

SHOT NOISE IN MESOSCOPIC SYSTEMS

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Abstract. This is a review of shot noise, the time-dependent fluctuations in the electrical current due to the discreteness of the electron charge, in small conductors. The shot-noise power can be smaller than that of a Poisson process as a result of correlations in the electron transmission imposed by the Pauli principle. This suppression takes on simple universal values in a symmetric double-barrier junction (suppression factor $\frac{1}{2}$), a disordered metal (factor $\frac{1}{3}$), and a chaotic cavity (factor $\frac{1}{4}$). Loss of phase coherence has no effect on this shot-noise suppression, while thermalization of the electrons due to electron-electron scattering increases the shot noise slightly. Sub-Poissonian shot noise has been observed experimentally. So far unobserved phenomena involve the interplay of shot noise with the Aharonov-Bohm effect, Andreev reflection, and the fractional quantum Hall effect.

1. Introduction

1.1. CURRENT FLUCTUATIONS

In 1918 Schottky [1] reported that in ideal vacuum tubes, where all sources of spurious noise had been eliminated, there remained two types of noise in the electrical current, described by him as the *Wärmeeffekt* and the *Schroteffekt*. The first type of noise became known as Johnson-Nyquist noise (after the experimentalist [2] and the theorist [3] who investigated it), or simply thermal noise. It is due to the thermal motion of the electrons and occurs in any conductor. The second type of noise is called shot noise,

caused by the discreteness of the charge of the carriers of the electrical current. Not all conductors exhibit shot noise.

Noise is characterized by its spectral density or power spectrum $P(\omega)$, which is the Fourier transform at frequency ω of the current-current correlation function [4, 5],

$$P(\omega) = 2 \int_{-\infty}^{\infty} dt e^{i\omega t} \langle \Delta I(t + t_0) \Delta I(t_0) \rangle. \quad (1)$$

Here $\Delta I(t)$ denotes the time-dependent fluctuations in the current at a given voltage V and temperature T . The brackets $\langle \dots \rangle$ indicate an ensemble average or, equivalently, an average over the initial time t_0 . Both thermal and shot noise have a white power spectrum — that is, the noise power does not depend on ω over a very wide frequency range. Thermal noise ($V = 0$, $T \neq 0$) is directly related to the conductance G by the fluctuation-dissipation theorem [6],

$$P = 4k_B T G, \quad (2)$$

as long as $\hbar\omega \ll k_B T$. Therefore, the thermal noise of a conductor does not give any new information.

Shot noise ($V \neq 0$, $T = 0$) is more interesting, because it gives information on the temporal correlation of the electrons, which is not contained in the conductance. In devices such as tunnel junctions, Schottky barrier diodes, p - n junctions, and thermionic vacuum diodes [4], the electrons are transmitted randomly and independently of each other. The transfer of electrons can be described by Poisson statistics, which is used to analyze events that are uncorrelated in time. For these devices the shot noise has its maximum value

$$P = 2eI \equiv P_{\text{POISSON}}, \quad (3)$$

proportional to the time-averaged current I . (We assume $I > 0$ and $V > 0$ throughout this review.) Equation (3) is valid for $\omega < \tau^{-1}$, with τ the width of a one-electron current pulse. For higher frequencies the shot noise vanishes. Correlations suppress the low-frequency shot noise below P_{POISSON} . One source of correlations, operative even for non-interacting electrons, is the Pauli principle, which forbids multiple occupancy of the same single-particle state. A typical example is a ballistic point contact in a metal, where $P = 0$ because the stream of electrons is completely correlated by the Pauli principle in the absence of impurity scattering. Macroscopic, metallic conductors have zero shot noise for a different reason, namely that inelastic electron-phonon scattering averages out the current fluctuations.

Progress in nanofabrication technology has revived the interest in shot noise, because nanostructures allow measurements to be made on “mesoscopic” length scales that were previously inaccessible. The mesoscopic

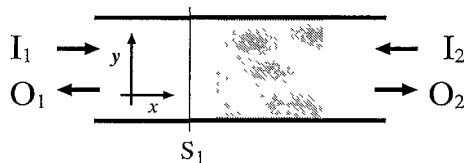


Figure 1. Schematic representation of the transport through the conductor. Incoming states (I) are scattered into outgoing states (O), by a scattering region (dashed). A cross section in lead 1 and its coordinates are indicated.

length scale is much greater than atomic dimensions, but small compared to the scattering lengths associated with various inelastic processes. Mesoscopic systems have been studied extensively through their conductance [7, 8, 9, 10]. Noise measurements are much more difficult, but the sensitivity of the experiments has made a remarkable progress in the last years. Some theoretical predictions have been observed, while others still remain an experimental challenge. This article is a review of the present status of the field, with an emphasis on the theoretical developments. We will focus on the scattering approach to electron transport, which provides a unified description of both conductance and shot noise. For earlier reviews, see Refs. [11, 12, 13]. For brief commentaries, see Refs. [14, 15].

1.2. SCATTERING THEORY

In his 1957 paper [16] Landauer discussed the problem of electrical conduction as a scattering problem. This has become a key concept in mesoscopic physics [7, 8]. The conductor is modeled as a scattering region, connected to electron reservoirs. The electrons inside each reservoir are assumed to be in thermal equilibrium. Incoming states, occupied according to the Fermi-Dirac distribution function, are scattered into outgoing states. At low temperatures the conductance is fully determined by the transmission matrix of electrons at the Fermi level. The two-terminal Landauer formula [7, 17] and its multi-terminal generalization [18, 19, 20, 21] constitute a general framework for the calculation of the conductance of a phase-coherent sample. A scattering theory of the noise properties of mesoscopic conductors was derived in Refs. [22, 23, 24, 25, 26, 27, 28]. The basic result is a relationship between the shot-noise power and the transmission matrix at the Fermi level, analogous to the Landauer formula for the conductance. Here we review the derivation of this result, following closely Büttiker's work [26, 28].

Two leads are connected to an arbitrary scattering region (see Fig. 1). Each lead contains N incoming and N outgoing modes at energy ε . We assume only elastic scattering so that energy is conserved. The incoming

and outgoing modes are related by a $2N \times 2N$ scattering matrix S

$$\begin{pmatrix} O_1 \\ O_2 \end{pmatrix} = S \begin{pmatrix} I_1 \\ I_2 \end{pmatrix}, \quad (4)$$

where I_1, O_1, I_2, O_2 are the N -component vectors denoting the amplitudes of the incoming (I) and outgoing (O) modes in lead 1 and lead 2. The scattering matrix can be decomposed in $N \times N$ reflection and transmission matrices,

$$S = \begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix} \equiv \begin{pmatrix} r & t' \\ t & r' \end{pmatrix}, \quad (5)$$

where the $N \times N$ matrix s_{ba} contains the amplitudes $s_{bn,am}$ from incoming mode m in lead a to outgoing mode n in lead b . Because of flux conservation S is a unitary matrix. Moreover, in the presence of time-reversal symmetry S is symmetric.

The current operator in lead 1 is given by

$$\hat{I}(t) = \frac{e}{\hbar} \sum_{\alpha, \beta} \int_0^\infty d\varepsilon \int_0^\infty d\varepsilon' I_{\alpha\beta}(\varepsilon, \varepsilon') \hat{a}_\alpha^\dagger(\varepsilon) \hat{a}_\beta(\varepsilon') e^{it(\varepsilon - \varepsilon')/\hbar}, \quad (6)$$

where $\hat{a}_\alpha^\dagger(\varepsilon)$ [$\hat{a}_\alpha(\varepsilon)$] is the creation [annihilation] operator of scattering state $\psi_\alpha(\mathbf{r}, \varepsilon)$. We have introduced the indices $\alpha \equiv (a, m)$, $\beta \equiv (b, n)$ and the coordinate $\mathbf{r} = (x, \mathbf{y})$. The matrix element $I_{\alpha\beta}(\varepsilon, \varepsilon')$ is determined by the value of the current at cross section S_1 in lead 1,

$$I_{\alpha\beta}(\varepsilon, \varepsilon') = \frac{1}{2} \int_{S_1} d\mathbf{y} \left\{ \psi_\alpha(\mathbf{r}, \varepsilon) [\hat{v}_x \psi_\beta(\mathbf{r}, \varepsilon')]^* + \psi_\beta^*(\mathbf{r}, \varepsilon') \hat{v}_x \psi_\alpha(\mathbf{r}, \varepsilon) \right\}. \quad (7)$$

Here, \hat{v}_x is the velocity operator in the x -direction. At equal energies, Eq. (7) simplifies to [26, 28]

$$I_{am,bn}(\varepsilon, \varepsilon) = \delta_{a1} \delta_{ab} \delta_{mn} - \sum_{p=1}^N s_{1p,am}(\varepsilon) s_{1p,bn}^*(\varepsilon). \quad (8)$$

The average current follows from

$$\langle \hat{a}_\alpha^\dagger(\varepsilon) \hat{a}_\beta(\varepsilon') \rangle = \delta_{\alpha\beta} \delta(\varepsilon - \varepsilon') f_a(\varepsilon), \quad (9)$$

where f_a is the Fermi-Dirac distribution function in reservoir a :

$$f_1(\varepsilon) = f(\varepsilon - E_F - eV), \quad (10a)$$

$$f_2(\varepsilon) = f(\varepsilon - E_F), \quad (10b)$$

$$f(x) = [1 + \exp(x/k_B T)]^{-1}, \quad (10c)$$

with Fermi energy E_F . The result is

$$\langle \hat{I}(t) \rangle = \frac{e}{h} \sum_{\alpha} \int_0^{\infty} d\varepsilon f_{\alpha}(\varepsilon) I_{\alpha\alpha}(\varepsilon, \varepsilon) = \frac{e}{h} \int_0^{\infty} d\varepsilon [f_1(\varepsilon) - f_2(\varepsilon)] \text{Tr} \mathbf{t}(\varepsilon) \mathbf{t}^{\dagger}(\varepsilon), \quad (11)$$

where we have substituted Eq. (8) and used the unitarity of \mathbf{S} . The linear-response conductance, $G \equiv \lim_{V \rightarrow 0} \langle I \rangle / V$, becomes

$$G = \frac{e^2}{h} \int_0^{\infty} d\varepsilon \left(-\frac{\partial f}{\partial \varepsilon} \right) \text{Tr} \mathbf{t}(\varepsilon) \mathbf{t}^{\dagger}(\varepsilon), \quad (12)$$

which at zero temperature simplifies to the Landauer formula

$$G = \frac{e^2}{h} \text{Tr} \mathbf{t} \mathbf{t}^{\dagger} = \frac{e^2}{h} \sum_{n=1}^N T_n. \quad (13)$$

Here \mathbf{t} is taken at E_F and $T_n \in [0, 1]$ is an eigenvalue of $\mathbf{t} \mathbf{t}^{\dagger}$. The conductance is thus fully determined by the transmission eigenvalues. Knowledge of the transmission eigenstates, each of which can be a complicated superposition of incoming modes, is not required.

