A CENTURY OF INFERENCE: 1837–1936*

Dedicated to the memory of Artur Rojszczak, 1968–2001

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1.

Just before my chosen period, in 1797, in a booklet bearing the splendid title *Wissenschaftslehre*, Johann Gottlieb Fichte observed, concerning the human act of knowledge and its object,

$$\frac{act}{object,}$$

that basically there are only two philosophical positions.[1] On the one hand you can determine the object by means of the act, and, on the other hand, you can determine the act by means of its object. In the first case you are an Idealist, and in the second you are a Realist (Fichte, being an Idealist himself, used the pejorative term *Dogmatist*), and never the two shall meet.

2.

My topic concerns *inference*, that is, acts of knowledge of a certain kind. An inference is an

> Urteilsfällung, die auf Grund schon früher gefällter Urteile nach logischen Gesetzen vollzogen wird,

Frege held, for one.[2]

On this view, then, an act of inference is nothing but a mediate act of judgement: in such an act new knowledge is obtained from previously known judgements. Consider the completely general *mode of inference* (German: *Schlussweise*) I:

$$\frac{J_1 \quad J_2 \quad \dots \quad J_k}{J} \quad .$$

A mediate act of knowledge according to this mode of inference takes the form (where the $J_1, \dots, J_k$ are the judgements of the prior acts of knowledge):

$$\frac{\begin{array}{cccc} | & | & \dots & | \\ J_1 & J_2 & \dots & J_k \end{array}}{J.} \quad .$$

In this Frege completely follows the epistemological tradition of his own *and* previous times.

At the beginning of my period, the scholastic patrimony still held sway in one form or other, for instance in Kant's Jäsche *Logik*. The following diagram may serve as a convenient starting point for my exposition:

The Traditional Structure of Logic:[3]		
Operation of the Intellect	(Mental) Product	(External) Sign
I *Simple Apprehension*	Concept, Idea, (Mental) Term	(Written/spoken) Term
II *Judging,* Composition/Division of two terms	Judgement, (Mental) Proposition: *S* is *P*	Assertion, (Written/spoken) Proposition
III *Reasoning,* Inferring	(Mental) Inference	(Written/spoken) Inference, Reasoning

There is a hierarchical order among the logical faculties of the intellect. Even though the third row is central to my topic, one has to deal also with the previous two, because inference makes use of judgements that are composed out of terms. The first major change in this order was made by Kant who reversed the order of priority between rows I and II:

Wir können alle Handlungen des Verstandes auf Urteile zurückführen, so daßder Verstand überhaupt als ein Vermögen zu urteilen vorgestellt werden kann.[4]

We should also note that the application of the act–object/product imagery after the fashion of the above diagram produces a tension in the traditional scheme. There the product of an act of inferring is a (mental) inference. An act of inference, however, is a mediate act of judgement, whence its object is a judgement made and not a mental inference. This, I think, is right: the *primary product of an act of inferring is not an inference, but a judgement made.* To this extent the traditional picture in the diagram is misleading and it is incumbent upon us to seek another home for the inferences.

Let me also take notice of some terminological matters. English is particularly rich in variants here: infer, illate, deduce, derive, and conclude may all be used more or less interchangeably. For what it is worth, the corresponding nouns seem to differ somewhat: a deduction, or derivation, might contain many steps, whereas normally an inference is one step only, from premisses to conclusion. A conclusion, on the other hand, is usually what is concluded and not the act of concluding it. In German we have *schliessen, Schluss(weise)* and *Schlusssatz* for, respectively (act of) inferring, (mode of) inference and conclusion, as well as the splendid verb *folgern* with such cognates as *Schlussfolgerung,* etc.

3.

