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Quantum theory of electromechanical noise and momentum transfer statistics

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A quantum-mechanical theory is developed for the statistics of momentum transferred to the lattice by conduction electrons Results for the electromechanical noise power in the semiclassical diffusive transport regime agree with a recent theory based on the Boltzmann-Langevin equation All moments of the transferred momentum are calculated for a single channel conductor with a localized scatterer, and compared with the known statistics of transmitted charge

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I. INTRODUCTION

Electrical current is the transfer of charge from one end of the conductor to the other The statistics of this charge transfer was investigated by Levitov and Lesovik [1] It is binomial for a single-channel conductor at zero temperature and double Poissonian at finite temperature in the tunneling regime [2] The second cumulant, the noise power, has been measured in a variety of systems [3] Ways of measuring the third cumulant have been proposed,[2,4] but not yet carried out

Electrical current also transfers momentum to the lattice The second cumulant, the electromechanical noise power, determines the mean-square displacement of an oscillator through which a current is driven It has been studied theoretically,[5–8] and is expected to lie within the range of sensitivity of nanomechanical oscillators [9] No theory exists for higher order cumulants of the transferred momentum (which would determine higher cumulants of the oscillator displacement) It is the purpose of the present paper to provide such a theory

In the context of charge transfer statistics there exist two approaches a fully quantum-mechanical approach using Keldysh Green functions[1,10] and a semiclassical approach using the Boltzmann-Langevin equation [11] Here we take the former approach, to arrive at a quantum theory of momentum transfer statistics As a test, we show that the second moment calculated from Keldysh Green functions coincides in the semiclassical limit with the result obtained from the Boltzmann Langevin equation by Shytov, Levitov, and one of the authors [8]

A calculation of the complete cumulant generating function of transferred momentum (or, equivalently, of oscillator displacement) is presented for the case of a single-channel conductor with a localized scatterer The generating function in this case can be written entirely in terms of the transmission probability Γ of the scatterer In the more general multichannel case one also needs a knowledge of the wave functions This is an essential difference from the charge transfer problem, which can be solved in terms of transmission eigenvalues for any number of channels At zero temperature the momentum statistics is binomial, just as for the charge At finite temperature it is multinomial, even in the limit $\Gamma \to 0$, different from the double-Poissonian distribution of charge

The outline of the paper is as follows In Sec II we formulate the problem in a way that is suitable for further analysis The key technical step in that section is a unitary transformation which eliminates the dependence of the electron-phonon coupling Hamiltonian on the (unknown) scattering potential of the disordered lattice The resulting coupling Hamiltonian contains the electron momentum flow and the phonon displacement In Sec II we use that Hamiltonian to derive a general formula for the generating function of the distribution of momentum transferred to a phonon (as well as the distribution of phonon displacements) It is the analog of the Levitov-Lesovik formula for the charge-transfer distribution [1] For a localized scatterer we can evaluate this statistics in terms of the scattering matrix We show how to do this in Sec IV, and give an application to a single-channel conductor in Sec V In Secs VI and VII we turn to the case that the scattering region extends throughout the conductor We follow the Keldysh approach to derive a general formula for the generating function, and check its validity by rederiving the result of Ref 8 We conclude in Sec VIII with an order-of-magnitude estimate of higher-order cumulants of the momentum-transfer statistics

II. FORMULATION OF THE PROBLEM

The excitation of a phonon mode by conduction electrons is described by the Hamiltonian

$$H=\Omega a^\dagger a+\sum_i \mathbf{p}_i^2/2m+\sum_i V[\mathbf{r}_i-Q\mathbf{u}(\mathbf{r}_i)], \qquad (2\ 1)$$

where we have set $\hbar=1$ The phonon mode has annihilation operator a, frequency Ω, mass M, and displacement $Q\mathbf{u}(\mathbf{r})$, where $Q=(2M\Omega)^{-1/2}(a+a^\dagger)$ is the amplitude operator The electrons have position $\mathbf{r}_i$, momentum $\mathbf{p}_i=-i\partial/\partial\mathbf{r}_i$, and mass m Electrons and phonons are coupled through the ion potential $V(\mathbf{r})$ We assume a zero magnetic field Electron-electron interactions and the interactions of electrons and phonons with an external electric field have also been omitted

We assume that electrons and phonons are uncoupled at time zero and measure moments of the observable A of the phonons after they have been coupled to the electrons for a time t The operator $A(a,a^\dagger)$ could be the amplitude Q of the phonon mode, its momentum $P=-i(M\Omega/2)^{1/2}(a-a^\dagger)$, or its energy $\Omega a^\dagger a$ The moment generating function for A is

$$\mathcal{F}(\xi)=\sum_{m=0}^{\infty}\frac{\xi^m}{m!}\langle A^m(t)\rangle=\mathrm{Tr}\,e^{\xi A}e^{-iHt}\rho e^{iHt}. \quad (2.2)$$

The initial density matrix $\rho=\rho_e\rho_p$ is assumed to factorize into an electron part and a phonon part.

We assume small displacements, so an obvious way to proceed would be to linearize $V(\mathbf{r}-Q\mathbf{u})$ with respect to the phonon amplitude Q. Such a procedure is complicated by the fact that the resulting coupling $-Q\mathbf{u}\cdot\nabla V$ of electrons and phonons depends on the ion potential V. Because of momentum conservation, it should be possible to find the momentum transferred by the electrons to the lattice without having to consider explicitly the force $-\nabla V$. In the semiclassical calculation of Ref. 8 that goal is achieved by the continuity equation for the flow of electron momentum. The unitary transformation that we now discuss achieves the same purpose in a fully quantum-mechanical framework.

What we need is a unitary operator U such that

$$U^{\dagger}V[\mathbf{r}-Q\mathbf{u}(\mathbf{r})]U=V(\mathbf{r}). \quad (2.3)$$

For constant $\mathbf{u}$ we have simply $U=\exp[-iQ\mathbf{u}\cdot\mathbf{p}]$. More generally, for space-dependent $\mathbf{u}$, we need to specify the operator ordering (denoted by colons $:\cdots:$) that all position operators $\mathbf{r}$ stand to the left of the momentum operators $\mathbf{p}$. We also need to include a Jacobian determinant $\|J\|$ to ensure unitarity of U. As shown in Appendix A, the desired operator is

$$U=\|J\|^{1/2}{:}e^{-iQ\mathbf{u}(\mathbf{r})\cdot\mathbf{p}}{:},\quad J_{\alpha\beta}=\delta_{\alpha\beta}-Q\partial_\alpha u_\beta(\mathbf{r}), \quad (2.4)$$

with $\partial_\alpha\equiv\partial/\partial r_\alpha$. All this was for a single electronic degree of freedom. The corresponding operator for many electrons is $U=\Pi_i U_i$, where U_i is given by Eq. (2.4) with $\mathbf{r}$, $\mathbf{p}$ replaced by $\mathbf{r}_i$, $\mathbf{p}_i$.

