq \frac{\varepsilon}{1-\varepsilon}. \end{aligned}$$

Similarly, for $m(j-1)$ large enough, say $m(j-1) \geq n_2$, we find, by $\lim_{n\to\infty}(f_{n+1}/f_n) = 1$, that $f_n\,\zeta(n) < \varepsilon/2$ for $p(j) \leq n \leq n(j)$ (using Lemma 3.2). Put $G_n(K) = G_{p(j)} \cdot \prod_{l=p(j)}^{n-1} (1 + f_k\,\zeta(n,K))/(1 - f_k\,\zeta(n+1,K))$ for $p(j) \leq n \leq n(j)$ and define functions $g, h : [0, n(j) - p(j)] \to \mathbf{R}_{\geq 0}$ by

$$g(K) = G_{n(j)}(K) \cdot \zeta(n(j), K), \qquad h(K) = f_{n(j)}/G_{n(j)}(K).$$

Then

$$0 \leq g(K)\,h(K) < \varepsilon/2 \quad \text{for} \quad 0 \leq K \leq p(j) - n(j)$$

provided that $m(j-1) \geq n_2$. In addition, g and h are monotonously non-increasing, and non-decreasing (in K), respectively. Since $\zeta(n,0) = \hat{\zeta}(n)$ for $n(j-1) < n \leq n(j)$ and $\{\hat{\zeta}(n)\}$ is not a subdominant solution of $\mathcal{F}(\{0\},\{f_n\})$ we have that $\sum_{k=n(j-1)+1}^{n(j)} f_k G_k(0)^{-1}$ converges as $n(j) \to \infty$, by $0 < f_n\,\hat{\zeta}(n) < \varepsilon/(1-\varepsilon) < 1$ and Proposition 2.3. So, if we choose $m(j-1)$ large enough, say $m(j-1) \geq n_3$, then $h(0) < \varepsilon/3$. Moreover, since $\Phi_j \to \infty$ as $m(j-1)$ (and so, $p(j)$) tends to infinity, we have $l_j < \frac{4}{3}\Gamma_{j-1}$ if $m(j-1)$ is larger than some number n_4. We now choose $m(j-1) \geq \max(n_1, n_2, n_3, n_4)$ and we choose K as follows: if $h(n(j) - p(j)) \leq \frac{3}{4}\varepsilon$ we set $K = n(j) - p(j)$, $\zeta(n) = \zeta(n,K)$ for $n = n(j-1)+1, \ldots, n(j)$. Then $f_{n(j)} G_{n(j)}^{-1} \leq \frac{3}{4}\varepsilon$ and $d_{n(j)} \geq \hat{d}_{n(j)}$. Hence, by (6.4) we have

$$\sum_{l=n(j-1)+1}^{n(j)} d_l^{\nu} \geq 1 \quad \text{and} \quad d_n^{\nu} \geq 0 \quad \text{for} \quad n(j-1) < n \leq n(j).$$

If $h(n(j) - p(j)) > \frac{3}{4}\varepsilon$, we may choose K such that $h(K) \leq \frac{3}{4}\varepsilon$, $g(K) < \frac{2}{3}$ and $0 \leq K < p(j) - n(j)$, by $h(0) < \varepsilon/3$. Then $\zeta(n) := \zeta(n,K)$ for $n(j-1) < n \leq n(j)$. Now again $f_{n(j)} G_{n(j)}^{-1} \leq \frac{3}{4}\varepsilon$ and

$$\begin{aligned}
d_{n(j)}^{\nu} &\geq (1-\varepsilon)\, G_{n(j)+1}(d_{n(j)} - \hat{d}_{n(j)}) \\
&\geq -(1-\varepsilon)\, G_{n(j)}\, \hat{d}_{n(j)} \cdot \frac{1 + f_{n(j)}\,\zeta(n(j))}{1 - f_{n(j)}\,\zeta(n(j)+1)} \\
&\geq -\frac{(1-\varepsilon^2)\, G_{n(j)} \cdot (\zeta(n(j)) - \zeta(n(j)+1) \cdot (1 + f_{n(j)}\,\zeta(n(j))))}{1 - f_{n(j)}/G_{n(j)}} \\
&\geq -\frac{(1-\varepsilon^2)}{1 - \frac{3}{4}\varepsilon} \cdot g(K) + (1-\varepsilon^2)\,\Gamma_j/l_{j+1} \\
&\geq -\frac{2}{3} \cdot \frac{1-\varepsilon^2}{1 - \frac{3}{4}\varepsilon} + \frac{3}{4}(1-\varepsilon^2) > \frac{1}{12} + \delta(\varepsilon)
\end{aligned}$$