The first serious breach in the traditional logical fortress was broached by one thoroughly steeped in the Scholastic patrimony, namely Bernard Bolzano, in another *Wissenschaftslehre* from 1837. This, however, is no puny pamphlet, but a monumental four-volume tome.[5] Like all good ideas the basic idea behind Bolzano's magisterial change is essentially simple: Bolzano revolutionizes logical theory by "objectivizing" the middle column of the traditional diagram. This objectivization consists in severing the left – and right-hand links to mind and language, thereby obtaining objective "Platonist" logical notions, for which Bolzano ironically adopts the Kantian '*an sich*' idiom. Thus, the (mental) terms become objective "ideas-in-themselves" (*Vorstellungen an sich*).[6] The judgements made, that is, the mental propositions, become propositions-in-themselves (*Sätze an sich*), that is, propositions in the modern, post-Russellian sense.[7] Finally, the mental inferences are replaced by *Ableitbarkeiten,* that is, relations of (logical) consequence between propositions-in-themselves.

The resulting change with respect to the form of judgement is particularly interesting. In place of the traditional bipartite Subject/copula/Predicate form Bolzano uses the unary form

C is true,

where C is a *Satz an sich*, that is, a proposition that serves as content of the judgement in question (WL §34).[8] The form of the proposition C, on the other hand, stays close to the traditional [S is P]. Bolzano uses [A has b], where A and b are *Vorstellungen an sich*, that is, (what corresponds to) objectivizations of the mental products of simple apprehensions, as canonical form for the objective propositions. Thus, he converts the traditional form of judgement into a form of *content*:

The proposition that the rose has redness is true

instead of

The rose is red.

Why Bolzano made this change in the form of *judgement* is not clear to me. That he wanted to give a foundation for logic is clear and also that he gave this in terms of objective *propositions*. I can offer the following argument. As Frege noted, assertion is the linguistic counterpart to judging.[9] Normally I assert by uttering a declarative sentence assertorically. Thus, for instance, by uttering 'Snow is white' I assert that snow is white. Propositions, the contents of judgements/assertions, on the other hand, are indicated by nominalized that-clauses. Thus,

that snow is white

indicates the proposition that snow is white. However, by uttering solely 'that snow is white' I cannot normally assert that snow is white. The nominalization needs to be completed with

... is true

in order to get a declarative sentence that is *behauptungsfähig*, that is, can be used to effect an assertion. Thus, a sole utterance of

that snow is white is true

can be used for effecting an assertion to the effect that snow is white, and so we have reached (what I call) the Bolzano form of judgement. This argument, to which I am quite partial myself, draws both upon language and judging, whence it would have been rejected by Bolzano.

Perhaps we might use the term 'statement' for what declaratives express: the nominalization of a declarative then stands for the proposition that serves as content of the statement that the declarative expresses. This terminology seems particularly apt in view of the legal use of statement for what is said by a witness. It is on the level of assertions, but we might not wish to join in the asserting thereof. The corresponding German term would be *Aussage*. This distinction between judgement/assertion, statement and proposition cannot be found in either Frege or Bolzano. Frege has only judgement and proposition,

whereas Bolzano's *Sätze an sich* sometimes seem to serve as my statements and sometimes as my propositions.

A judgement of the form [proposition C is true] is *richtig* (correct), and is then said to be a piece of knowledge (an *Erkenntnis*), when the proposition C really is true. Bolzano reduces the correctness ("truth") of a judgement to the (propositional) truth of its content (WL §36).

Under the Bolzano form of judgement, the form of inference I is transformed into the form I':

$$\frac{A_1 \text{ is true} \quad A_2 \text{ is true} \quad \ldots \quad A_k \text{ is true}}{C \text{ is true}}$$

For Bolzano, such an inference I' is valid when a relation of logical consequence – *eine Ableitbarkeit* – obtains between the propositions that serve as premisses, respectively conclusion, of the inference in question, that is, when the consequence, or using current terminology introduced by Hertz and Gentzen, *sequent S*:

$$A_1, A_2, \ldots, A_k \Rightarrow C$$

holds "logically", and where such logical holding is explained in terms of the familiar variation of non-logical sub-propositional parts that was used by Bolzano in his *Variationslogik*.[10]