Hamiltonian (2.1) transforms as $U^{\dagger}HU=H_0+H_{\mathrm{int}}$, with

$$H_0=\Omega a^{\dagger}a+\sum_i[\mathbf{p}_i^2/2m+V(\mathbf{r}_i)], \quad (2.5a)$$

$$H_{\mathrm{int}}=-QF-\frac{1}{M}P\Pi+\mathcal{O}(\mathbf{u}^2). \quad (2.5b)$$

Here F is the driving force of the phonon mode,

$$F=\frac{1}{4m}\sum_i[u_{\alpha\beta}(\mathbf{r}_i)p_{i\alpha}p_{i\beta}+p_{i\alpha}u_{\alpha\beta}(\mathbf{r}_i)p_{i\beta}]+\mathrm{H.c}, \quad (2.6)$$

and Π is the total electron momentum,

$$\Pi=\tfrac{1}{2}\sum_i\mathbf{u}(\mathbf{r}_i)\cdot\mathbf{p}_i+\mathrm{H.c.}, \quad (2.7)$$

weighted with the (dimensionless) mode profile $\mathbf{u}(\mathbf{r})$. We have defined the shear tensor $u_{\alpha\beta}=\frac{1}{2}(\partial_\alpha u_\beta+\partial_\beta u_\alpha)$. The abbreviation H.c. indicates the Hermitian conjugate and a summation over repeated Cartesian indices α,β is implied.

The interaction Hamiltonian H_{int} is now independent of the ion potential, as desired. In the first term $-QF$ we recognize the momentum flux tensor, while the second term $-P\Pi/M$ is an inertial contribution to the momentum transfer. The inertial contribution is of relative order $\Omega\lambda/v_F$ (λ being the wavelength of the phonon and v_F the Fermi velocity of the electrons) and typically $\ll 1$. In what follows we will neglect it. We also neglect the terms in H_{int} of second and higher order in $\mathbf{u}$, which contribute to order λ_F/L to the generating function (with L the length scale on which $\mathbf{u}$ varies). These higher order interaction terms account for the momentum uncertainty of an electron upon a position measurement by the phonon.

If we apply the unitary transformation U to generating function (2.2), we need to transform not only H but also $A\rightarrow U^{\dagger}AU=\tilde{A}$ and $\rho\rightarrow U^{\dagger}\rho U=\tilde{\rho}$, resulting in

$$\mathcal{F}(\xi)=\mathrm{Tr}\,e^{\xi\tilde{A}}e^{-it(H_0+H_{\mathrm{int}})}\tilde{\rho}e^{it(H_0+H_{\mathrm{int}})}. \quad (2.8)$$

In Appendix A we show that, quite generally, the distinction between ρ,A and $\tilde{\rho},\tilde{A}$ is irrelevant in the limit of a long detection time t, and we will therefore ignore this distinction in what follows.

If $\mathbf{u}$ is smooth on the scale of λ_F, so that gradients of $u_{\alpha\beta}$ can be neglected, one can apply the effective mass approximation to Hamiltonian (2.5). The ion potential $V=V_{\mathrm{lat}}+V_{\mathrm{imp}}$ is decomposed into a contribution V_{lat} from the periodic lattice and a contribution V_{imp} from impurities and boundaries that break the periodicity. The effects of V_{lat} can be incorporated in an effective mass m^* (assumed to be deformation independent[12,13]) and a corresponding quasimomentum p^*. The unperturbed Hamiltonian takes the usual form

$$H_0=\Omega a^{\dagger}a+\sum_i[\mathbf{p}_i^{*2}/2m^*+V_{\mathrm{imp}}(\mathbf{r}_i)]. \quad (2.9)$$

As shown in Appendix B, the force operator in H_{int} is then expressed through the flow of quasi-momentum,

$$F=\frac{1}{m^*}\sum_i p_{i\alpha}^* u_{\alpha\beta}(\mathbf{r}_i)p_{i\beta}^*, \quad (2.10)$$

whereas the inertial contribution is still given by Eq. (2.7) in terms of the true electron momentum.

III. MOMENTUM TRANSFER STATISTICS

A. Generating function

A massive phonon mode absorbs the momentum that electrons transfer to it without changing its displacement. We may therefore define a statistics of momentum transfer to the phonons without back action on the electrons by choosing the observable $A=P=-i(M\Omega/2)^{1/2}(a-a^{\dagger})$ in Eq. (2.2) and taking the limit $M\rightarrow\infty$, $\Omega\rightarrow 0$ at fixed $M\Omega$. We assume that the phonon mode is initially in the ground state, so that $a\rho_p=0$.

We transform to the interaction picture by means of the identity

$$e^{iH_0t}e^{-iHt}=\mathcal{T}\exp\left[-i\int_0^t dt' H_{\mathrm{int}}(t')\right], \quad (3.1)$$

where $\mathcal{T}$ denotes time ordering (earlier times to the right of later times) of the time-dependent operator $H_{\rm int}(t) = e^{iH_0 t}H_{\rm int}e^{-iH_0 t}$. In the massive phonon limit we have $H_{\rm int}(t)=-QF(t)$ with time-independent Q (since Q commutes with H_0 when $\Omega\to 0$). Equation (2.2) takes the form

$$\mathcal{F}(\xi)=\langle \mathcal{T}_\pm \exp[-iQK_-(t)]e^{\xi P}\exp[iQK_+(t)]\rangle, \tag{3.2}$$

where $K_\pm(t)=\int_0^t dt_\pm F(t_\pm)$ and $\mathcal{T}_\pm$ denotes the Keldysh time ordering: times t_- to the left of times t_+, earlier t_- to the left of later t_-, earlier t_+ to the right of later t_+.

Taking the expectation value of the phonon degree of freedom we find

$$\mathcal{F}(\xi)=e^{\xi^2 M\Omega/2}\left\langle \mathcal{T}_\pm \exp\left[\tfrac{1}{2}\xi(K_-+K_+)-\frac{(K_+-K_-)^2}{4M\Omega}\right]\right\rangle. \tag{3.3}$$

The factor $\exp(\xi^2 M\Omega/2)$ originates from the uncertainty $(M\Omega)^{1/2}$ of the momentum of the phonon mode in the ground state (vacuum fluctuations). It is a time-independent additive contribution to the second cumulant, and we can omit it for long detection times. The quadratic term $\propto K_\pm^2/M\Omega$ becomes small for a small uncertainty $(M\Omega)^{-1/2}$ of the displacement in the ground state. It describes a back action of the phonon mode on the electrons that persists in the massive phonon limit. (A similar effect is known in the context of charge counting statistics.[14]) This term may be of importance in some situations, but we will not consider it here, assuming that the electron dynamics is insensitive to the vacuum fluctuations of the phonon mode.

With these simplifications we arrive at a formula for the momentum transfer statistics,

$$\mathcal{F}(\xi)=\langle \mathcal{T}_\pm \exp[\tfrac{1}{2}\xi K_-(t)]\exp[\tfrac{1}{2}\xi K_+(t)]\rangle, \tag{3.4}$$

that is of the same form as the formula for charge counting statistics due to Levitov and Lesovik.[1]

$$\mathcal{F}_{\rm charge}(\xi)=\langle \mathcal{T}_\pm \exp[\tfrac{1}{2}\xi J_-(t)]\exp[\tfrac{1}{2}\xi J_+(t)]\rangle. \tag{3.5}$$

The role of the integrated current $J(t)=\int_0^t dt' I(t')$ is taken in our problem by the integrated force $K(t)$.

