Weakly pushed nature of "pulled" fronts with a cutoff

Debabrata Panja and Wim van Saarloos
Instituut Lorentz, Universiteit Leiden, Postbus 9506, 2300 RA Leiden, The Netherlands
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The concept of pulled fronts with a cutoff ϵ has been introduced to model the effects of the discrete nature of the constituent particles on the asymptotic front speed in models with continuum variables (pulled fronts are the fronts that propagate into an unstable state, and have an asymptotic front speed equal to the linear spreading speed v^* of small linear perturbations around the unstable state). In this paper, we demonstrate that the introduction of a cutoff actually makes such pulled fronts weakly pushed. For the nonlinear diffusion equation with a cutoff, we show that the longest relaxation times τ_m that govern the convergence to the asymptotic front speed and profile, are given by $\tau_m^{-1} \simeq [(m+1)^2-1]\pi^2/\ln^2\epsilon$, for $m=1,2,\ldots$.

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I. INTRODUCTION

Pulled fronts are fronts that propagate into an unstable state, for which the propagation dynamics is essentially as if they are being pulled along by the growth and spreading of small perturbations about the unstable state, into which the front propagates. Concretely, this means that their asymptotic speed v_{as} is equal to the linear spreading speed v^* of perturbations around the unstable state, $v_{as}=v^*$ [1–6]. Fronts that propagate into an unstable state but for which $v_{as}>v^*$ are often termed "pushed." The name stems from the intuitive idea [7,8] that in this regime, the dynamics in the nonlinear front region or the bulk region behind the front actually *drives* the front propagation: effectively it pushes the front from behind, and the front moves with a speed that is higher than the natural speed with which small perturbations about the unstable state spread by themselves ahead of the front.

It is clear from the definition that the concept of a pulled front essentially pertains to a continuum formulation of the relevant dynamical variables. The linear spreading speed v^* is defined and calculated in practice by considering perturbations of *arbitrarily small amplitude* about the unstable state of the dynamical equations; the value of v^* then follows from an asymptotic analysis of the linearized dynamical equations [5]. However, in all cases, in which one cannot ignore the fact that matter is made of discrete particles, one cannot perturb the unstable state by any arbitrary small amount, because this amount must be at least one "quantum" of particle large.

To model this discrete nature of the constituent particles by means of a continuum equation, Brunet and Derrida [6] studied the nonlinear diffusion equation

$$\frac{\partial \phi}{\partial t} = \frac{\partial^2 \phi}{\partial x^2} + f(\phi) \tag{1}$$

with a cutoff ϵ in the growth term $f(\phi)$,

$$f(\phi) = \Theta(\phi-\epsilon)[\phi-\phi^n], \quad n>1, \quad \text{e.g., } n=2 \text{ or } 3. \tag{2}$$

Without the cutoff, i.e., for $\epsilon=0$, this equation is the well-known nonlinear diffusion equation, which has been used since long as the simplest model to study front propagation into an unstable state [9–11]. Brunet and Derrida [6] found that the asymptotic front speed v_{as} goes as

$$v_{as}=v_\epsilon \simeq v^* - \frac{\pi^2}{\ln^2\epsilon}, \tag{3}$$

where $v^*=2$ is the asymptotic speed of the corresponding pulled front of Eq. (1) for $\epsilon=0$. The above formula shows that the front speed v_ϵ converges very slowly to the asymptotic speed v^*; this illustrates that unlike pushed fronts, pulled fronts are very sensitive to small changes in the dynamics of the phase into which they propagate.

In comparing with stochastic models of particles on a lattice, Brunet and Derrida associated the cutoff ϵ with $1/N$, where N is the average number of particles in a correlation region in the saturation phase behind the front [6]. Although the validity of this identification has been the matter of some debate, it appears that Eq. (3) with $\epsilon=1/N$ does give the proper asymptotic correction to the front speed even for very large N. We refer to the literature [6,12–15] for a further discussion of the applicability of these ideas to stochastic models.

It is intuitively clear that as soon as we introduce this cutoff, fronts that are pulled for $\epsilon=0$ must actually become weakly pushed as soon as $\epsilon>0$. After all, any perturbation around the value $\phi=0$ does not start to grow until the local ϕ value crosses ϵ, so strictly speaking, the linear spreading velocity $v^*(\epsilon)$ of *arbitrarily small* linear perturbations about the state $\phi=0$ vanishes. As $v_\epsilon>v^*(\epsilon)=0$, one clearly must have a weakly pushed front. With this idea in mind, it is natural to address the convergence of the front speed to the asymptotic value, since it is well known that the speed of pulled fronts relaxes algebraically slowly to the asymptotic value v^* [3–6], while pushed fronts normally have exponential relaxation to their asymptotic speed.