As we know, the consequence S holds logically precisely when the corresponding implication

$$A_1 \& A_2 \& \ldots \& A_k \supset C$$

is a *logical* truth. In effect, we can say that Bolzano reduces the validity of mode I' of inference to the logical truth of a certain proposition. The first Bolzano-reduction reduced the epistemic notion of correctness (*Richtigkeit*) to the alethic notion of truth. This second Bolzano-reduction similarly reduces an epistemic notion, namely inferential validity, to an alethic notion, albeit a more complex one than propositional truth.[11] Also, it is much to Bolzano's credit that, contrary to most later logical practitioners, his theory allows also for *material* consequence (by considering variation with respect to a *Vorstellung* that does not occur in the antecedents or in the succedent). That is, the sequent $A \Rightarrow B$ holds (materially) when the implication(al proposition) $A \supset B$ is true. In order to obtain the truth of the proposition B, it is enough to know the judgement [A is true] and that the sequent $A \Rightarrow B$ holds. The *logical* holding of the sequent, that is, the preservation of truth under all variations of non-logical sub-propositional parts, is not needed.

In my opinion, the magnitude and richness of Bolzano's achievement in the foundations of logical theory can hardly be exaggerated.

4.

About 35 years after Bolzano an alternative form of judgement was proposed by another priest-philosopher steeped in the tradition, namely Franz Brentano. Brentano used as his form(s)

α IS (exists)

as well as

α IS NOT (does not exist),

where α is a concept, or *term*.[12] The four categorical judgements from the tradition can be reduced to these forms. For instance,

No α are β

can be expressed as

An α which is β does not exist.

Later, during the first two decades of our century, Brentano came to reject his earlier realist views in a series of brief writings that were posthumously published in a volume with the splendid title *Wahrheit und Evidenz*.[13] In particular he criticised the notion of a blind judgement and, *contra* Bolzano, we might extend his criticism also to blind inference. Blindness of judgement (and inference) is so called after Plato, *Rep.* 506c:

> [...] opinions divorced from knowledge, are ugly things [...]. The best of them are blind. Or do you think that those who hold some correct opinion without evidence differ appreciably from blind men who go the right way?[14]

Under the above two Bolzano reductions, the correctness of judgements and inferences depend solely on the truth-behaviour of propositions under variability of suitable sub-propositional parts. A judgement is *richtig, ist eine Erkenntnis*, in complete independence of whether we know it or not, depending *solely* on whether its content is a true proposition or not, and similarly, an inference is valid depending solely on whether the matching implication is a logical truth or not. In no way does it depend on whether the conclusion drawn actually is known, or whether it can be known from knowledge of the premisses, or whether the premisses are knowable: the only thing that matters is whether a certain implication is a logical truth. I for one share Brentano's misgivings concerning the correctness of such blind judgement and inference; without epistemological warrant the inference is not allowed.[15]

While I have to agree with Brentano concerning the blindness that results through the Bolzano-reductions, I must put on the record that Brentano was singularly ungenerous in his treatment of Bolzano:

> Wenn ich unter solchen Umständen auf Bolzano aufmerksam machte, so geschah dies, [...], keineswegs, um den jungen Leuten Bolzano als Lehrer und Führer zu empfehlen. Was sie von ihm, das dürfte ich mir sagen, konnten sie besser von mir lernen [...]

Und wie gesagt, wie ich selbst von Bolzano nie auch nur einen einzigen Satz entnommen habe, so habe ich auch niemals meinen Schülern glaubhaft gemacht, daß sie dort eine wahre Bereicherung ihrer philosophischen Erkenntnis gewinnen würden.

Brentano to Hugo Bergmann, June 1, 1909.[16]

A spirit equally ungenerous might counter by observing that Brentano's foremost logical achievement, namely the reduction of the four kinds of categorical judgement to his two kinds of existential judgement, can be found wholesale already in Bolzano.[17]
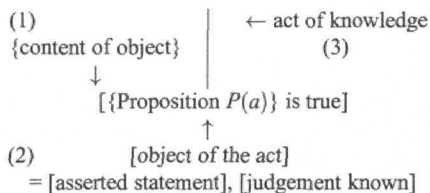
5.