B. Relation to displacement statistics

Cumulants $\langle\langle \Delta P(t)\rangle\rangle$ of the momentum transferred in a time t are obtained from the cumulant generating function $\ln \mathcal{F}(\xi)=\Sigma_n \langle\langle \Delta P(t)^n\rangle\rangle \xi^n/n!$. Cumulants $\langle\langle F(\omega)^n\rangle\rangle$ of the Fourier transformed force $F(\omega)=\int dt\, e^{i\omega t}F(t)$ then follow from the relation $\Delta P(t)=\int_0^t dt'\, F(t')$. The limit $t\to\infty$ of a long detection time corresponds to the low-frequency limit

$$\left\langle\left\langle \prod_{i=1}^{n} F(\omega_i)\right\rangle\right\rangle \to 2\pi\delta\left(\sum_{i=1}^{n}\omega_i\right)\lim_{t\to\infty}\frac{1}{t}\langle\langle \Delta P(t)^n\rangle\rangle. \tag{3.6}$$

Cumulants of the Fourier transformed displacement $Q(\omega)$ of the oscillator follow from the phenomenological equation of motion

$$Q(\omega)=R(\omega)F(\omega), \quad R(\omega)=\frac{1}{M}(\Omega^2-\omega^2-i\omega\Omega/\mathcal{Q})^{-1}, \tag{3.7}$$

where $\mathcal{Q}$ is the quality factor of the oscillator. Since the force noise is white until frequencies that are typically $\gg\Omega$, one has, in a good approximation,

$$\left\langle\left\langle \prod_{i=1}^{n} Q(\omega_i)\right\rangle\right\rangle = 2\pi\delta\left(\sum_{i=1}^{n}\omega_i\right)\prod_{j=1}^{n} R(\omega_j)\lim_{t\to\infty}\frac{1}{t}\langle\langle \Delta P(t)^n\rangle\rangle. \tag{3.8}$$

Optical or magnetomotive detection of the vibration, as in Refs. 15–17, measures the probability distribution $P(Q)$ of the displacement at any given time. The cumulants of $P(Q)$ are obtained by a Fourier transformation of Eq. (3.8):

$$\langle\langle Q^n\rangle\rangle=\mathcal{R}_n\lim_{t\to\infty}\frac{1}{t}\langle\langle \Delta P(t)^n\rangle\rangle, \tag{3.9}$$

$$\mathcal{R}_n=\int\frac{d\omega_1}{2\pi}\cdots\int\frac{d\omega_n}{2\pi}R(\omega_1)\cdots R(\omega_n)2\pi\delta\left(\sum_{i=1}^{n}\omega_i\right) = \int_{-\infty}^{\infty}dt\, R(t)^n. \tag{3.10}$$

For $\mathcal{Q}\gg 1$ the odd moments can be neglected, while the even moments are given by

$$\mathcal{R}_{2k}\approx\frac{1}{2k}(M\Omega)^{-2k}\frac{\mathcal{Q}}{\Omega}, \quad k\ll\mathcal{Q}. \tag{3.11}$$

C. Validity of the massive phonon approximation

These results were obtained in the massive phonon limit. Let us estimate how large M should be, for the simplest case of the scattering of an electron (mass m, velocity v_F) by a barrier (mass M, velocity Q). Finite M corrections appear because a reflected electron transfers to the barrier not only a momentum $2p_F$ but also an energy $\delta E\simeq 2p_F Q$. This energy transfer effectively changes the voltage drop over the barrier by an amount $\delta V=\delta E/e$, because reflected electrons suffer this energy change whereas transmitted electrons do not.

A voltage drop δV creates a feedback loop. The current is changed by $\delta I=G\delta V$, and hence the force on the barrier is changed by $\delta F=(2p_F/e)\delta I$, hence the velocity of the barrier is changed by $\delta Q=i\omega R(\omega)\delta F=4i\omega(p_F/e)^2R(\omega)GQ$ (in a Fourier representation). The feedback may be neglected if $\delta Q\ll Q$ at the resonance frequency Ω (where it is strongest). Since $R(\Omega)=i\mathcal{Q}/M\Omega^2$ the requirement for negligible feedback, and therefore for the validity of the massive phonon approximation, is

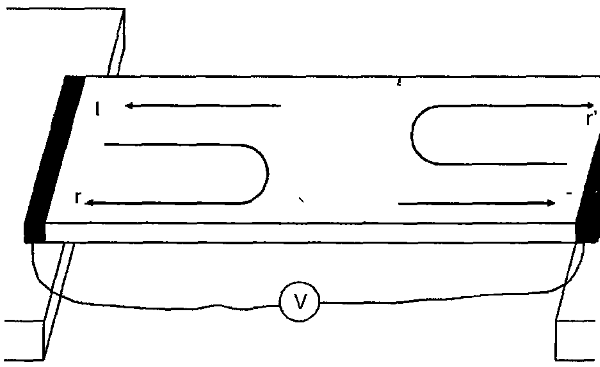

FIG. 1. Sketch of a freely suspended wire. The matrices t,t' and r,r' describe transmission and reflection by a localized scatterer (shaded). A voltage V drives a current through the conductor, exciting a vibration.

$$\alpha = Q\frac{Gh}{e^2}\frac{E_F}{\hbar\Omega}\frac{m}{M} \ll 1. \tag{3.12}$$

The left-hand side of this inequality is the product of three large ratios (the quality factor, the dimensionless conductance, and the ratio of Fermi energy over phonon energy) and one small ratio (the electron mass over the mass of the resonator). For typical parameter values of a single-channel conductor one has $Gh/e^2<1$, $M=10^{-20}$ kg, $\Omega/2\pi=5$ GHz, and $E_F/\hbar=0.5\cdot 10^{15}$ Hz, yielding $\alpha<10^{-3}$ for $Q=10^3$.

IV. EVALUATION IN TERMS OF THE SCATTERING MATRIX

The Levitov-Lesovik formula [Eq. (3.5)] for the charge transfer statistics can be evaluated in terms of the scattering matrix of the conductor,[1,18,19] without an explicit knowledge of the scattering states. This is possible because the current operator depends only on the asymptotic form of the scattering states, far from the scattering region. Formula (3.4) for the momentum-transfer statistics can be evaluated in a similar way, but only if the mode profile $\mathbf{u}(\mathbf{r})$ is approximately constant over the scattering region.

To this end, we first write force operator (2.6) in second quantized form using a basis of scattering states $\psi_{n,\varepsilon}(\mathbf{r})$:

$$F(t)=\int\int\frac{d\varepsilon d\varepsilon'}{2\pi}\sum_{n,n'} e^{i(\varepsilon-\varepsilon')t}c_n^\dagger(\varepsilon)M_{nn'}(\varepsilon,\varepsilon')c_{n'}(\varepsilon'), \tag{4.1}$$

$$M_{nn'}(\varepsilon,\varepsilon')=\frac{1}{m}\int d\mathbf{r}\,\psi_{n,\varepsilon}^{*}(p_\alpha u_{\alpha\beta}p_\beta + [[u_{\alpha\beta},p_\alpha],p_\beta])\psi_{n',\varepsilon'}. \tag{4.2}$$

The operator $c_n(\varepsilon)$ annihilates an electron in the nth scattering channel at energy ε. The mode index n runs from 1 to N (or from $N+1$ to $2N$) for waves incident from the left (or from the right). (See Fig. 1 for a diagram of the geometry, and see Ref. 20 for the analogous representation of the current operator.) The commutator $[[u_{\alpha\beta},p_\alpha],p_\beta]$ can be neglected if $\mathbf{u}$ is smooth on the scale of the wavelength (hence if $\lambda_F/L\ll 1$).