In order to evaluate the shot-noise power we substitute the current operator (6) into Eq. (1) and determine the expectation value. We use the formula [28]

$$\langle \hat{a}_1^{\dagger} \hat{a}_2 \hat{a}_3^{\dagger} \hat{a}_4 \rangle - \langle \hat{a}_1^{\dagger} \hat{a}_2 \rangle \langle \hat{a}_3^{\dagger} \hat{a}_4 \rangle = \delta_{14} \delta_{23} f_1 (1 - f_2) \equiv \Delta_{1234}, \quad (14)$$

where *e.g.* δ_{12} stands for $\delta_{\alpha\beta} \delta(\varepsilon - \varepsilon')$. Equation (14) shows that there are cross correlations between different scattering states. Although this bears no effect on the time-averaged current, it is essential for the current fluctuations. For the noise power one finds

$$\begin{aligned} P(\omega) &= 2 \frac{e^2}{h^2} \sum_{\alpha, \beta, \gamma, \delta} \int_{-\infty}^{\infty} dt \int_0^{\infty} d\varepsilon \int_0^{\infty} d\varepsilon' \int_0^{\infty} d\varepsilon'' \int_0^{\infty} d\varepsilon''' e^{i(\hbar\omega + \varepsilon - \varepsilon')t/\hbar} \\ &\times I_{\alpha\beta}(\varepsilon, \varepsilon') I_{\gamma\delta}(\varepsilon'', \varepsilon''') \Delta_{\alpha\beta\gamma\delta}(\varepsilon, \varepsilon', \varepsilon'', \varepsilon''') \\ &= 2 \frac{e^2}{h} \sum_{\alpha, \beta} \int_0^{\infty} d\varepsilon I_{\alpha\beta}(\varepsilon, \varepsilon + \hbar\omega) I_{\beta\alpha}(\varepsilon + \hbar\omega, \varepsilon) f_{\alpha}(\varepsilon) [1 - f_{\beta}(\varepsilon + \hbar\omega)]. \end{aligned} \quad (15)$$

The low-frequency limit is found by substitution of Eq. (8),

$$\begin{aligned} P &= 2 \frac{e^2}{h} \int_0^{\infty} d\varepsilon \left\{ [f_1(1 - f_2) + f_2(1 - f_1)] \text{Tr} \mathbf{t} \mathbf{t}^{\dagger} (1 - \mathbf{t} \mathbf{t}^{\dagger}) \right. \\ &\quad \left. + [f_1(1 - f_1) + f_2(1 - f_2)] \text{Tr} \mathbf{t} \mathbf{t}^{\dagger} \mathbf{t} \mathbf{t}^{\dagger} \right\}, \end{aligned} \quad (16)$$

where we have again used the unitarity of S .

Equation (16) allows us to evaluate the noise for various cases. Below we will assume that eV and $k_B T$ are small enough to neglect the energy dependence of the transmission matrix, so that we can take t at $\varepsilon = E_F$. Let us first determine the noise in equilibrium, *i.e.* for $V = 0$. Using the relation $f(1 - f) = -k_B T \partial f / \partial \varepsilon$ we find

$$P = 4k_B T \frac{e^2}{h} \text{Tr} t t^\dagger = 4k_B T \frac{e^2}{h} \sum_{n=1}^N T_n, \quad (17)$$

which is indeed the Johnson-Nyquist formula (2). For the shot-noise power at zero temperature we obtain

$$P = 2eV \frac{e^2}{h} \text{Tr} t t^\dagger (1 - t t^\dagger) = 2eV \frac{e^2}{h} \sum_{n=1}^N T_n (1 - T_n). \quad (18)$$

Equation (18), due to Büttiker [26], is the multi-channel generalization of the single-channel formulas of Khlus [22], Lesovik [24], and Yurke and Kochanski [25]. One notes, that P is again only a function of the transmission eigenvalues.

It is clear from Eq. (18) that a transmission eigenstate for which $T_n = 1$ does not contribute to the shot noise. This is easily understood: At zero temperature there is a non-fluctuating incoming electron stream. If there is complete transmission, the transmitted electron stream will be noise free, too. If T_n decreases, the transmitted electron stream deviates in time from the average current. The resulting shot noise P is still smaller than P_{Poisson} , because the transmitted electrons are correlated due to the Pauli principle. Only if $T_n \ll 1$, the transmitted electrons are uncorrelated, yielding full Poisson noise (see Sec. 1.3.1). Essentially, the non-fluctuating occupation number of the incoming states is a consequence of the electrons being fermions. In this sense, the suppression below the Poisson noise is due to the Pauli principle. On the other hand, one must realize that the noise suppression is not an exclusive property of fermions. It occurs for any incoming beam with a non-fluctuating occupation number, for example a photon number state [29].

The generalization of Eq. (18) to the non-zero voltage, non-zero temperature case is [28, 30]

$$P = 2 \frac{e^2}{h} \sum_{n=1}^N \left[2k_B T T_n^2 + T_n (1 - T_n) eV \coth(eV/2k_B T) \right]. \quad (19)$$

The crossover from the thermal noise (17) to the shot noise (18) depends on the transmission eigenvalues.

As a final remark, we mention that in the above derivations the absence of spin and valley degeneracy has been assumed for notational convenience. It can be easily included. For a two-fold spin degeneracy this results in the replacement of the e^2/h prefactors [such as in Eqs. (13) and (18)] by $2e^2/h$. From now on, we simply use $G_0 \equiv$ degeneracy factor $\times e^2/h$ as the unit of conductance and $P_0 \equiv 2eVG_0$ as the unit of shot-noise power.

1.3. TWO SIMPLE APPLICATIONS

The above results are valid for conductors with arbitrary (elastic) scattering. If the transmission eigenvalues are known, the conduction and noise properties can be readily calculated. Below, this is illustrated for two simple systems. More complicated conductors are discussed in Secs. 2–4.

1.3.1. Tunnel barrier

In a tunnel barrier, electrons have a very small probability of being transmitted. We model this by taking $T_n \ll 1$, for all n . Substitution into the formula for the shot noise (18) and the Landauer formula for the conductance (13) yields $P = P_{\text{Poisson}}$ at zero temperature. For arbitrary temperature we obtain from Eq. (19),

$$P = \coth(eV/2k_B T) P_{\text{Poisson}} . \quad (20)$$

This equation, due to Pucel [31], describes the crossover from thermal noise to full Poisson noise. For tunnel barriers this crossover is governed entirely by the ratio $eV/k_B T$ and not by details of the conductor. This behavior has been observed in various systems, see *e.g.* Refs. [32, 33]. Electron-electron interactions can lead to modification of Eq. (20), see Ref. [34, 35, 36].

1.3.2. Quantum Point Contact

A point contact is a narrow constriction between two pieces of conductor. If the width W of the constriction is much smaller than the mean free path of the bulk material, but much greater than the Fermi wave length λ_F , the conductance is given by the Sharvin formula [37], which in two dimensions reads $G = G_0 2W/\lambda_F$. In such a *classical* point contact the shot noise is absent, as found by Kulik and Omel'yanchuk [38]. In a *quantum* point contact W is comparable to λ_F . Experimentally, a quantum point contact can be formed in a two-dimensional electron gas in an (Al,Ga)As heterostructure [8]. The constriction is defined by depletion of the electron gas underneath metal gates on top of the structure. Upon changing the gate voltage V_g , the width W is varied. The conductance displays a stepwise increase in units of G_0 as a function of V_g [39, 40]. This is caused by the discrete number $N_0 = \text{Int}[2W/\lambda_F]$ of modes at the Fermi energy which fit

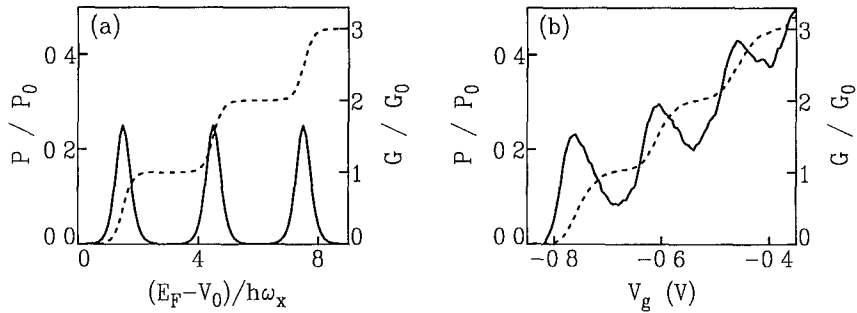


Figure 2 (a) Conductance G (dashed line) and shot-noise power P (full line) versus Fermi energy of a two-dimensional quantum point contact, according to the saddle-point model, with $\omega_y = 3\omega_x$ (b) Experimentally observed G and P versus gate voltage V_g (unpublished data from Reznikov *et al* similar to the experiment of Ref [49], but at a lower temperature $T = 0.4$ K)

into the constriction width. As a result, N_0 transmission eigenvalues equal 1, the others 0, yielding a quantized conductance according to Eq. (13).

Lesovik has predicted that the shot noise in a quantum point contact is distinct from a classical point contact [24]. At the conductance plateaus the shot noise is absent, as follows from Eq. (18). However, in between the plateaus, where the conductance increases by G_0 , there is a transmission eigenvalue which is between 0 and 1. As a consequence, the shot noise has a peak. We illustrate this behavior with a model by Büttiker [41] of a two-dimensional saddle-point potential,

$$V(x, y) = V_0 - \frac{1}{2}m\omega_x^2x^2 + \frac{1}{2}m\omega_y^2y^2, \quad (21)$$

where V_0 is the potential at the saddle point, and ω_x and ω_y determine the curvatures. The transmission eigenvalues at the Fermi energy are [41]

$$T_n = [1 + \exp(-2\pi\varepsilon_n/\hbar\omega_x)]^{-1}, \quad \varepsilon_n \equiv E_F - V_0 - (n - \frac{1}{2})\hbar\omega_y. \quad (22)$$

Results for the conductance and the shot-noise power are displayed in Fig. 2a. The shot noise peaks in between the conductance plateaus and is absent on the plateaus. For large N , the peaks in the shot noise become negligible with respect to the Poisson noise, in agreement with the classical result [38]. More theoretical work on noise in quantum point contacts is given in Refs. [42, 43, 44, 45].