With respect to judgement Frege concurs with Bolzano. His formulation from the famous essay *Über Sinn und Bedeutung* could not be more explicit:

Ein Urteil ist mir nicht das bloße Fassen eines Gedanken, sondern die Anerkennung seiner Wahrheit.[18]

However, with respect to the notion of consequence between propositions Frege represents a retrograde step in comparison with Bolzano. It does not play a significant role in his writings.[19]

Frege's major innovation over and above Bolzano lies in his use of a better form of content; his propositions (or "Thoughts") do not take the form [*A* has *b*], but are of the form $P(a)$, that is, function applied to argument. This essentially mathematical notion Frege had readily available owing to his training. Most of his achievements can be seen to flow naturally from this basic change, but to document this fully would take a course of lectures rather than a section in one lecture.

THE REALIST (BOLZANO-FREGE) THEORY[20]

(1)   ← act of knowledge
{content of object}   (3)
    ↓
        [{Proposition $P(a)$} is true]
                    ↑
(2)   [object of the act]
    = [asserted statement], [judgement known]

Michael Dummett's views on Frege and Wittgenstein can be summarised as follows:

1) Wittgenstein is nearly always right, except when he disagrees with Frege.[21]

2) Frege is even more right, except when he is obviously wrong.

According to Dummett, inference is one such topic where Frege is obviously wrong:

> Frege's account of inference allows no place for a[n] [...] act of supposition. *Gentzen later had the highly successful idea of formalizing inference so as to* leave a place for the introduction of hypotheses [...]

Indeed,

> [i]t can be said of Gentzen that it was he who showed how proof theory should be done.[22]

In particular, Frege's view that we can only draw inferences from true, nay, known premisses has come in for much flak and even ridicule.[23] I think, to the contrary, that he was absolutely right, given his Aristotelian views on the use of logic in demonstrations. Frege's uses logic to *establish* truths, mathematical theorems. He is not concerned with the so-called logical truth of propositions, but with how we obtain further knowledge by proceeding from theorem to theorem.[24] We do that by proceeding from known truths to a novel insight by drawing a valid inference. This is consistently the practice in the *Grundgeseteze* where every premiss of an inference is prefixed with the *Urteilsstrich*. Thus premisses of inferences are asserted, and assertion effects a claim to knowledge.

A mode of inference is valid, if the conclusion can be known, given that the premisses are known. In the *Begriffsschrift* Frege spends a number of pages showing that his primitive rules have this property. Inferential validity is not preservation of truth, that is, the (possibly logical) holding of consequence – between propositions – but the preservation of knowability from premiss statements to conclusion statement.

6.

The remainder of my paper will largely be spent on an attempt to adjudicate between Frege and Gentzen. This will take one more preliminary, though, concerning truth. Frege, notoriously, held that truth is *sui generis* and indefinable.[25] Towards the end of the first decade of the century G. E. Moore and Bertrand Russell offered novel analyses of truth that made use of (various versions of) a so-called truth-maker analysis.[26] These proceed by putting

proposition C is true = there exists a truth-maker for C.

Among the candidates for truth-makers we find facts, complexes and states of affairs. The most celebrated truth-maker analysis is that of Wittgenstein's *Tractatus*, where an atomic proposition C presents a *Sachverhalt* S_C such that
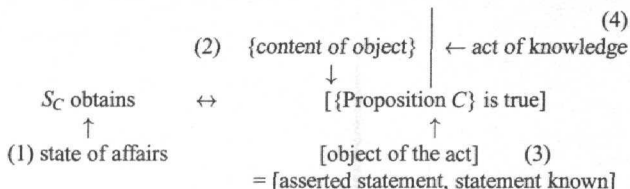
C is true iff S_C obtains (*besteht*),

where truth for complex propositions is obtained recursively from this atomic case. Here the crucial notion of "obtaining" is ontologically primitive. The

bivalence of the ontological *obtains* induces bivalence for propositional truth, whence the logic is classical.