We assume that the derivative $u_{\alpha\beta}$ of the mode profile vanishes in the scattering region, so that for the scattering states we may use the asymptotic form

$$\psi_{n,\varepsilon}(\mathbf{r})=\phi_{n,\varepsilon}^{\rm in}(\mathbf{r})+\sum_m S_{mn}(\varepsilon)\phi_{m,\varepsilon}^{\rm out}(\mathbf{r}) \tag{4.3}$$

in terms of incident and outgoing waves $\phi_{n,\varepsilon}^{\rm in,out}$ (normalized to unit current) and the scattering matrix $S_{mn}(\varepsilon)$. Since we are neglecting the Lorentz force we may assume that $\phi_{m,\varepsilon}^{\rm out}=\phi_{m,\varepsilon}^{\rm in*}$. The scattering matrix has the block structure

$$S=\begin{pmatrix} r & t' \\ t & r' \end{pmatrix}, \tag{4.4}$$

with $N\times N$ transmission and reflection matrices t, t', and r,r'. These matrices are related by unitarity ($S=S^\dagger$) and possibly also by time-reversal symmetry ($S=S^T$).

The operator $p_\alpha u_{\alpha\beta}p_\beta$ will couple only weakly the incident to the outgoing waves, provided $\mathbf{u}$ is smooth on the scale of λ_F, and we neglect this coupling. The matrix M then separates into incident and outgoing parts:

$$M(\varepsilon,\varepsilon')=M^{\rm in}(\varepsilon,\varepsilon')+S^\dagger(\varepsilon)M^{\rm out}(\varepsilon,\varepsilon')S(\varepsilon'). \tag{4.5}$$

The matrices $M^{\rm in}$ and $M^{\rm out}$ are defined as in Eq. (4.2) with ψ replaced by $\phi^{\rm in}$ and $\phi^{\rm out}$, respectively. (They are Hermitian and related by $M^{\rm out}=M^{\rm in*}$.) These two matrices vary with energy on the scale of the Fermi energy E_F, while the scattering matrix S has a much stronger energy dependence (on the scale of the Thouless energy). We may therefore replace $M^{\rm in}$, $M^{\rm out}$ by their value at $\varepsilon=\varepsilon'=E_F$ and assume that the energy dependence of M is given entirely by the scattering matrix.

The force operator can similarly be separated into $F=F^{\rm in}+F^{\rm out}$, where $F^{\rm in}$ and $F^{\rm out}$ are defined as in Eq. (4.1) with the matrix M replaced by $M^{\rm in}$ and $S^\dagger M^{\rm out}S$, respectively. We now proceed in the same way as in Ref. 19 for the current operator, by noting that the analyticity of $S(\varepsilon)$ in the upper half of the complex plane implies simple commutation relations

$$[F^{\rm in}(t),F^{\rm in}(t')]=0,\quad [F^{\rm out}(t),F^{\rm out}(t')]=0,\quad \forall t,t',$$
$$[F^{\rm in}(t),F^{\rm out}(t')]=0 \quad \text{if } t>t'. \tag{4.6}$$

It follows that the Keldysh time ordering $\mathcal{T}_\pm$ of the force operators is the same as the so-called input-output ordering, defined by moving the operators $F_{\rm in}(t_-)$ to the left and $F_{\rm in}(t_+)$ to the right of all other operators—irrespective of the value of the time arguments. The reason for preferring input-output ordering over time ordering is that Fourier transformation from time to energy commutes with the former ordering but not with the latter.

In the limit $t\to\infty$ different energies become uncoupled, and the cumulant generating function takes the simple form

$$\ln \mathcal{F}(\xi) = \frac{t}{2\pi}\int d\varepsilon \, \ln\langle e^{F^{\rm in}(\varepsilon)\xi/2} e^{F^{\rm out}(\varepsilon)\xi} e^{F^{\rm in}(\varepsilon)\xi/2}\rangle, \quad (4.7)$$

entirely analogous to the input-output ordered formula for charge transfer.[19] The Fourier transformed force is defined as

$$F^{\rm in}(\varepsilon) = c^\dagger(\varepsilon) M^{\rm in}(\varepsilon,\varepsilon) c(\varepsilon), \quad (4.8a)$$

$$F^{\rm out}(\varepsilon) = c^\dagger(\varepsilon) S^\dagger(\varepsilon) M^{\rm out}(\varepsilon,\varepsilon) S(\varepsilon) c(\varepsilon). \quad (4.8b)$$

(The operators c_n have been collected in a vector c.)

The matrices $M^{\rm in,out}$ are block diagonal,

$$M^{\rm in} = M^{\rm out} = \begin{pmatrix} M_L & 0 \\ 0 & M_R \end{pmatrix}, \quad (4.9)$$

but the $N\times N$ matrices $M_{L,R}$ are in general not diagonal themselves. They take a simple form for a longitudinal phonon mode, when $\mathbf{u}$ is a function of x in the x direction (along the conductor), so that $u_{\alpha\beta}(\mathbf{r}) = \delta_{\alpha x}\delta_{\beta x} u'(x)$. The commutator $[[u',p_x],p_x]$ does not contribute because $\phi_n^{\rm in,out}$ is an eigenstate of p_x (with eigenvalue $p_n^{\rm in} = -p_n^{\rm out} \equiv p_n$). Hence for a longitudinal vibration one has

$$(M_L)_{nn'} = \delta_{nn'} |p_n| (u_0 - u_L), \quad (4.10a)$$

$$(M_R)_{nn'} = \delta_{nn'} |p_n| (u_R - u_0). \quad (4.10b)$$

The value of $u(x)$ in the scattering region is denoted by u_0, while u_L and u_R denote the values at the left and right ends of the conductor. The more complex situation of a transverse phonon mode, when the matrices $M_{L,R}$ are no longer diagonal, is treated in Ref. 21.

We are now ready to calculate the expectation value in Eq. (4.7). We assume that the incident waves originate from reservoirs in thermal equilibrium at temperature T, with a voltage difference V between the left and right reservoir. The Fermi function in the left (right) reservoir is f_L (f_R). We collect the Fermi functions in a diagonal matrix f and write

$$\langle c_n^\dagger(\varepsilon) c_{n'}(\varepsilon')\rangle = f_{nn'}(\varepsilon)\delta(\varepsilon-\varepsilon'), \quad f = \begin{pmatrix} f_L & 0 \\ 0 & f_R \end{pmatrix}. \quad (4.11)$$

All other expectation values of c and $c^\dagger$ vanish. We evaluate Eq. (4.7) with help of the determinantal identity

$$\left\langle \prod_i \exp(c^\dagger A_i c)\right\rangle = \left|\left| 1 - f + f\prod_i e^{A_i}\right|\right|, \quad (4.12)$$

valid for an arbitrary set of matrices A_i, and the identity

$$\exp(S^\dagger A S) = S^\dagger e^A S, \quad (4.13)$$

valid for unitary S. The result is

$$\ln \mathcal{F}(\xi) = \frac{t}{2\pi}\int d\varepsilon \, \ln||1 - f + f e^{\xi M^{\rm in}} S^\dagger(\varepsilon) e^{\xi M^{\rm out}} S(\varepsilon)||, \quad (4.14)$$

where we have also used that the two matrices $M^{\rm in}$ and f commute.

