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Quantum limit of the laser line width in chaotic cavities and statistics of residues of scattering matrix poles

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Abstract

The quantum-limited line width of a laser cavity is enhanced above the Schawlow–Townes value by the Petermann factor K, due to the non-orthogonality of the cavity modes We derive the relation between the Petermann factor and the residues of poles of the scattering matrix and investigate the statistical properties of the Petermann factor for cavities in which the radiation is scattered chaotically For a single scattering channel we determine the complete probability distribution of K and find that the average Petermann factor $\langle K \rangle$ depends non-analytically on the area of the opening, and greatly exceeds the most probable value For an arbitrary number N of scattering channels we calculate $\langle K \rangle$ as a function of the decay rate Γ of the lasing mode We find for $N \gg 1$ that for typical values of Γ the average Petermann factor $\langle K \rangle \propto \sqrt{N} \gg 1$ is parametrically larger than unity

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1. Introduction

Laser action selects a mode in a cavity and enhances the output intensity in this mode by a non-linear feedback mechanism Vacuum fluctuations of the electromagnetic field ultimately limit the narrowing of the emission spectrum [1] The quantum-limited line

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width, or Schawlow–Townes line width,

$$\delta\omega_{\mathrm{ST}} = \tfrac{1}{2}\Gamma^2/I\,, \tag{1.1}$$

is proportional to the square of the decay rate Γ of the lasing cavity mode and inversely proportional to the output power I (in units of photons/s). This is a lower bound for the line width when Γ is much less than the line width of the atomic transition and when the lower level of the transition is unoccupied. Many years after the work of Schawlow and Townes it was realized [2–4] that the true fundamental limit is larger than Eq. (1.1) by a factor K that characterizes the non-orthogonality of the cavity modes. This excess noise factor, or Petermann factor, has generated an extensive literature [4–10].

Apart from its importance for cavity lasers, the Petermann factor is of fundamental significance in the more general context of scattering theory. A lasing cavity mode is associated with a pole of the scattering matrix in the complex frequency plane. We will show that the Petermann factor is proportional to the squared modulus of the residue of this pole. Poles of the scattering matrix also determine the position and height of resonances of nuclei, atoms, and molecules [11]. Powerful numerical tools that give access to poles even deep in the complex plane have been developed recently [12]. They can be used to determine the residues of the poles as well. Our work is of relevance for these more general studies, beyond the original application to cavity lasers.

Existing theories of the Petermann factor deal with cavities in which the scattering is essentially one-dimensional, because the geometry has a high degree of symmetry. For such cavities the framework of ray optics provides a simple way to solve the problem in a good approximation [6,7]. This approach breaks down if the light propagation in the cavity becomes chaotic, either because of an irregular shape of the boundaries (like for the cavity depicted in Fig. 1) or because of randomly placed scatterers. The method of random-matrix theory is well-suited for such chaotic cavities [13,14]. Instead of considering a single cavity, one studies an ensemble of cavities with small variations in shape and size, or position of the scatterers. The distribution of the scattering matrix in this ensemble is known. Recent work has provided a detailed knowledge on the statistics of the poles [15–19]. Much less is known about the residues [20–22]. In this work we fill the remaining gap to a considerable extent.

The outline of this paper is as follows. In Section 2 we derive the connection between the Petermann factor and the residue of the pole of the lasing mode. The residue in turn is seen to be characteristic for the degree of non-orthogonality of the modes. In this way we make contact with the existing literature on the Petermann factor [9,10].

In Section 3 we study the single-channel case of a scalar scattering matrix. This applies to a cavity that is coupled to the outside via a small opening of area $\mathscr{A} \lesssim \lambda^2/2\pi$ (with λ the wavelength of the lasing mode). For preserved time-reversal symmetry (the relevant case in optics) we find that the ensemble average of $K-1$ depends *non-analytically* $\propto T \ln T^{-1}$ on the transmission probability T through the opening, so that it is beyond the reach of perturbation theory even if $T \ll 1$. We present a

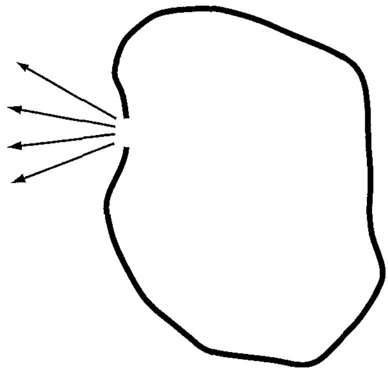

Fig. 1. Chaotic cavity that radiates light from a small opening.

complete resummation of the perturbation series that overcomes this obstacle. We derive the conditional distribution $P(K)$ of the Petermann factor at a given decay rate Γ of the lasing mode, valid for any value of T. The most probable value of $K-1$ is $\propto T$. Hence it is parametrically smaller than the average.

In a cavity with such a small opening the deviations of K from unity are very small. For larger deviations we study, in Section 4, the multi-channel case of an $N \times N$ scattering matrix, which corresponds to an opening of area $\mathscr{A} \approx N\lambda^2/2\pi$. The lasing mode acquires a decay rate Γ of order $\Gamma_0 = NT\Delta/2\pi$ (with Δ the mean spacing of the cavity modes). We compute the mean Petermann factor as a function of Γ for broken time-reversal symmetry, which is technically simpler than the case of preserved time-reversal symmetry, but qualitatively similar. We find a parametrically large mean Petermann factor $K \propto \sqrt{N}$.

Our conclusions are given in Section 5. The main results of Sections 3 and 4 have been reported in Refs. [23,24], respectively.

2. Relationship between Petermann factor and residue

Modes of a closed cavity, in the absence of absorption or amplification, are eigenvalues ω_n of a Hermitian operator H. This operator can be chosen real if the system possesses time-reversal symmetry (symmetry index $\beta = 1$), otherwise it is complex ($\beta = 2$). For a chaotic cavity, H can be modeled by an $M \times M$ Hermitian matrix with independent Gaussian distributed elements

$$P(H) \propto \exp\left[-\frac{\beta M}{4\mu^2}\operatorname{tr} H^2\right] \tag{2.1}$$

(For $\beta = 1$ (2), this is the Gaussian orthogonal (unitary) ensemble [14].) The mean density of eigenvalues is the Wigner semicircle

$$\rho(\omega) = \frac{M}{2\pi\mu^2}\sqrt{4\mu^2 - \omega^2} \tag{2.2}$$

The mean mode spacing at the center $\omega=0$ is $\Delta=\pi\mu/M$. (The limit $M\to\infty$ at fixed spacing Δ of the modes is taken at the end of the calculation.)

A small opening in the cavity is described by a real, non-random $M\times N$ coupling matrix W, with N the number of scattering channels transmitted through the opening. (For an opening of area $\mathscr{A}$, $N\simeq 2\pi\mathscr{A}/\lambda^2$ at wavelength λ.) Modes of the open cavity are complex eigenvalues (with negative imaginary part) of the non-Hermitian matrix

$$\mathscr{H}=H-\mathrm{i}\pi WW^{\mathrm{T}}. \tag{2.3}$$

In absence of amplification or absorption, the scattering matrix S at frequency ω is related to $\mathscr{H}$ by [11,25]

$$S=\mathbb{1}-2\pi\mathrm{i}W^{\dagger}(\omega-\mathscr{H})^{-1}W. \tag{2.4}$$

The scattering matrix is a unitary (and symmetric, for $\beta=1$) random $N\times N$ matrix, with poles at the eigenvalues of $\mathscr{H}$. It enters the input–output relation

$$a_m^{\mathrm{out}}(\omega)=\sum_{n=1}^{N}S_{mn}(\omega)a_n^{\mathrm{in}}(\omega), \tag{2.5}$$

which relates the annihilation operators a_m^{out} of the scattering states that leave the cavity to the annihilation operators a_n^{in} of states that enter the cavity. The indices n,m label the scattering channels.

We now assume that the cavity is filled with a homogeneous amplifying medium (constant amplification rate $1/\tau_a$ over a large frequency window $\Omega_a=L\Delta$, $L\gg N$). This adds a term $\mathrm{i}/2\tau_a$ to the eigenvalues, shifting them upwards towards the real axis. The scattering matrix

$$S=\mathbb{1}-2\pi\mathrm{i}W^{\dagger}(\omega-\mathscr{H}-\mathrm{i}/2\tau_a)^{-1}W \tag{2.6}$$

is then no longer unitary, and the input–output relation changes to [26,27]

$$a_m^{\mathrm{out}}(\omega)=\sum_{n=1}^{N}S_{mn}(\omega)a_n^{\mathrm{in}}(\omega)+\sum_{n=1}^{N}Q_{mn}(\omega)b_n^{\dagger}(\omega). \tag{2.7}$$

All operators fulfill the canonical bosonic commutation relations $[a_n(\omega),a_m^{\dagger}(\omega')]=\delta_{nm}\delta(\omega-\omega')$. As a consequence,

$$Q(\omega)Q^{\dagger}(\omega)=S(\omega)S^{\dagger}(\omega)-\mathbb{1}. \tag{2.8}$$

The operators b describe the spontaneous emission of photons in the cavity and have expectation value

$$\langle b_n^{\dagger}(\omega)b_m(\omega')\rangle=\delta_{nm}\delta(\omega-\omega')f(\omega,T), \tag{2.9}$$

with $f(\omega,T)=[\exp(\hbar\omega/k_BT)-1]^{-1}$ the Bose–Einstein distribution function at frequency ω and temperature T.

In the absence of external illumination ($\langle a^{\dagger\mathrm{in}}a^{\mathrm{in}}\rangle=0$), the photon current per frequency interval,

$$I(\omega)=\frac{1}{2\pi}\sum_{m=1}^{N}\langle a_m^{\mathrm{out}\dagger}(\omega)a_m^{\mathrm{out}}(\omega)\rangle, \tag{2.10}$$

is related to the scattering matrix by Kirchhoff's law [22,23]

$$I(\omega) = f(\omega, T)\frac{1}{2\pi}\mathrm{tr}\,[\mathbb{1} - S^{\dagger}(\omega)S(\omega)]\,. \tag{2.11}$$

For ω near the laser transition we may replace f by the population inversion factor $N_{\mathrm{up}}/(N_{\mathrm{low}} - N_{\mathrm{up}})$, where N_{up} and N_{low} are the mean occupation numbers of the upper and lower levels of the transition In this way the photon current can be written in the form

$$I(\omega) = \frac{1}{2\pi}\frac{N_{\mathrm{up}}}{N_{\mathrm{up}} - N_{\mathrm{low}}}\mathrm{tr}\,[S^{\dagger}(\omega)S(\omega) - \mathbb{1}]\,, \tag{2.12}$$

that is suitable for an amplifying medium. (Alternatively, one can associate a negative temperature to an amplifying medium)

The lasing mode is the eigenvalue $\Omega - \mathrm{i}\Gamma/2$ closest to the real axis, and the laser threshold is reached when the decay rate Γ of this mode equals the amplification rate $1/\tau_a$ Near the laser threshold we need to retain only the contribution from the lasing mode (say mode number l) to the scattering matrix (2 6),

$$S_{nm} = -2\pi\mathrm{i}\frac{(W^{\dagger}U)_{nl}(U^{-1}W)_{lm}}{\omega - \Omega + \mathrm{i}\Gamma/2 - \mathrm{i}/2\tau_a}\,, \tag{2 13}$$

where U is the matrix of right eigenvectors of $\mathscr{H}$ (no summation over l is implied) The photon current near threshold takes the form

$$I(\omega) = \frac{2\pi N_{\mathrm{up}}}{N_{\mathrm{up}} - N_{\mathrm{low}}}\frac{(U^{\dagger}WW^{\dagger}U)_{ll}(U^{-1}WW^{\dagger}U^{-1\dagger})_{ll}}{(\omega - \Omega)^2 + \frac{1}{4}(\Gamma - 1/\tau_a)^2}\,. \tag{2.14}$$