The prediction by Lesovik of this quantum size-effect in the shot noise formed a challenge for experimentalists. An early experiment was done by Li *et al.* [46]. However, a difficulty in the interpretation was that the frequency was not high enough to distinguish between shot noise and resistance fluctuations, which are also quite sensitive to changes in the width of

the point contact [47, 48]. Recent experiments at much higher frequencies by Reznikov *et al.* [49] (see also Ref. [50]) and with very elaborate shielding by Kumar *et al.* [51] have unambiguously demonstrated the occurrence of suppressed shot noise on the conductance plateaus. Experimental data of Reznikov *et al.* are shown in Fig. 2b.

1.4. KINETIC THEORY

The scattering theory of Sec. 1.2 fully takes into account the phase coherence of the electron wave function. If phase coherence is not essential, one can use instead a semiclassical kinetic theory. The word “semiclassical” means that classical mechanics is combined with the quantum-mechanical Pauli principle. A semiclassical kinetic theory for shot noise has been developed by Kulik and Omel’yanchuk for a point contacts[38], by Nagaev for a diffusive conductor [52], and by the authors for an arbitrary conductor [53, 54]. (Refs. [53, 54] correct Ref. [55].)

The theory is based on an extension of the Boltzmann equation to include fluctuations of the distribution function [56, 57, 58]. By analogy with the Langevin equation in the theory of stochastic processes, this fluctuating Boltzmann equation is called the *Boltzmann-Langevin equation*. We give a brief summary of the method.

The fluctuating distribution function $f(\mathbf{r}, \mathbf{k}, t)$ in the conductor equals $(2\pi)^d$ times the density of electrons with position \mathbf{r} , and wave vector \mathbf{k} , at time t . The average over time-dependent fluctuations $\langle f \rangle \equiv \bar{f}$ obeys the Boltzmann equation,

$$\left(\frac{d}{dt} + \mathcal{S} \right) \bar{f}(\mathbf{r}, \mathbf{k}, t) = 0, \quad (23a)$$

$$\frac{d}{dt} \equiv \frac{\partial}{\partial t} + \mathbf{v} \cdot \frac{\partial}{\partial \mathbf{r}} + \mathcal{F} \cdot \frac{\partial}{\hbar \partial \mathbf{k}}. \quad (23b)$$

The derivative (23b) (with $\mathbf{v} = \hbar \mathbf{k} / m$) describes the classical motion in the force field $\mathcal{F}(\mathbf{r}) = -e\partial\phi(\mathbf{r})/\partial\mathbf{r} + e\mathbf{v} \times \mathbf{B}(\mathbf{r})$, with electrostatic potential $\phi(\mathbf{r})$ and magnetic field $\mathbf{B}(\mathbf{r})$. The term $\mathcal{S}\bar{f}$ accounts for the stochastic effects of scattering. In the case of impurity scattering, the scattering term equals

$$\mathcal{S}f(\mathbf{r}, \mathbf{k}, t) = \int d\mathbf{k}' W_{\mathbf{k}\mathbf{k}'}(\mathbf{r}) [f(\mathbf{r}, \mathbf{k}, t) - f(\mathbf{r}, \mathbf{k}', t)]. \quad (24)$$

The kernel $W_{\mathbf{k}\mathbf{k}'}(\mathbf{r})$ is the transition rate for scattering from \mathbf{k} to \mathbf{k}' , which may in principle also depend on \mathbf{r} .

We consider the stationary situation, where \bar{f} is independent of t . The time-dependent fluctuations $\delta f \equiv f - \bar{f}$ satisfy the Boltzmann-Langevin

equation [56, 57],

$$\left(\frac{d}{dt} + \mathcal{S}\right) \delta f(\mathbf{r}, \mathbf{k}, t) = j(\mathbf{r}, \mathbf{k}, t), \quad (25)$$

where j is a fluctuating source term representing the fluctuations induced by the stochastic nature of the scattering. The flux j has zero average, $\langle j \rangle = 0$, and covariance

$$\langle j(\mathbf{r}, \mathbf{k}, t) j(\mathbf{r}', \mathbf{k}', t') \rangle = (2\pi)^d \delta(\mathbf{r} - \mathbf{r}') \delta(t - t') J(\mathbf{r}, \mathbf{k}, \mathbf{k}'). \quad (26)$$

The delta functions ensure that fluxes are only correlated if they are induced by the same scattering process. The flux correlator J depends on the type of scattering and on \bar{f} , but not on δf . Due to the Pauli principle the scattering possibilities of an incoming state depend on the occupation of possible outgoing states. As a consequence, J is roughly proportional to $\bar{f}(1 - \bar{f}')$. The precise correlator J for the impurity-scattering term (24) has been derived by Kogan and Shul'man [57]. Scattering by a tunnel barrier corresponds to another correlator [53, 54].

The kinetic theory can be applied to calculate various noise properties, including the effects of electron-electron and electron-phonon scattering [59, 60, 61]. In Refs. [53, 54] a general formula for the shot-noise power has been derived from Eqs. (25) and (26). Further discussion of the kinetic theory is outside the scope of this review. In the following Section, we discuss an alternative method to calculate the effects of phase breaking and other scattering processes.

1.5. PHASE BREAKING, THERMALIZATION, AND INELASTIC SCATTERING

Noise measurements require rather high currents, which enhance the rate of scattering processes other than purely elastic scattering. The phase-coherent transmission approach of Sec. 1.2 is then no longer valid. The effects of dephasing and inelastic scattering on the shot noise have been studied in Refs. [52, 54, 59, 60, 61, 62, 63, 64, 65, 66, 67]. Below, we discuss a model [54, 64] in which the conductor is divided in separate, phase-coherent parts connected by charge-conserving reservoirs. This model includes the following types of scattering:

- *Quasi-elastic scattering.* Due to weak coupling with external degrees of freedom the electron-wave function gets dephased, but its energy is conserved. In metals, this scattering is caused by fluctuations in the electromagnetic field [68].
- *Electron heating.* Electron-electron scattering exchanges energy between the electrons, but the total energy of the electron gas is con-

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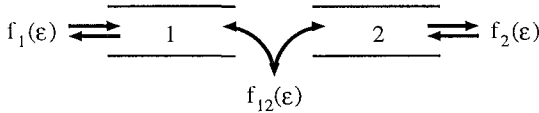


Figure 3. Additional scattering inside the conductor is modeled by dividing it in two parts and connecting them through another reservoir. The electron distributions in the left and the right reservoir, $f_1(\epsilon)$ and $f_2(\epsilon)$, are Fermi-Dirac distributions. The distribution $f_{12}(\epsilon)$ in the intermediate reservoir depends on the type of scattering.

served. The distribution function is therefore assumed to be a Fermi-Dirac distribution at a temperature above the lattice temperature.

- *Inelastic scattering.* Due to electron-phonon interactions the electrons exchange energy with the lattice. The electrons emerging from the reservoir are distributed according to the Fermi-Dirac distribution, at the lattice temperature T .

The model is depicted in Fig. 3. The conductors 1 and 2 are connected via a reservoir with distribution function $f_{12}(\epsilon)$. The time-averaged current I_m through conductor $m = 1, 2$ is given by

$$I_1 = (G_1/e) \int d\epsilon [f_1(\epsilon) - f_{12}(\epsilon)], \quad (27a)$$

$$I_2 = (G_2/e) \int d\epsilon [f_{12}(\epsilon) - f_2(\epsilon)]. \quad (27b)$$

The conductance $G_m \equiv 1/R_m = G_0 \sum_{n=1}^N T_n^{(m)}$, with $T_n^{(m)}$ the n -th transmission eigenvalue of conductor m . We assume small eV and $k_B T$, so that the energy dependence of the transmission eigenvalues can be neglected.

Current conservation requires that $I_1 = I_2 \equiv I$. The total resistance of the conductor is given by Ohm's law,

$$R = R_1 + R_2, \quad (28)$$

for all three types of scattering that we consider. Our model is not suitable for transport in the ballistic regime or in the quantum Hall effect regime, where a different type of “one-way” reservoirs is required [69, 70].

The time-averaged current (27) depends on the average distribution $f_{12}(\epsilon)$ in the reservoir between conductors 1 and 2. In order to calculate the current fluctuations, we need to take into account that this distribution varies in time. We denote the time-dependent distribution by $\tilde{f}_{12}(\epsilon, t)$. The fluctuating current through conductor 1 or 2 causes electrostatic potential fluctuations $\delta\phi_{12}(t)$ in the reservoir, which enforce charge neutrality. In Ref. [64], the reservoir has a Fermi-Dirac distribution $f_{12}(\epsilon, t) = f[\epsilon - E_F - eV_{12} - e\delta\phi_{12}(t)]$, with $E_F + eV_{12}$ the average electrochemical potential

equation [56, 57],

$$\left(\frac{d}{dt} + \mathcal{S}\right) \delta f(\mathbf{r}, \mathbf{k}, t) = j(\mathbf{r}, \mathbf{k}, t), \quad (25)$$

where j is a fluctuating source term representing the fluctuations induced by the stochastic nature of the scattering. The flux j has zero average, $\langle j \rangle = 0$, and covariance

$$\langle j(\mathbf{r}, \mathbf{k}, t) j(\mathbf{r}', \mathbf{k}', t') \rangle = (2\pi)^d \delta(\mathbf{r} - \mathbf{r}') \delta(t - t') J(\mathbf{r}, \mathbf{k}, \mathbf{k}'). \quad (26)$$

The delta functions ensure that fluxes are only correlated if they are induced by the same scattering process. The flux correlator J depends on the type of scattering and on \bar{f} , but not on δf . Due to the Pauli principle the scattering possibilities of an incoming state depend on the occupation of possible outgoing states. As a consequence, J is roughly proportional to $\bar{f}(1 - \bar{f}')$. The precise correlator J for the impurity-scattering term (24) has been derived by Kogan and Shul'man [57]. Scattering by a tunnel barrier corresponds to another correlator [53, 54].