At zero temperature $f_L = \theta(E_F + eV - \varepsilon)$ and $f_R = \theta(E_F - \varepsilon)$. The energy range $\varepsilon < E_F$, where $f_L = f_R = 1$, contributes only to the first moment, while the energy range $E_F < \varepsilon < E_F + eV$, where $f_L = 1$ and $f_R = 0$, contributes to all moments. For small voltages we may neglect the energy dependence of $S(\varepsilon)$ in that range. Using the block structure [Eqs. (4.4) and (4.9)], of S, $M^{\rm in,out}$ the generating function for the second and higher cumulants takes the form

$$\ln \mathcal{F}(\xi) = \frac{eVt}{2\pi}\ln||r^\dagger e^{\xi M_L} r + t^\dagger e^{\xi M_R} t|| + \mathcal{O}(\xi). \quad (4.15)$$

[By $\mathcal{O}(\xi)$ we mean terms linear in ξ.] This determinant cannot be simplified further without knowledge of S. That is a major complication relative to the analogous formula for the charge-transfer statistics,[1] which can be cast entirely in terms of the transmission eigenvalues Γ_n (eigenvalues of $tt^\dagger$):

$$\ln \mathcal{F}_{\rm charge}(\xi) = \frac{t}{2\pi}\int d\varepsilon \sum_n \ln[1 + \Gamma_n(e^{e\xi}-1) f_L(1-f_R) + \Gamma_n(e^{-e\xi}-1) f_R(1-f_L)]. \quad (4.16)$$

In the case of momentum transfer, eigenvalues and eigenvectors both play a role.

V. APPLICATION TO A ONE-DIMENSIONAL CONDUCTOR

A. Straight wire

Further simplification of Eqs. (4.14) and (4.15) is possible if the conductor is so narrow that it supports only a single propagating mode to the left and right of the scattering region ($N=1$). The scattering matrix then consists of scalar transmission and reflection coefficients t,t' and r,r' (related to each other by unitarity). We consider the case of a longitudinal vibration with

$$M^{\rm in} = M^{\rm out} = p_F \begin{pmatrix} u_0 - u_L & 0 \\ 0 & u_R - u_0 \end{pmatrix} \quad (5.1)$$

[cf. Eq. (4.10)]. Because of unitarity the result depends only on the transmission probability $\Gamma = |t|^2 = |t'|^2 = 1 - |r|^2 = 1 - |r'|^2$,

$$\begin{aligned}\ln \mathcal{F}(\xi) = \frac{t}{2\pi}\int d\varepsilon \, \ln[1 &+ (e^{2\xi p_F(u_R - u_L)} - 1) f_L f_R \\ &+ \Gamma(e^{\xi p_F(u_R - u_L)} - 1)[f_L(1-f_R) + f_R(1-f_L)] \\ &+ (1-\Gamma)(e^{2\xi p_F(u_0 - u_L)} - 1) f_L(1-f_R) \\ &+ (1-\Gamma)(e^{2\xi p_F(u_R - u_0)} - 1) f_R(1-f_L)]. \end{aligned} \quad (5.2)$$

At zero temperature this simplifies further to

$$\ln\mathcal{F}(\xi) = \frac{eVt}{2\pi}\ln[1 + \Gamma e^{\xi p_F(u_R + u_L - 2u_0)} - \Gamma] + \mathcal{O}(\xi). \quad (5.3)$$

The zero-temperature statistics [Eq. (5.3)] is binomial, just as for the charge. [The generating function $\mathcal{F}_{\rm charge}(\xi)$ at $T=0$ is obtained from Eq. (5.3) after substitution of $p_F(u_R$

$+u_L-2u_0$) by e, cf Eq (4 16)] At finite temperatures one has the multinomial statistics [Eq (5 2)], made up of stochastically independent elementary processes with more than two possible outcomes The elementary processes may be characterized by the numbers $(n^L_{\rm in},n^R_{\rm in})\in\{0,1\}$ of electrons incident on the scatterer from the left, right and the numbers $(n^L_{\rm out},n^R_{\rm out})\in\{0,1\}$ of outgoing electrons to the left, right The non-vanishing probabilities $P[(n^L_{\rm in},n^R_{\rm in})\rightarrow(n^L_{\rm out},n^R_{\rm out})]$ of scattering events evaluate to

$$\begin{aligned}
P[(0,0)\rightarrow(0,0)]&=(1-f_L)(1-f_R),\\
P[(0,1)\rightarrow(0,1)]&=(1-f_L)f_R(1-\Gamma),\\
P[(0,1)\rightarrow(1,0)]&=(1-f_L)f_R\Gamma,\\
P[(1,0)\rightarrow(1,0)]&=f_L(1-f_R)(1-\Gamma),\\
P[(1,0)\rightarrow(0,1)]&=f_L(1-f_R)\Gamma,\\
P[(1,1)\rightarrow(1,1)]&=f_Lf_R
\end{aligned} \tag{5 4}$$

These probabilities appear in generating function (5 3), multiplied by exponentials of ξ times the amount of transferred momentum

A longitudinal vibration of a straight wire clamped at both ends would correspond to $u_L=u_R=0$ and $u_0\neq 0$ In that special case Eq (5 2) is equivalent to Eq (4 16) for $\mathcal{F}_{\rm charge}(\xi)$ under the substitution $\Gamma\rightarrow 1-\Gamma$, $2p_Fu_0\rightarrow e$ In this case the multinomial statistics becomes a double-Poissonian in the limit $\Gamma\rightarrow 0$, corresponding to two independent Poisson processes originating from the left and right reservoirs [2] A longitudinal vibration is difficult to observe, in contrast to a transverse vibration which can be observed optically[15,16] or magnetomotively [17] However, the direct excitation of a transverse mode is not possible in a single-channel conductor, while in a multichannel conductor (width W) it is smaller than the excitation of a longitudinal mode by a factor (W/L) [2,21] So it would be desirable to find a way of coupling longitudinal electron motion to transverse vibration modes In the following subsection we discuss how this can be achieved by bending the wire

B. Bent wire

The bending of the wire is described as explained in Ref 22, by means of a vector $\mathbf{\Omega}(s)$ that rotates the local coordinate system $\mathbf{e}_x(s)$, $\mathbf{e}_y(s)$, and $\mathbf{e}_z(s)$ as one moves an infinitesimal distance ds along the wire $\delta\mathbf{e}_\alpha=\mathbf{\Omega}\times\mathbf{e}_\alpha\delta s$ The local coordinate x is along the wire and y,z are perpendicular to it The component $\Omega_{\|}$ of $\mathbf{\Omega}$ along the wire describes a torsion (with $|\Omega_{\|}|$ the torsion angle per unit length), while the perpendicular component $\Omega_\perp$ describes the bending (with $|\Omega_\perp|^{-1}$ the radius of curvature)

The momentum operators and wave functions, written in local coordinates, depend on the bending by terms of order $\lambda_F|\mathbf{\Omega}|$, which we assume to be $\ll 1$ These quantities may therefore be evaluated for a straight wire ($\mathbf{\Omega}=0$) The dependence on the bending of the strain tensor is of order $L|\mathbf{\Omega}|$ and can not be neglected For interaction Hamiltonian (2 5)

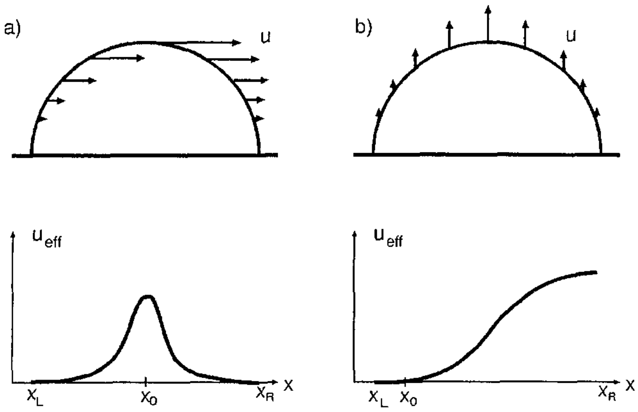

FIG 2 Two vibration modes in a bent wire (top) and the corresponding longitudinal displacements $\mathbf{u}_{\rm eff}$ in the straight wire (bottom)

we need $\nabla\mathbf{u}$ in the global coordinate system It is obtained by differentiating the local coordinates of $\mathbf{u}$ as well as the local basis vectors A bent wire can then be represented by a straight wire with an effective displacement $\mathbf{u}_{\rm eff}$ related to $\mathbf{u}$ (in local coordinates) by

$$\frac{\partial}{\partial x}\mathbf{u}_{\rm eff}=\frac{\partial}{\partial x}\mathbf{u}+\mathbf{\Omega}\times\mathbf{u}, \tag{5 5a}$$

$$\frac{\partial}{\partial y}\mathbf{u}_{\rm eff}=\frac{\partial}{\partial y}\mathbf{u},\quad \frac{\partial}{\partial z}\mathbf{u}_{\rm eff}=\frac{\partial}{\partial z}\mathbf{u} \tag{5 5b}$$