This is a Lorentzian with full width at half maximum $\delta\omega = \Gamma - 1/\tau_a$ The coupling matrix W can be eliminated by writing

$$-\pi(U^{\dagger}WW^{\dagger}U)_{ll} = \mathrm{Im}(U^{\dagger}\mathscr{H}U)_{ll} = -\frac{\Gamma}{2}(U^{\dagger}U)_{ll}\,, \tag{2 15a}$$

$$-\pi(U^{-1}WW^{\dagger}U^{-1\dagger})_{ll} = \mathrm{Im}(U^{-1}\mathscr{H}U^{-1\dagger})_{ll} = -\frac{\Gamma}{2}(U^{-1}U^{-1\dagger})_{ll}\,. \tag{2.15b}$$

The total output current is found by integrating over frequency,

$$I = (U^{\dagger}U)_{ll}(U^{-1}U^{-1\dagger})_{ll}\frac{N_{\mathrm{up}}}{N_{\mathrm{up}} - N_{\mathrm{low}}}\frac{\Gamma^2}{\delta\omega}\,. \tag{2 16}$$

Comparison with the Schawlow–Townes value (1 1) shows that

$$\delta\omega = 2K\frac{N_{\mathrm{up}}}{N_{\mathrm{up}} - N_{\mathrm{low}}}\delta\omega_{\mathrm{ST}}\,, \tag{2 17}$$

where the Petermann factor K is identified as

$$K = (U^{\dagger}U)_{ll}(U^{-1}U^{-1\dagger})_{ll} \geqslant 1 \tag{2 18}$$

For time-reversal symmetry, we can choose $U^{-1} = U^{\mathrm{T}}$, and find $K = [(UU^{\dagger})_{ll}]^2$. The factor of 2 in the relation between $\delta\omega$ and $\delta\omega_{\mathrm{ST}}$ occurs because we have computed the laser line width in the linear regime just below the threshold, instead of far above the threshold The effect of the non-linearities above threshold is to suppress the amplitude fluctuations while leaving the phase fluctuations intact [28], hence the simple factor of two reduction of the line width The factor $N_{\rm up}/(N_{\rm up} - N_{\rm low})$ accounts for the extra noise due to an incomplete population inversion The remaining factor K is due to the non-orthogonality of the cavity modes [3,4], since $K = 1$ if U is unitary

3. Single scattering channel

Relation (2 18) serves as the starting point for a calculation of the statistics of the Petermann factor in an ensemble of chaotic cavities In this section we consider the case $N = 1$ of a single scattering channel, for which the coupling matrix W reduces to a vector $\boldsymbol{\alpha} = (W_{11}, W_{21}, \ldots, W_{M1})$ The magnitude $|\boldsymbol{\alpha}|^2 = (M\Delta/\pi^2)w$, where $w \in [0, 1]$ is related to the transmission probability T of the single scattering channel by $T = 4w(1+w)^{-2}$ [29] We assume a basis in which H is diagonal (eigenvalues ω_q, right eigenvectors $|q\rangle$, left eigenvectors $\langle q|$) In this basis the entries α_q remain real for $\beta = 1$, but become complex numbers for $\beta = 2$ Since the eigenvectors $|q\rangle$ point into random directions, and since the fixed length of $\boldsymbol{\alpha}$ becomes an irrelevant constraint in the limit $M \to \infty$, each real degree of freedom in α_q is an independent Gaussian distributed number [14] The squared modulus $|\alpha_q|^2$ has probability density

$$P(|\alpha_q|^2) = \frac{1}{2\pi|\alpha_q|^2}\left(\frac{2\pi^3|\alpha_q|^2}{w\Delta}\right)^{\beta/2} \exp\left[-\frac{\beta\pi^2}{2w\Delta}|\alpha_q|^2\right] \quad (3\ 1)$$

Eq (3 1) is a χ^2-distribution with β degrees of freedom and mean $\Delta w/\pi^2$

We first determine the distribution of the decay rate Γ of the lasing mode, following Ref [30] Since the lasing mode is the mode closest to the real axis, its decay rate is much smaller than the typical decay rate of a mode, which is $\simeq T\Delta$ Then we calculate the conditional distribution and mean of the Petermann factor for given Γ The unconditional distribution of the Petermann factor is found by folding the conditional distribution with the distribution of Γ, but will not be considered here

3 1 Decay rate of the lasing mode

The amplification with rate $1/\tau_a$ is assumed to be effective over a window $\Omega_a = L\Delta$ containing many modes The lasing mode is the mode within this window that has the smallest decay rate Γ For such small decay rates we can use first-order perturbation theory to obtain the decay rate of mode q,

$$\Gamma_q = 2\pi|\alpha_q|^2 \quad (3\ 2)$$

The χ^2 distribution (3 1) of the squared moduli $|\alpha_q|^2$ translates into a χ^2 distribution of the decay rates

$$P(\Gamma) \propto \Gamma^{(2-\beta)/2} \exp\left(-\frac{\beta\pi\Gamma}{4w\Delta}\right) \quad (3\ 3)$$

Ignoring correlations, we may obtain the decay rate of the lasing mode by considering the L decay rates as independent random variables drawn from the distribution $P(\Gamma)$ The distribution of the smallest among the L decay rates is then given by

$$P_L(\Gamma) = LP(\Gamma)\left[1 - \int_0^{\Gamma} \mathrm{d}\Gamma'\, P(\Gamma')\right]^{L-1} \tag{3 4}$$

For small rates Γ we can insert distribution (3 3) and obtain

$$P_L(\Gamma) \approx \frac{1}{\sqrt{\Gamma}} \exp\left(-\frac{L\pi\Gamma}{4w\Delta}\right)\left[\mathrm{erf}\left(\frac{\pi\Gamma}{4w\Delta}\right)\right]^{L-1}, \quad \beta = 1\,, \tag{3 5a}$$

$$P_L(\Gamma) \approx \exp\left(-\frac{L\pi\Gamma}{2w\Delta}\right), \quad \beta = 2 \tag{3 5b}$$

Here $\mathrm{erf}(x) = 2\pi^{-1/2}\int_0^x \mathrm{d}y \exp(-y^2)$ is the error function The decay rate of the lasing mode decreases with increasing width of the amplification window as $\Gamma \sim w\Delta(\Omega_a/\Delta)^{-2/\beta} \ll w\Delta$

3 2 First-order perturbation theory

If the opening is much smaller than a wavelength, then a perturbation theory in $\boldsymbol{\alpha}$ seems a natural starting point We assign the index l to the lasing mode, and write the perturbed right eigenfunction $|l\rangle' = \sum_q d_q|q\rangle$ and the perturbed left eigenfunction $\langle l|' = \sum_q e_q\langle q|$, in terms of the eigenfunctions of H The coefficients are $d_q = U_{ql}/U_{ll}$ and $e_q = U_{lq}^{-1}/U_{ll}^{-1}$, i e , we do not normalize the perturbed eigenfunctions but rather choose $d_l = e_l = 1$

To leading order the lasing mode remains at $\Omega = \omega_l$ and has width

$$\Gamma = 2\pi|\alpha_l|^2 \tag{3 6}$$

The coefficients of the wave function are

$$d_q = \mathrm{i}\frac{\pi\alpha_q\alpha_l^*}{\omega_q - \omega_l}, \qquad e_q = \mathrm{i}\frac{\pi\alpha_q^*\alpha_l}{\omega_q - \omega_l} \tag{3 7}$$

The Petermann factor of the lasing mode follows from Eq (2 18),

$$\begin{aligned} K &= \frac{(1 + \sum_{q\neq l}|d_q|^2)(1 + \sum_{q\neq l}|e_q|^2)}{|1 + \sum_{q\neq l} d_q e_q|^2} \\ &\approx 1 + \sum_{q\neq l}|d_q - e_q^*|^2, \end{aligned} \tag{3 8}$$

where we linearized with respect to Γ because the lasing mode is close to the real axis From Eq (3 7) one finds

$$K = 1 + (2\pi|\alpha_l|)^2 \sum_{q\neq l} \frac{|\alpha_q|^2}{(\omega_l - \omega_q)^2} \tag{3 9}$$

We seek the distribution $P(K)$ and the average $\langle K\rangle_{\Omega\,\Gamma}$ of K for a given value of Ω and Γ

For $\beta=1$, the probability to find an eigenvalue at ω_q given that there is an eigenvalue at ω_l vanishes *linearly* for small $|\omega_q - \omega_l|$, as a consequence of eigenvalue repulsion constrained by time-reversal symmetry. Since expression (3.9) for K diverges *quadratically* for small $|\omega_q - \omega_l|$, we conclude that $\langle K\rangle_{\Omega,\Gamma}$ does not exist in perturbation theory.[1] This severely complicates the problem.

3.3. Summation of the perturbation series

To obtain a finite answer for the average Petermann factor we need to go beyond perturbation theory. By a complete summation of the perturbation series we will in this section obtain results that are valid for all values $T \leqslant 1$ of the transmission probability. Our starting point are the exact relations

$$d_q z_l = \omega_q d_q - i\pi\alpha_q \sum_p \alpha_p^* d_p , \tag{3.10a}$$

$$e_q z_l = \omega_q e_q - i\pi\alpha_q^* \sum_p \alpha_p e_p , \tag{3.10b}$$

between the complex eigenvalues z_q of $\mathscr{H}$ and the real eigenvalues ω_q of H. Distinguishing between $q = l$ and $q \neq l$, we obtain three recursion relations

$$z_l = \omega_l - i\pi|\alpha_l|^2 - i\pi\alpha_l \sum_{q\neq l} \alpha_q^* d_q , \tag{3.11a}$$

$$i d_q = \frac{\pi\alpha_q}{z_l - \omega_q}\left(\alpha_l^* + \sum_{p\neq l} \alpha_p^* d_p\right) , \tag{3.11b}$$

$$i e_q = \frac{\pi\alpha_q^*}{z_l - \omega_q}\left(\alpha_l + \sum_{p\neq l} \alpha_p e_p\right). \tag{3.11c}$$

We now use the fact that z_l is the eigenvalue closest to the real axis. We may therefore assume that z_l is close to the unperturbed value ω_l and replace the denominator $z_l - \omega_q$ in Eq. (3.11c) by $\omega_l - \omega_q$. That decouples the recursion relations, which may then be solved in closed form:

$$z_l = \omega_l - i\pi|\alpha_l|^2(1 + i\pi A)^{-1} , \tag{3.12a}$$

$$i d_q = \frac{\pi\alpha_q\alpha_l^*}{\omega_l - \omega_q}(1 + i\pi A)^{-1} , \tag{3.12b}$$

$$i e_q = \frac{\pi\alpha_q^*\alpha_l}{\omega_l - \omega_q}(1 + i\pi A)^{-1} . \tag{3.12c}$$

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[1] For broken time-reversal symmetry there is no divergence. We can use the known two-point correlation function $R(\omega_l, \omega_q)$ of the Gaussian unitary ensemble to obtain $\langle K\rangle_{\Omega,\Gamma} = 1 + \frac{1}{3}\pi T\Gamma/\Delta$ for $T \ll 1$.