The kinetic theory can be applied to calculate various noise properties, including the effects of electron-electron and electron-phonon scattering [59, 60, 61]. In Refs. [53, 54] a general formula for the shot-noise power has been derived from Eqs. (25) and (26). Further discussion of the kinetic theory is outside the scope of this review. In the following Section, we discuss an alternative method to calculate the effects of phase breaking and other scattering processes.

1.5. PHASE BREAKING, THERMALIZATION, AND INELASTIC SCATTERING

Noise measurements require rather high currents, which enhance the rate of scattering processes other than purely elastic scattering. The phase-coherent transmission approach of Sec. 1.2 is then no longer valid. The effects of dephasing and inelastic scattering on the shot noise have been studied in Refs. [52, 54, 59, 60, 61, 62, 63, 64, 65, 66, 67]. Below, we discuss a model [54, 64] in which the conductor is divided in separate, phase-coherent parts connected by charge-conserving reservoirs. This model includes the following types of scattering:

- *Quasi-elastic scattering.* Due to weak coupling with external degrees of freedom the electron-wave function gets dephased, but its energy is conserved. In metals, this scattering is caused by fluctuations in the electromagnetic field [68].
- *Electron heating.* Electron-electron scattering exchanges energy between the electrons, but the total energy of the electron gas is con-

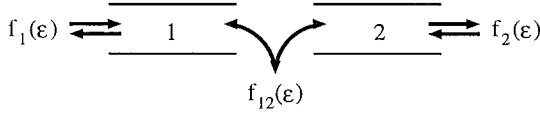


Figure 3. Additional scattering inside the conductor is modeled by dividing it in two parts and connecting them through another reservoir. The electron distributions in the left and the right reservoir, $f_1(\varepsilon)$ and $f_2(\varepsilon)$, are Fermi-Dirac distributions. The distribution $f_{12}(\varepsilon)$ in the intermediate reservoir depends on the type of scattering.

served. The distribution function is therefore assumed to be a Fermi-Dirac distribution at a temperature above the lattice temperature.

- *Inelastic scattering.* Due to electron-phonon interactions the electrons exchange energy with the lattice. The electrons emerging from the reservoir are distributed according to the Fermi-Dirac distribution, at the lattice temperature T .

The model is depicted in Fig. 3. The conductors 1 and 2 are connected via a reservoir with distribution function $f_{12}(\varepsilon)$. The time-averaged current I_m through conductor $m = 1, 2$ is given by

$$I_1 = (G_1/e) \int d\varepsilon [f_1(\varepsilon) - f_{12}(\varepsilon)], \quad (27a)$$

$$I_2 = (G_2/e) \int d\varepsilon [f_{12}(\varepsilon) - f_2(\varepsilon)]. \quad (27b)$$

The conductance $G_m \equiv 1/R_m = G_0 \sum_{n=1}^N T_n^{(m)}$, with $T_n^{(m)}$ the n -th transmission eigenvalue of conductor m . We assume small eV and $k_B T$, so that the energy dependence of the transmission eigenvalues can be neglected.

Current conservation requires that $I_1 = I_2 \equiv I$. The total resistance of the conductor is given by Ohm's law,

$$R = R_1 + R_2, \quad (28)$$

for all three types of scattering that we consider. Our model is not suitable for transport in the ballistic regime or in the quantum Hall effect regime, where a different type of "one-way" reservoirs is required [69, 70].

The time-averaged current (27) depends on the average distribution $f_{12}(\varepsilon)$ in the reservoir between conductors 1 and 2. In order to calculate the current fluctuations, we need to take into account that this distribution varies in time. We denote the time-dependent distribution by $\tilde{f}_{12}(\varepsilon, t)$. The fluctuating current through conductor 1 or 2 causes electrostatic potential fluctuations $\delta\phi_{12}(t)$ in the reservoir, which enforce charge neutrality. In Ref. [64], the reservoir has a Fermi-Dirac distribution $f_{12}(\varepsilon, t) = f[\varepsilon - E_F - eV_{12} - e\delta\phi_{12}(t)]$, with $E_F + eV_{12}$ the average electrochemical potential

in the reservoir. As a result, it is found that the shot-noise power P of the entire conductor is given by [64]

$$R^2 P = R_1^2 P_1 + R_2^2 P_2. \quad (29)$$

In other words, the voltage fluctuations add. The noise powers of the two segments depend solely on the time-averaged distributions [26, 28],

$$P_m = 2G_m \int d\varepsilon [f_m(1 - f_m) + f_{12}(1 - f_{12})] + 2S_m \int d\varepsilon (f_m - f_{12})^2, \quad (30)$$

where $S_m \equiv G_0 \sum_{n=1}^N T_n^{(m)}(1 - T_n^{(m)})$. The analysis of Ref. [64] is easily generalized to arbitrary distribution f_{12} . Then, we have $\tilde{f}_{12}(\varepsilon, t) = f_{12}[\varepsilon - e\delta\phi_{12}(t)]$. It follows that Eqs. (29) and (30) remain valid, but $f_{12}(\varepsilon)$ may be different. Let us determine the shot noise for the three types of scattering.

Quasi-elastic scattering. Here, it is not just the total current which must be conserved, but the current in each energy range. This requires

$$f_{12}(\varepsilon) = \frac{G_1 f_1(\varepsilon) + G_2 f_2(\varepsilon)}{G_1 + G_2}. \quad (31)$$

We note that Eq. (31) implies the validity of Eq. (28). Substitution of Eq. (31) into Eqs. (29) and (30) yields at zero temperature the result [54]:

$$P = \left(R_1^4 S_1 + R_2^4 S_2 + R_1 R_2^2 + R_1^2 R_2 \right) R^{-3} P_{\text{Poisson}}. \quad (32)$$

Electron heating. We model electron-electron scattering, where energy can be exchanged between the electrons at constant total energy. We assume that the exchange of energies establishes a Fermi-Dirac distribution $f_{12}(\varepsilon)$ at an electrochemical potential $E_F + eV_{12}$ and an elevated temperature T_{12} . From current conservation it follows that

$$V_{12} = (R_2/R) V. \quad (33)$$

Conservation of the energy of the electron gas requires that T_{12} is such that no energy is absorbed or emitted by the reservoir. This implies

$$T_{12}^2 = T^2 + \frac{V^2}{\mathcal{L}_0} \frac{R_1 R_2}{R^2}, \quad (34)$$

with the Lorentz number $\mathcal{L}_0 \equiv \frac{1}{3}(\pi k_B/e)^2$. At zero temperature in the left and right reservoir and for $R_1 = R_2$ we have $k_B T_{12} = (\sqrt{3}/2\pi)eV \simeq$

0.28eV. For the shot noise at $T = 0$, we thus obtain using Eqs. (29) and (30) the result [54]:

$$\begin{aligned}
 P = & \left\{ R_1^3 S_1 + R_2^3 S_2 + \frac{1}{\pi} \sqrt{3R_1 R_2} \left[R_1(1 - R_1 S_1) + R_2(1 - R_2 S_2) \right. \right. \\
 & + 2R_1^2 S_1 \ln \left(1 + e^{-\pi \sqrt{R_1/3R_2}} \right) \\
 & \left. \left. + 2R_2^2 S_2 \ln \left(1 + e^{-\pi \sqrt{R_2/3R_1}} \right) \right] \right\} R^{-2} P_{\text{Poisson}} . \quad (35)
 \end{aligned}$$

Inelastic scattering. The distribution function of the intermediate reservoir is the Fermi-Dirac distribution at the lattice temperature T , with an electrochemical potential $\mu_{12} \equiv E_F + eV_{12}$, where V_{12} is given by Eq. (33). This reservoir absorbs energy, in contrast to the previous two cases. The zero-temperature shot-noise power is given by [64]:

$$P = \left(R_1^3 S_1 + R_2^3 S_2 \right) R^{-2} P_{\text{Poisson}} . \quad (36)$$

This model will be applied to double-barrier junctions, chaotic cavities, and disordered conductors in Secs. 2–4. Quite generally, we will find that quasi-elastic scattering has no effect on the shot noise, while electron heating leads to a small enhancement of the shot noise. Inelastic scattering suppresses the shot noise in most cases, but not in the double-barrier junction.

1.6. STATISTICS OF TRANSMITTED CHARGE

The conductance is a measure for the average number of electrons transmitted per unit time. The shot noise quantifies the variance of the transmitted charge. Levitov and Lesovik have studied the full distribution function of charge transmitted through a mesoscopic conductor [71, 72]. This function $P_q(t)$ gives the probability that exactly q electrons have been transmitted during a given time interval t . An alternative way to describe this distribution is through its characteristic function $\chi(\lambda, t)$. They are mutually related according to [73]

$$\chi(\lambda, t) = \sum_{q=0}^{\infty} P_q(t) e^{iq\lambda} , \quad P_q(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\lambda e^{-iq\lambda} \chi(\lambda, t) . \quad (37)$$

The average number of electrons transmitted during a time t is given by

$$\overline{q(t)} = \sum_{q=0}^{\infty} q P_q(t) = \lim_{\lambda \rightarrow 0} \frac{\partial}{\partial \lambda} \chi(\lambda, t) . \quad (38)$$

More generally, one can express the k -th moment $\mu_k(t)$ of the distribution by

$$\mu_k(t) \equiv \overline{q^k(t)} = \lim_{\lambda \rightarrow 0} \left(\frac{\partial}{i\partial\lambda} \right)^k \chi(\lambda, t). \quad (39)$$

The average current is simply $I = e\mu_1(t)/t$ and the noise power equals $P = 2e^2 \lim_{t \rightarrow \infty} \text{var } q(t)/t = 2e^2 \lim_{t \rightarrow \infty} [\mu_2(t) - \mu_1^2(t)]/t$.

Levitov and Lesovik [71] have computed the characteristic function at zero temperature and at small voltage V . The result

$$\chi_N(\lambda, t) = \prod_{n=1}^N [(e^{i\lambda} - 1)T_n + 1]^{G_0 V t / e} \quad (40)$$

is the characteristic function of the binomial or Bernoulli distribution: In scattering channel n , the charge transmitted in a time interval t is due to $G_0 V t / e$ independent attempts to transmit an electron, each time with a probability T_n . The fact that only one electron (within a single channel) can be transmitted during a time $e / G_0 V$ is due to the Pauli principle. Only if $T_n \ll 1$ for all n , Eq. (40) reduces to the characteristic function of a Poisson process. Otherwise, the electrons are transmitted according to sub-Poissonian statistics.