THE TRUTH-MAKER REDUCTION IN WITTGENSTEIN'S *Tractatus*

$$
\begin{array}{ccc}
 & & (4) \\
(2) & \{\text{content of object}\} \bigm| & \leftarrow \text{act of knowledge} \\
 & \downarrow \bigm| & \\
S_C \text{ obtains} \quad \leftrightarrow & [\{\text{Proposition } C\} \text{ is true}] & \\
\uparrow & \uparrow & \\
(1) \text{ state of affairs} & [\text{object of the act}] \quad (3) & \\
 & = [\text{asserted statement, statement known}] &
\end{array}
$$

However, a truth-maker analysis is not committed to a realist stance. According to the intuitionist critique of classical semantics the quantifier laws are not evident under the classical truth-value semantics. In other words, the legitimacy of $\forall$-formation is disputed for classical semantics. Since mathematics is replete with quantifiers it is incumbent upon the constructivist to offer another notion of proposition. This was implicit in Kronecker and Brouwer, and was made explicit in the semantical writings of Brouwer's pupil Arend Heyting during the thirties. With the benefit of hindsight, Heyting's work can be seen as a truth-maker analysis:

proposition C is true = $\text{Proof}(C)$ exists.

The notion of a *proof of a proposition*, or "proof-object" for short, is novel with intuitionism; previously all proving had been at the level of judgements and not at that of their contents.[27] Furthermore, the notion of existence that is at issue here is not that of realist "obtaining" or something similar. On the contrary, it is the constructivist notion of existence, which was made fully explicit by Hermann Weyl, who held that an existential judgement was only an *Urteilsabstrakt* that had to be grounded in a real judgement, for instance:

construction c is a proof for proposition A.

Thus the form of judgement [C is true] is reduced back to a certain judgement of the traditional [S is P] form; namely

c is a truth-maker for C.[28]

7.

With this apparatus I am now ready to treat of 'the great works' of Frege and Gentzen. There is a crucial difference between Gentzen and the other logicians treated of so far: Gentzen belongs to the period *after* the metamathematical revolution. He received his logical training within the circle around Hilbert and so, strictly speaking, his formal languages are uninterpreted. Its

formulae are (meta)mathematical objects of study to be spoken *about*, rather than by means of, and he belongs to Jean van Heijenoort's paradigm of logic as calculus, whereas the previous logicians all belong to the earlier paradigm of logic as language.[29] According to Dummett it was Gentzen who showed how proof theory should be done. The proper evaluation of this claim depends strongly on how to interpret "proof theory". Three readings suggest themselves: (i) a formal machinery that induces a "syntactic" consequence relation for an uninterpreted formalism, and which ideally has to coincide with that determined by a certain given formal semantics, (ii) contribution(s) to the Hilbert programme, and (iii) that branch of epistemology that treats of demonstrative knowledge. With respect to the first two readings Dummett is certainly right. However, when our aim is a comparison with Frege, reading (iii) is the relevant one.

Frege's system was an interpreted one; a well-formed formula in the *Begriffsschrift* does express a proposition in virtue of being a name of a truth-value.[30] Thus, in order to carry out a fair comparison, the basic notions of Gentzen's system must be supplied with careful meaning-explanations converting the formulae into propositions. In particular, if we are going to convert his (meta)mathematical production-systems for strings of (meta)mathematical objects into a meaningful language, not only the terms and formulae should be interpreted; also the derivation trees should be assigned semantic values. Finally, corresponding to Frege's use of the *Urteilsstrich*, an account must be given at the pragmatic level of the notion of an assertion and how it interacts with that of assumption.

Gentzen actually gave *two* formulations of his natural deduction systems that incorporate the idea of assumptions in different ways. First there is the standard variant from his 1933 Göttingen dissertation.[31] Here the derivable objects are formulae, which may depend on certain assumptions, and several rules serve to discharge such open assumptions. A derivation thus takes the general form:

$$D: \quad \begin{matrix} A_1, & A_2, & \dots, & A_k, \\ & \bullet & & \bullet \\ & \bullet & \bullet & \\ & & \bullet & \\ & & C. & \end{matrix}$$

The other form, from the first consistency-proof, makes use of *Sequenzen*, which German term is translated into English as *sequents*.[32] Why Gentzen left out *Kon* is a mystery to me; he took his term and the notion from the earlier works of Paul Hertz, but surely the sequents are consequence(-relation)s. The sequent $A_1, A_2, \dots, A_k \Rightarrow C$ should be read:

C is true, when $A_1, A_2, \dots, A_k$ are true.

