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Brightness of a phase-conjugating mirror behind a random medium

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Abstract. – A random-matrix theory is presented for the reflection of light by a disordered medium backed by a phase-conjugating mirror Two regimes are distinguished, depending on the relative magnitude of the inverse dwell time of a photon in the disordered medium and the frequency shift acquired at the mirror The qualitatively different dependence of the reflectance on the degree of disorder in the two regimes suggests a distinctive experimental test for cancellation of phase shifts in a random medium

A phase-conjugating mirror has the remarkable ability to cancel phase shifts between incident and reflected light [1]-[3] This cancellation is used in optics to correct for wave front distortions [4] A plane wave which has been distorted by an inhomogeneous medium is reflected at a phase-conjugating mirror, after traversing the medium for a second time, the original undistorted wave front is recovered It is as if the reflected wave were the time reverse of the incident wave

Complete wave front reconstruction is possible if the distorted wave front remains approximately planar, because perfect time reversal upon reflection holds only in a narrow range of angles of incidence for realistic systems (For the hypothetical case of perfect time reversal at all angles, see refs [5], [6]) For this reason phase-conjugating mirrors have been studied mainly in combination with weakly inhomogeneous media An exception is formed by the experimental work of McMichael, Ewbank, and Vachss [7], who measured the intensity of the reconstructed wave front for a strongly inhomogeneous medium (small transmission probability T_0), and found that it was proportional to T_0^2, in agreement with the theoretical prediction of Gu and Yeh [8] If $T_0 \ll 1$, the intensity of the reconstructed wave is much smaller than the total reflected intensity The total reflected intensity was not studied previously, perhaps because it was believed that the diffusive illumination resulting from a strongly inhomogeneous medium would render the effect of phase conjugation insignificant In this paper we show that, on the contrary, phase conjugation has a large effect on the total reflected intensity, even if $T_0 \ll 1$

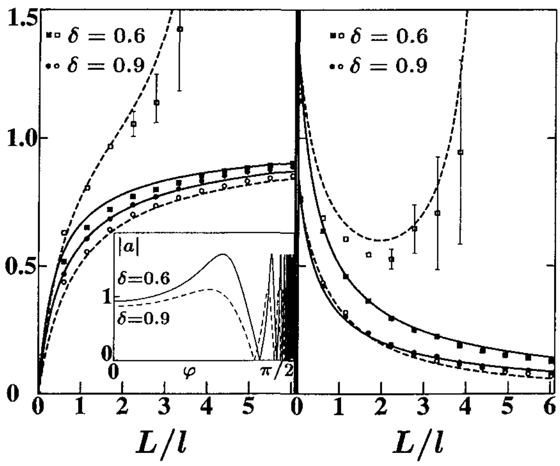

Fig. 2. – Average reflectances $\langle R_\pm \rangle$ as a function of L/l for $\alpha = \pi/4$ and $\delta = 0.6$, 0.9. The dashed curves are the incoherent result, given by eq. (13). The solid curves are the coherent result, given by eq. (14) for $L/l \gtrsim 3$. Data points are results from numerical simulations (open symbols for the incoherent regime, filled symbols for the coherent regime). Error bars are the statistical uncertainty of the average over 150 disorder configurations. (When the error bar is not shown it is smaller than the size of the marker.) The inset is a plot of the absolute value of the reflection amplitude a of the phase-conjugating mirror, given by eq. (9c).

which is unitary (because of flux conservation) and symmetric (because of time-reversal invariance). (In contrast, $\mathbf{r}_{\mathrm{PCM}}$ is not flux conserving.) Without loss of generality the reflection and transmission matrices of the disordered region can be decomposed as [9]

$$\mathbf{r}_{11}(\omega_\pm) = i\mathbf{U}_\pm\sqrt{\boldsymbol{\rho}_\pm}\,\mathbf{U}_\pm^{\mathrm{T}}, \qquad \mathbf{t}_{21}(\omega_\pm) = \mathbf{V}_\pm\sqrt{\boldsymbol{\tau}_\pm}\,\mathbf{U}_\pm^{\mathrm{T}}, \tag{11a}$$

$$\mathbf{r}_{22}(\omega_\pm) = i\mathbf{V}_\pm\sqrt{\boldsymbol{\rho}_\pm}\,\mathbf{V}_\pm^{\mathrm{T}}, \qquad \mathbf{t}_{12}(\omega_\pm) = \mathbf{U}_\pm\sqrt{\boldsymbol{\tau}_\pm}\,\mathbf{V}_\pm^{\mathrm{T}}. \tag{11b}$$

Here $\mathbf{U}_\pm$ and $\mathbf{V}_\pm$ are $N \times N$ unitary matrices, and $\boldsymbol{\tau}_\pm \equiv 1 - \boldsymbol{\rho}_\pm$ is a diagonal matrix with the transmission eigenvalues $T_{\pm,n} \in [0,1]$ on the diagonal.