The distribution function of transmitted charge has been determined for a normal-metal-superconductor point contact by Muzykantskii and Khmel'nitskii [74], for a disordered conductor by Lee, Levitov, and Yakovets [75], and for a double-barrier junction by one of the authors [76].

2. Double-Barrier Junction

2.1. RESONANT TUNNELING

In 1973 Tsu and Esaki predicted the occurrence of a negative differential resistance due to resonant tunneling through two tunnel barriers in series inside a semiconductor heterostructure [77]. The experimental observation [78] opened a large field of research. The study of noise in resonant tunneling is a recent development, sparked by the demonstration by Li *et al.* [79] that the shot noise in an (Al,Ga)As double-barrier junction may vary between one-half (for equal barrier heights) and the full Poisson noise (for very unequal barrier heights). This observation, confirmed by other experiments [80, 81, 82], has inspired many theoreticians [83, 84, 85, 86, 87, 88, 89]. Below we will only consider the zero-frequency, low-voltage limit, in order to treat the double-barrier junction on the same footing as the other systems described in this review. We assume high tunnel barriers with mode-independent transmission probabilities $\Gamma_1, \Gamma_2 \ll 1$.

The transmission eigenvalues through the two barriers in series are given by a Fabry-Perot type of formula,

$$T_n = \frac{\Gamma_1 \Gamma_2}{2 - \Gamma_1 - \Gamma_2 - 2\sqrt{1 - \Gamma_1 - \Gamma_2} \cos \phi_n}, \quad (41)$$

where ϕ_n is the phase accumulated in one round trip between the barriers. The density $\rho(T) \equiv \langle \sum_n \delta(T - T_n) \rangle$ of the transmission eigenvalues follows from the uniform distribution of ϕ_n between 0 and 2π [90],

$$\rho(T) = \frac{N\Gamma_1\Gamma_2}{\pi(\Gamma_1 + \Gamma_2)} \frac{1}{\sqrt{T^3(T_+ - T)}}, \quad T \in [T_-, T_+], \quad (42)$$

$\rho(T) = 0$ otherwise, with $T_- = \Gamma_1\Gamma_2/\pi^2$ and $T_+ = 4\Gamma_1\Gamma_2/(\Gamma_1 + \Gamma_2)^2$. The density (42) is plotted in Fig. 4a.

The average conductance,

$$\langle G \rangle = G_0 \int_0^1 dT \rho(T) T = G_0 N \frac{\Gamma_1 \Gamma_2}{\Gamma_1 + \Gamma_2}, \quad (43)$$

is just the series conductance of the two tunnel conductances. The resonances are averaged out by taking a uniform distribution of the phase shifts ϕ_n . Physically, this averaging corresponds either to an average over weak disorder in the region between the barriers, or to a summation over a large number of modes if the separation between the barriers is large compared to the Fermi wave length, or to an applied voltage larger than the width of the resonance.

For the shot-noise power one obtains

$$\langle P \rangle = P_0 \int_0^1 dT \rho(T) T(1 - T) = \frac{\Gamma_1^2 + \Gamma_2^2}{(\Gamma_1 + \Gamma_2)^2} P_{\text{Poisson}}, \quad (44)$$

using Eqs. (42) and (43). This result was first derived by Chen and Ting [83]. For asymmetric junctions, one barrier dominates the transport and the shot noise equals the Poisson noise. For symmetric junctions, the shot noise gets suppressed down to $\langle P \rangle = \frac{1}{2} P_{\text{Poisson}}$ for $\Gamma_1 = \Gamma_2$. The theoretical result (44) is in agreement with the experimental observations [79, 80, 81, 82].

The suppression of the shot noise below P_{Poisson} in symmetric junctions is a consequence of the *bimodal* distribution of transmission eigenvalues, as plotted in Fig. 4a. Instead of all T_n 's being close to the average transmission probability, the T_n 's are either close to 0 or to 1. This reduces the sum $T_n(1 - T_n)$. A similar suppression mechanism exists for shot noise in chaotic cavities and in disordered conductors, see Secs. 3 and 4.

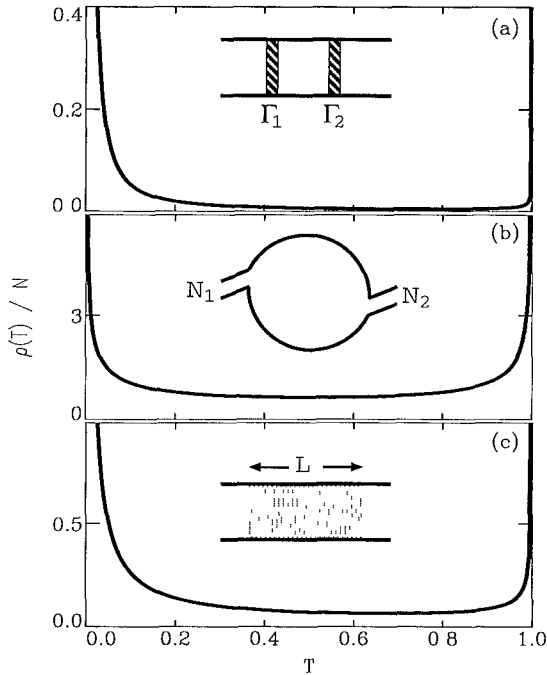


Figure 4. The distribution $\rho(T)$ of transmission eigenvalues T for (a) a double-barrier junction, according to Eq. (41) with $\Gamma_1 = \Gamma_2 = 0.01$; (b) a chaotic cavity, according to Eq. (46) with $N_1 = N_2 \equiv N$; and (c) a disordered wire, according to Eq. (51) with $L = 20\ell$. Each structure has a bimodal distribution.

Phase coherence is not essential for the occurrence of suppressed shot noise. Davies *et al.* obtained the result (44) from a model of incoherent sequential tunneling [84]. The method of Sec. 1.5 (with $G_m = S_m = G_0 N \Gamma_m$ for $m = 1, 2$) shows that both quasi-elastic scattering [see Eq. (32)] and inelastic scattering [see Eq. (36)] do not modify Eq. (44). Thermalization of the electrons in the region between the barriers enhances the shot noise, as follows from Eq. (35). For $\Gamma_1 = \Gamma_2$ we find

$$P = \left[\frac{1}{2} + \frac{\sqrt{3}}{\pi} \ln \left(1 + e^{-\pi/\sqrt{3}} \right) \right] P_{\text{Poisson}} \simeq 0.58 P_{\text{Poisson}}, \quad (45)$$

which is slightly above the one-half suppression in the absence of thermalization. More theoretical work on the influence of internal scattering and of dephasing on the shot noise in double-barrier junctions is contained in Refs. [91, 92, 93, 94, 95].

2.2. COULOMB BLOCKADE

The suppression of the shot noise described in the previous Section is due to correlations induced by the Pauli principle. Coulomb interactions are

another source of correlations among the electrons. A measure of the importance of Coulomb repulsion is the charging energy $E_C = e^2/2C$ of a single electron inside the conductor with a capacitance C . In open conductors, where C is large, charging effects are expected to be negligible [96]. For closed conductors, such as a double-barrier junction, E_C can be larger than $k_B T$, in which case charging effects have a pronounced influence on the conduction [97]. If $eV < E_C$, conduction through the junction is suppressed. This is known as the Coulomb blockade. At $eV > E_C$, one electron at a time can tunnel into the junction. The next electron can follow, only after the first electron has tunneled out of the junction. This is the single-electron tunneling regime. The theory of shot noise in single-electron tunneling devices has been developed by Korotkov *et al.* [98], Hershfield *et al.* [99], and others [100, 101, 102, 103, 104, 105, 106].

Experiments have been reported by Birk, de Jong, and Schönenberger [107]. Here, the double-barrier junction was formed by a scanning-tunneling microscope positioned above a metal nanoparticle on an oxidized substrate. Due to the small size of the particle, $E_C \geq 1000k_B T$, at $T = 4$ K. The relative heights of the two tunnel barriers can be modified by changing the tip-particle distance. Experimental results for an asymmetric junction are plotted in Fig. 5. The I - V characteristics display a stepwise increase of the current with the voltage. (Rotating the plot 90° yields the usual presentation of the ‘Coulomb staircase.’) At small voltage, $I \simeq 0$ due to the Coulomb blockade. At each subsequent step in I , the number of excess electrons in the junction increases by one. The measured shot noise oscillates along with the step structure in the I - V curve. The full shot-noise level $P = P_{\text{Poisson}}$ is reached at each plateau of constant I . In between, P is suppressed down to $\frac{1}{2}P_{\text{Poisson}}$. The experimental data are in excellent agreement with the theory of Ref. [99].

A qualitative understanding of the periodic shot-noise suppression caused by the Coulomb blockade goes as follows: On a current plateau in the I - V curve, the number of electrons in the junction is constant for most of the time. Only during a very short instance an excess electron occupies the junction, leading to the transfer of one electron. This fast transfer process is dominated by the highest tunnel barrier. Since the junction is asymmetric, Poisson noise is expected. The situation is different for voltages where there is a step in the I - V curve. Here, two charge states are degenerate in total energy. If an electron tunnels into the junction, it may stay for a longer time, during which tunneling of the next electron is forbidden. Both barriers are thus alternately blocked. This leads to a correlated current, yielding a suppression of the shot noise.