These observations motivate us to investigate here the slowest relaxation modes of the stability spectrum of fronts for the nonlinear diffusion equation (1), with a cutoff ϵ in the

growth term (2). We calculate these relaxation modes explicitly for small ϵ, and find that the slowest relaxation times τ_m are given by

$$\tau_m^{-1} \simeq \frac{[(m+1)^2-1]\pi^2}{\ln^2 \epsilon}, \quad m=1,2\ldots. \tag{4}$$

Hence, the relaxation times of the front velocity and profile approach zero as $\epsilon \to 0$, but only logarithmically slowly. Just like the corrections to the front speed for practical values of ϵ are often significant, so is the exponential relaxation, for example, for $\epsilon = 10^{-5}$, the longest relaxation time τ_1 is about 4.48. Thus, while in the absence of a cutoff the front speed is approached very slowly, only as $3/2t$ where t is the time [3–6], with a realistic value of ϵ, the front speed converges relatively quickly to the asymptotic value.

II. STABILITY ANALYSIS OF THE ASYMPTOTIC FRONT SOLUTION

A. The stability operator

The asymptotic shape of the front is a uniformly translating front solution $\phi_\epsilon(x,t)$ which is a function of *only* the comoving coordinate $\xi = x - v_\epsilon t$, and which is obtained by solving the ordinary differential equation

$$-v_\epsilon \frac{d\phi_\epsilon(\xi)}{d\xi} = \frac{d^2\phi_\epsilon(\xi)}{d\xi^2} + f(\phi_\epsilon(\xi)). \tag{5}$$

In carrying out the linear stability analysis of this front solution, it is convenient to follow the standard route of transforming the linear eigenvalue problem into a Schrödinger eigenvalue problem [2,5]. We consider a function $\phi(x,t)$, which is infinitesimally different from $\phi_\epsilon(\xi) \equiv \phi_\epsilon(x - v_\epsilon t)$ in the comoving frame, i.e., $\phi(x,t) = \phi_\epsilon(x - v_\epsilon t) + \eta(\xi,t)$. Upon linearizing the dynamical equation in the comoving frame, one finds that the function $\eta(x,t) \equiv \eta(\xi,t)$ obeys the following equation:

$$\frac{\partial \eta}{\partial t} = v_\epsilon \frac{\partial \eta}{\partial \xi} + \frac{\partial^2 \eta}{\partial \xi^2} + \frac{\delta f(\phi)}{\delta \phi}\Big|_{\phi=\phi_\epsilon} \eta. \tag{6}$$

Since this equation is linear in η, the question of stability can be answered by studying the spectrum of the temporal eigenvalues. To this end, we express $\eta(\xi,t)$ as

$$\eta(\xi,t) = e^{-Et} e^{-v_\epsilon \xi/2} \psi_E(\xi), \tag{7}$$

which converts Eq. (6) to the following one-dimensional Schrödinger equation for a particle in a potential (with $\hbar^2/2m = 1$),

$$\left[-\frac{d^2}{d\xi^2} + \frac{v_\epsilon^2}{4} - \frac{\delta f(\phi)}{\delta \phi}\bigg|_{\phi=\phi_\epsilon} \right] \psi_E(\xi) = E \psi_E(\xi). \tag{8}$$

In Eq. (8), the quantity

$$V(\xi) = \left[\frac{v_\epsilon^2}{4} - \frac{\delta f(\phi)}{\delta \phi}\bigg|_{\phi=\phi_\epsilon} \right]$$

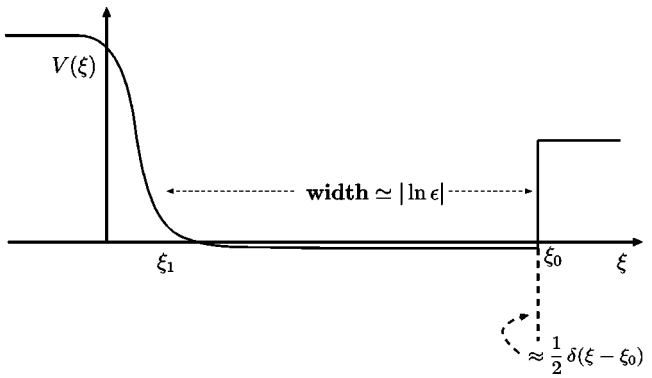

FIG. 1. The potential $V(\xi)$ in the Schrödinger operator obtained in the stability analysis.