The second term on the right-hand side of Eq (5 5a) accounts for the centrifugal force exerted by an electron moving along the bent wire It rotates a transverse mode, with $\mathbf{u}$ pointing in radial direction, into a fictitious longitudinal mode with $u_{{\rm eff}\,x}$ of order $L|\Omega_\perp|$ Note that in order for $\partial u_{{\rm eff}\,x}/\partial x$ to be nonzero, the displacement $\mathbf{u}$ needs to induce a stretching/compression of the wire Only then is $\mathbf{e}_x\cdot\partial\mathbf{u}/\partial x=\partial u_x/\partial x+\mathbf{e}_x\cdot(\mathbf{\Omega}\times\mathbf{u})=\partial u_{{\rm eff}\,x}/\partial x\neq 0$

Figure 2 shows two vibration modes in a bent wire with the corresponding longitudinal component $u_{{\rm eff}\,x}$ of the effective displacement To apply the formulas of Sec V A we need $u_L=u_{{\rm eff}\,x}(x_L)$, $u_R=u_{{\rm eff}\,x}(x_R)$, and $u_0=u_{{\rm eff}\,x}(x_0)$ The first mode, Fig 2(a), has $u_L=u_R=0$ and $u_0\neq 0$ It measures the amount of electron momentum that has been transferred to the scatterer (located at x_0) The statistics of this process is equivalent to the charge-transfer statistics [Eq (4 16)], as mentioned at the end of the previous subsection

The second mode, Fig 2(b), has $u_L=0$, $u_R\neq 0$, and $u_0\ll u_R$ (assuming that the scatterer is located much closer to the left reservoir than to the right reservoir) It measures the amount of momentum transferred from the left to the right reservoir Its statistics reads

$$\ln\mathcal{F}(\xi)=\frac{t}{2\pi}\int d\varepsilon\,\ln\{1+(e^{2\xi p_F u_R}-1)[f_R-\Gamma f_R(1-f_L)] +\Gamma(e^{\xi p_F u_R}-1)[f_L(1-f_R)+f_R(1-f_L)]\} \tag{5 6}$$

It cannot be reduced to the charge transfer statistics [Eq. (4.16)] by a substitution of variables, and in particular does not reduce to a double Poissonian in the limit $\Gamma\to 0$. (It remains multinomial in this limit.) Comparing the second cumulant $C^{(2)}$ of momentum with the second cumulant $C^{(2)}_{\rm charge}$ of charge [the terms of order ξ^2 in Eqs. (4.16) and (5.6)], we find (setting $u_R\equiv 1$)

$$C^{(2)}-(p_F/e)^2C^{(2)}_{\rm charge}=\frac{2}{\pi}tp_F^2k_BT(1-\Gamma) \tag{5.7}$$

The difference vanishes at zero temperature, in accordance with Eq. (5.3). It is independent of the voltage (as long as the energy dependence of Γ can be ignored), so the difference is an equilibrium property.

Equation (5.7) can be given a physical interpretation by grouping the electrons to the right of the scattering region into $n_>$ right movers and $n_<$ left movers. The momentum transfer to the right reservoir is proportional to the sum $n_> + n_<$, while the charge transfer is proportional to the difference $n_>-n_<$, hence

$$\begin{aligned}C^{(2)}-\frac{p_F^2}{e^2}C^{(2)}_{\rm charge}&\propto\langle\langle(n_>+n_<)^2\rangle\rangle-\langle\langle(n_>-n_<)^2\rangle\rangle\\&=4(\langle n_>n_<\rangle-\langle n_>\rangle\langle n_<\rangle)\end{aligned} \tag{5.8}$$

We see that the difference measures correlations between left and right-moving electrons. Such correlations are due to electrons that are backscattered with probability $1-\Gamma$. Equation (5.7) describes the variance in the number of such backscattered electrons, given that electrons in an energy range k_BT leave the right reservoir independently of each other.

VI. EVALUATION IN TERMS OF THE KELDYSH GREEN FUNCTION

A scattering approach as in Sec. IV is not possible if the displacement $\mathbf{u}(\mathbf{r})$ varies in the scattering region. Time ordering then no longer reduces to input-output ordering, and we need the Keldysh technique to make progress.[23] Following the analogous formulation of the charge counting statistics,[10] we write the generating function (3.4) as a single exponential of an integral along the Keldysh time contour

$$\mathcal{F}(\xi)=\left\langle \mathcal{T}_\pm\exp\left[\tfrac{1}{2}\xi\int_0^t dt'\int d\mathbf{r}\,F_\pm(\mathbf{r},t')\right]\right\rangle, \tag{6.1a}$$

$$F_\pm(\mathbf{r},t)=\frac{1}{m}\sum_{\sigma=\pm}\psi^\dagger_\sigma(\mathbf{r},t)p_\alpha u_{\alpha\beta}(\mathbf{r})p_\beta\psi_\sigma(\mathbf{r},t) \tag{6.1b}$$

We have written the force operator in second quantized form, as in Eq. (4.1), but do not assume that the electron field operator $\psi_\pm(\mathbf{r},t)\equiv\psi(\mathbf{r},t_\pm)$ takes its asymptotic form in terms of incident and outgoing states.[24]

The generating function can be expressed in terms of the Keldysh Green function G:

$$\frac{d}{d\xi}\ln\mathcal{F}(\xi)=\frac{i}{2m}\sum_{\sigma=\pm}\sigma\int_0^t dt'\int d\mathbf{R}\,u_{\alpha\beta}(\mathbf{R})\frac{\partial^2}{\partial r_\alpha\partial r_\beta}G_{\sigma\sigma}(\mathbf{R},\mathbf{r},t',t',\xi)|_{\mathbf{r}=0} \tag{6.2}$$

The Green function $G_{\sigma\sigma'}$ is a 2×2 matrix in the indices $\sigma,\sigma'\in\{+,-\}$ that assure the correct time ordering of the operators. It is defined by

$$G_{\sigma\sigma'}(\mathbf{R},\mathbf{r},t,t',\xi)=\frac{-i\sigma\left\langle\mathcal{T}_\pm\psi_\sigma(\mathbf{R}+\tfrac{1}{2}\mathbf{r},t)\psi^\dagger_{\sigma'}(\mathbf{R}-\tfrac{1}{2}\mathbf{r},t')\exp\left[\tfrac{1}{2}\xi\int_0^t dt'\int d\mathbf{r}'F_\pm(\mathbf{r}',t')\right]\right\rangle}{\left\langle\mathcal{T}_\pm\exp\left[\tfrac{1}{2}\xi\int_0^t dt'\int d\mathbf{r}'F_\pm(\mathbf{r}',t')\right]\right\rangle} \tag{6.3}$$

VII. APPLICATION TO A DIFFUSIVE CONDUCTOR

We apply the formalism of Sec. VI to the example of diffusive electron transport through a freely suspended disordered wire. The semiclassical calculation of the transverse momentum noise in this geometry was done in Ref. 8, so we can compare results.