An essential requirement for the Coulomb blockade is that $G \lesssim e^2/h$. For larger G the quantum-mechanical charge fluctuations in the junction

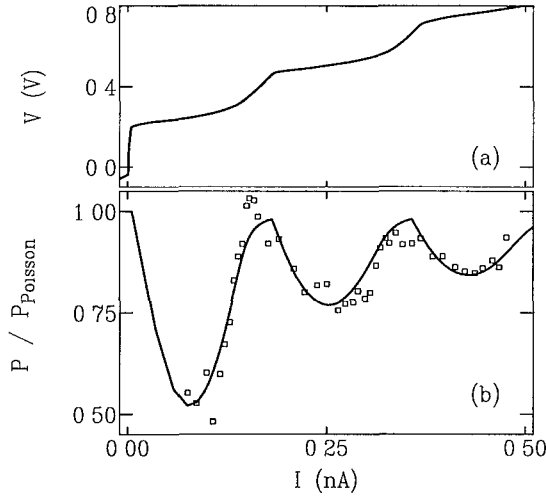


Figure 5 Experimental results by Birk *et al* [107] in the single-electron tunneling regime. The double-barrier junction consists of a tip positioned above a nanoparticle on a substrate. (a) Experimental voltage V versus current I . (b) Shot-noise power P versus I . Squares: experiment, solid line: theory of Hershfield *et al* [99].

become big enough to overcome the Coulomb blockade. The next Section will discuss shot noise in a quantum dot, without including Coulomb interactions. This is justified as long as $G \gtrsim e^2/h$. For smaller G , the quantum dot behaves essentially as the double-barrier junction considered above.

3. Chaotic Cavity

A cavity of sub-micron dimensions, etched in a semiconductor is called a quantum dot. The transport properties of the quantum dot can be measured by coupling it to two electron reservoirs, and bringing them out of equilibrium. We consider the generic case that the classical motion in the cavity can be regarded as chaotic, as a result of scattering by randomly placed impurities or by irregularly shaped boundaries. Then transport quantities are insensitive to microscopic properties of the quantum dot, such as the shape of the cavity and the degree of disorder.

A theory of transport through a chaotic cavity can be based on the single assumption that the scattering matrix of the system is uniformly distributed in the unitary group [108, 109]. This is the “circular ensemble” of random-matrix theory [110, 111]. The assumption of a uniform distribution of the scattering matrix is valid if the coupling to the electron reservoirs occurs via two *ballistic* point contacts, with a conductance $G_m = G_0 N_m$, $m = 1, 2$. (Otherwise a more general distribution, known as the “Poisson kernel” applies [112].) The presence of time-reversal symmetry is accounted for

by restricting the scattering matrix to the subset of symmetric unitary matrices. This is known as the circular orthogonal ensemble (labeled by the index $\beta = 1$). If any unitary matrix is equally probable, the ensemble is called circular unitary ($\beta = 2$).

To compute the statistics of transport properties in a quantum dot one needs to know the distribution of the transmission eigenvalues in the circular ensemble. In the most general case $N_1 \neq N_2$, the transmission matrices t_{12} and t_{21} are rectangular. The two matrix products $t_{12}t_{12}^\dagger$ and $t_{21}t_{21}^\dagger$ contain a common set of $\min(N_1, N_2)$ non-zero transmission eigenvalues. Only these contribute to the transport properties. For $N_i \gg 1$ the distribution $\rho(T)$ of the transmission eigenvalues is [113]

$$\rho(T) = \frac{1}{\pi} (N_1 N_2)^{1/2} \frac{1}{T} \left[\frac{T - T_-}{(1 - T)(1 - T_-)} \right]^{1/2}, \quad T \in [T_-, 1], \quad (46)$$

$\rho(T) = 0$ otherwise, with $T_- = (N_1 - N_2)^2 / (N_1^2 + N_2^2)$. This density is plotted in Fig. 4b.

The average conductance,

$$\langle G \rangle = G_0 \int_0^1 dT \rho(T) T = G_0 \frac{N_1 N_2}{N_1 + N_2}, \quad (47)$$

is the series conductance of the two point contacts. The average shot-noise power,

$$\langle P \rangle = P_0 \int_0^1 dT \rho(T) T(1 - T) = \frac{N_1 N_2}{(N_1 + N_2)^2} P_{\text{Poisson}}, \quad (48)$$

is smaller than the Poisson noise. For two identical point contacts the suppression factor is one quarter [109], to be compared with the one-half suppression in a double-barrier junction (see Sec. 2.1) and the one-third suppression in a disordered wire (see Sec. 4.1).

The result (48) does not require phase coherence, as follows from Eq. (32) using $S_m = 0$ for $m = 1, 2$. However, it is affected by thermalization of the electrons and also by inelastic scattering. From Eq. (35) we find in the case of complete thermalization,

$$P = \frac{\sqrt{3N_1 N_2}}{\pi(N_1 + N_2)} P_{\text{Poisson}}. \quad (49)$$

For $N_1 = N_2$, this yields $P = (\sqrt{3}/2\pi) P_{\text{Poisson}} \simeq 0.28 P_{\text{Poisson}}$. As follows from Eq. (36), inelastic scattering suppresses the shot noise completely.

4. Disordered Metal

4.1. ONE-THIRD SUPPRESSION

We now turn to transport through a diffusive conductor of length L much greater than the mean free path ℓ , in the metallic regime ($L \ll$ localization length). The average conductance is given by the Drude formula,

$$\langle G \rangle = G_0 \frac{N\ell}{L}, \quad (50)$$

up to small corrections of order G_0 (due to weak localization). The mean free path $\ell = a_d \ell_{\text{tr}}$ equals the transport mean free path ℓ_{tr} times a numerical coefficient, which depends on the dimensionality d of the Fermi surface ($a_2 = \pi/2$, $a_3 = 4/3$).

From Eq. (50) one might surmise that for a diffusive conductor all the transmission eigenvalues are of order ℓ/L , and hence $\ll 1$. This would imply the shot-noise power $P = P_{\text{Poisson}}$ of a Poisson process. This surmise is completely incorrect, as was first pointed out by Dorokhov [114], and later by Imry [115] and by Pendry, MacKinnon, and Roberts [116]. A fraction ℓ/L of the transmission eigenvalues is of order unity (open channels), the others being exponentially small (closed channels). For $\ell \ll L \ll N\ell$, the density of the T_n 's is given by [114]

$$\rho(T) = \frac{N\ell}{2L} \frac{1}{T\sqrt{1-T}}, \quad T \in [T_-, 1], \quad (51)$$

$\rho(T) = 0$ otherwise, with $T_- = 4e^{-2L/\ell}$. The density $\rho(T)$, plotted in Fig. 4c, is again bimodal with peaks near unit and zero transmission. Dorokhov [114] obtained Eq. (51) from a scaling equation, which describes the evolution of $\rho(T)$ on increasing L in a wire geometry [117, 118]. A derivation for other geometries has been given by Nazarov [119].

One easily checks that the bimodal distribution (51) leads to the Drude conductance (50). For the average shot-noise power it implies

$$\langle P \rangle = P_0 \frac{N\ell}{3L} = \frac{1}{3} P_{\text{Poisson}}. \quad (52)$$

This suppression of the shot noise by a factor one-third is *universal*, in the sense that it does not depend on the specific geometry nor on any intrinsic material parameter (such as ℓ). The one-third suppression was discovered by Beenakker and Büttiker [64] in the way described above, and by others using different methods [52, 119, 120, 121]. Nagaev's theory [52] is based on the semiclassical Boltzmann-Langevin equation, see Sec. 1.4. One might therefore infer that there is also a semiclassical derivation of the bimodal

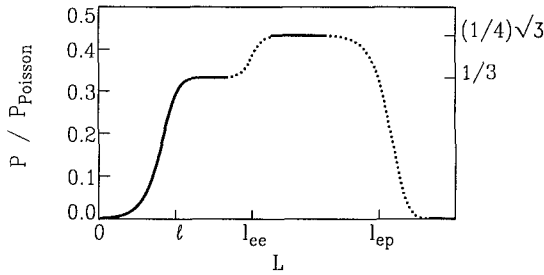


Figure 6. The shot-noise power P of a disordered metallic wire as a function of its length L , as predicted by theory. Indicated are the elastic mean free path ℓ , the electron-electron scattering length l_{ee} and the electron-phonon scattering length l_{ep} . Dotted lines are interpolations (after Ref. [127]).

distribution of transmission eigenvalues, which is the key ingredient of the quantum-mechanical theory. Such a derivation is given in Ref. [122].

4.2. DEPENDENCE ON WIRE LENGTH

The one-third suppression of the shot noise breaks down if the conductor becomes too short or too long. Upon decreasing the length of the conductor, when L becomes comparable to ℓ , the electron transport is no longer diffusive, but enters the ballistic regime. Then the shot noise is suppressed more strongly, according to [120]

$$P = \frac{1}{3} [1 - (1 + L/\ell)^{-3}] P_{\text{Poisson}}. \quad (53)$$

For $L \ll \ell$ there is no shot noise, as in a ballistic point contact [38]. Equation (53) is exact for a special model of one-dimensional scattering, but holds more generally within a few percent [54]. A Monte-Carlo simulation in a wire geometry [123] is in good agreement with Eq. (53). The crossover of the shot noise from the ballistic to the diffusive regime is plotted in Fig. 6. Upon increasing L at constant cross section of the conductor, one enters the localized regime. Here, even the largest transmission eigenvalue is exponentially small [114], so that $P = P_{\text{Poisson}}$. Shot noise in one-dimensional chains for various models of disorder has been studied in Ref. [124].

Experimentally, the crossover from the metallic to the localized regime is usually not reached, because phase coherence is broken when L is still much smaller than the localization length $N\ell$. In the remainder of this Section, we apply the method of Sec. 1.5 to determine the effect of phase breaking and other inelastic scattering events on the shot noise in a disordered metal [54]. We divide the conductor into M segments connected by reservoirs, taking the continuum limit $M \rightarrow \infty$. The electron distribution at position x is denoted by $f(\varepsilon, x)$. At the ends of the conductor $f(\varepsilon, 0) = f_1(\varepsilon)$ and

$f(\varepsilon, L) = f_2(\varepsilon)$, *i.e.* the electrons are Fermi-Dirac distributed at temperature T and with electrochemical potential $\mu(0) = E_F + eV$ and $\mu(L) = E_F$, respectively. It follows from Eqs. (29) and (30) that the noise power is given by

$$P = \frac{4}{RL} \int_0^L dx \int_0^\infty d\varepsilon f(\varepsilon, x)[1 - f(\varepsilon, x)], \quad (54)$$

a formula first obtained by Nagaev [52]. We evaluate Eq. (54) for the three types of scattering discussed in Sec. 1.5.

