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# Theory of directed localization in one dimension

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We present an analytical solution of the delocalization transition that is induced by an imaginary vector potential in a disordered chain [N. Hatano and D. R. Nelson, Phys. Rev. Lett. **77**, 570 (1996)]. We compute the relation between the real and imaginary parts of the energy in the thermodynamic limit, as well as finite-size effects. The results are in good agreement with numerical simulations for weak disorder (in which the mean free path is large compared to the wavelength). [S0163-1829(97)51032-0]

In a recent paper,[1] Hatano and Nelson have demonstrated the existence of a mobility edge in a disordered ring with an imaginary vector potential. A non-Hermitian Hamiltonian containing an imaginary vector potential arises from the study of the pinning of vortices by columnar defects in a superconducting cylinder.[2] Their discovery of a delocalization transition in one- and two-dimensional systems has generated considerable interest,[3-5] since all states are localized by disorder in one and two dimensions if the vector potential is real. Localization in this specific kind of non-Hermitian quantum mechanics is referred to as "directed localization,"[3] because the imaginary vector potential breaks the symmetry between left-moving and right-moving particles, without breaking time-reversal symmetry.

The analytical results of Ref. 1 consist of expressions for the mobility edge and for the stretched-exponential relaxation of delocalized states, and a solution of the one-dimensional problem with a single impurity. Here we go further, by solving the many-impurity case in one dimension. Most of the technical results which we will need were derived previously in connection with the problem of localization in the presence of an imaginary *scalar* potential. Physically, these two problems are entirely different: an imaginary vector potential singles out a direction in space, while an imaginary scalar potential singles out a direction in time. A negative imaginary part of the scalar potential corresponds to absorption and a positive imaginary part to amplification. One might surmise that amplification could cause a delocalization transition, but in fact all states remain localized in one dimension in the presence of an imaginary scalar potential.[6,7]

Following Ref. 1 we consider a disordered chain with the single-particle Hamiltonian

$$\mathcal{H}=-\frac{w}{2}\sum_j \left(e^{ha}c_{j+1}^\dagger c_j+e^{-ha}c_j^\dagger c_{j+1}\right)+\sum_j V_j c_j^\dagger c_j \tag{1}$$

The operators $c_j^\dagger$ and $c_j$ are creation and annihilation operators, $a$ is the lattice constant, and $w$ the hopping parameter. The random potential $V_j$ is chosen independently for each site, from a distribution with zero mean and variance $u^2$. For weak disorder (mean free path much larger than the wavelength), higher moments of the distribution of $V_j$ are not relevant. The Hamiltonian is non-Hermitian because of the real parameter $h$, corresponding to the imaginary vector potential. The chain of length $L$ is closed into a ring, and the problem is to determine the eigenvalues $\varepsilon$ of $\mathcal{H}$. If $\varepsilon$ is an eigenvalue of $\mathcal{H}$, then also $\varepsilon^*$ is one — because $\mathcal{H}$ is real. Real $\varepsilon$ corresponds to localized states, while complex $\varepsilon$ corresponds to extended states.[1]

To solve this problem, we reformulate it in terms of the $2\times 2$ transfer matrix $M_h(\varepsilon)$ of the chain, which relates wave amplitudes at both ends.[8] The energy $\varepsilon$ is an eigenvalue of $\mathcal{H}$ if and only if $M_h(\varepsilon)$ has an eigenvalue of 1. The use of the transfer matrix is advantageous, because the effect of the imaginary vector potential is just to multiply $M$ with a scalar,

$$M_h(\varepsilon)=e^{hL}M_0(\varepsilon). \tag{2}$$

The energy spectrum is therefore determined by

$$\det[1-e^{hL}M_0(\varepsilon)]=0. \tag{3}$$

Time-reversal symmetry implies $\det M_0=1$. Hence the determinantal Eq. (3) is equivalent to[11]

$$\mathrm{tr}M_0(\varepsilon)=2\cosh hL. \tag{4}$$

We seek the solution in the limit $L\to\infty$.

