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Distribution of the reflection eigenvalues of a weakly absorbing chaotic cavity

C.W.J. Beenakker[a], P.W. Brouwer[b,*]

[a] *Instituut-Lorentz Universiteit Leiden P O Box 9506 2300 RA Leiden The Netherlands*
[b] *Laboratory of Atomic and Solid State Physics Cornell University Ithaca NY 14853-2501 USA*

Abstract

The scattering-matrix product $SS^\dagger$ of a weakly absorbing medium is related by a unitary transformation to the time-delay matrix without absorption It follows from this relationship that the eigenvalues of $SS^\dagger$ for a weakly absorbing chaotic cavity are distributed according to a generalized Laguerre ensemble © 2001 Elsevier Science B V All rights reserved

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1. Problem

The purpose of this note is to answer a question raised by Kogan et al [1], concerning the statistics of the eigenvalues of the scattering-matrix product $SS^\dagger$ for an absorbing optical cavity with chaotic dynamics *Without* absorption the scattering matrix S is an $N \times N$ unitary matrix, hence $SS^\dagger$ is simply the unit matrix *With* absorption the eigenvalues $R_1, R_2, \ldots, R_N$ of $SS^\dagger$ are real numbers between 0 and 1 What is the probability distribution $P(\{R_n\})$ of these reflection eigenvalues in an ensemble of chaotic cavities?

In principle, this problem can be solved by starting from the known distribution of S in the absence of absorption (which is Dyson's circular ensemble [2]), and incorporating the effects of absorption by a fictitious lead [3] What has been calculated in this way is the distribution $P(\{R_n\})$ for small N [3] and the density $\rho(R) = \langle \sum_n \delta(R - R_n) \rangle$ for large N [4,5] These results have a complicated form, quite unlike those familiar from the classical ensembles of random-matrix theory [6] For example, in the presence of time-reversal symmetry the distribution for $N = 2$ is given by Ref [3]

$$\begin{aligned} P(R_1,R_2) &= (1-R_1)^{-4}(1-R_2)^{-4} \\ &\times \exp[-\gamma(1-R_1)^{-1} - \gamma(1-R_2)^{-1}] \\ &\times |R_1 - R_2| [\gamma^2(1 - e^{2\gamma} + \gamma + \gamma e^{2\gamma}) \\ &+ \gamma(R_1 + R_2 - 2)(\tfrac{3}{2} - \tfrac{3}{2}e^{2\gamma} + 2\gamma + \gamma e^{2\gamma} + \gamma^2) \\ &+ (1-R_1)(1-R_2)(3 - 3e^{2\gamma} + \tfrac{9}{2}\gamma \\ &+ \tfrac{3}{2}\gamma e^{2\gamma} + 3\gamma^2 + \gamma^3)], \end{aligned} \tag{1}$$

* Corresponding author Fax +1-607-255-6428
E-mail address brouwer@ccmr cornell edu (P W Brouwer)

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where γ is the ratio of the mean dwell time τ_d inside the cavity [1] and the absorption time τ_a.

The situation is simpler for reflection from an absorbing disordered waveguide. In the limit that the length of the waveguide goes to infinity, the distribution of the reflection eigenvalues becomes that of the Laguerre ensemble, after a transformation of variables from R_n to $\lambda_n = R_n(1-R_n)^{-1} \geqslant 0$. The (unnormalized) distribution is given by [7,8]

$$P(\{\lambda_n\}) \propto \prod_{i<j} |\lambda_i - \lambda_j|^\beta \times \prod_k \exp[-\gamma(\beta N + 2 - \beta)\lambda_k], \tag{2}$$

where now $\gamma = \tau_s/\tau_a$ contains the scattering time τ_s of the disorder. The integer $\beta = 1(2)$ in the presence (absence) of time-reversal symmetry. The eigenvalue density is given by a sum over Laguerre polynomials, hence the name "Laguerre ensemble" [6].

Kogan et al. [1] used a maximum entropy assumption [9] to argue that a chaotic cavity is also described by the Laguerre ensemble, but in the variables R_n instead of the λ_n's. Their maximum entropy distribution is

$$P(\{R_n\}) \propto \prod_{i<j} |R_i - R_j|^\beta \prod_j \exp(-aR_j) \tag{3}$$

The coefficient a in the exponent is left undetermined [2]. Comparison with computer simulations gave good agreement for strong absorption, but not for weak absorption [1]. This is unfortunate since the weak-absorption regime ($\gamma \ll 1$) is likely to be the most interesting for optical experiments. Although we know from the exact small-N results [3] that no simple distribution exists in the entire range of γ, one might hope for a simple eigenvalue distribution for small γ. What is it?

2. Solution

Absorption with rate $1/\tau_a$ is equivalent to a shift in frequency ω by an imaginary amount $\delta\omega = i/2\tau_a$. If we denote by $S(\omega)$ the scattering matrix with absorption and by $S_0(\omega)$ the scattering matrix without absorption, then $S(\omega) = S_0(\omega + i/2\tau_a)$. For weak absorption we can expand

$$S_0(\omega + i/2\tau_a) \approx S_0(\omega) + \frac{i}{2\tau_a}\frac{d}{d\omega}S_0(\omega) = S_0(\omega)\left[1 - \frac{1}{2\tau_a}Q(\omega)\right], \tag{4}$$

where $Q = -iS_0^\dagger\, dS_0/d\omega$ is the time-delay matrix [10]. Since S_0 is unitary, Q is Hermitian. The eigenvalues of Q, the delay times $\tau_1, \tau_2, \ldots, \tau_N$, are real positive numbers. Eq. (4) implies that, for weak absorption,

$$S(\omega)S^\dagger(\omega) = S_0(\omega)\left[1 - \frac{1}{\tau_a}Q(\omega)\right]S_0^\dagger(\omega) \tag{5}$$