Quasi-elastic scattering. Current conservation and the absence of inelastic scattering requires

$$f(\varepsilon, x) = \frac{L-x}{L} f(\varepsilon, 0) + \frac{x}{L} f(\varepsilon, L). \quad (55)$$

The electron distribution at $x = L/2$ is plotted in the inset of Fig. 7. Substitution of Eq. (55) into Eq. (54) yields [52]

$$P = \frac{2}{3} [4k_B T G + eI \coth(eV/2k_B T)]. \quad (56)$$

At zero temperature the shot noise is one-third of the Poisson noise. The same result follows from the phase-coherent theory [Eqs. (19) and (51)], demonstrating that quasi-elastic scattering has no effect on the shot noise. The temperature dependence of P is plotted in Fig. 7.

Electron heating. The electron-distribution function is a Fermi-Dirac distribution with a spatially dependent electrochemical potential $\mu(x)$ and temperature $T_e(x)$,

$$f(\varepsilon, x) = \left\{ 1 + \exp \left[\frac{\varepsilon - \mu(x)}{k_B T_e(x)} \right] \right\}^{-1}, \quad (57a)$$

$$\mu(x) = E_F + \frac{L-x}{L} eV, \quad (57b)$$

$$T_e(x) = \sqrt{T^2 + (x/L)[1 - (x/L)]V^2/\mathcal{L}_0}, \quad (57c)$$

cf. Eqs. (33) and (34). Equations (54) and (57) yield for the noise power the result [59, 60, 125]

$$P = 2k_B T G + 2eI \left[\frac{2\pi}{\sqrt{3}} \left(\frac{k_B T}{eV} \right)^2 + \frac{\sqrt{3}}{2\pi} \right] \arctan \left(\frac{\sqrt{3}}{2\pi} \frac{eV}{k_B T} \right), \quad (58)$$

plotted in Fig. 7. In the limit $eV \gg k_B T$ one finds

$$P = \frac{1}{4} \sqrt{3} P_{\text{Poisson}} \simeq 0.43 P_{\text{Poisson}}. \quad (59)$$

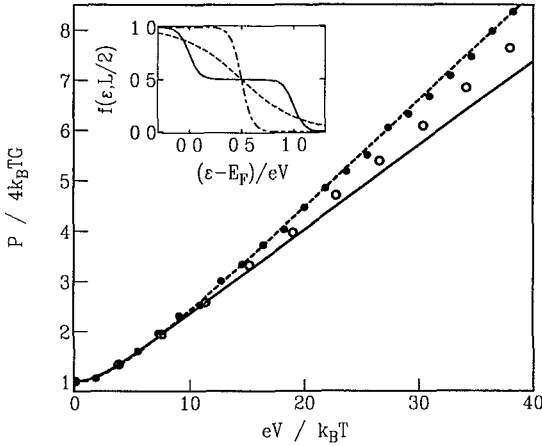


Figure 7. The noise power P versus voltage V for a disordered wire in the presence of quasi-elastic scattering [solid curve, from Eq. (56)] and of electron heating [dashed curve, from Eq. (58)]. The inset gives the electron distribution in the middle of the wire at $k_B T = \frac{1}{20} eV$. The distribution for inelastic scattering is included for comparison (dash-dotted). Experimental data of Steinbach, Martinis, and Devoret [127] on silver wires at $T = 50$ mK are indicated for length $L = 1 \mu\text{m}$ (circles) and $L = 30 \mu\text{m}$ (dots).

Electron-electron scattering increases the shot noise above $\frac{1}{3} P_{\text{Poisson}}$ because the exchange of energies makes the current less correlated.

Inelastic scattering. The electron-distribution function is given by

$$f(\varepsilon, x) = \left\{ 1 + \exp \left[\frac{\varepsilon - \mu(x)}{k_B T} \right] \right\}^{-1} \quad (60)$$

with $\mu(x)$ according to Eq. (57b). We obtain from Eqs. (54) and (60) that the noise power is equal to the Johnson-Nyquist noise (2) for arbitrary V . The shot noise is thus completely suppressed by inelastic scattering [59, 60, 62, 64, 65, 66].

The dependence of the shot-noise power on the length of a disordered conductor is plotted in Fig. 6. The phase coherence length (between ℓ and ℓ_{ee}) does not play a role. An early experimental demonstration of sub-Poissonian shot noise in a wire defined in an (Al,Ga)As heterostructure was reported by Liefink *et al.* [126]. The measurements were in agreement with theory, but lacked the precision needed to discriminate between the elastic and the hot-electron value. More accurate experiments by Steinbach, Martinis, and Devoret [127] on silver wires are shown in Fig. 7. The noise in a wire of $L = 30 \mu\text{m}$ is in excellent agreement with the hot-electron result (58). For the $L = 1 \mu\text{m}$ wire the noise crosses over to the elastic result (56), without quite reaching it.

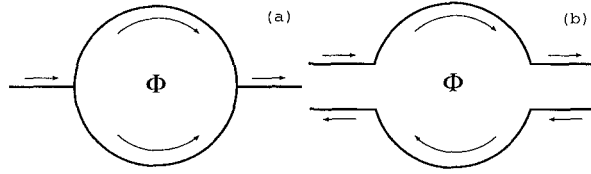


Figure 8. Feynman paths enclosing a magnetic flux Φ . The paths in (a) correspond to the same electron, the paths in (b) to two different electrons.

5. Aharonov-Bohm Effect

Since the current operator is a one-particle observable, the shot-noise power (given by the current-current correlator) is a *two-particle* transport property. This is an essential difference with the conductance, which is a one-particle property. The distinction between one-particle and two-particle properties is relevant even without Coulomb interactions between the electrons, because of the quantum-mechanical exchange interaction. A striking demonstration of the two-particle nature of shot noise, discovered by Büttiker [128, 129], occurs in the Aharonov-Bohm effect.

The Aharonov-Bohm effect in electrical conduction is a periodic oscillation of the conductance of a ring (or cylinder) as a function of the enclosed magnetic flux Φ . (For reviews, see Refs. [130, 131].) The fundamental periodicity of the oscillation is h/e , because a flux increment of an integer number of flux quanta changes by an integer multiple of 2π the phase difference between Feynman paths along the two arms of the ring (see Fig. 8a). Since the conductance is a one-particle property, the two interfering Feynman paths must belong to the same electron, which on entering the ring has a probability to traverse the ring either clock-wise or counter-clock-wise. For a maximal amplitude of the conductance oscillations the two probabilities should be approximately equal. The Lorentz force causes an electron to traverse the ring preferentially in one of the two directions. This is why the Aharonov-Bohm effect is suppressed by a strong magnetic field [8].

Shot noise can exhibit Aharonov-Bohm oscillations which persist in a strong magnetic field. The Lorentz force guides the right-moving electron along the upper arc of the ring, and the left-moving electron along the lower arc (see Fig. 8b). The absolute value squared $|\psi_1|^2$, $|\psi_2|^2$ of each of the two single-particle wave functions ψ_1, ψ_2 does not depend on Φ , hence the conductance is Φ -independent. However, the absolute value squared of the two-particle wave function $|\Psi(\mathbf{r}_1, \mathbf{r}_2)|^2 = |\psi_1(\mathbf{r}_1)\psi_2(\mathbf{r}_2) - \psi_1(\mathbf{r}_2)\psi_2(\mathbf{r}_1)|^2$ does depend on Φ . The flux sensitivity is an exchange effect, which vanishes only if the two single-particle wave functions do not overlap.

To measure the flux sensitivity in the shot noise due to the exchange effect one can not simply use the ring geometry of Fig. 8: Because each electron is fully transmitted the shot noise vanishes completely in a strong magnetic field. (The case of weak magnetic field has been studied in Ref. [132].) Büttiker [129] has suggested a four-terminal configuration, where the correlator of the current at two terminals is measured using the other two terminals as current sources. Lesovik and Levitov [133] have proposed a two-terminal configuration, but with a time-dependent magnetic field. Ideally, the shot noise should show a flux sensitivity while the time-averaged current should not. Observation of this effect remains an experimental challenge.

6. Cooper Pairs

6.1. NORMAL-METAL-SUPERCONDUCTOR JUNCTIONS

If a normal metal is connected to a superconductor, the dissipative normal current is converted into dissipationless supercurrent. This conversion goes through a process called Andreev reflection [134]: Incoming electrons are reflected into outgoing holes, with the transfer of a Cooper pair into the superconductor. Since the elementary charge transfer now involves a charge $2e$ instead of e , one might expect a doubling of the shot noise in an NS junction. Let us see how this follows from the theory [22, 74, 135].

We assume low temperatures and an applied voltage eV smaller than the excitation gap Δ in the superconductor, so that the electrons and holes are confined to the normal metal. The scattering from incoming into outgoing states is described by the $2N \times 2N$ reflection matrix R ,

$$\begin{pmatrix} O_e \\ O_h \end{pmatrix} = R \begin{pmatrix} I_e \\ I_h \end{pmatrix}, \quad R = \begin{pmatrix} r_{ee} & r_{eh} \\ r_{he} & r_{hh} \end{pmatrix}, \quad (61)$$

where I_e, I_h, O_e, O_h are the N -component vectors denoting the amplitudes of the incoming (I) and outgoing (O) electron (e) and hole (h) modes. The reflection matrix R can be decomposed in $N \times N$ submatrices, where *e.g.* r_{he} contains the reflection amplitudes from incoming electrons into outgoing holes. The conductance [136, 137, 138] and the shot-noise power [135] are given by

$$G_{\text{NS}} = 2G_0 \text{Tr} r_{he} r_{he}^\dagger = 2G_0 \sum_{n=1}^N \mathcal{R}_n, \quad (62)$$

$$P_{\text{NS}} = 4P_0 \text{Tr} r_{he} r_{he}^\dagger (1 - r_{he} r_{he}^\dagger) = 4P_0 \sum_{n=1}^N \mathcal{R}_n (1 - \mathcal{R}_n), \quad (63)$$

with \mathcal{R}_n an eigenvalue of $r_{he} r_{he}^\dagger$, evaluated at the Fermi energy.