plays the role of the potential. If we now denote by ξ_0 the coordinate of the point where $\phi_\epsilon(\xi) = \epsilon$, then for the nonlinearity (2) the potential $V(\xi)$ is easily seen to have the form

$$V(\xi) = \left[\frac{v_\epsilon^2}{4} - 1 + n\phi_\epsilon^{n-1}(\xi) \right] \Theta(\xi_0 - \xi) - \frac{1}{v_\epsilon} \delta(\xi - \xi_0) + \frac{v_\epsilon^2}{4} \Theta(\xi - \xi_0). \tag{9}$$

The δ-function in Eq. (9) appears from the functional derivative in Eq. (6), since there is a discontinuity of magnitude ϵ in $f(\phi)$ at $\phi = \epsilon$. This discontinuity contributes an amount equal to

$$\phi_\epsilon \frac{d\Theta(\phi_\epsilon - \epsilon)}{d\phi_\epsilon} = \phi_\epsilon \delta(\phi_\epsilon - \epsilon) = \frac{\epsilon}{|\phi_\epsilon'(\xi_0)|} \delta(\xi - \xi_0) \tag{10}$$

to $V(\xi)$. If we combine this with the fact that $|\phi_\epsilon'(\xi_0)| = \epsilon v_\epsilon$, which follows immediately from the fact that one simply has $\phi_\epsilon(\xi) = \epsilon e^{-v_\epsilon(\xi - \xi_0)}$ for $\xi \geq \xi_0$, one obtains the δ-function term in the potential given in Eq. (9).

The form of the potential $V(\xi)$ is sketched in Fig. 1. Notice that $\phi_\epsilon(\xi)$ is a monotonically increasing function from ϵ at ξ_0 *towards the left*, asymptotically reaching the value 1 as $\xi \to -\infty$. As a result, for $\xi < \xi_0$, $V(\xi)$ also increases monotonically towards the left, from $v_\epsilon^2/4 - 1 + n\epsilon^{n-1} \simeq -\pi^2/\ln^2 \epsilon$ at $\xi = \xi_{0-}$, to $(n - \pi^2/\ln^2 \epsilon) \approx n$ as $\xi \to -\infty$. At ξ_0, there is an attractive δ-function potential of strength $1/v_\epsilon \approx 1/2$ and a finite step of height 1. The crucial feature for the stability analysis below is the fact that $V(\xi)$ stays remarkably flat at a value $-\pi^2/\ln^2 \epsilon$ over a distance $(\xi_0 - \xi_1) \simeq |\ln \epsilon|$, and then on the left of ξ_1, it increases to the value $\approx n$, over a distance of order unity. As argued in Sec. II B, this is a consequence of the nature of the solution $\phi_\epsilon(\xi)$.

If there are negative eigenvalues of the above Schrödinger equation, then according to Eq. (7), $\eta(\xi,t)$ grows in time in the comoving frame, i.e., the front solution $\phi_\epsilon(\xi)$ is unstable. On the other hand, if there are no negative eigenvalues, then the asymptotic front shape is stable, and the spectrum of the eigenvalues then determines the nature of the relaxation of $\phi(x,t)$ to the solution $\phi_\epsilon(\xi)$.

The full spectrum in general depends on the boundary conditions imposed on the eigenfunctions ψ_E. Here we consider only localized perturbations, for which we need to have $\eta(\xi,t)\to 0$ as $\xi\to\pm\infty$. Due to the exponential factor in Eq. (7), any eigenfunction ψ_E that vanishes as $\xi\to\infty$ is consistent with vanishing η towards the right [16]. However, for $\xi\to-\infty$, the eigenfunctions ψ_E need to vanish exponentially fast with a sufficiently large exponent, so that when it is combined with the exponentially diverging term $e^{-v_\epsilon/2}$, they are still consistent with the requirement that η vanishes for $\xi\to-\infty$. For the lowest "energy" eigenvalues, which we will investigate below, we will demonstrate that these requirements are obeyed.