This rendering, which is taken from Gentzen's first publication (on Hertz's systems) could have been offered by Bolzano.[33] A variant reading of the sequent is:

C is true under the assumptions $A_1, A_2, \ldots, A_k$.[34]

In the 1936 version of Natural Deduction the derivable objects are sequents and the derivation trees start, not with assumptions, but with axioms of the form $A \Rightarrow A$, which correspond to the making of assumptions in the 1933 version.[35]

Let us consider his arithmetical language: I grant Dummett that if we treat it as an interpreted language, rather than as an uninterpreted object of study (Hilbert calculus), we obtain mathematical propositions. How should the derivation trees be interpreted? One problem with standard predicate-calculus languages, such as that of Gentzen, is that the well-formed formulae do double duty. They serve as formalistic *simulacra* for *two* notions, namely mathematical propositions and theorems proved, that is, judgements known; on the one hand, they are what composite formulae are built from, that is, propositional analogues, and, on the other hand, they are what is derived by proof trees, that is, the analogues of theorems. An attempt to redress this using force indicators of the same kind as Frege's *Urteilsstrich* rapidly gets into trouble. Consider the following example D':

$$
\frac{\dfrac{[A]}{B} \qquad A}{B.}
\quad A \supset B
$$

This example is well known from one of Dag Prawitz's "reductions": the implication $A \supset B$ has been introduced and is immediately eliminated again.[36] The minor premiss of the $\supset E$ is the formula A. In an interpreted calculus this has to be an assertion whereas the implication antecedent A in the major premiss of the $\supset E$ is neither asserted nor assumed.[37] It constitutes no problem, however, since it occurs as a propositional part of the major premiss, which, as a whole, is asserted. The assumption formula A, on the other hand, is assumed; so perhaps we might make do with one assumption indicator and one assertion indicator. If we consider the premiss of the $\supset I$, however, stalemate results. It is certainly not asserted outright, nor is it assumed, so how should it be interpreted? First, we begin by taking seriously that what is asserted are not propositions, but statements. The formulae are propositions. Appending

(is) true

to a proposition A produces a statement

A true.

The result is a tree D'':

$$
\begin{array}{cc}
[A]? & | \\
| & | \\
B? & | \\
\hline
A \supset B \text{ true} \quad A \text{ true}
\end{array}
$$

$$B \text{ true.}$$

With the assumption formula A, and the formula B that depends on it, we are at a loss. Consequently, in Gentzen derivations, what in general is ascribed to the conclusion formulae of derivation trees is not truth outright, but *dependent* truth:

  ... is true, given that the assumptions $A_1, A_2, \ldots, A_k$ are true, or

  ... true (A_1 true, A_2 true, $\ldots$, A_k true)

for short.

The diagram can then be completed into D''':

$$
\begin{array}{cc}
A \text{ true } (A \text{ true}) & | \\
| & | \\
B \text{ true } (A \text{ true}) & | \\
\hline
A \supset B \text{ true} \quad A \text{ true}
\end{array}
$$

$$B \text{ true.}$$

Here we infer the categorical truth of the implication $A \supset B$ from the *dependent* truth of the proposition B. Remark further that *all* statements, also those ascribing *dependent* truth to propositions, are asserted here. "The assumption of A" corresponds to an assertion that A is true, dependent on the truth of A. This tree D''', however, is not the original Gentzen proof-tree. In particular, the *assumptions have disappeared*.

The original, unadorned, proof tree D', we can say, then shows the truth of the proposition B in virtue of being a notation for a proof(-object) for the proposition B. In fact, the whole diagram can best be read as

  D' is a proof of the proposition B.

The 1936 version, on the other hand, when interpreted, has to have a novel form of judgement, namely

  sequent S holds.