We have defined

$$A = \sum_{q \neq l} |\alpha_q|^2 (\omega_l - \omega_q)^{-1} \tag{3.13}$$

The decay rate of the lasing mode is

$$\Gamma = -2\,\mathrm{Im}\, z_l = 2\pi |\alpha_l|^2 (1 + \pi^2 A^2)^{-1} \tag{3.14}$$

From Eq. (3.8) we find

$$K = 1 + \frac{2\pi\Gamma}{\Delta} \frac{B}{1 + \pi^2 A^2}, \tag{3.15}$$

with

$$B = \Delta \sum_{q \neq l} |\alpha_q|^2 (\omega_l - \omega_q)^{-2} \tag{3.16}$$

The problem is now reduced to a calculation of the joint probability distribution $P(A,B)$. This problem is closely related to the level curvature problem of random-matrix theory [31–33]. The calculation is presented in Appendix A. The result is

$$P(A,B) = \frac{\pi}{24} \left(\frac{8}{\pi w} \right)^{\beta/2} \frac{(\pi^2 A^2 + w^2)^\beta}{B^{2+3\beta/2}} \exp\left[-\frac{\beta w}{2B} \left(\frac{\pi^2 A^2}{w^2} + 1 \right) \right] \tag{3.17}$$

3.4. Probability distribution of the Petermann factor

From Eqs. (3.1), (3.14), (3.15), and (3.17) we can compute the probability distribution

$$P(K) = \langle Z \rangle^{-1} \left\langle \delta\left(K - 1 - \frac{2\pi\Gamma}{\Delta} \frac{B}{1 + \pi^2 A^2} \right) Z \right\rangle, \tag{3.18a}$$

$$Z = \delta(\Omega - \omega_l)\, \delta\left(\Gamma - \frac{2\pi |\alpha_l|^2}{1 + \pi^2 A^2} \right), \tag{3.18b}$$

of K at fixed Γ and Ω by averaging over $|\alpha_l|^2$, A, and B. In principle one should also require that the decay rates of modes $q \neq l$ are bigger than Γ, but this extra condition becomes irrelevant for $\Gamma \to 0$. The average of Z over $|\alpha_l|^2$ with Eq. (3.1) yields a factor $(1 + \pi^2 A^2)^{\beta/2}$. (Only the behavior of $P(|\alpha_l|^2)$ for small $|\alpha_l|^2$ matters, because we concentrate on the lasing mode.) After integration over B the distribution can be expressed as a ratio of integrals over A,

$$\begin{aligned} P(K) = {} & \frac{(2\pi)^{2\beta}}{3\beta} \frac{\Delta w}{\Gamma} \left(\frac{(K-1)\Delta}{w\Gamma} \right)^{2-3\beta/2} \\ & \times \int_0^\infty \mathrm{d}A\, \frac{(1 + \pi^2 A^2/w^2)^\beta}{(1 + \pi^2 A^2)^{1+\beta}} \exp\left[-\frac{\beta\pi w\Gamma (1 + \pi^2 A^2/w^2)}{(K-1)\Delta(1 + \pi^2 A^2)} \right] \\ & \times \left(\int_0^\infty \mathrm{d}A \frac{(1 + \pi^2 A^2)^{\beta/2}}{(1 + \pi^2 A^2/w^2)^{1+\beta/2}} \right)^{-1} \end{aligned} \tag{3.19}$$

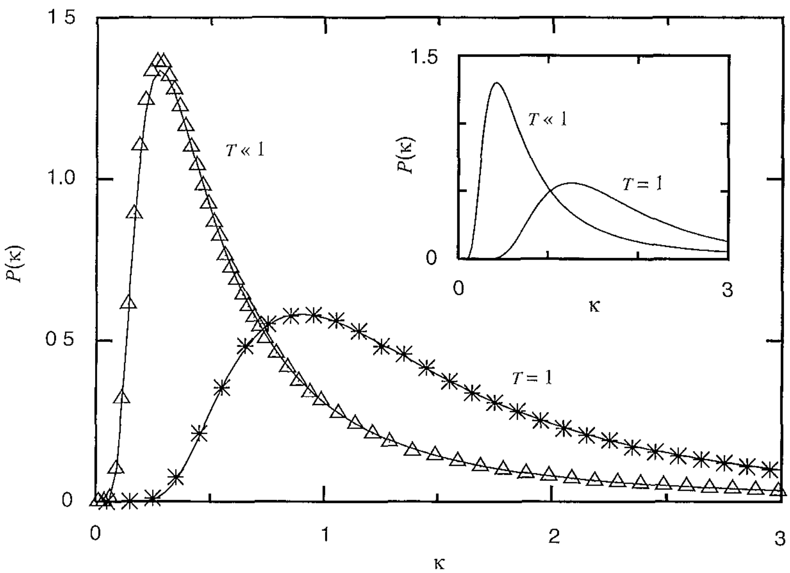

Fig 2 Probability distribution of the rescaled Petermann factor $\kappa = (K-1)\Delta/\Gamma T$ for $T=1$ and $T \ll 1$, in the presence of time-reversal symmetry The solid curves follow from Eqs (3 20) (with $\beta = 1$) and (3 21a) The data points follow from a numerical simulation of the random-matrix model The inset shows the results (3 20) (with $\beta = 2$) and (3 21b) for broken time-reversal symmetry

We introduce the rescaled Petermann factor $\kappa = (K-1)\Delta/\Gamma T$. A simple result for $P(\kappa)$ follows for $T = 1$,

$$P(\kappa) = \frac{4\beta\pi^{2\beta}}{3\kappa^{2+3\beta/2}} \exp\left[-\frac{\beta\pi}{\kappa}\right] , \tag{3.20}$$

and for $T \ll 1$,

$$P(\kappa) = \frac{\pi}{12\kappa^2}\left(1 + \frac{\pi}{2\kappa}\right)\exp\left[-\frac{\pi}{4\kappa}\right], \quad \beta = 1 , \tag{3.21a}$$

$$P(\kappa) = \frac{\pi}{8\sqrt{2\kappa^5}}\left(1 + \frac{2\pi}{3\kappa} + \frac{\pi^2}{3\kappa^2}\right)\exp\left[-\frac{\pi}{2\kappa}\right], \quad \beta = 2 . \tag{3.21b}$$

As shown in Fig. 2, the distributions are very broad and asymmetric, with a long tail towards large κ.

To check our analytical results we have also done a numerical simulation of the random-matrix model, generating a large number of random matrices H and computing K from Eq. (2.18). As one can see from Fig. 2, the agreement with the theoretical predictions is flawless.

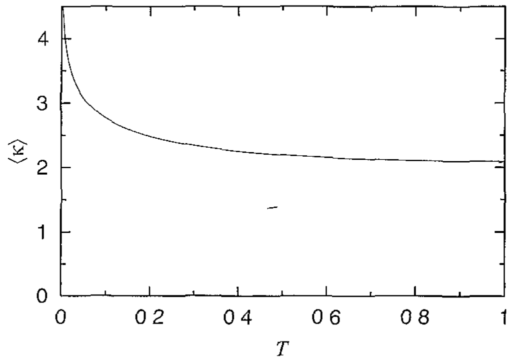

Fig. 3. Average of the rescaled Petermann factor κ as a function of transmission probability T. The solid curve is the result (3.22) in the presence of time reversal symmetry, the dashed curve is the result (3.24) for broken time reversal symmetry. For small T the solid curve diverges $\propto \ln T^{-1}$ while the dashed curve has the finite limit of $\pi/3$. For $T=1$ both curves reach the value $2\pi/3$.

3.5. Mean Petermann factor

The distribution (3.19) gives for preserved time-reversal symmetry ($\beta=1$) the mean Petermann factor

$$\langle K\rangle_{\Omega,\Gamma} = 1 - \frac{\Gamma}{\Delta}\frac{2\pi}{3}\frac{G^{22}_{22}\left(w^2\,\middle|\,\begin{matrix}0 & 0\\ -\frac{1}{2} & -\frac{1}{2}\end{matrix}\right)}{G^{22}_{22}\left(w^2\,\middle|\,\begin{matrix}-\frac{1}{2} & \frac{1}{2}\\ -1 & 0\end{matrix}\right)}, \tag{3.22}$$

in terms of the ratio of two Meijer G-functions. We have plotted the result in Fig. 3, as a function of $T=4w(1+w)^{-2}$.

It is remarkable that the average K depends *non-analytically* on T, and hence on the area of the opening. (The transmission probability T is related to the area $\mathscr{A}$ of the opening by $T \simeq \mathscr{A}^3/\lambda^6$ for $T \ll 1$ [34].) For $T \ll 1$, the average approaches the form

$$\langle K\rangle_{\Omega,\Gamma} = 1 + \frac{\pi}{6}\frac{T\Gamma}{\Delta}\ln\frac{16}{T}. \tag{3.23}$$

The most probable (or modal) value of $K-1 \simeq T\Gamma/\Delta$ is parametrically smaller than the mean value (3.23) for $T \ll 1$. The non-analyticity results from the relatively weak eigenvalue repulsion in the presence of time-reversal symmetry. If time-reversal symmetry is broken, then the stronger quadratic repulsion is sufficient to overcome the ω^{-2} divergence of perturbation theory (3.9) and the average K becomes an analytic function of T. For this case, we find from Eq. (3.19) the mean Petermann factor

$$\langle K\rangle_{\Omega,\Gamma} = 1 + \frac{\Gamma}{\Delta}\frac{4\pi w}{3(1+w^2)}, \tag{3.24}$$

shown dashed in Fig. 3.

4. Many scattering channels

For arbitrary number of scattering channels N the coupling matrix W is an $M \times N$ rectangular matrix. The square matrix $\pi W^\dagger W$ has N eigenvalues $(M\Delta/\pi)w_n$. The transmission coefficients of the eigenchannels are

$$T_n = \frac{4w_n}{(1+w_n)^2}. \tag{4.1}$$

A single hole of area $\mathscr{A} \gg \lambda^2$ (at wavelength λ) corresponds to $N \simeq 2\pi\mathscr{A}/\lambda^2$ fully transmitted scattering channels, with all $T_n = w_n = 1$ the same.

As in the single-channel case, we first determine the distribution of the decay rate Γ of the lasing mode. This decay rate is smaller than the typical decay rate $\Gamma_0 = TN\Delta/2\pi$ of the non-lasing modes. Then we calculate the mean Petermann factor $\langle K \rangle$ for given Γ and investigate its behavior for the atypically small decay rates of the lasing mode.

4.1. Decay rate of the lasing mode

The distribution of decay rates $P(\Gamma)$ has been calculated by Fyodorov and Sommers. For broken time-reversal symmetry the result is [17,18]

$$P(\Gamma) = \frac{\pi}{\Delta}\mathscr{F}_1\left(\frac{\pi}{\Delta}\Gamma\right)\mathscr{F}_2\left(\frac{\pi}{\Delta}\Gamma\right), \tag{4.2a}$$

$$\mathscr{F}_1(y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}x\, \mathrm{e}^{-\mathrm{i}xy}\prod_{n=1}^{N}\frac{1}{g_n - \mathrm{i}x}, \tag{4.2b}$$

$$\mathscr{F}_2(y) = \frac{1}{2}\int_{-1}^{1} \mathrm{d}x\, \mathrm{e}^{-xy}\prod_{n=1}^{N}(g_n + x), \tag{4.2c}$$

where $g_n = -1 + 2/T_n$. For identical $g_n \equiv g$ the two functions $\mathscr{F}_1$ and $\mathscr{F}_2$ simplify to

$$\mathscr{F}_1(y) = \frac{1}{(N-1)!}y^{N-1}\mathrm{e}^{-gy}, \tag{4.3a}$$

$$\mathscr{F}_2(y) = \sum_{n=0}^{N}(-1)^n\binom{N}{n}g^{N-n}\frac{\mathrm{d}^n}{\mathrm{d}y^n}\left(\frac{\sinh y}{y}\right), \tag{4.3b}$$

and a convenient form of the distribution function is

$$P(\Gamma) = \frac{\Delta}{2\pi\Gamma^2(N-1)!}\int_{N(1-T)\Gamma/\Gamma_0}^{N\Gamma/\Gamma_0} \mathrm{d}x\, x^N \mathrm{e}^{-x}. \tag{4.4}$$

The behavior of $P(\Gamma)$ for various numbers N of fully transmitted ($T = 1$) scattering channels is illustrated in Fig. 4.