B. Shape of $\phi_\epsilon(\xi)$ and the zero mode of the stability operator

From the form in the potential, it is clear that the lowest "energy" eigenmodes, i.e., the slowest relaxation eigenmodes, are the ones that are confined to the bottom of the potential. This is the region where the nonlinear terms proportional to ϕ^{n-1} are negligible, and which is often called the "leading edge" of the front profile. For $\epsilon\ll 1$, the solution of $\phi_\epsilon(\xi)$ in this leading edge is given by [6]

$$\phi_\epsilon(\xi)\approx\frac{|\ln\epsilon|}{\pi}\sin[z_i\xi]e^{-z_r\xi}\quad\text{for}\quad\xi_1\lesssim\xi\lesssim\xi_0\simeq|\ln\epsilon|$$
$$=\epsilon e^{-v_\epsilon(\xi-\xi_0)}\quad\text{for}\quad\xi\gtrsim\xi_0. \tag{11}$$

Here, $z_i\approx\pi/|\ln\epsilon|$ and $z_r=1+O(\epsilon^2)$. The values of $\phi_\epsilon(\xi)$ and $d\phi_\epsilon/d\xi$ are continuous at $\xi=\xi_0$, and $\phi_\epsilon(\xi_0)=\epsilon$. Although Eqs. (11) and (12) suggest at first sight that the $\phi_\epsilon(\xi)$ has a node at $\xi=0$, Eq. (11) is only valid in the leading edge, and $\phi_\epsilon(\xi)$ crosses over to other behavior around ξ_1, which makes the front solution $\phi_\epsilon(\xi)$ a monotonically decreasing function of ξ. The value of ξ_1 is set by the criterion that around ξ_1 the nonlinear terms of $f(\phi_\epsilon(\xi))$ start to become significant, just like ξ_1 marks the point where the potential $V(\xi)$ crosses over from the asymptotic value on the left to the bottom value. The coordinate ξ_1, therefore, is more or less fixed; on the other hand, ξ_0 asymptotically diverges as $\simeq|\ln\epsilon|$ for small ϵ, making $(\xi_0-\xi_1)$ also diverge as $\simeq|\ln\epsilon|$. This is an immediate consequence of the overall exponential decay of $\phi_\epsilon(\xi)$ in ξ at the leading edge.

From the form in the potential, it is clear that the lowest "energy" eigenmodes, i.e., the slowest relaxation eigenmodes, are the ones that are confined to the bottom of the potential. We notice that among these modes, invariably there is a zero mode of the stability operator that is associated with the uniformly translating front solution of a dynamical equation, e.g., Eq. (1): since $\phi_\epsilon(\xi)$ and $\phi_\epsilon(\xi+a)$ are solutions of Eq. (5) for any arbitrary a, we find by expanding to first order in a that $\psi_0(\xi)=e^{v_\epsilon\xi/2}d\phi_\epsilon/d\xi$ is a solution of Eq. (8) with eigenvalue $E=0$. From the result (11) for the asymptotic front solution, we then immediately get *to dominant order*

$$\psi_0\sim\sin z_i\xi,\quad z_i\simeq\pi/|\ln\epsilon|,\quad\xi_1\lesssim\xi\lesssim\xi_0. \tag{12}$$

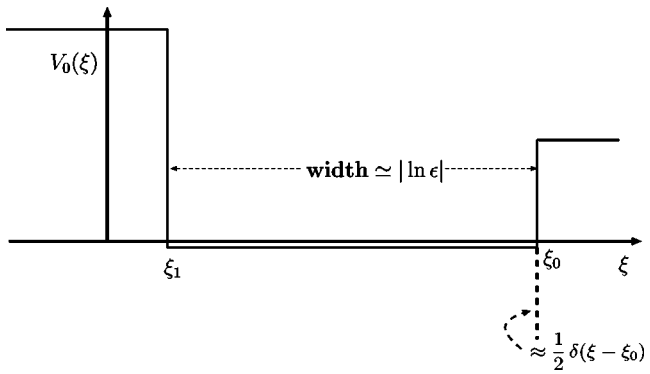

FIG. 2. The approximate potential $V_0(\xi)$ that can be used for calculating the low-lying modes for large widths of the bottom well, i.e., for small ϵ.

Furthermore, since $\phi_\epsilon(\xi)$ is a monotonically decreasing function of ξ, the solution $\psi_0(\xi)=e^{v_\epsilon\xi/2}d\phi_\epsilon/d\xi$ is nodeless. Since we know from elementary quantum mechanics that the nodeless eigenfunction has the lowest eigenvalue, this implies that all the *other* eigenvalues of Eq. (8) are positive, i.e., the solution $\phi_\epsilon(\xi)$ is stable.

The spectrum of eigenvalues of Eq. (8) for $E>0$, therefore, is going to determine the decay property of localized perturbations $\eta(\xi,t)$ in time. We notice that for $E>v_\epsilon^2/4\approx 1$, the value of the potential on the far right, the spectrum of eigenvalues will be continuous. However, we are particularly interested in the smallest eigenvalues $E_m>0$ for small m, since these are the eigenmodes that decay the slowest in time. These are the eigenvalues associated with bound states in the potential well.