Since $M_0$ is the transfer matrix in the absence of the imaginary vector potential ($h=0$), we can use the results in the literature on localization in conventional one-dimensional systems (having an Hermitian Hamiltonian).[9] The four matrix elements of $M_0$ are given in terms of the reflection amplitudes $r$, $r'$ and the transmission amplitude $t$ by

$$(M_0)_{11}=-(1/t)\det S,\quad (M_0)_{12}=r'/t,$$
$$(M_0)_{21}=-r/t,\quad (M_0)_{22}=1/t, \tag{5}$$

where $\det S=rr'-t^2$ is the determinant of the scattering matrix. (There is only a single transmission amplitude because of time-reversal symmetry, so that transmission from left to right is equivalent to transmission from right to left.) The transmission probability $T=|t|^2$ decays exponentially in the large-$L$ limit, with decay length $\xi$:

$$-\lim_{L\to\infty}L^{-1}\ln T=\xi^{-1}. \tag{6}$$

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The energy dependence of $\xi$ is known for weak disorder, such that $|k|\xi \gg 1$, where the complex wave number $k$ is related to $\varepsilon$ by the dispersion relation

$$\varepsilon = -w\cos ka. \tag{7}$$

For real $k$, the decay length is the localization length $\xi_0$, given by[10]

$$\xi_0 = a(w/u)^2 \sin^2(\,\mathrm{Re}\,ka). \tag{8}$$

(Since $\xi_0$ is of the order of the mean free path $\ell$, the condition of weak disorder requires $\ell$ large compared to the wavelength.) For complex $k$, the decay length is shorter than $\xi_0$, regardless of the sign of Im$k$, according to[6,7]

$$\xi^{-1} = \xi_0^{-1} + 2|\,\mathrm{Im}k|. \tag{9}$$

We use these results to simplify Eq. (4). Upon taking the logarithm of both sides of Eq. (4), dividing by $L$ and taking the limit $L \to \infty$, one finds

$$|h| - \tfrac{1}{2}\xi^{-1} = \lim_{L\to\infty} L^{-1} \ln|1 - \det S|, \tag{10}$$

where we have used $L^{-1}\ln f \to L^{-1}\ln|f|$ as $L \to \infty$ for any complex function $f(L)$. For complex $k$, the absolute value of det$S$ is either $<1$ (for Im$k>0$) or $>1$ (for Im$k<0$). As a consequence, $\ln|1-\det S|$ remains bounded for $L\to\infty$, so that the right-hand side of Eq. (10) vanishes. Substituting Eq. (9), we find that complex wave numbers $k$ satisfy

$$|\,\mathrm{Im}k| = |h| - \tfrac{1}{2}\xi_0^{-1}. \tag{11}$$

Together with the expression (8) for the localization length $\xi_0$, this is a relation between the real and imaginary parts of the wave number. Using the dispersion relation (7), and noticing that the condition $|k|\xi \gg 1$ for weak disorder implies $|\,\mathrm{Im}k| \ll |\,\mathrm{Re}k|$, we can transform Eq. (11) into a relation between the real and imaginary parts of the energy,

$$|\,\mathrm{Im}\varepsilon| = |h|a\sqrt{w^2 - (\,\mathrm{Re}\varepsilon)^2} - \frac{u^2}{2\sqrt{w^2-(\,\mathrm{Re}\varepsilon)^2}}. \tag{12}$$

The support of the density of states in the complex plane consists of the closed curve (12) plus two line segments on the real axis,[12] extending from the band edge $\pm w$ to the mobility edge $\pm\varepsilon_c$. The real eigenvalues are identical to the eigenvalues at $h=0$, up to exponentially small corrections. The energy $\varepsilon_c$ is obtained by putting Im$\varepsilon=0$ in Eq. (12), or equivalently be equating[1] $2\xi_0$ to $1/|h|$, hence

$$\varepsilon_c = (w^2 - u^2/2|h|a)^{1/2}. \tag{13}$$

The delocalization transition at $\varepsilon_c$ exists for $|h| > h_c = \tfrac{1}{2}u^2/w^2 a$.