For long detection times we may assume that the Green function (6.3) depends only on the difference $\tau=t-t'$ of the time arguments. A Fourier transform gives

$$G_{\sigma\sigma'}(\mathbf{R},\mathbf{p},\varepsilon,\xi)=\int d\mathbf{r}\int d\tau e^{-i\mathbf{p}\cdot\mathbf{r}-i\varepsilon\tau}G_{\sigma\sigma'}(\mathbf{R},\mathbf{r},\tau,\xi) \tag{7.1}$$

We write $\mathbf{p}=|\mathbf{p}|\mathbf{n}$ and use the fact that in the semiclassical limit the Green function is peaked as a function of the absolute value $|\mathbf{p}|$ of the momentum. Integration over this variable yields the semiclassical Green function[23]

$$G_{\sigma\sigma'}(\mathbf{R},\mathbf{n},\varepsilon,\xi)=\frac{i}{\pi}\int d\varepsilon'\,G_{\sigma\sigma'}(\mathbf{R},\mathbf{n}\sqrt{2m\varepsilon'},\varepsilon,\xi) \tag{7.2}$$

We next make the diffusion approximation, expanding the $\mathbf{n}$ dependence in spherical harmonics:

$$\begin{aligned}G_{\sigma\sigma'}(\mathbf{R},\mathbf{n},\varepsilon,\xi)&=G^{(0)}_{\sigma\sigma'}(\mathbf{R},\varepsilon,\xi)+n_\alpha G^{(1)}_{\alpha\sigma\sigma'}(\mathbf{R},\varepsilon,\xi)\\&\quad+(n_\alpha n_\beta-\tfrac{1}{3}\delta_{\alpha\beta})G^{(2)}_{\alpha\beta\sigma\sigma'}(\mathbf{R},\varepsilon,\xi)\end{aligned} \tag{7.3}$$

Substituting Eq. (7.3) into Eq. (6.2) we find

$$\frac{d}{d\xi}\ln F(\xi)=\tfrac{1}{2}tE_F\nu\sum_{\sigma=\pm}\sigma\int d\varepsilon\int d\mathbf{R}u_{\alpha\beta}(\mathbf{R})\times\left[\frac{1}{3}\delta_{\alpha\beta}G^{(0)}_{\sigma\sigma}(\mathbf{R},\varepsilon,\xi)+\frac{2}{15}G^{(2)}_{\alpha\beta\sigma\sigma}(\mathbf{R},\varepsilon,\xi)\right], \tag{7.4}$$

where $\nu=p_F^2/2\pi^2 v_F$ is the density of states.

The equation of motion for the semiclassical Green function in the diffusion approximation is derived in the same way as for the charge statistics.[10] We find

$$2ln_\alpha\frac{\partial}{\partial R_\alpha}G+[G^{(0)},G]+\xi p_F l u_{\alpha\beta}n_\alpha n_\beta[\tau_3,G]=0. \tag{7.5}$$

The length l is the mean free path, assuming isotropic impurity scattering. The commutators [,] are taken with respect to the Keldysh indices σ,σ', and τ_3 is the third Pauli matrix in these indices. The Green function satisfies the normalization condition $G^2=1$ that is respected by differential equation (7.5). The boundary conditions at the left and right ends of the wire are[10]

$$G_L=\begin{pmatrix}1-2f_L & 2f_L\\ 2-2f_L & 2f_L-1\end{pmatrix},\quad G_R=\begin{pmatrix}1-2f_R & 2f_R\\ 2-2f_R & 2f_R-1\end{pmatrix}. \tag{7.6}$$

By projecting Eq. (7.5) onto spherical harmonics we find that, to leading order in l/L, the second harmonic $G^{(2)}$ depends only on the zeroth harmonic $G^{(0)}$:

$$G^{(2)}_{\alpha\beta}=\frac{\xi}{2}p_F l\left(u_{\alpha\beta}-\frac{1}{3}\delta_{\alpha\beta}u_{\gamma\gamma}\right)G^{(0)}[\tau_3,G^{(0)}][1+\mathcal{O}(l/L)^2]. \tag{7.7}$$

Combining this relation with Eq. (7.4) we see that the momentum statistics of a transverse mode, with $u_{xx}=0$, $u_{xy}\neq 0$, follows from

$$\frac{d}{d\xi}\ln F(\xi)=\frac{\xi}{30}tp_F lE_F\nu\sum_{\sigma\,\alpha\,\beta}\int d\varepsilon\int d\mathbf{R}u^2_{\alpha\beta}\times(\tau_3 G^{(0)}[\tau_3,G^{(0)}])_{\sigma\sigma}. \tag{7.8}$$

It remains to compute $G^{(0)}$. To calculate $\ln\mathcal{F}$ to order ξ^2, that is to calculate the variance $C^{(2)}$ of the force noise, it is sufficient to know $G^{(0)}$ for $\xi=0$. The solution to unperturbed diffusion equation (7.5) is known,[10]

$$G^{(0)}(\mathbf{R},\varepsilon,\xi=0)=\begin{pmatrix}1-2f(\mathbf{R},\varepsilon) & 2f(\mathbf{R},\varepsilon)\\ 2-2f(\mathbf{R},\varepsilon) & 2f(\mathbf{R},\varepsilon)-1\end{pmatrix}, \tag{7.9}$$

where $f(\mathbf{R},\varepsilon)=f_L(\varepsilon)+(x/L)[f_R(\varepsilon)-f_L(\varepsilon)]$. (The coordinate x runs along the wire, from $x=0$ to $x=L$.) We find

$$C^{(2)}=t\frac{16}{15}p_F lE_F\nu A\int_0^L dx d\varepsilon\, u^2_{xy}(x)f(x,\varepsilon)[1-f(x,\varepsilon)], \tag{7.10}$$

with A the cross-sectional area of the wire. This is the same result as in Ref. 8.

More complicated networks of diffusive wires, including tunnel barriers or point contacts, can be treated in the same way. In such situations the unperturbed Green function $G^{(0)}(\mathbf{R},\varepsilon,\xi=0)$ can be determined using Nazarov's circuit theory,[24] and then substituted into Eq. (7.8).

VIII. CONCLUSION

We conclude by estimating the order of magnitude of the cumulants of the displacement distribution $P(Q)$ of a vibrating current-carrying wire. For an oscillator with a large quality factor only the even order cumulants $\langle\langle Q^{2k}\rangle\rangle$ are appreciable, given in good approximation by

$$\langle\langle Q^{2k}\rangle\rangle\approx\frac{1}{2k}(M\omega_0)^{-2k}\frac{Q}{\omega_0}\lim_{t\to\infty}\frac{1}{t}\langle\langle\Delta P(t)^{2k}\rangle\rangle, \tag{8.1}$$

cf. Eqs. (3.9) and (3.11). The cumulants of transferred momentum ΔP have been calculated for a single-channel conductor with a localized scatterer in Sec. V. At zero temperature one has

$$\lim_{t\to\infty}\frac{1}{t}\langle\langle\Delta P(t)^{2k}\rangle\rangle=\frac{eV}{2\pi\hbar}p_F^{2k}(u_R+u_L-2u_0)^{2k}\frac{d^{2k}}{d\xi^{2k}}\ln[1+\Gamma(e^{\xi}-1)]|_{\xi=0}, \tag{8.2}$$

cf. Eq. (5.3). (We have reinserted Planck's constant $\hbar$ for clarity.)