The result for preserved time-reversal symmetry is a bit more involved [19]. Fortunately, we can draw all important conclusions from the results for broken time-reversal symmetry, on which we will concentrate here.

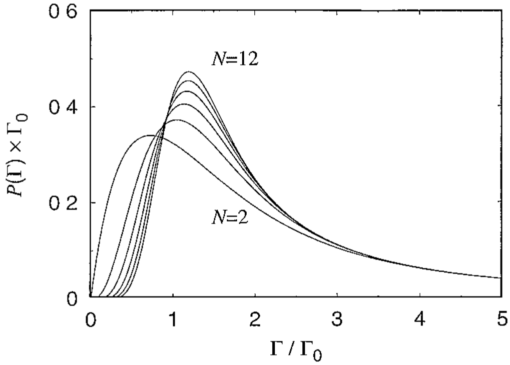

Fig 4 Decay rate distribution $P(\Gamma)$ of a chaotic cavity with an opening that supports $N=2$ 4 6 8 10 12 fully transmitted scattering channels Computed from Eq (4 2) for the case of broken time reversal symmetry

For large N, the distribution $P(\Gamma)$ becomes non-zero only in the interval $\Gamma_0 < \Gamma < \Gamma_0/(1-T)$, where it is equal to [35,36]

$$P(\Gamma)=\frac{\Gamma_0}{T\Gamma^2}, \quad \Gamma_0 < \Gamma < \Gamma_0/(1-T) \tag{4 5}$$

This limit is β-independent The smallest decay rate Γ_0 corresponds to the inverse mean dwell time in the cavity

We are interested in the "good cavity" regime, where the typical decay rate Γ_0 is small compared to the amplification bandwidth Ω_a From $\Gamma_0 = TN\Delta/2\pi$ it follows that the number $L \simeq \Omega_a/\Delta$ of amplified modes is then much larger than TN In this regime the decay rate of the lasing mode (the smallest among the L decay rates in the frequency window Ω_a) drops below Γ_0 The asymptotic result (4 5) cannot be used in this case, since it does not describe accurately the tail $\Gamma \lesssim \Gamma_0$ Going back to the exact result (4 2) we find for the tail of the distribution the expression

$$P(\Gamma)=\frac{\pi}{NT^2\Delta}[1+\operatorname{erf}(u)]+\mathcal{O}(N^{-3/2}), \tag{4 6}$$

where we have defined $u=\sqrt{N/2}(\Gamma/\Gamma_0-1)$ The distribution $P_L(\Gamma)$ of the lasing mode follows from $P(\Gamma)$ by means of Eq (3 4) We find that it has a pronounced maximum at a value $u_{\max}$ determined by

$$\frac{\exp(-u_{\max}^2)}{[1+\operatorname{erf}(u_{\max})]^2}=\frac{L-1}{\sqrt{2N}}\frac{\sqrt{\pi}(g+1)}{4} \tag{4 7}$$

For $L \gg \sqrt{N}$ (and hence also in the good cavity regime) we find $u_{\max} \sim -\sqrt{\ln L} < 0$, and the deviation of Γ from Γ_0 is of order $\Delta\sqrt{N} \ll \Gamma_0$ (as long as $L \ll e^N$)

4 2 Mean Petermann factor

Eigenfunction correlations of non-Hermitian operators have been studied in Refs [20 22] The eigenfunction autocorrelator considered in these studies is directly connected to the Petermann factor K Ref [20] provides a convenient expression of the mean Petermann factor,

$$M\pi\langle K\rangle_{\Omega,\Gamma}\rho(\omega)=\lim_{\varepsilon\to 0^+}\left\langle\left(\mathrm{tr}\frac{\varepsilon}{(\omega-\mathscr{H})(\omega^*-\mathscr{H}^\dagger)+\varepsilon^2}\right)^2\right\rangle \tag{4.8}$$

In Ref. [20] this average has been calculated perturbatively for $N\gg 1$, with the result

$$\langle K\rangle_{\Omega,\Gamma}\approx -N\left(\frac{\Gamma}{\Gamma_0}-1\right)\left(\frac{(1-T)\Gamma}{\Gamma_0}-1\right) \tag{4.9}$$

for $\Gamma_0<\Gamma<\Gamma_0/(1-T)$. This result is at the same level of approximation as Eq. (4.5) for the distribution of the decay rates, i.e., it does not describe the range $\Gamma\lesssim\Gamma_0$ of atypically small decay rates. Since that is precisely the range that we need for the Petermann factor, we cannot use the existing perturbative results. We have calculated the mean Petermann factor non-perturbatively for any Γ and N, assuming broken time-reversal symmetry. The derivation is given in Appendix B. The final result for the mean Petermann factor is

$$\langle K\rangle_{\Omega,\Gamma}=1+\frac{2S(\pi\Gamma/\Delta)}{\mathscr{F}_1(\pi\Gamma/\Delta)\mathscr{F}_2(\pi\Gamma/\Delta)}, \tag{4.10a}$$

$$S(y)=-\int_0^y \mathrm{d}y'\,\mathscr{F}_1(y')\frac{\partial}{\partial y'}\mathscr{F}_2(y'), \tag{4.10b}$$

with $\mathscr{F}_1$ and $\mathscr{F}_2$ given in Eq. (4.2). For identical $g_n\equiv g$ we can use Eq. (4.3) and obtain by successive integrations by parts

$$S(y)=\sum_{n=0}^{N-1}\frac{(-1)^n}{n!}y^n\frac{\mathrm{d}^n}{\mathrm{d}y^n}\left\{\mathrm{e}^{-gy}\frac{\mathrm{d}}{\mathrm{d}y}\left(\frac{\sinh y}{y}\right)\right\} \tag{4.11}$$

For $N=1$ and $\Gamma\ll\Delta$ we recover the single-channel result (3.24) of the previous section. In what follows we will continue to assume for simplicity that all g_n's are equal to a common value g.

The large-N behavior can be conveniently studied from the expression

$$S(y)=-\frac{1}{4y^2(N-1)!}\int_{y(g-1)}^{y(g+1)}\mathrm{d}x\,x^{N-1}\mathrm{e}^{-x}[x-(g-1)y][x-(g+1)y], \tag{4.12}$$

because the integral permits a saddle-point approximation. For $\Gamma>\Gamma_0$ we recover Eq. (4.9), but now we can also study the precise behavior of the mean Petermann factor for $\Gamma\lesssim\Gamma_0$, hence also for decay rates relevant for the lasing mode. The results will again be presented in terms of the rescaled parameter $u=\sqrt{N/2}(\Gamma/\Gamma_0-1)$. We expand the integrands in Eqs. (4.4) and (4.12) around the saddle point at $x=N$ (which coincides with the upper integration limit at $\Gamma=\Gamma_0$) and keep the first non-Gaussian correction. This yields

$$\begin{aligned}\langle K\rangle_{\Omega,\Gamma}&=T\sqrt{2N}[F(u)+u]-T(g-1)u^2\\&+TF(u)\left[(3-g)u+\tfrac{4}{3}u^3+\tfrac{4}{3}(1+u^2)F(u)\right]\\&+\mathcal{O}(N^{-1/2}),\end{aligned} \tag{4.13a}$$

$$F(u) = \frac{\exp(-u^2)}{\sqrt{\pi}[1+\mathrm{erf}(u)]} \tag{4 13b}$$

For $\Gamma = \Gamma_0$ ($u = 0$) this simplifies to

$$\langle K \rangle_{\Omega\, \Gamma\ \Gamma_0} = T\left(\sqrt{\frac{2N}{\pi}} + \frac{4}{3\pi}\right) \tag{4 14}$$

We see that the mean Petermann factor varies on the same scale of Γ as the decay-rate distribution $P(\Gamma)$, Eq (4 6) However, while $P(\Gamma)$ decays exponentially for $u \ll -1$, the mean Petermann factor displays an algebraic tail

$$\langle K \rangle_{\Omega\, \Gamma} = -\frac{T\sqrt{N}}{u\sqrt{2}} + 1 - T + \mathcal{O}(u^{-2}) \tag{4 15}$$

For an amplification window $\Omega_a = L\Delta$ with $L \gg \sqrt{N}$ we found in Section 4 1 that the decay rate Γ of the lasing mode drops below Γ_0 (the rescaled parameter $u_{\max} \sim -\sqrt{\ln L}$) Still, the mean Petermann factor

$$\langle K \rangle_{\Omega\, \Gamma} \sim \sqrt{\frac{N}{\ln L}} \tag{4 16}$$

remains parametrically larger than unity (as long as $L \ll \sqrt{N}\mathrm{e}^N$)

We now compare our analytical findings with the results of numerical simulations We generated a large number of random matrices $\mathcal{H}$ with dimension $M=120$ ($M=200$) for $N = 2, 4, 6, 8$ ($N = 10, 12$) fully transmitted scattering channels ($g = T = 1$) Fig 5 shows the mean K at given Γ We find excellent agreement with our analytical result (4 10)

The behavior $\langle K \rangle \sim \sqrt{N}$ at $\Gamma = \Gamma_0$ is shown in Fig 6 The inset depicts the distribution of K at $\Gamma = \Gamma_0$ for $N = 10$, which only can be accessed numerically We see that the mean Petermann factor is somewhat larger than the most probable (or modal) value

4 3 Preserved time-reversal symmetry

In the derivation of the mean Petermann factor for broken time-reversal symmetry Appendix B) it turned out that the final result is formally connected to the expression for the decay-rate distribution $P(\Gamma)$, in as much as both expressions are built from the factors $\mathcal{T}_1$ (involving non-compact bosonic degrees of freedom of the saddle-point manifold) and $\mathcal{T}_2$ (involving compact bosonic degrees of freedom of that manifold) We tried to translate this description to the case of preserved time-reversal symmetry ($\beta = 1$), by operating in the same way on the compact and non-compact factors of the expression of Ref [19], but could obtain a satisfactory result only for $N = 2$,

$$\langle K \rangle = \frac{1}{2\Gamma_0} \frac{\Gamma(\Gamma - \Gamma_0)\exp(\Gamma/\Gamma_0) + \Gamma_0^2 \sinh(\Gamma/\Gamma_0)}{\Gamma\cosh(\Gamma/\Gamma_0) - \Gamma_0 \sinh(\Gamma/\Gamma_0)} \tag{4 17}$$

In Fig 7 this expression is compared to the result of a numerical simulation

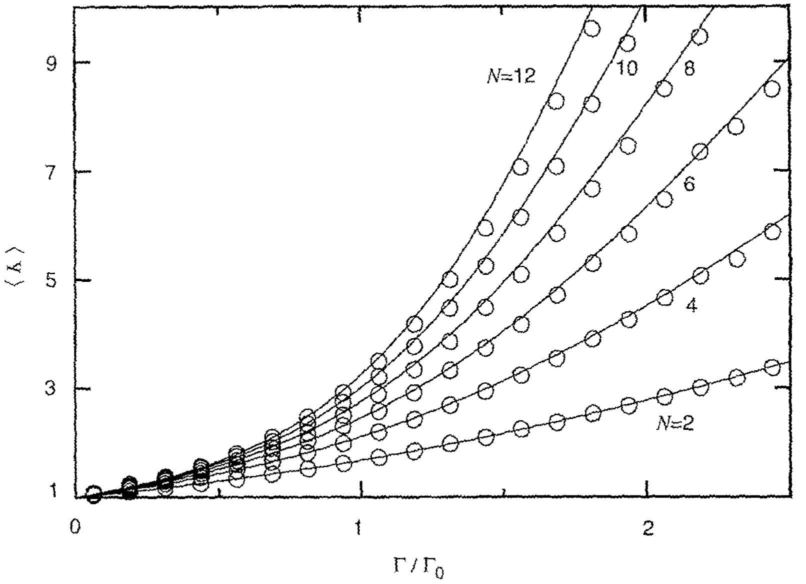

Fig. 5. Average Petermann factor $\langle K\rangle$ as a function of the decay rate Γ for different values N of fully transmitted scattering channels. The solid curves are the analytical result (4.10), the data points are obtained by a numerical simulation. Time-reversal symmetry is broken.