Combining eqs. (8)-(11), we find expressions for $R_\pm$ in terms of $\boldsymbol{\tau}_\pm$ and $\boldsymbol{\Omega} = \mathbf{V}_-^\dagger \mathbf{a} \mathbf{V}_+$:

$$R_- = N^{-1}\,\mathrm{tr}\,\boldsymbol{\tau}_-\boldsymbol{\Omega}\left(1 - \sqrt{\boldsymbol{\rho}_+}\,\boldsymbol{\Omega}^{\mathrm{T}}\sqrt{\boldsymbol{\rho}_-}\,\boldsymbol{\Omega}\right)^{-1}\boldsymbol{\tau}_+\left(1 - \boldsymbol{\Omega}^\dagger\sqrt{\boldsymbol{\rho}_-}\,\boldsymbol{\Omega}^*\sqrt{\boldsymbol{\rho}_+}\right)^{-1}\boldsymbol{\Omega}^\dagger. \tag{12}$$

The expression for R_+ is similar (but more lengthy). To compute the averages $\langle R_\pm \rangle$, we have to average over $\boldsymbol{\tau}_\pm$ and $\mathbf{V}_\pm$. We make the isotropy approximation [9] that the matrices $\mathbf{V}_\pm$ are uniformly distributed over the unitary group $\mathcal{U}(N)$. For $\tau_{\mathrm{dwell}}\Delta\omega \ll 1$ we may identify $\mathbf{V}_+ = \mathbf{V}_-$, while for $\tau_{\mathrm{dwell}}\Delta\omega \gg 1$ the matrices $\mathbf{V}_+$ and $\mathbf{V}_-$ are independent. In each case the average over $\mathcal{U}(N)$ with $N \gg 1$ can be done using the large-N expansion of ref. [13]. The remaining average over $T_{\pm,n}$ can be done using the known density $\rho(T)$ of the transmission eigenvalues in a disordered medium [9].

In the incoherent regime ($\tau_{\mathrm{dwell}}\Delta\omega \gg 1$) the result is

$$\langle R_- \rangle = T_0^2 A[1-(1-T_0)^2A^2]^{-1}, \qquad \langle R_+ \rangle = 1 - T_0 + T_0^2(1-T_0)A^2[1-(1-T_0)^2A^2]^{-1}, \tag{13}$$

where $T_0 = (1 + 2L/\pi l)^{-1}$ is the transmittance at frequency ω_0 of the disordered medium in the large-N limit [14]. The quantity $A = N^{-1}\,\mathrm{tr}\,\mathbf{a}\mathbf{a}^\dagger$ is the modal average of the reflectance of the phase conjugating mirror ($A \to \int_0^{\pi/2} \mathrm{d}\phi\, |a(\phi)|^2 \cos\phi$ for $N \to \infty$). Equation (13) can also be obtained within the framework of radiative transfer theory, in which interference effects in the disordered medium are disregarded [15].

In fig. 2 we have plotted the result (13) for $\langle R_\pm \rangle$ in the incoherent regime (dashed curves). For $A > 1$ (corresponding to $\delta < 0.78$) the reflectance $\langle R_- \rangle$ has a minimum at $L/l = \frac{1}{2}\pi(A^2-1)^{-1}$, and both $\langle R_+ \rangle$ and $\langle R_- \rangle$ diverge at $L/l = \frac{1}{2}\pi(A-1)^{-1}$. This divergence is preempted by depletion of the pump beams in the phase-conjugating mirror, and signals the breakdown of a stationary solution to the scattering problem. For $A < 1$ (corresponding to $\delta > 0.78$) $\langle R_- \rangle$ tends to 0 as L^{-2} for $L \to \infty$, while $\langle R_+ \rangle$ approaches 1 as L^{-1}.