This is a most natural generalisation of the form of judgement

  proposition A is true

in that we demand a verification also here. In order to have the right to assert that S holds I must have a function f such that when a_1 is a proof of $A_1, \ldots, a_k$

is a proof of A_k, then $f(a_1, \ldots, a_k)$ is a proof of C. In particular, we must note that the three statements

$A \supset B$ true, which demands a proof-object formed by $\supset I$,

B true (A true), which demands a dependent proof-object $b : B$, given that $x : A$, and

$A \Rightarrow B$ holds, which demands a function $f : (A)B$,

are equivalent (that is, when one is correct, so are the other two), but they differ in meaning, of course, since their respective assertion-conditions are different.[38]

The 1933 derivations, when interpreted, become (dependent) proof-objects for propositions. What about the 1936 derivation trees? When a mathematician proves a theorem, he obtains an object of knowledge, namely the theorem proved, but he also leaves a trace.[39] This may have a subjective, and often physical, character: coffee cups, scrap paper, cigarette stubs, or what have you. This trace will not enable us to demonstrate the theorem ourselves. Another kind of trace, though, is a carefully written out blue-print for how to proceed in order to demonstrate the theorem. Traces of this kind are published in mathematical texts and are commonly called proofs, or demonstrations. The derivations in the (1936) sequential formulation of Natural Deduction are such blueprints for acts that demonstrate their conclusion statements.

Finally, then, what about Dummett on Gentzen and Frege? Is he right? Bolzano had a splendid theory of consequence, whereas Frege was more right about inference than he is given credit for. However, only with Gentzen (1936) do we get a formulation of logic that does equal justice to alethic consequence among propositions as well as to epistemic inference from judgement(s) to judgement. Thus, strictly speaking, Dummett was right in what he said, but he was wrong in what he meant; the use of assumption formulae does not convince in a fair comparison with Frege.

Notes

1. Fichte (1797).

2. Frege (1906, p. 387). (My) English translation: An inference is an "Act of judgement, which is made, according to logical laws, on the basis of judgements already made."

3. The diagram is based on a similar one in Maritain (1946, p. 6), but is reasonably standard. Maritain's source, and also that of other Neo-Thomists, is the splendid *Ars Logica* by John of St. Thomas.

4. *KrV A69*. (My) English translation: "All operations of the intellect can be reduced to judgements, so that in general the intellect can be seen as a capacity for judging."

5.   Possibly Bolzano wanted to demonstrate, by means of an example, what a proper, realist *Wissenschaftslehre* should be in reaction to the earlier idealist effort of Fichte. The latter, however, is hardly mentioned in Bolzano's severely realist treatise and Kant is the only idealist foe worthy of. Bolzano's exposition has surely benefited from the fortuitous circumstance that it could be developed in adversity to a (the) major idealist philosopher of the age. This is brought out is sharp relief when one compares the early realist writings of Moore and Russell with the depth and width of the pellucid *Wissenschaftslehre*: their idealist adversaries were Bradley, Bosanquet, Green, and Joachim.

6.   To my mind this unfortunate terminology – *Vorstellung an sich* – comes close to a *contradictio in adiecto*.

7.   Russell (1903, Appendix A) might be responsible for sanctioning the unfortunate use of the term proposition for the Fregean *Gedanken*.

8.   In the discussion after my Cracow lecture, Wolfgang Künne objected to my ascribing the [proposition *A* is true] form of judgement to Bolzano. Künne is certainly right in that all of *A*, *A* is true, [*A* is true] is true, [[*A* is true] is true] is true, etc., are *propositions* (rather than judgements), which observation constitutes the basis for Bolzano's "proof" that there are infinitely many different truths (WL §32). Indeed, in §35 Bolzano states that 'jedes Urtheil auch ein Satz ist'. So every judgement is a proposition, but is it also a proposition-in-itself? My reading is certainly compatible with §36, where we find: 'ein jedes Urtheil, das einen wahren Satz enthält', that is, every judgement that *contains* a true proposition (my emphasis). So on that reading the judgement would be a mental or linguistic statement that a proposition is true, and it is correct if the proposition whose truth is stated actually is a truth.