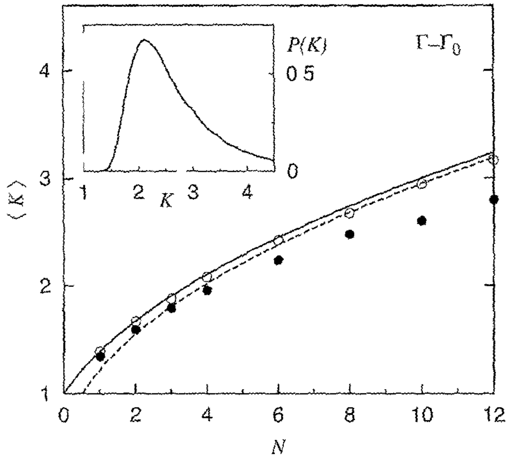

Fig. 6. Average of the Petermann factor K at $\Gamma=\Gamma_0$ as function of the number N of fully transmitted scattering channels. The analytical result (4.10) for broken time reversal symmetry (full curve) is compared with the result of a numerical simulation (open circles for broken time reversal symmetry, filled circles for preserved time reversal symmetry). The dashed line is the large N result (4.14). The inset shows the distribution of K at $\Gamma=\Gamma_0$ for $N=10$.

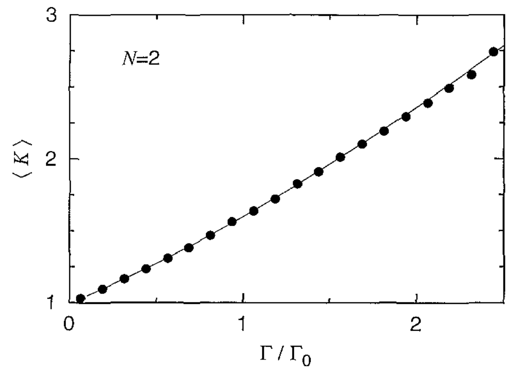

Fig 7 Theoretical expectation (4 17) (full curve) and the result of a numerical simulation (data points) for the average Petermann factor K in the presence of time-reversal symmetry, as a function of the decay rate Γ for 2 fully transmitted scattering channels.

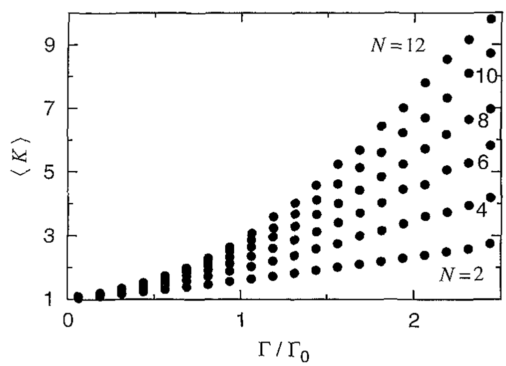

Fig 8 Results of a numerical simulation of the average Petermann factor $\langle K \rangle$ in the presence of time-reversal symmetry, as a function of the decay rate Γ for N fully transmitted scattering channels

For larger numbers of channels we can draw our conclusions from the numerical results that are presented in Fig. 8. Interestingly enough the data points for N channels are close to the results for broken time-reversal symmetry with $N/2$ channels, when the decay rate is given in units of Γ_0. This is illustrated for $N = 8$ in Fig. 9. Such a rule of thumb (motivated by the number of real degrees of freedom that enter the non-Hermitian part of $\mathcal{H}$) was already known for the decay rate distribution (inset in Fig. 9). Hence the Petermann factor for the lasing mode should again display a sub-linear growth with increasing channel number N. This expectation is indeed confirmed by the numerical simulations, see the filled circles in Fig. 6.

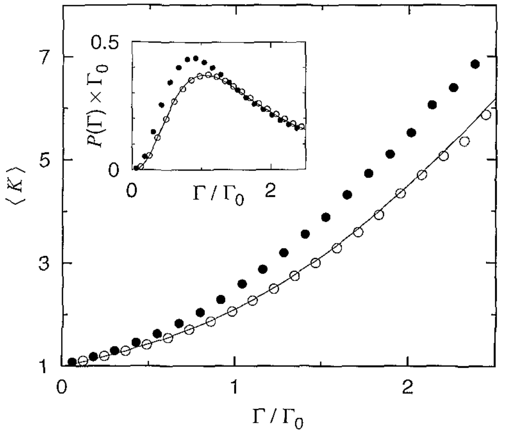

Fig 9 Average Petermann factor $\langle K\rangle$ for $N=4$, $\beta=2$ [open circles result of a numerical simulation, curve Eq (4 10)] and for $N=8$, $\beta=1$ (filled circles· result of a numerical simulation) The parameter Γ_0 equals $N\Delta/2\pi$ in both cases, so it is twice as large for $\beta=2$ as for $\beta=1$ The inset depicts the probability distribution of Γ

5. Discussion

The Petermann factor K enters the fundamental lower limit of the laser line width due to vacuum fluctuations and is a measure of the non-orthogonality of cavity modes. We related the Petermann factor to the residue of the scattering-matrix pole that pertains to the lasing mode and computed statistical properties of K in an ensemble of chaotic cavities. The technical complications that had to be overcome arise from the fact that laser action selects a mode which has a small decay rate Γ, and hence belongs to a pole that lies anomalously close to the real axis. Parametrically large Petermann factors $\propto \sqrt{N}$ arise when the number N of scattering channels is large. For a single scattering channel the mean Petermann factor depends non-analytically on the transmission probability T.

The quantity K is also of fundamental significance in the general theory of scattering resonances, where it enters the width-to-height relation of resonance peaks and determines the scattering strength of a quasi-bound state with given decay rate Γ. If we write the scattering matrix (2.6) in the form

$$S_{nm} = \delta_{nm} + \sigma_n\sigma'_m(\omega - \Omega + \mathrm{i}\Gamma/2)^{-1}\,, \tag{5.1}$$

then the scattering strengths σ_n, σ'_m are related to Γ by a sum rule. For resonances close to the real axis ($\Gamma \ll \Delta$) the relation is

$$\sum_{n,m} |\sigma_n\sigma'_m|^2 = \Gamma^2\,. \tag{5.2}$$

For poles deeper in the complex plane, however, the sum rule has to be replaced by

$$\sum_{n,m} |\sigma_n\sigma'_m|^2 = K\Gamma^2, \quad K \geqslant 1\,. \tag{5.3}$$

The method of filter diagonalization (or harmonic inversion) that was used in Ref [12] to obtain for the H_3^+ molecular ion the location of poles even deep in the complex plane can also be employed to determine the corresponding residues, and hence K

The parameter K defined in Eq (2 18) appears as a measure of mode non-orthogonality also in problems outside of scattering theory These problems involve non-Hermitian operators that are not of the form (2 3) [21,22] Many applications share the common feature that they can be addressed statistically by an ensemble description, and that the physically relevant modes lie at the boundary of the complex eigenvalue spectrum The non-perturbative statistical methods reported in this paper should prove useful in the investigation of some of these problems as well

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Appendix A. Joint distribution of A and B

We calculate the joint distribution $P(A,B)$ [Eq (3 17)] of the quantities A [Eq (3 13)] and B [Eq (3 16)] by generalizing the theory of Ref [33] We give the lasing mode ω_l the new index M and assume that it lies at the center of the semicircle (2 2), $\omega_M = 0$ Other choices just renormalize the mean modal spacing Δ, which we can set to $\Delta = 1$ The quantities A and B are then of the form

$$A = \sum_{m=1}^{M-1} \frac{|\alpha_m|^2}{\omega_m}, \qquad B = \sum_{m=1}^{M-1} \frac{|\alpha_m|^2}{\omega_m^2} \tag{A 1}$$

The joint probability distribution of A and B,

$$P(A,B) = \left\langle \delta\left(A - \sum_{m=1}^{M-1} \frac{|\alpha_m|^2}{\omega_m}\right) \delta\left(B - \sum_{m=1}^{M-1} \frac{|\alpha_m|^2}{\omega_m^2}\right) \right\rangle , \tag{A 2}$$

is obtained by averaging over the variables $\{|\alpha_m|^2, \omega_m\}$ The quantities $|\alpha_m|^2$ are independent numbers with probability distribution (3 1) The joint probability distribution of the eigenfrequencies $\{\omega_m\}$ of the closed cavity is the eigenvalue distribution of the Gaussian ensembles (2 1) of random-matrix theory,

$$P(\{\omega_m\}) \propto \prod_{i<j} |\omega_i - \omega_j|^\beta \exp\left[-\frac{\beta M}{4\mu^2} \sum_k \omega_k^2\right] \tag{A 3}$$

Our choice $\Delta = 1$ translates into $\mu = M/\pi$

The joint probability distribution of the eigenvalues $\{\omega_m\}$ ($m=1,\ldots,M-1$) is found by setting $\omega_M=0$ in Eq. (A.3). It factorizes into the eigenvalue distribution of $M-1$ dimensional Gaussian matrices H' [again distributed according to Eq. (2.1)], and the term $\prod_{j=1}^{M-1}|\omega_j|^\beta=|\det H'|^\beta$.

In the first step of our calculation, we use the Fourier representation of the δ-functions in Eq. (A.2) and write

$$P(A,B)\propto\left\langle\int_{-\infty}^{\infty}\mathrm{d}x\int_{-\infty}^{\infty}\mathrm{d}y\,\mathrm{e}^{\mathrm{i}xA+\mathrm{i}yB}\prod_{m=1}^{M-1}\int_0^\infty\mathrm{d}|\alpha_m|^2P(|\alpha_m|^2)\right.$$
$$\left.\times\exp\left[-\mathrm{i}x\sum_{m=1}^{M-1}\frac{|\alpha_m|^2}{\omega_m}-\mathrm{i}y\sum_{m=1}^{M-1}\frac{|\alpha_m|^2}{\omega_m^2}\right]\right\rangle, \tag{A.4}$$

where the average refers to the variables $\{\omega_m\}$. The integrals over $|\alpha_m|^2$ can be performed, resulting in

$$P(A,B)\propto\int\mathrm{d}x\int\mathrm{d}y\,\mathrm{e}^{\mathrm{i}xA+\mathrm{i}yB}\left\langle\frac{\det H'^{2\beta}}{\det[H'^2+2\mathrm{i}w(xH'+y)/\pi^2\beta]^{\beta/2}}\right\rangle, \tag{A.5}$$

where the average is now over the Gaussian ensemble of H'-matrices. It is our goal to relate this average to autocorrelators of the secular polynomial of Gaussian distributed random matrices, given in Refs. [37,38].