The situation is entirely different in the coherent regime ($\tau_{\mathrm{dwell}}\Delta\omega \ll 1$). The complete result is a complicated function of L/l (plotted in fig. 2, solid curves). For $L/l \to \infty$ the result takes the simpler form

$$\langle R_- \rangle = 2T_0\, \mathrm{Re}\frac{a_0^*(a_0^2-1)}{a_0^2 - a_0^{*2}} \operatorname{artgh} a_0, \qquad \langle R_+ \rangle = 1 - 2T_0\, \mathrm{Re}\frac{a_0^*(a_0^2-1)}{a_0^2 - a_0^{*2}} \operatorname{artgh} a_0^*, \tag{14a}$$

where the complex number a_0 is determined by

$$\int_0^{\pi/2} \mathrm{d}\phi\, \frac{\cos\phi\; a(\phi)}{1 - a_0\, a(\phi)} = \frac{a_0}{1-a_0^2}. \tag{14b}$$

When $\delta \to 0$, $a_0 \to 1.284 - 0.0133\,i$ for $\alpha = \pi/4$. Both $\langle R_- \rangle$ and $\langle R_+ \rangle$ have a monotonic L-dependence, tending to 0 and 1, respectively, as $1/L$ for $L \to \infty$.

To test the analytical predictions of random-matrix theory, we have carried out numerical simulations. The Helmholtz equation $\left(\nabla^2 + \varepsilon\omega_\pm^2/c^2\right)\mathcal{E} = 0$ is discretised on a square lattice (lattice constant d, length L, width W). The relative dielectric constant ε fluctuates from site to site between $1 \pm \delta\varepsilon$. Using the method of recursive Green functions [16], we compute the scattering matrix $\mathbf{S}$ of the disordered medium at frequencies ω_+ and ω_-. The reflection matrix $\mathbf{r}_{\mathrm{PCM}}$ of the phase-conjugating mirror is calculated by discretising eq. (2). From $\mathbf{S}(\omega_\pm)$ and $\mathbf{r}_{\mathrm{PCM}}$ we obtain the reflection matrix $\mathbf{r}$ of the entire system, and from eq. (8) the reflectances $R_\pm$.

We took $W = 51\,d$, $\delta\varepsilon = 0.5$, $\alpha = \pi/4$, and varied δ and L. For the coherent regime we took $\omega_+ = \omega_- = 1.252\,c/d$, and for the incoherent regime $\omega_+ = 1.252\,c/d$, $\omega_- = 1.166\,c/d$. These parameters correspond to $N_+ = 22$, $l_+ = 15.5\,d$ at frequency ω_+. (The mean free path is determined from the transmittance of the disordered region.) In the incoherent regime we have $N_- = 20$, $l_- = 20.1\,d$. For comparison with the analytic theory, where the difference between N_+ and N_- and between l_+ and l_- is neglected, we use the values N_+ and l_+. Results for the average reflectances are shown in fig. 2, and are in good agreement with the analytical predictions.

A striking feature of the coherent regime is the absence of the minimum in $\langle R_- \rangle$ as a function of L/l for $A > 1$. A qualitative explanation for the disappearance of the reflectance minimum goes as follows. To first order in L/l, disorder reduces the intensity of light reflected with frequency shift $2\Delta\omega$, because some light is scattered back before it can reach the phase-conjugating mirror and undergo a frequency shift. To second order in L/l, disorder increases the intensity because it traps the light near the mirror, where it is amplified by interaction with the pump beams. This explains the initial decrease of $\langle R_- \rangle$ followed by an increase in the incoherent regime. The decrease persists in the coherent regime, because resonant transmission through the disordered region makes trapping inefficient. The resonant transmission is the result of constructive interference of multiply scattered light, which is made possible by phase-shift cancellation.

In conclusion, we have studied the interplay of optical phase-conjugation and multiple scattering in a random medium. The theoretical prediction of a reflectance minimum provides a clear signature for experimentalists in search for effects of phase-shift cancellation in strongly inhomogeneous media. The random-matrix approach presented here is likely to have a broad range of applicability, as in the analogous electronic problem [9], [10]. One direction for future research is to include a second phase-conjugating mirror opposite the first, with a different phase of the coupling constant. Such a system is the optical analogue of a Josephson junction [12], and it would be interesting to see how far the analogy goes.

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