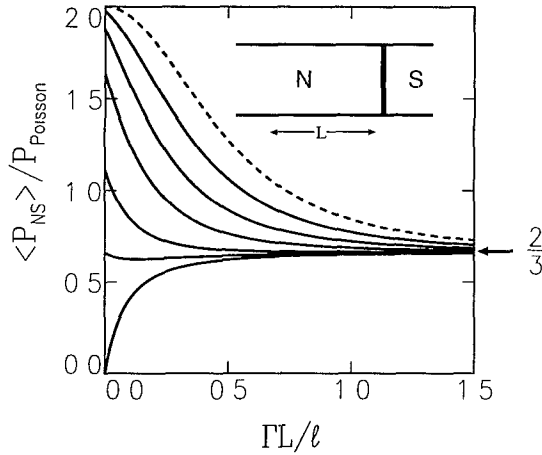


Figure 9 The shot-noise power $\langle P_{NS} \rangle$ of a disordered NS junction with a barrier at the NS interface (shown in the inset) as a function of its length L , for barrier transparencies $\Gamma = 1, 0.9, 0.8, 0.6, 0.4, 0.2, \ll 1$ from bottom to top. For $L = 0$, $\langle P_{NS} \rangle$ varies with Γ according to Eq. (67). If L increases it approaches the limiting value $\langle P_{NS} \rangle = \frac{4}{3}eI$ for each Γ (after Ref. [135]).

The eigenvalue \mathcal{R}_n can be related to the scattering properties of the normal region through the Bogoliubov-de Gennes equation [139], which is a 2×2 matrix Schrödinger equation for electron and hole wave functions. In the presence of time-reversal symmetry, \mathcal{R}_n can be expressed in terms of the transmission eigenvalue T_n of the normal region [140]:

$$\mathcal{R}_n = T_n^2 (2 - T_n)^{-2}. \quad (64)$$

Substitution into Eqs. (62) and (63) yields [135, 140]

$$G_{NS} = G_0 \sum_{n=1}^N \frac{2T_n^2}{(2 - T_n)^2}, \quad (65)$$

$$P_{NS} = P_0 \sum_{n=1}^N \frac{16T_n^2(1 - T_n)}{(2 - T_n)^4}. \quad (66)$$

As in the normal state, scattering channels which have $T_n = 0$ or $T_n = 1$ do not contribute to the shot noise. However, the way in which partially transmitting channels contribute is entirely different from the normal state result (18).

For a planar tunnel barrier ($T_n = \Gamma$ for all n) one finds [22]

$$P_{NS} = P_0 N \frac{16\Gamma^2(1 - \Gamma)}{(2 - \Gamma)^4} = \frac{8(1 - \Gamma)}{(2 - \Gamma)^2} P_{\text{Poisson}}, \quad (67)$$

which for $\Gamma \ll 1$ simplifies to $P_{\text{NS}} = 4eI = 2P_{\text{Poisson}}$. This can be interpreted as an uncorrelated current of $2e$ -charged particles.

Since Eq. (66) is valid for arbitrary scattering region, we can easily determine the average shot-noise power for a double-barrier junction in series with a superconductor,

$$\langle P_{\text{NS}} \rangle = \left(2 - \frac{5\Gamma_1^2\Gamma_2^2}{(\Gamma_1^2 + \Gamma_2^2)^2} \right) P_{\text{Poisson}}, \quad (68)$$

for a chaotic cavity in series with a superconductor,

$$\langle P_{\text{NS}} \rangle = \frac{x^2}{(1+x)^2(\sqrt{1+x}-1)} P_{\text{Poisson}}, \quad (69)$$

with $x = 4N_1N_2/(N_1 + N_2)^2$, and for a disordered NS junction [135],

$$\langle P_{\text{NS}} \rangle = \frac{2}{3} P_{\text{Poisson}} \quad (70)$$

The average was computed with the densities (42), (46), and (51) of transmission eigenvalues, respectively. The shot noise in a disordered NS junction with a barrier at the NS interface is plotted in Fig. 9. It makes the connection between the results (67) and (70). We have indeed found that for a high tunnel barrier and for a disordered NS junction the shot noise is doubled with respect to the normal-state results. For other systems, the relation is more complicated.

More theoretical work on shot noise in NS systems is given in Refs. [141, 142, 143]. The effects of the Coulomb blockade on the shot noise in low-capacitance NSN junctions are described in Refs. [144, 145]. With one inconclusive exception [146], no experimental observation of shot noise in NS junctions has been reported, yet.

6.2. JOSEPHSON JUNCTIONS

A Josephson junction contains two normal-metal–superconductor interfaces, with a phase difference ϕ of the superconducting order parameter. Such an SNS junction sustains a current $I(\phi)$ in equilibrium, *i.e.* even if the voltage difference V between the superconductors vanishes. Since this supercurrent is a ground-state property, it can not fluctuate by itself. (It exhibits no shot noise.) Time-dependent fluctuations result from quasiparticles which are excited at any finite temperature T . Their zero-frequency power density $P(\phi)$ is related to the linear-response conductance $G(\phi) = \lim_{V \rightarrow 0} \partial I(\phi)/\partial V$ of the Josephson junction in the same way as in the normal state,

$$P(\phi) = 4k_{\text{B}}TG(\phi), \quad (71)$$

cf. Eq. (2). A remarkable difference with Johnson-Nyquist noise in a normal metal is that $P(\phi)$ may actually increase with decreasing temperature, because of the rapid increase of $G(\phi)$ when $T \rightarrow 0$. Because thermal noise falls outside the scope of our review, we do not discuss this topic further. The interested reader is referred to Refs. [147, 148].

7. Quantum Hall Effect

In a strong magnetic field the scattering channels of a two-dimensional electron gas consist of *edge states*. Edge states at opposite edges propagate in opposite directions. In the absence of scattering from one edge to the other, each of the scattering channels at the Fermi level is transmitted with probability $T_n = 1$. This is the regime of the (integer) quantum Hall effect. (For reviews, see Refs. [8, 149].) The conductance $G = (e^2/h) \sum_n T_n$ shows plateaus at integer multiples of e^2/h as a function of magnetic field. The implication for the shot-noise power $P \propto \sum_n T_n(1 - T_n)$ is that it should vanish on the plateaus, similar to the situation in a quantum point contact, see Sec. 1.3.2.

Büttiker [26, 28] considered the noise in the four-terminal conductor depicted in Fig. 10. It is assumed that the transmission probability of all edge channels but one is reduced to zero by means of a gate across the conductor. The remaining non-zero transmission probability is denoted by T . A current flows between contacts 1 and 2 (voltage difference V), while contacts 3 and 4 are voltage probes. This four-terminal configuration requires a generalization of the two-terminal formulas of Sec. 1.2. The current I_a in contact a is related to the voltages V_b at the contact b by the scattering matrix [18],

$$I_a = \frac{e^2}{h} \left[V_a \text{Tr} \left(1 - s_{aa}^\dagger s_{aa} \right) - \sum_{b \neq a} V_b \text{Tr} s_{ab}^\dagger s_{ab} \right], \quad (72)$$

where we have assumed zero temperature. The correlator

$$P_{ab} = 2 \int_{-\infty}^{\infty} dt \langle \Delta I_a(t + t_0) \Delta I_b(t_0) \rangle \quad (73)$$

of the current fluctuations in contacts a and b is given by [26, 28]

$$P_{ab} = e \frac{e^2}{h} \sum'_{c,d} (V_c - V_d) \left(\text{Tr} s_{ac}^\dagger s_{ad} s_{bd}^\dagger s_{bc} + \text{Tr} s_{ad}^\dagger s_{ac} s_{bc}^\dagger s_{bd} \right). \quad (74)$$

The prime in the summation over c and d means a restriction to terms with $V_c > V_d$. The two-terminal formula (18) is recovered if $c, d = a, b$. Application to the four-terminal geometry of Fig. 10 shows that P_{ab} vanishes if a

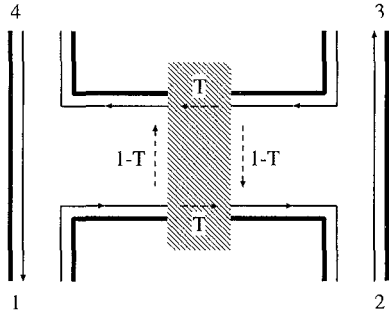


Figure 10 Four-terminal conductor in the regime of the quantum Hall effect. The spatial location of the edge states is indicated, as well as their direction of motion (solid lines with arrows). A barrier (shaded region) causes scattering from one edge to the other (dashed lines). The corresponding scattering probabilities are indicated.

or b equals 1 or 3, while [26, 28]

$$P_{22} = P_{44} = -P_{24} = 2eV \frac{e^2}{h} T(1 - T). \quad (75)$$

Equation (75) assumes that the voltages on all terminals are fixed, while the current fluctuates in time. Usually, one measures voltage fluctuations at fixed currents. Current and voltage are linearly related by Eq. (72), which at low frequencies holds both for the time-average and for the fluctuations. The resulting voltage-noise power measured between contacts 1 and 4 or between contacts 2 and 3 is zero. The noise power measured between any other pair of contacts equals $2eV(h/e^2)(1 - T)/T$ [26, 28]. The voltage fluctuations diverge as $T \rightarrow 0$ with increasing barrier height. Experiments in support of the edge-channel description of shot noise in the quantum Hall effect have been reported by Washburn *et al.* [150].

If the magnetic field becomes so strong that only a single edge channel remains at the Fermi level, one enters the regime of the *fractional* quantum Hall effect. Plateaus in the conductance now occur at e^2/mh , $m = 1, 3, 5, \dots$ (and at other odd-denominator fractions of e^2/h as well) [149]. The quasi-particle excitations of the electron gas have a fractional charge $e^* = e/m$. Since the shot noise, in contrast to the conductance, is sensitive to the charge of the carriers, one might hope to be able to find evidence for fractionally charged quasi-particles in shot-noise measurements. The theory has been developed in Refs. [151, 152, 153]. For very low barrier heights, Poisson noise with a fractional charge e^* is expected in the backscattered current ($I_{\max} - I_1$), with $I_{\max} = (e^*/h)eV$ the current in the absence of a barrier. This yields for the shot noise

$$P_{22} = 2e^*(I_{\max} - I_1). \quad (76)$$

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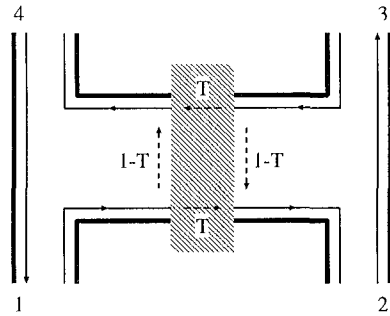


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$$P_{22} = 2e^*(I_{\max} - I_1). \quad (76)$$

For high barriers the usual Poisson noise $P = 2eI_1$ is recovered. Experiments remain to be done.

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