The determinant in the denominator can be expressed as a Gaussian integral,

$$P(A,B)\propto\int\mathrm{d}x\int\mathrm{d}y\,\mathrm{e}^{\mathrm{i}xA+\mathrm{i}yB}\int\mathrm{d}\mathbf{z}\int\mathrm{d}H'\det H'^{2\beta}$$
$$\times\exp\left[-\frac{\beta\pi^2}{4M}\mathrm{tr}\,H'^2-\mathbf{z}^\dagger\left(H'^2+\frac{2\mathrm{i}w}{\beta\pi^2}(xH'+y)\right)\mathbf{z}\right], \tag{A.6}$$

where the $M-1$ dimensional vector $\mathbf{z}$ is real (complex) for $\beta=1$ (2). Since our original expression did only depend on the eigenvalues of H', the formulation above is invariant under orthogonal (unitary) transformations of H', and we can choose a basis in which $\mathbf{z}$ points into the direction of the last basis vector (index $M-1$). Let us denote the Hamiltonian in the block form

$$H'=\begin{pmatrix}V & \mathbf{h}\\ \mathbf{h}^\dagger & g\end{pmatrix}. \tag{A.7}$$

Here V is a $(M-2)\times(M-2)$ matrix, g a number, and $\mathbf{h}$ a $(M-2)$ dimensional vector. In this notation,

$$P(A,B)\propto\int\mathrm{d}x\int\mathrm{d}y\,\mathrm{e}^{\mathrm{i}xA+\mathrm{i}yB}\int\mathrm{d}\mathbf{z}\int\mathrm{d}g\int\mathrm{d}V\int\mathrm{d}\mathbf{h}$$
$$\times\det[V^{2\beta}(g-\mathbf{h}^\dagger V^{-1}\mathbf{h})^{2\beta}]$$
$$\times\exp\left[-\frac{\beta\pi^2}{4M}(g^2+2|\mathbf{h}|^2+\mathrm{tr}\,V^2)\right]$$
$$\times\exp\left[-|\mathbf{z}|^2\left(g^2+|\mathbf{h}|^2+\frac{2\mathrm{i}w}{\beta\pi^2}(xg+y)\right)\right]. \tag{A.8}$$

The integrals over x and y give δ-functions,

$$P(A,B) \propto \int \mathrm{d}\mathbf{z} \int \mathrm{d}g \int \mathrm{d}V \int \mathrm{d}\mathbf{h} \det[V^{2\beta}(g - \mathbf{h}^\dagger V^{-1}\mathbf{h})^{2\beta}]$$
$$\times \exp\left[-\frac{\beta\pi^2}{4M}(g^2 + 2|\mathbf{h}|^2 + \operatorname{tr} V^2) - |\mathbf{z}|^2(g^2 + |\mathbf{h}|^2)\right]$$
$$\times \delta(A - gB)\delta(B - 2w|\mathbf{z}|^2/\beta\pi^2) \tag{A.9}$$

We then integrate over g and $\mathbf{z}$,

$$P(A,B) \propto \int \mathrm{d}V\, \mathrm{d}\boldsymbol{h} \det\left[V^{2\beta}\left(\frac{A}{B} - \boldsymbol{h}^\dagger V^{-1}\boldsymbol{h}\right)^{2\beta}\right] B^{(\beta/2)(M-1)-2}$$
$$\times \exp\left[-\frac{\beta\pi^2}{4M}(2|\boldsymbol{h}|^2 + \operatorname{tr} V^2) - \frac{\beta\pi^2 B}{2w}\left(\frac{A^2}{B^2} + |\boldsymbol{h}|^2\right)\right] . \tag{A.10}$$

We already anticipated $B \gg 1/M$ and omitted in the exponent a term $-\beta\pi^2 A^2/4MB^2$

The integral over $\boldsymbol{h}$ can be interpreted as an average over Gaussian random variables with variance

$$h^2 \equiv \langle |h_i|^2 \rangle = \frac{1}{\pi^2}\frac{1}{B/w + 1/M} \approx \frac{w}{\pi^2 B}\left(1 - \frac{w}{MB}\right) . \tag{A 11}$$

For the stochastic interpretation one also has to supply the normalization constants proportional to

$$h^{\beta(M-2)} = \left(\frac{w}{\pi^2 B}\right)^{\beta(M-2)/2} \exp\left[-\frac{\beta w}{2B}\right] . \tag{A 12}$$

The integral over V is another Gaussian average, and thus

$$P(A,B) \propto Q_\beta B^{(\beta/2)-2} \exp\left[-\frac{\beta w}{2B}\left(1 + \frac{\pi^2 A^2}{w^2}\right)\right] , \tag{A 13a}$$

$$Q_\beta = \left\langle \det\left[V^{2\beta}\left(\frac{A}{B} - \boldsymbol{h}^\dagger V^{-1}\boldsymbol{h}\right)^{2\beta}\right]\right\rangle . \tag{A 13b}$$

After averaging over $\boldsymbol{h}$, one has now to consider for $\beta = 1$

$$Q_1 = \left\langle \det\left[V^2 \frac{A^2}{B^2} + h^4 V^2[(\operatorname{tr} V^{-1})^2 + 2\operatorname{tr} V^{-2}]\right]\right\rangle , \tag{A.14}$$

where only the even terms in V have been kept. The ratio of coefficients in this polynomial in A/B can be calculated from the autocorrelator [38]

$$G_1(\omega,\omega') = \frac{\langle \det(V+\omega)(V+\omega')\rangle}{\langle \det V^2 \rangle}$$
$$= -\frac{3}{\pi^2 x}\frac{\mathrm{d}}{\mathrm{d}x}\frac{\sin \pi x}{\pi x}\bigg|_{x=\omega-\omega'} \tag{A.15}$$

of the secular polynomial of Gaussian distributed real matrices V. This is achieved by expressing the products of traces and determinants through secular coefficients, and these then as derivatives of the secular determinant,

$$\frac{\langle \det V^2 (\operatorname{tr} V^{-1})^2\rangle}{\langle \det V^2\rangle} = \frac{\partial^2}{\partial\omega\partial\omega'} G_1(\omega,\omega')\Big|_{\omega=\omega'=0} = -\frac{\partial^2}{\partial\omega^2} G_1(\omega,0)\Big|_{\omega=0} = \frac{\pi^2}{5}\,, \tag{A.16a}$$

$$\frac{2\langle \det V^2 (\operatorname{tr} V^{-2})\rangle}{\langle \det V^2\rangle} = -\,4\frac{\partial^2}{\partial\omega^2} G_1(\omega,0)\Big|_{\omega=0}\,. \tag{A.16b}$$

[We used the translational invariance of $G(\omega,\omega')$.] Eqs. (A.11) and (A.15) yield

$$Q_1 \propto \frac{A^2}{B^2} + \frac{w^2}{\pi^2 B^2}\,. \tag{A.17}$$

For $\beta = 2$, the average over $\mathbf{h}$ yields the expression

$$Q_2 \propto \frac{A^4}{B^4} + q_1 h^4 \frac{A^2}{B^2} + q_2 h^8\,, \tag{A.18a}$$

$$q_1 = 6\langle \det V^4 [(\operatorname{tr} V^{-1})^2 + \operatorname{tr} V^{-2}]\rangle\,, \tag{A.18b}$$

$$q_2 = \langle \det V^4 [(\operatorname{tr} V)^{-4} + 6 \operatorname{tr} V^{-2} (\operatorname{tr} V^{-1})^2 + 8 \operatorname{tr} V^{-1} \operatorname{tr} V^{-3} + 6 \operatorname{tr} V^{-4} + 3 (\operatorname{tr} V^{-2})^2]\rangle\,. \tag{A.18c}$$

The coefficients can now be computed from the four-point correlator of the Gaussian unitary ensemble [37]:

$$\begin{aligned} G_2(\omega_1,\omega_2,\omega_3,\omega_4) &= \frac{\langle \det(V+\omega_1)(V+\omega_2)(V+\omega_3)(V+\omega_4)\rangle}{\langle \det V^4\rangle} \\ &= \frac{3}{2\pi^4}\Bigg[\frac{\cos\pi(\omega_1+\omega_2-\omega_3-\omega_4)}{(\omega_1-\omega_3)(\omega_1-\omega_4)(\omega_2-\omega_3)(\omega_2-\omega_4)} \\ &\quad + \frac{\cos\pi(\omega_1+\omega_3-\omega_2-\omega_4)}{(\omega_1-\omega_2)(\omega_1-\omega_4)(\omega_3-\omega_2)(\omega_3-\omega_4)} \\ &\quad + \frac{\cos\pi(\omega_1+\omega_4-\omega_3-\omega_2)}{(\omega_1-\omega_3)(\omega_1-\omega_2)(\omega_4-\omega_3)(\omega_4-\omega_2)}\Bigg]\,, \end{aligned} \tag{A.19a}$$

$$G_2(\omega,0,0,0) = \frac{3}{\pi^3\omega^3}(\sin\pi\omega - \pi\omega\cos\pi\omega)\,, \tag{A.19b}$$

$$G_2(\omega,\omega,0,0) = \frac{3}{2\pi^4\omega^4}(\cos 2\pi\omega - 1 + 2\pi^2\omega^2)\,. \tag{A.19c}$$

In this case

$$q_1 = \frac{\partial^2}{\partial\omega^2}[6G_2(\omega,\omega,0,0) - 18G_2(\omega,0,0,0)]\bigg|_{\omega=0}$$
$$= 2\pi^2 , \tag{A.20a}$$

$$q_2 = \frac{\partial^4}{\partial\omega^4}[10G_2(\omega,\omega,0,0) - 15G_2(\omega,0,0,0)]\bigg|_{\omega=0}$$
$$= \pi^2 , \tag{A.20b}$$

which gives

$$Q_2 \propto Q_1^2 . \tag{A.21}$$

Collecting results we obtain Eq. (3.17), where we also included the normalization constant.