We conclude that the matrix product $SS^\dagger$ is related to the time-delay matrix Q by a unitary transformation. This relationship is a generalization to $N > 1$ of the result of Ramakrishna and Kumar [11] for $N = 1$ (when the unitary transformation becomes a simple identity). Because a unitary transformation leaves the eigenvalues unchanged, one has $R_n = 1 - \tau_n/\tau_a$, or equivalently, $\lambda_n = \tau_a/\tau_n$ (since $\lambda_n \to (1-R_n)^{-1}$ for weak absorption).

The probability distribution of the delay times in a chaotic cavity has recently been calculated, first for $N = 1$ [12,13] and later for any N [14,15]. The corresponding distribution of the reflection eigenvalues for weak absorption is a generalized Laguerre ensemble in the variables λ_n,

$$P(\{\lambda_n\}) \propto \prod_{i<j} |\lambda_i - \lambda_j|^\beta \times \prod_j \lambda_k^{\beta N/2} \exp[-\tfrac{1}{2}\gamma\beta N\lambda_j] \tag{6}$$

The eigenvalue density is given in terms of *generalized* Laguerre polynomials, hence the name. The corresponding distribution of the reflection eigenvalues is

$$P(\{R_n\}) \propto \prod_{i<j} |R_i - R_j|^\beta \times \prod_j \frac{\exp[-\tfrac{1}{2}\gamma\beta N(1-R_j)^{-1}]}{(1-R_j)^{2-\beta+3\beta N/2}} \tag{7}$$

The first moment of this distribution is $N^{-1}\langle\sum_n R_n\rangle = 1 - \gamma$, independent of β. One can check that Eq. (7)

[1] The mean dwell time is related to the mean frequency interval Δ of the cavity modes by $\tau_d = 2\pi/N\Delta$, so that $\gamma = 2\pi(\tau_a N\Delta)^{-1}$. The definition of γ used in Ref. [3] differs by a factor N.

[2] We have verified that the theory of Ref. [3] agrees with Eq. (3) for strong absorption ($\gamma \gg 1$) with coefficient $a = \frac{1}{2}\gamma\beta N$.

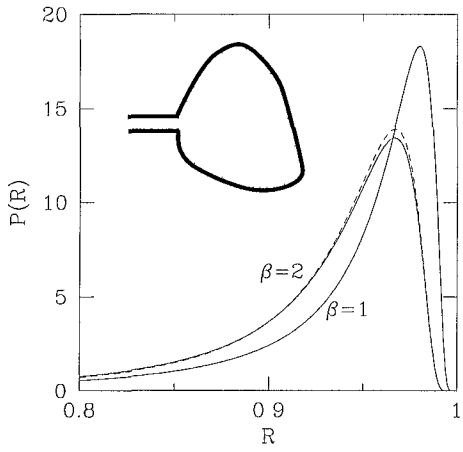

Fig. 1. Probability distribution of the reflectance of a weakly absorbing chaotic cavity that is coupled to the outside via a single-mode waveguide (inset). The solid curves have been computed from Eq. (8) for $\gamma = 0.1$ (hence $\langle R\rangle = 1 - \gamma = 0.9$). The dashed curve is the exact $\beta = 2$ result from Eq. (9).

is the small-γ asymptote of the exact result (1) for $N = 2$, $\beta = 1$.

In the case $N = 1$ of a single scattering channel, the distribution (7) reduces to

$$P(R) = \frac{(\gamma\beta/2)^{1+\beta/2}}{\Gamma(1+\beta/2)}(1-R)^{-2-\beta/2} \times \exp[-\tfrac{1}{2}\gamma\beta(1-R)^{-1}], \quad (8)$$

including the normalization constant. We have plotted this function in Fig. 1 for $\gamma = 0.1$ and $\beta = 1, 2$. It is totally different from the exponential distribution $P(R) \propto \exp(-aR)$ of Ref. [1]. For comparison, we have also included in Fig. 1 the exact $N = 1$ result [16] (which is known only for $\beta = 2$):

$$P(R) = (1-R)^{-3}\exp[-\gamma(1-R)^{-1}] \times [\gamma(e^{\gamma} - 1) + (1 + \gamma - e^{\gamma})(1-R)]. \quad (9)$$

It is indeed close to the small-γ asymptote (8).

3. Conclusion

Summarizing, the distribution of the reflection eigenvalues of a weakly absorbing chaotic cavity is the generalized Laguerre ensemble (6) in the parameterization $\lambda_n = R_n(1-R_n)^{-1}$. The Laguerre ensemble (3) in the variables R_n, following from the maximum entropy assumption [1], is only valid for strong absorption. For intermediate absorption strengths the distribution is not of the form of the Laguerre ensemble in any parameterization, cf. Eq. (1). In contrast, the distribution of a long disordered waveguide is the Laguerre ensemble (2) for all absorption strengths.

The relationship between the reflection eigenvalues for weak absorption and the delay times implies that the delay times τ_n for reflection from a disordered waveguide of infinite length are distributed according to Eq. (2) if one substitutes $\gamma\lambda_n \to \tau_s/\tau_n$. The implications of this Laguerre ensemble for the delay times will be discussed elsewhere.

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