9.   (1918, p. 62). The key observation on which my argument is based can be found in Frege (1892, p. 35), but it is used for another purpose there.

10.   Lack of space, and the level of abstraction at which I want to move, prevents me from doing full justice to Bolzano here on at least three scores: (i) I ignore the demand of compatibility that he imposes on the antecedents in consequence relations; (ii) I leave out the relation of *Abfolge*, with its concomitant grounding proof-trees (§220), and (iii) I do not treat of the intricate elaboration concerning the "*mediation*" (*Vermittlung*) of judgements (§300.7).

11.   In this Bolzano was followed by virtually the entire modern tradition in *classical* logic. Similar accounts of validity can be found in Wittgenstein's *Tractatus*, in Ajdukiewicz (1934), as well as in the writings of Quine, just to mention some out of many, many places.

12.   Brentano presented his theory in lectures at Würzburg 1870–1, and repeatedly until 1895 at Vienna.

13.   Brentano (1930).

14.   I am indebted to Per Martin-Löf for drawing my attention to this passage.

15.   (1930, IV:4, pp. 140–143).

16.   Bergmann (1966, pp. 307–308).

17.   Berg (1962, p. 55) lists the relevant WL passages.

18.   (1892, p. 34): 'A judgement is not the mere grasping of a Thought, but the recognition of its truth.' In Bs p. 2 and p. 4 the suggestion is made that the form of judgement is [… *ist eine Thatsache*], where the blank has to be filled with an "*Umstand*". Thus the form of the judgement made through an assertoric utterance of 'Snow is white' is [The circumstance that snow is white is a fact]. In place of circumstance, Frege also allows for "*Satz*". In (1918, p. 74) he identifies *fact* with *true Thought*, which yields the final reformulation [The proposition that snow is white is true].

19.   Note, though, that an iterated Frege-conditional can equally well be read as a Bolzano-Gentzen sequent (which expresses a consequence relation between propositions). This has recently been remarked by Franz von Kutschera and Peter Schröder-Heister. However, the point was made with maximum clarity already by Pavel Tichy (1988, p. 252). A notion close to Bolzano's *formale Abfolge* is found in Frege (1906, III). In my (2000) I suggest that this is no accident, but that Frege actually read Bolzano in 1905/06.

20.   The numerals within brackets indicate the order of conceptual priority of the notions and their respective correctness-notions: truth of content, correctness of the product, and rightness of the act. For Bolzano and Frege, the truth of propositions is *the* key notion of logic.

21.   Dummett (1981, p. 33).

22.   Dummett (1973, p. 309 and p. 435, respectively).

23. For a more balanced view, as well as references to Frege, see Currie (1987).

24. As far as I know, logical truth occurs only in (1923, p. 50), the very latest of Frege's articles, and then only as a result of Frege's struggles with Wittgenstein's *Tractatus*.

25. (1918, p. 60).

26. For terminology and background, see Mulligan, Simons, and Smith (1984).

27. Concerning proof-objects and their use in constructive semantics, see my (1983), (1993), and (2000a).

28. Documentation for the claims of this section can be found in a number of my papers, for example, (1994), (1997), and (1999).

29. See my (1999, pp. 140–141) for a discussion of the Van Heijenoort paradigm from the present perspective.

30. Or would have done so, had Frege's semantical elaboration in §§ 29–31 of the *Grundgesetze* actually worked. Dummett is trivially right if we take the inconsistency of Frege's system into account; that, however, was not the point at issue but the proper way of keeping the books of logical deduction.

31. Published as (1934–35).

32. (1936).

33. (1932, p. 320).

34. (1936, p. 512).

35. Dummett's (1977, Ch. 4) exposition of normalization and other matters actually uses the (1936) formulation of Natural Deduction.

36. (1965, p. 37).

37. For this "Frege point", see Geach (1965).

38. All three statements, it should be noted, are refuted by the same counterexample, namely a pair of proof-objects for A and $\neg B$, respectively.

39. I owe the notion of trace to Per Martin-Löf. It is treated of at some length in my (1993) and (2000a).

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