Appendix B. Derivation of Eq. (4.10) for the mean Petermann factor

The computation of the mean Petermann factor from expression (4.8) is facilitated by the fact that it can be obtained from the same generating function [18,39],

$$\Psi(\omega_1,\omega_2,u_1,u_2,\varepsilon) = \left\langle \frac{\det[(\omega-\mathscr{H})(\omega^*-\mathscr{H}^\dagger) - (u_1-\mathrm{i}\varepsilon)(u_2-\mathrm{i}\varepsilon)]}{\det[(\omega-\mathscr{H})(\omega^*-\mathscr{H}^\dagger) - (u_1+\mathrm{i}\varepsilon)(u_2+\mathrm{i}\varepsilon)]} \right\rangle , \tag{B.1}$$

as the distribution function

$$\rho(\omega) = \lim_{\varepsilon\to 0^+} \left\langle \mathrm{tr}\, \frac{\varepsilon}{(\omega^*-\mathscr{H}^\dagger)(\omega-\mathscr{H})+\varepsilon^2} \frac{\varepsilon}{(\omega-\mathscr{H})(\omega^*-\mathscr{H}^\dagger)+\varepsilon^2} \right\rangle \tag{B.2}$$

of poles in the complex plane. (The distribution of poles is related to the distribution of decay rates by $P(\Gamma) = \frac{1}{2}\Delta\rho(\omega)|_{\omega=\Omega-\mathrm{i}\Gamma/2}$.) The relations are

$$\pi\rho(\omega) = \lim_{\varepsilon\to 0^+} \left(\frac{\partial^2}{\partial\omega_2\partial\omega_2^*} + \frac{1}{2}\frac{\partial^2}{\partial\omega_2\partial\omega_1^*} + \frac{1}{2}\frac{\partial^2}{\partial\omega_1\partial\omega_2^*} \right)$$
$$\times\, \Psi(\omega_1,\omega_2,0,0,\varepsilon)|_{\omega_1=\omega_2=\omega} , \tag{B.3}$$

$$M\pi\langle K\rangle_{\Omega,\Gamma}\rho(\omega) = -\lim_{\varepsilon\to 0^+} \frac{1}{4}\frac{\partial}{\partial u_1}\frac{\partial}{\partial u_2}\Psi(\omega,\omega,u_1,u_2,\varepsilon)\bigg|_{u_1=u_2=0} . \tag{B.4}$$

Most of the analysis runs therefore in parallel with the calculation of $\rho(\omega)$ in Ref. [18]. We restrict ourselves to the case of broken time-reversal symmetry, where the algebra is less involved.

The ratio of determinants in Eq. (B.1) can be written as a superdeterminant, which in turn can be expressed as a Gaussian integral over bosonic and fermionic variables

$$\Psi(\omega,\omega,\varepsilon,u_1,u_2)=(-1)^M\langle \mathrm{Sdet}^{-1}(A)\rangle=(-1)^M\left\langle\int \mathrm{d}\boldsymbol{\Psi}^\dagger\int \mathrm{d}\boldsymbol{\Psi}\; \mathrm{e}^{\mathrm{i}\boldsymbol{\Psi}^\dagger A\boldsymbol{\Psi}}\right\rangle . \tag{B.5}$$

The matrix A is

$$\begin{aligned}A&=\begin{pmatrix}\omega-\mathscr{H} & 0 & \mathrm{i}\varepsilon+u_1 & 0\\ 0 & \omega-\mathscr{H} & 0 & -\mathrm{i}\varepsilon+u_1\\ -\mathrm{i}\varepsilon-u_2 & 0 & -\omega^*+\mathscr{H}^\dagger & 0\\ 0 & -\mathrm{i}\varepsilon+u_2 & 0 & \omega^*-\mathscr{H}^\dagger\end{pmatrix}\\ &=(\Omega-H)\otimes\hat{L}+\mathrm{i}\left(\pi W^\dagger W-\frac{\Gamma}{2}\right)\otimes\hat{\sigma}_z\hat{L}\\ &\quad-\mathrm{i}\varepsilon\otimes\hat{\sigma}_x\hat{L}+\hat{u}\hat{\sigma}_x\hat{L} .\end{aligned} \tag{B.6}$$

The vector $\boldsymbol{\Psi}=\boldsymbol{\Psi}_1\oplus\boldsymbol{\Psi}_2\oplus\boldsymbol{\Psi}_3\oplus\boldsymbol{\Psi}_4$ is a $4M$-dimensional supervector consisting of two M-dimensional bosonic entries $\boldsymbol{\Psi}_\alpha$ with $\alpha=1$ and 3, supplemented by two M-dimensional fermionic entries with $\alpha=2$ and 4. We encounter the four-dimensional supermatrices $\hat{L}=\mathrm{diag}(1,1,-1,1)$, $\hat{u}=\mathrm{diag}(-u_1,u_1,-u_2,u_2)$, and $\hat{\sigma}_i=\sigma_i\otimes \mathbb{1}_2$, where σ_i are the usual Pauli matrices [e.g. $\hat{\sigma}_z=\mathrm{diag}(1,1,-1,-1)$].

The linear appearance of H in the exponent of Eq. (B.5) facilitates the ensemble average with the distribution function (2.1), since the integral over the independent components of H factorizes, and each single integral is Gaussian. The result is

$$\langle\exp[-\mathrm{i}\boldsymbol{\Psi}^\dagger H\otimes\hat{L}\boldsymbol{\Psi}]\rangle=\exp\left[-\frac{\mu^2M}{2}\mathrm{Str}(\hat{L}\hat{R})^2\right], \tag{B.7a}$$

$$\hat{R}_{\alpha\beta}=\frac{1}{M}\boldsymbol{\Psi}_\alpha\cdot\boldsymbol{\Psi}_\beta^\dagger . \tag{B.7b}$$

The order of $\hat{R}$ in the exponent is reduced from quadratic to linear by a Hubbard–Stratonovich transformation, based on the identity

$$\exp\left[-\frac{\mu^2M}{2}\,\mathrm{Str}(\hat{L}\hat{R})^2\right]=\int \mathrm{d}\hat{S}\exp\left[-M\,\mathrm{Str}\left(\frac{\hat{S}^2}{2}-\mathrm{i}\mu\hat{S}\hat{L}\hat{R}\right)\right] . \tag{B.8}$$

The integral over $\boldsymbol{\Psi}$ and $\boldsymbol{\Psi}^\dagger$ is again Gaussian and results in

$$\Psi=\int \mathrm{d}\hat{S}\exp\left[-M\,\mathrm{Str}\left(\frac{\hat{S}^2}{2}+\ln\hat{S}\right)\right]\mathrm{Sdet}^{-1}(1+C), \tag{B.9a}$$

$$C=\left(\Omega+\mathrm{i}\left(\pi W^\dagger W-\frac{\Gamma}{2}\right)\otimes\hat{\sigma}_z-\mathrm{i}\varepsilon\hat{\sigma}_x+\hat{u}\hat{\sigma}_x\right)\frac{1}{\mu\hat{S}} . \tag{B.9b}$$

One now can write $\mathrm{Sdet}^{-1}(1+C)=\exp[-\mathrm{Str}\ln(1+C)]$ and expand the logarithm to first order in Γ, ε, and the source term J; in addition we set $\Omega=0$ and pass from the generating function to the mean Petermann factor according to Eq. (B.4). This gives

$$M\pi\langle K\rangle_{\Omega,\Gamma}\rho(\omega) = -\frac{1}{4}\frac{\pi^2}{\Delta^2}\int \mathrm{d}\hat{S}\exp\left[-M\,\mathrm{Str}\left(\frac{\hat{S}^2}{2}+\ln\hat{S}\right)\right.$$
$$\left.+\mathrm{i}\frac{y}{2}\mathrm{Str}\,\hat{\sigma}_z\hat{S}^{-1}+\mathrm{i}\frac{\varepsilon'}{2}\mathrm{Str}\,\hat{\sigma}_x\hat{S}^{-1}\right]$$
$$\times \mathrm{tr}_{12}\,\hat{\sigma}_x\hat{S}^{-1}\,\mathrm{tr}_{34}\,\hat{\sigma}_x\hat{S}^{-1}\prod_{n=1}^{N}\mathrm{Sdet}^{-1}(\mathbb{1}_4+\mathrm{i}w_n\hat{\sigma}_z\hat{S}^{-1}) \qquad \text{(B.10)}$$

The traces $\mathrm{tr}_{ij}\,A = A_{ii}+A_{jj}$ operate only on the indicated subspaces. We introduced the rescaled variables $y=-2\pi\,\mathrm{Im}\,\omega/\Delta = \pi\Gamma/\Delta$ and $\varepsilon' = 2\pi\varepsilon/\Delta$. In what follows we will write ε instead of ε'.

The condition $M \gg 1$ justifies a saddle-point approximation. The main contribution to the preceding integral comes from points for which the first part of the exponent is minimal, that is from the solutions of

$$\frac{1}{\hat{S}}+\hat{S}=0 \;\Leftrightarrow\; \hat{S}^2=-1. \qquad \text{(B.11)}$$

With $\hat{S}=\mathrm{i}\hat{Q}$, the solutions fulfill $\hat{Q}^2=1$. As inherited from the definition of $\hat{R}$ in Eq. (B.7b), $\hat{Q}\hat{L}$ is a Hermitian matrix and $\hat{Q}=\hat{T}^{-1}\hat{Q}_{\mathrm{diag}}\hat{T}$ can be diagonalized by a pseudounitary supermatrix $\hat{T}\in \mathrm{U}(1,1/2)$ (these matrices fulfill $\hat{T}^{\dagger}\hat{L}\hat{T}=\hat{L}$). The largest manifold which respects the definiteness requirements on $\hat{Q}$ is obtained by the choice $\hat{Q}_{\mathrm{diag}}=\hat{\sigma}_z$. However, rotations in the block $\alpha=1,3$ and in the block $\alpha=2,4$ leave $\hat{Q}$ invariant, the saddle-point manifold is hence covered exactly once if we take the $\hat{T}$ matrices from the coset space $\mathrm{U}(1,1/2)/\mathrm{U}(1/1)\times\mathrm{U}(1/1)$.

A convenient parameterization of the coset space has been given by Efetov [40],

$$\hat{T}=\begin{pmatrix}U^{-1}&0\\0&V^{-1}\end{pmatrix}\exp\begin{pmatrix}0&\frac{1}{2}\mathrm{diag}(\theta_1,\mathrm{i}\theta_2)\\ \frac{1}{2}\mathrm{diag}(\theta_1,\mathrm{i}\theta_2)&0\end{pmatrix}\begin{pmatrix}U&0\\0&V\end{pmatrix}, \qquad \text{(B.12a)}$$

$$U=\begin{pmatrix}\mathrm{e}^{\mathrm{i}\phi_1}&0\\0&\mathrm{e}^{\mathrm{i}\phi_2}\end{pmatrix}\begin{pmatrix}1+\rho\rho^*/2&\rho\\ \rho^*&1+\rho^*\rho/2\end{pmatrix}, \qquad \text{(B.12b)}$$

$$V=\begin{pmatrix}1-\sigma\sigma^*/2&\mathrm{i}\sigma\\ \mathrm{i}\sigma^*&1-\sigma^*\sigma/2\end{pmatrix}, \qquad \text{(B.12c)}$$

with bosonic variables θ_1, θ_2, ϕ_1, and ϕ_2, and fermionic variables ρ, ρ^*, σ, and σ^*. We introduce $\lambda_1=\cosh\theta_1$ and $\lambda_2=\cos\theta_2$. In this parameterization

$$\mathrm{Str}\,\hat{\sigma}_z\hat{Q}=2(\lambda_1-\lambda_2), \qquad \text{(B.13a)}$$

$$\begin{aligned}\mathrm{Str}\,\hat{\sigma}_x\hat{Q}=&-\sinh\theta_1\mathrm{e}^{\mathrm{i}\phi_1}[(1+\rho\rho^*/2)(1-\sigma\sigma^*/2)-\mathrm{i}\rho\sigma^*]\\&+\sinh\theta_1\mathrm{e}^{-\mathrm{i}\phi_1}[(1+\rho\rho^*/2)(1-\sigma\sigma^*/2)-\mathrm{i}\sigma\rho^*]\\&+\mathrm{i}\sin\theta_2\mathrm{e}^{\mathrm{i}\phi_2}[(1+\rho^*\rho/2)(1-\sigma^*\sigma/2)-\mathrm{i}\rho^*\sigma]\\&-\mathrm{i}\sin\theta_2\mathrm{e}^{-\mathrm{i}\phi_2}[(1+\rho^*\rho/2)(1-\sigma^*\sigma/2)-\mathrm{i}\sigma^*\rho],\end{aligned} \qquad \text{(B.13b)}$$

$$\mathrm{tr}_{12}\,\sigma_x\hat{Q} = -\sinh\theta_1 e^{i\phi_1}[(1+\rho\rho^*/2)(1-\sigma\sigma^*/2) - i\sigma^*\rho] \\ -i\sin\theta_2 e^{i\phi_2}[(1+\rho^*\rho/2)(1-\sigma^*\sigma/2) - i\sigma\rho^*]\,, \tag{B.13c}$$

$$\mathrm{tr}_{34}\,\sigma_x\hat{Q} = \sinh\theta_1 e^{-i\phi_1}[(1+\rho\rho^*/2)(1-\sigma\sigma^*/2) - i\rho^*\sigma] \\ +i\sin\theta_2 e^{-i\phi_2}[(1+\rho^*\rho/2)(1-\sigma^*\sigma/2) - i\rho\sigma^*]\,, \tag{B.13d}$$

$$\mathrm{Sdet}^{-1}[\mathbb{1}_4 + w_n\hat{\sigma}_z\hat{Q}] = \frac{g_n+\lambda_2}{g_n+\lambda_1}\,. \tag{B.13e}$$