where $\delta(\varepsilon) \to 0$ as $\varepsilon \to 0$ (we may assume that $\Gamma_i/l_{i+1} > \frac{3}{4}$ for all i). So we choose ε so small that $|\delta(\varepsilon)| < \frac{1}{12}$. Then $\hat{d}_{n(j)}^{\nu} \geq 0$, so that $\hat{d}_n^{\nu} \geq 0$ for $n(j-1) < n \leq n(j)$ and $\sum_{l=n(j-1)+1}^{n(j)} \hat{d}_l^{\nu} \geq 1$. Moreover, we have $f_{n(j)}/G_{n(j)} \to 0$ as $j \to \infty$. (In fact, we have only constructed a solution $\{\zeta(n)\}_{n \geq N}$ for N so large that $|\delta(\varepsilon)| < \frac{1}{12}$. Of course, this solution can be completed to a solution $\{\zeta(n)\}_{n\geq 0}$ with $\{\zeta(n)\} \geq \{0\}$ and $\hat{d}_n \geq 0$ for all n.)

It remains to show that $\nu(\{\zeta(n)\},\{f_n\})$ is indeed an affine renormalization. Since we have $\{G_n\} \geq \{0\}$, $\lim_{n\to\infty} f_n\,\zeta(n) = 0$, the recurrence $\mathcal{F}(\{\hat{d}_n\},\{f_n\})$, with $\hat{d}_n = \zeta(n) - \zeta(n+1) - f_n\,\zeta(n)\,\zeta(n+1)$, is of converging type and by

$G_{p(j)}(\tilde{z}_{p(j)} - \zeta(p(j))) = 1$ (with $\{\tilde{z}_n\}$ as in (6.3)), we infer from (2.3) that $\sum_{k=N}^{\infty} f_k\, G_k^{-1} = \infty$, so that $\{\zeta(n)\}$ is the subdominant solution of $\mathcal{F}(\{d_n\},\{f_n\})$. We now treat the general case. Thus, let $\{d_n\}$ be an arbitrary real sequence with $\lim_{n\to\infty} d_n f_n = 0$. Put $d_n' = \min(d_n, 0)$. Then $\lim_{n\to\infty} d_n' f_n = 0$ and $\{d_n'\} \le \{0\}$. Hence $\mathcal{F}(\{d_n'\},\{f_n\})$ is of converging type (by Proposition 3.5(a)) and defines an affine renormalization $\nu' = \nu(\{\eta(n)\},\{f_n\})$. Then $\{d_n^{\nu'}\} \ge 0$ and $\mathcal{F}(\{d_n^{\nu'}\},\{f_n^{\nu'}\})$ is of diverging type, by Proposition 4.1. Thus we can find an affine renormalization $\nu'' = \nu(\{\eta'(n)\},\{f_n^{\nu'}\})$ such that $\sum_{n=N}^{\infty} (d_n^{\nu'})^{\nu''} = \infty$ and $\{(d_n^{\nu'})^{\nu''}\} \ge \{0\}$. By Corollary 4.3, there is an affine renormalization $\nu''' = \nu(\{\eta'(n)\},\{f_n\})$ such that $\{(d_n^{\nu'})^{\nu''}\} = \{d_n^{\nu'''}\}$. This concludes the proof. □

§7. SUBDOMINANCE FOR LINEAR RECURRENCES RELATED TO CONVERGING TYPE RECURRENCES

In this section we study the relationship between linear recurrences and $PSL(2,\mathbf{R})$-recurrences. In §1 we saw that a linear recurrence (1.3a) can be reduced to a linear recurrence (1.4) in almost all cases, and particularly in the case that the limit recurrence has characteristic polynomial $(X-a)^2$, $a \neq 0$. We further saw that the study of (1.4) is intimately related to the study of (the solutions of) (1.5), a $PSL(2,\mathbf{R})$-recurrence. We recall that, in the case that $\lim_{n\to\infty} c(n) = 0$, $\lim_{n\to\infty}(u_{n+1}/u_n) = 1$ for all solutions $\{u_n\} \neq \{0\}$ of (1.4) if and only if $\mathcal{F}(\{-c(n)\},\{1\})$ is of converging type. In the opposite case, $\lim_{n\to\infty}(u_{n+1}/u_n)$ does not exist for any solution $\{u_n\}$.

We call a solution $\{v_n\} \neq \{0\}$ of a linear second-order recurrence subdominant if $\lim_{n\to\infty}(v_n/u_n) = 0$ for all solutions $\{u_n\}$ that are linearly independent with $\{v_n\}$. If such a solution exists then $\lim_{n\to\infty}(u_n'/u_n)$ exists (including infinity as a possible value) for all non-trivial solutions $\{u_n\},\{u_n'\}$. In what follows we show that convergence of $\mathcal{F}(\{-c(n)\},\{1\})$ implies the existence of a subdominant solution for (1.4), whereas the converse is not true, as a counter-example will show.