C. Lowest eigenmodes and eigenvalues for $\epsilon\ll 1$

As $\epsilon\to 0$, the bottom well of the potential becomes very wide: its width diverges as $|\ln\epsilon|$. As we know from elementary quantum mechanics, the lowest "energy" eigenfunctions then become essentially sine or cosine waves in the potential well with small wave numbers k and correspondingly small "energy" eigenvalues.

Based on the fact that the potential $V(\xi)$ on the left rises over length scales of order unity, we now make an approximation. In the limit that the bottom well is very wide and the k values of the bound state eigenmodes very small, it becomes an increasingly good and an asymptotically correct approximation to view the left wall of the well simply as a steep step, as sketched in Fig. 2—we thus approximate the potential by

$$V_0(\xi)=n[1-\Theta(\xi)]-\frac{\pi^2}{\ln^2\epsilon}\Theta(\xi)[1-\Theta(\xi-\xi_0)]$$
$$-\frac{1}{2}\delta(\xi-\xi_0)+\Theta(\xi-\xi_0). \tag{13}$$

On the right hand side, there is an attractive δ-function potential at the point where the potential shows a step to a value close to 1. It is easy to check that the prefactor of the

δ-function of 1/2 is not strong enough to give rise to bound states with $E<0$, and as a result, for very small values of ϵ, the low-lying eigenmodes approach sine waves with nodes at the position of the walls of the potential [17]

$$\psi_m \simeq \sin[k_m(\xi-\xi_1)]. \tag{14}$$

The condition that these solutions have nodes at the right edge of the well then yields

$$k_m \simeq \frac{(m+1)\pi}{\xi_0-\xi_1} \simeq \frac{(m+1)\pi}{|\ln\epsilon|}, \tag{15}$$

implying that the corresponding eigenvalues are given by

$$E_m \simeq \frac{[(m+1)^2-1]\pi^2}{\ln^2\epsilon}, \quad m=0,1,2,\ldots. \tag{16}$$

Here, the first term between square brackets comes from the ''kinetic energy'' term k^2, while the second term originates from the value of the potential at the bottom.

Note that for $m=0$, the eigenmode $\sin k_0$ with eigenvalue E_0 is indeed the same as the zero eigenmode of Eq. (12) with $k_0=z_i$, which we calculated from the shape of the front solution ϕ_ϵ in the leading edge. Besides verifying the consistency of our approach, this also confirms that there are no corrections to Eq. (16) for $m=0$: for $m=0$ it will yield an eigenvalue zero to all orders in ϵ. Therefore, the smallest nonzero eigenvalue, which governs the relaxation of the front velocity and profile to the asymptotic ones, is E_1 with relaxation time τ_1 given by

$$\tau_1^{-1}=E_1 \simeq \frac{3\pi^2}{\ln^2\epsilon}. \tag{17}$$

Equation (16) also confirms that as $\epsilon\to 0$, the gap between the spectral lines decreases as $\ln^{-2}\epsilon$, which is consistent with the fact that for a pulled front $\epsilon=0$ and the spectrum becomes gapless. Also notice that for the eigenvalues in Eq. (16), the corresponding eigenmodes $\psi_E(\xi)$ decay as $\exp(-\sqrt{n}|\xi|)$ for $\xi\to-\infty$ and as $\exp(-v_\epsilon\xi/2)$ for $\xi\to\infty$, which make $e^{v_\epsilon\xi/2}\psi_E(\xi)$ go to zero for $\xi\to\pm\infty$, satisfying the boundary conditions discussed previously at the end of Sec. II A.

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[16] The fact that eigenfunctions ψ_E, which diverge as $\xi\to\infty$, are allowed, means that there are admissible eigenfunctions that are not in the Hilbert space of the Schrödinger operator. See, e.g., Ref. [5] for further discussion on this point.
[17] More explicitly, if we write the solution within the well as $A\sin[k(\xi-\xi_1)+B]$, and for $\xi>\xi_0$ as $A_+e^{-(\xi-\xi_0)}$ (which is correct to lowest order in ϵ), then we get from the boundary conditions at ξ_0: $2k\cot[k(\xi_0-\xi_1)+B]=-1$; likewise, at the left boundary, of the well, we get $k\cot B=$const, where the constant is determined by the size of the potential step. For $\xi_0-\xi_1\gg 1$, the small-k solutions are those stated in the text with $B\to 0$.