In Fig. 1(a), the analytical theory is compared with a numerical diagonalization of the Hamiltonian (1). The numerical finite-$L$ results are consistent with the large-$L$ limit (dashed curve). To leading order in $1/L$, fluctuations of Im$\varepsilon$ around the large-$L$ limit (12) are governed by fluctuations of the transmission probability $T$. [Fluctuations of $L^{-1}\ln T$ are of order $L^{-1/2}$, while the other fluctuating contributions to Eq. (4) are of order $L^{-1}$.] The variance of $\ln T$ for large $L$ is known,[13]

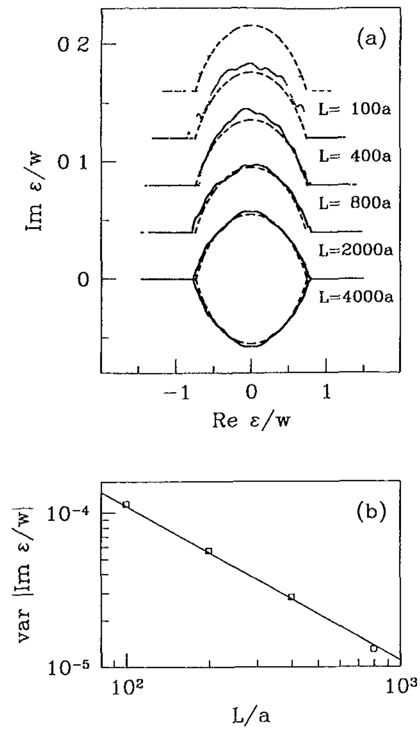


FIG. 1. (a) Data points: eigenvalues of the Hamiltonian (1), for parameter values $ha=0.1$, $u/w=0.3$, and for five values of $L/a$. Dashed curves: analytical large-$L$ limit, given by Eq. (12). (Except for the case $L=4000a$, spectra are offset vertically and only eigenvalues with Im$\varepsilon \geq 0$ are shown.) (b) Variance of the imaginary part of the eigenvalues as a function of the sample length, for Re$\varepsilon\approx 0$ and for the same parameter values as in (a). The data points are the numerical results for 1000 samples. The solid line is the analytical result (15).

$$\mathrm{var}\ln T = \frac{2L}{\xi_0} + 8L|\,\mathrm{Im}k|e^{4\xi_0|\mathrm{Im}k|}\,\mathrm{Ei}(-4\xi_0|\,\mathrm{Im}k|), \tag{14}$$

where Ei is the exponential integral. Equating $|\,\mathrm{Im}k| = |h| + \tfrac{1}{2}L^{-1}\ln T$, we find $\mathrm{var}|\,\mathrm{Im}k| = \tfrac{1}{4}L^{-2}\,\mathrm{var}\ln T$ and thus

$$\mathrm{var}|\,\mathrm{Im}\varepsilon| = \frac{a^2}{2L\xi_0}[1 + 2\gamma e^{2\gamma}\,\mathrm{Ei}(-2\gamma)][w^2 - (\,\mathrm{Re}\varepsilon)^2], \tag{15}$$

where $\gamma = 2|h|\xi_0 - 1$. In Fig. 1(b) we see that Eq. (15) agrees well with the results of the numerical diagonalization. The fluctuations $\Delta$ Im$\varepsilon$ are correlated over a range $\Delta$ Re$\varepsilon$ which is large compared to $\Delta$ Im$\varepsilon$ itself, their ratio $\Delta$ Im$\varepsilon/\Delta$ Re$\varepsilon$ decreasing $\propto L^{-1/2}$. This explains why the complex eigenvalues for a specific sample appear to lie on a smooth curve [see Fig. 1(a)]. This curve is sample specific and fluctuates around the large-$L$ limit (12).

In conclusion, we have presented an analytical theory for the delocalization transition in a single-channel disordered wire with an imaginary vector potential. We find good agreement with numerical diagonalizations, both for the relation between the real and imaginary parts of the energy in the