Proposition 7.1. *The linear recurrence* (1.4) *with* $\lim_{n\to\infty} c(n) = 0$ *has a subdominant solution if and only if for some solution* $\{u_n\} \neq \{0\}$ *the sum* $\sum_{n=N}^{\infty} \tilde{G}_n$ *converges in* $\mathbf{P}^1(\mathbf{R})$, *where*

$$\tilde{G}_n = \left(\prod_{k=N}^{n} \frac{2u_k - u_{k+1}}{u_{k+1}}\right) \quad (n \ge N).$$

In this case, $\sum_{n=N}^{\infty} \tilde{G}_n$ *converges for all solutions* $\{u_n\} \neq \{0\}$ *of* (1.4). (*If* $u_{n+1} = 0$, *then we omit* $\tilde{G}_n, \tilde{G}_{n+1}$ *in the summation and replace* $(2u_{n+2} - u_{n+3})/(u_{n+1})$ *by* $1 - c_{n+1}$ *in* $\tilde{G}_m$ *for* $m > n+1$).

Proof. Given $\{u_n\}$, (1.4) has a basis of solutions $\{u_n\},\{v_n\}$ with $u_n = v_n \cdot \sum_{l=N}^{n-1} \tilde{G}_n$. Further use the fact that $\lim_{n\to\infty} u_n/v_n$ exists in $\mathbf{P}^1(\mathbf{R})$ if and only if (1.4) has a subdominant solution. □

Corollary 7.2. *If $\mathcal{F}(\{-c(n)\},\{1\})$ is of converging type, then the corresponding linear recurrence* (1.4) *has a subdominant solution.*

Proof. By Proposition 2.3, $\sum_{n=N}^{\infty}(\prod_{l=N}^{n-1}(1+\zeta_m)/(1-\zeta_{m+1}))^{-1}$ converges in $\mathbf{P}^1(\mathbf{R})$ for every solution $\{\zeta_n\}$ of $\mathcal{F}(\{-c(n)\},\{1\})$. Setting $\zeta_n=(u_{n+1}/u_n)-1$, we have $\tilde{G}_n=(1-\zeta_N)/(1+\zeta_n)\cdot\prod_{m=N}^{n-1}(1-\zeta_{m+1})/(1+\zeta_m)$ $(n\ge N)$. By $\lim_{n\to\infty}\zeta_n=0$, the assertion follows. □

Remark 7.1. In particular, it follows from Corollary 7.2 that if (1.4) has a solution $\{u_n\}$ with $\lim_{n\to\infty}(u_{n+1}/u_n)=1$, then it has a subdominant solution. The following example shows that the converse is not generally true.

Let $\{\nu(j)\}$ be an increasing sequence of natural numbers such that $\nu(j+1)-\nu(j)\sim a^j$ for some $a>1$ $(j\to\infty)$. We define a sequence of non-negative real numbers $\{c(n)\}$ such that $c(n)=0$ if n is not one of the numbers $\nu(j)$, and $\mathcal{F}(\{-c(n)\},\{1\})$ has a solution $\{z_n\}$ such that $z_{\nu(j)+k}^{-1}=m_j+k$ $(0<k\le\nu(j+1)-\nu(j))$ with $m_j\to-\infty$ and $m_j+\nu(j+1)-\nu(j)\to\infty$ as $j\to\infty$. Then $c(\nu(j))<0$ and $\lim_{n\to\infty}c(n)=0$. Put $G_n=\prod_{l=0}^{n}(1+z_l)/(1-z_l)$ $(n\in\mathbf{N})$ and $\Gamma_j=G_{\nu(j)}$ $(j\in\mathbf{N})$. Then

$$G_{\nu(j)+k}=\Gamma_j\cdot\prod_{l=1}^{k}\frac{m_j+l+1}{m_j+l-1}=\Gamma_j\cdot\frac{(m_j+k)(m_j+k+1)}{m_j(m_j+1)}$$

and

$$\sum_{n=\nu(j)+1}^{\nu(j)+k}G_n^{-1}=\Gamma_j^{-1}\cdot\frac{m_jk}{m_j+k+1}.$$

If we choose $m_j=-c_j(\nu(j+1)-\nu(j))$ such that $2m_j\equiv 1 \bmod 2$ and $c_j=c+\mathrm{o}(1)$, $0<c<(a+1)^{-1}$, then