The integration measure is

$$d\hat{Q} = \frac{d\lambda_1\, d\lambda_2\, d\phi_1\, d\phi_2\, d\rho^*\, d\rho\, di\sigma^*\, di\sigma}{(2\pi)^2(\lambda_1-\lambda_2)^2}\,. \tag{B.14}$$

In order to integrate over the fermionic variables we have to expand in these quantities and only keep the term in which all four variables appear linearly. The angle ϕ_2 appears in the pre-exponential factor as well as in the exponential term $\exp(-\varepsilon\sin\theta_2\sin\phi_2)$. We expand the exponential and integrate over ϕ_2. Only terms of order $\varepsilon^n\sinh^m\theta_1$ with $n\leqslant m$ survive the limit $\varepsilon\to 0$. We discard all other terms and obtain

$$-4\frac{\Delta^2}{\pi^2}\langle K\rangle_{\Omega,\Gamma}\rho(\omega) \\ = \lim_{\varepsilon\to 0}\int_1^\infty d\lambda_1\int_{-1}^{1} d\lambda_2\,\frac{1}{(\lambda_1-\lambda_2)^2}\int_0^{2\pi}\frac{d\phi_1}{2\pi}D \\ \times\exp[-i\varepsilon\sqrt{\lambda_1^2-1}\sin\phi_1 + y(\lambda_1-\lambda_2)]\prod_{n=1}^{N}\frac{g_n+\lambda_2}{g_n+\lambda_1}\,, \tag{B.15a}$$

$$D = -i\varepsilon\sinh\theta_1\sin\phi_1(2\sin^2\theta_2 + \tfrac{9}{4}\sinh^2\theta_1) \\ + \frac{\varepsilon^2}{4}\sinh^2\theta_1[\sinh^2\theta_1\cos^2\phi_1 - (3\cos^2\phi_1 + 5\sin^2\phi_1)\sin^2\theta_2] \\ + i\varepsilon^3\sinh^3\theta_1\sin\phi_1\sin^2\theta_2(-\tfrac{9}{16}\sin^2\phi_1 - \tfrac{13}{16}\cos^2\phi_1) \\ + \tfrac{1}{16}\varepsilon^4\sinh^4\theta_1\sin^2\phi_1\sin^2\theta_2\,. \tag{B.15b}$$

It is convenient to bring the factor D into a form which involves ϕ_1 only in the combination $z_1 = -i\sinh\theta_1\sin\phi_1$, because such terms can be expressed as derivatives with respect to ε of the exponential $\exp(\varepsilon z_1)$ appearing in Eq. (B.15a). This goal can be achieved by integrating by parts all terms that involve $\cos\phi_1$. Effectively this amounts to the substitutions $\varepsilon\sinh\theta_1\sin\phi_1\cos^2\phi_1 \to i(\sin^2\phi_1 - \cos^2\phi_1)$ and $\varepsilon\sinh\theta_1\cos^2\phi_1 \to i\sin\phi_1$, resulting in

$$D = \varepsilon z_1(\tfrac{7}{2}\sin^2\theta_2 + 2\sinh^2\theta_1) + \tfrac{1}{2}\varepsilon^2 z_1^2\sin^2\theta_2 - \tfrac{1}{2}\varepsilon^3 z_1^3\sin^2\theta_2\,. \tag{B.16}$$

Mathematically these expressions are quite similar to those obtained for the decay-rate distribution in Ref. [18]. By a simple substitution rule that relates to each other the terms of different order in ε, we now rewrite D in a way that allows to make direct contact to Ref [18], yielding a result in terms of the two functions $\mathscr{F}_{1\,2}$ given in Eq (4 2) As in Ref [18] we express the factors $(g_n + \lambda_1)^{-1}$ as an integral of exponential functions

$$\frac{1}{g_n + \lambda_1} = \int_0^\infty \mathrm{d}s_n \exp[-s_n(g_n + \lambda_1)] \tag{B 17}$$

We also write $(\lambda_1 - \lambda_2)^2 = \int_0^\infty \mathrm{d}x\, x \exp[-x(\lambda_1 - \lambda_2)]$ Then the integrations over θ_1 and ϕ_1 can be performed, and ε only appears in a factor

$$\Phi(\varepsilon, y') = \frac{\exp[-\sqrt{\varepsilon^2 + y'^2}]}{\sqrt{\varepsilon^2 + y'^2}}\,, \tag{B 18}$$

with $y' = y - x - \sum_n s_n$ The limiting value for $\varepsilon \to 0$ of the derivatives

$$\varepsilon^n \frac{\partial^n}{\partial \varepsilon^n} \Phi(\varepsilon, y') = C_n \delta(y'), \quad C_1 = -C_2 = C_3/2 = -2\,, \tag{B 19}$$

amounts in Eq (B 16) to the substitutions $\varepsilon^3 z_1^3 \to 2\varepsilon z_1$ and $\varepsilon^2 z_1^2 \to -\varepsilon z_1$, which gives $D = 2\varepsilon z_1(\lambda_1^2 - \lambda_2^2)$ As a result, we obtain

$$\left[\frac{\Delta}{\pi} P(\Gamma)\right] \langle K \rangle_{\Omega\, K} = I_0(\pi\Gamma/\Delta) + 2I_1(\pi\Gamma/\Delta)\,, \tag{B 20}$$

$$I_1(y) = -\frac{1}{4} \lim_{\varepsilon \to 0} \varepsilon \frac{\partial}{\partial \varepsilon} \int_1^\infty \mathrm{d}\lambda_1 \int_{-1}^1 \mathrm{d}\lambda_2 \frac{\lambda_2^1}{(\lambda_1 - \lambda_2)^1}$$
$$\times J_0(\varepsilon\sqrt{\lambda_1^2 - 1}) \exp[y(\lambda_1 - \lambda_2)] \prod_{n=1}^N \frac{g_n + \lambda_2}{g_n + \lambda_1}\,, \tag{B 21}$$

where J_0 is a Bessel function By comparing expressions with Ref [18], we recognize that $I_0(y) = \mathscr{F}_1(y)\mathscr{F}_2(y) = (\Delta/\pi)P(\Gamma = \Delta y/\pi)$ [cf Eq (4 2)], while

$$I_1(y) = -\int_0^y \mathrm{d}y'\, \mathscr{F}_1(y') \frac{\partial}{\partial y'} \mathscr{F}_2(y') \tag{B 22}$$

This concludes the derivation of the final result (4 10)

References

[1] A L Schawlow, C H Townes Phys Rev 112 (1958) 1940
[2] K Petermann IEEE J Quant Electron 15 (1979) 566
[3] A E Siegman, Phys Rev A 39 (1989) 1253
[4] A E Siegman, Phys Rev A 39 (1989) 1264
[5] Y J Cheng, C G Fanning, A E Siegman, Phys Rev Lett 77 (1996) 627
[6] M A van Eijkelenborg, Å M Lindberg, M S Thijssen, J P Woerdman Phys Rev Lett 77 (1996) 4314
[7] M A van Eijkelenborg, Å M Lindberg M S Thijssen, J P Woerdman, Phys Rev A 55 (1997) 4556
[8] M Brunel, G Ropars, A Le Floch, F Bretenaker, Phys Rev A 55 (1997) 4563
[9] P Grangier, J P Poizat Eur Phys J D 1 (1998) 97
[10] A E Siegman, unpublished
[11] C Mahaux, H A Weidenmuller, Shell-Model Approach to Nuclear Reactions, North Holland Amsterdam, 1969

[12] V A Mandelshtam, H S Taylor, Phys Rev Lett 78 (1997) 3274
[13] F Haake, Quantum Signatures of Chaos, Springer, Berlin, 1991
[14] M Mehta, Random Matrices, Academic, New York, 1990
[15] V Sokolov, G Zelevinsky, Phys Lett B 202 (1988) 10
[16] V Sokolov, G Zelevinsky, Nucl Phys A 504 (1989) 562
[17] Y V Fyodorov, H -J Sommers, Pis'ma Zh Eksp Teor Fiz 63 (1996) 970 [JETP Lett 63 (1996) 1026]
[18] Y V Fyodorov, H -J Sommers, J Math Phys 38 (1997) 1918
[19] H -J Sommers, Y V Fyodorov, M Titov, J Phys A 32 (1999) L77
[20] R A Janik, W Noerenberg, M A Nowak, G Papp, I Zahed, preprint (cond-mat/9902314)
[21] J T Chalker, B Mehlig, Phys Rev Lett 81 (1998) 3367
[22] B Mehlig, J T Chalker, preprint (cond-mat/9906279)
[23] M Patra, H Schomerus, C W J Beenakker, Phys Rev A 61 (2000) 23810
[24] K M Frahm, H Schomerus, M Patra, C W J Beenakker, Europhys Lett 49 (2000) 48
[25] J J M Verbaarschot, H A Weidenmuller, M R Zirnbauer, Phys Rep 129 (1985) 367
[26] J Jeffers, N Imoto, R Loudon, Phys Rev A 47 (1993) 3346
[27] C W J Beenakker, Phys Rev Lett 81 (1998) 1829
[28] P Goldberg, P W Milonni, B Sundaram, Phys Rev A 44 (1991) 1969
[29] C W J Beenakker, Rev Mod Phys 69 (1997) 731
[30] T S Misirpashaev, C W J Beenakker, Phys Rev A 57 (1998) 2041
[31] F von Oppen, Phys Rev Lett 73 (1994) 798
[32] F von Oppen, Phys Rev E 51 (1995) 2647
[33] Y V Fyodorov, H -J Sommers, Z Phys B 99 (1995) 123
[34] H A Bethe, Phys Rev 66 (1944) 163
[35] F Haake, F Izrailev, N Lehmann, D Saher, H -J Sommers, Z Phys B 88 (1992) 359
[36] N Lehmann, D Saher, V V Sokolov, H -J Sommers, Nucl Phys A 582 (1995) 223
[37] A V Andreev, B D Simons, Phys Rev Lett 75 (1995) 2304
[38] S Kettemann, D Klakow, U Smilansky, J Phys A 30 (1997) 3643
[39] H -J Sommers, A Crisanti, H Sompolinsky, Y Stein, Phys Rev Lett 60 (1988) 1895
[40] K B Efetov, Adv Phys 32 (1983) 53