$$\Gamma_{j+1}=\Gamma_j\cdot\frac{(1-c_j)(1+m_j-m_jc_j^{-1})}{-c_j(m_j+1)}\sim\left(\frac{1-c}{c}\right)^2\Gamma_j\quad(j\to\infty)$$

and furthermore,

$$\sum_{n=1}^{\nu(j)}G_n^{-1}=\sum_{l=0}^{j-1}\frac{\Gamma_l^{-1}m_l(\nu(l+1)-\nu(l))}{m_l+1+\nu(l+1)-\nu(l)}\quad\text{converges as } j\to\infty$$

and

$$\left|\sum_{l=1}^{k}G_{\nu(j)+l}^{-1}\right|=\left|\Gamma_j^{-1}\cdot\frac{m_jk}{m_j+k+1}\right|\le m_j(2m_j+3)\,\Gamma_j^{-1}\to 0\quad\text{as } j\to\infty.$$

Hence, $\sum_{n=0}^{\infty}G_n^{-1}$ converges, so that, by Proposition 7.1, (1.4) has indeed a subdominant solution. □

The problems treated in this paper are closely connected to the classical results of Poincaré and Perron. If $\lim_{n\to\infty}c(n)=c>0$ in (1.4), then the Theorem of Poincaré and Perron says that $\lim_{n\to\infty}(u_{n+1}/u_n)$ exists for all non-trivial solutions $\{u_n\}$ of (1.4), and, even more, for every zero α of $P(X)=X^2-2X+(1-c)$ there is a solution $\{u_n\}$ such that $\lim_{n\to\infty}(u_{n+1}/u_n)=\alpha$.

(See e.g. [4], [7], [8], [10].) This fact is no longer true if $c \leq 0$, which is due to the fact that in this case the zeros of P have equal moduli. The first counterexamples were given by Perron (in [9]).

In addition, the matter is intimately related to the convergence of continued fractions (see e.g. [4], Ch. 7). If $\{v_n^{(1)}\}$ and $\{v_n^{(2)}\}$ are the two solutions of (1.4) with $v_1^{(1)} = 1$, $v_2^{(1)} = 0$, $v_1^{(2)} = 0$, $v_2^{(2)} = 1$, then

$$\frac{u_{n+2}}{v_{n+2}} = 2 \cdot \frac{q(1)|}{|\,1} + \cdots + \frac{q(n)|}{|\,1} =: \mathbf{K}_{m=1}^{n} \frac{q(m)}{1}$$

where $q(n) = (c(n) - 1)/4$ $(n \in \mathbf{N})$. So, the continued fraction $\mathbf{K}_{m=1}^{\infty}(q(m)/1)$ converges, i.e. $\lim_{n\to\infty} \mathbf{K}_{m=1}^{n}(q(m)/1)$ is a real number or is infinity, precisely if (1.4) has a subdominant solution. Moreover, in this case the subdominant solution can be expressed by means of the 'queues' $\mathbf{K}_{m=n}^{\infty}(q(m)/1)$ of the continued fraction. Namely, if we set $y_n = \mathbf{K}_{m=n}^{\infty}(q(m)/1)$, we have $y_n = q(n)/(1 + y_{n+1})$, so that $\{(-2)^{n-1} \cdot y_{n-1} \cdot \cdots \cdot y_1\} = \{w_n\}$ is a solution of (1.4) if $y_1 \neq \infty$. In fact, it is a subdominant solution, since $w_1 = 1$, $w_2 = -2y_1$, whence $w_n = v_n^{(1)} - 2y_1 v_n^{(2)}$ $(n \in \mathbf{N})$, and hence $\lim_{n\to\infty}(w_n/v_n^{(2)}) = 0$. Similarly, if $y_1 = \infty$, then $\{w_n\} = \{(-2)^{n-1} \cdot y_{n-1} \cdot \cdots \cdot y_2\}$ is a subdominant solution of (1.4), since by $w_2 = 1$ and $w_2 = -2y_2 = 2$ it follows that $\{w_n\} = \{v_n^{(2)}\}$ and $\lim_{n\to\infty}(v_n^{(2)}/v_n^{(1)}) = 0$ by assumption. Thus we see that the problem concerning the convergence of $\mathbf{K}_{m=1}^{\infty}(a_m/1)$ for $\lim_{n\to\infty} a_n = -\frac{1}{4}$ is equivalent to the question if (1.4) with $\lim_{n\to\infty} c(n) = 0$ has a subdominant solution. Besides [4] and [5], where the matter is treated from the latter point of view, we refer to [1], [2], [3] for a treatment from the former point of view, i.e. as a problem about continued fractions.

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