Combining Eqs. (8.1) and (8.2) we see that in order of magnitude $\langle\langle Q^{2k}\rangle\rangle\simeq(eVQ/\hbar\omega_0)(p_F/M\omega_0)^{2k}$. Inserting parameter values (following Ref. 7) $V=1$ mV, $Q=10^3$, $\omega_0/2\pi=5$ GHz, $p_F=2\times10^{-24}$ Ns, and $M=10^{-20}$ kg, we estimate

$$\langle\langle Q^{2k}\rangle\rangle^{1/2k}\approx10^{4/2k}\times10^{-4}\ \text{Å}. \tag{8.3}$$

Detectors with a 10^{-4}-Å sensitivity have been proposed.[25] For a measurement of higher-order cumulants one would want cumulants of different order to be of roughly the same magnitude. This can be achieved by choosing the number $eVQ/\hbar\omega_0$ not too large. For the parameters chosen above, $\langle\langle Q^4\rangle\rangle^{1/4}/\langle\langle Q^2\rangle\rangle^{1/2}\approx0.1$.

The theory presented in this work is more than a framework for the calculation of higher order cumulants in the momentum transfer statistics. It also provides for a formalism to treat quantum effects in electromechanical noise. A first application, to quantum size effects in a constriction, has been realized.[21] Other applications, including resonant tunneling, superconductivity, and interaction effects, are envisaged.

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APPENDIX A: DERIVATION OF UNITARY TRANSFORMATION (2.4)

We demonstrate that the operator U given in Eq. (2.4) has the desired property [Eq. (2.3)] of eliminating the phonon displacement from the ion potential. By expanding the exponential in Eq. (2.4) we calculate the effect of U on a one-electron and one-phonon wave function in the position space representation:

$$U\psi(\mathbf{r},q)=\|J\|^{1/2}\psi[\mathbf{r}-q\mathbf{u}(\mathbf{r}),q]. \tag{A1}$$

We prove Eq. (2.3) by calculating matrix elements:

$$\begin{aligned}\langle\psi_1|U^\dagger V[\mathbf{r}-Q\mathbf{u}(\mathbf{r})]U|\psi_2\rangle &=\int d\mathbf{r}\int dq\|J\|\psi_1^*[\mathbf{r}-q\mathbf{u}(\mathbf{r}),q]\\ &\quad\times V[\mathbf{r}-q\mathbf{u}(\mathbf{r})]\psi_2[\mathbf{r}-q\mathbf{u}(\mathbf{r}),q]\\ &=\int d\tilde{\mathbf{r}}\int dq\,\psi_1^*(\tilde{\mathbf{r}},q)V(\tilde{\mathbf{r}})\psi_2(\tilde{\mathbf{r}},q)\\ &=\langle\psi_1|V|\psi_2\rangle.\end{aligned} \tag{A2}$$

The unitarity of U follows as the special case $V\equiv 1$.

We now justify the replacement of $\tilde{\rho}=U^\dagger\rho U$ with ρ and $\tilde{A}=U^\dagger A U$ by A in generating function (2.8), in the limit of a long detection time t. Since Q commutes with U, it is sufficient to consider $A=P$. [Then $\tilde{A}=A(Q,\tilde{P})$ in the more general case that A is a function of both Q and P.] To first order in the displacement one has

$$\tilde{P}=P-\Pi+\mathcal{O}(\mathbf{u}^2). \tag{A3}$$

The difference between $\tilde{P}$ and P is of the order of the total momentum Π inside the wire, which is t independent in a stationary state. Since the expectation value (as well as higher cumulants) of P increases linearly with t, we can neglect the difference between $\tilde{P}$ and P for large t.

To justify the replacement of $\tilde{\rho}$ by ρ we note that the effect of U on the initial state is to shift the electron coordinates by the local phonon displacement [cf. Eq. (A1)]. This initial shift has only a transient effect and can be neglected for large t.

APPENDIX B: EFFECTIVE MASS APPROXIMATION

We start with Hamiltonian (2.5) with $V=V_{\rm lat}+V_{\rm imp}$. In the absence of any deformation of the periodic lattice one has, in the effective mass approximation,

$$\frac{1}{2m}\mathbf{p}^2+V_{\rm lat}(\mathbf{r})=\frac{1}{2m^*}\mathbf{p}^{*2}. \tag{B1}$$

The quasimomentum operator $\mathbf{p}^*$ is defined in terms of the Bloch function $g(\mathbf{r})$ by $\mathbf{p}^*=-ig\nabla g^{-1}$. We seek a similar approximation to the same Hamiltonian in a distorted lattice, assuming that $\mathbf{u}$ is sufficiently smooth that we can neglect derivatives of the shear tensor $u_{\alpha\beta}$. Hamiltonian (2.5) (for one electron) then has the form

$$H=\frac{1}{2m}p_\alpha(\delta_{\alpha\beta}-2Qu_{\alpha\beta})p_\beta+V_{\rm lat}+V_{\rm imp}-\frac{1}{M}P\Pi+\Omega a^\dagger a. \tag{B2}$$

For small displacements Q the real symmetric matrix $X_{\alpha\beta}=\delta_{\alpha\beta}-2Qu_{\alpha\beta}$ is positive definite. We can therefore factorize $\mathbf{X}=\mathbf{T}\mathbf{T}^T$, with $\mathbf{T}$ real. We change coordinates to $\tilde{\mathbf{r}}=\mathbf{T}^{-1}\mathbf{r}$, and find

$$H=-\frac{1}{2m}\frac{\partial}{\partial\tilde{r}_\alpha}\frac{\partial}{\partial\tilde{r}_\alpha}+V_{\rm lat}(\mathbf{T}\tilde{\mathbf{r}})+V_{\rm imp}(\mathbf{T}\tilde{\mathbf{r}})-\frac{1}{M}P\Pi+\Omega a^\dagger a. \tag{B3}$$

We now make the assumption of a deformation independent effective mass,[12,13] that is to say we assume that the Hamiltonian with the distorted lattice potential $V_{\rm lat}(\mathbf{T}\tilde{\mathbf{r}})$ is approximated as in Eq. (B1) with distorted Bloch functions, but the same effective mass m^*. Hence

$$H=-\frac{1}{2m^*}\left[g(\mathbf{T}\tilde{\mathbf{r}})\frac{\partial}{\partial\tilde{r}_\alpha}\frac{1}{g(\mathbf{T}\tilde{\mathbf{r}})}\right]^2+V_{\rm imp}(\mathbf{T}\tilde{\mathbf{r}})-\frac{1}{M}P\Pi+\Omega a^\dagger a. \tag{B4}$$

Transforming back to the original coordinates we arrive at the Hamiltonian

$$H=\frac{1}{2m}p_\alpha^*(\delta_{\alpha\beta}-2Qu_{\alpha\beta})p_\beta^*+V_{\rm imp}-\frac{1}{M}P\Pi+\Omega a^\dagger a, \tag{B5}$$

given in Sec. II.

[1] L. S. Levitov and G. B. Lesovik, Pis'ma Zh. Eksp. Teor. Fiz. 58, 225 (1993) [JETP Lett. 58, 230 (1993)]; L. S. Levitov, H. Lee, and G. B. Lesovik, J. Math. Phys. 37, 4845 (1996).
[2] L. S. Levitov and M. Reznikov, cond-mat/0111057 (unpublished).
[3] Ya. M. Blanter and M. Büttiker, Phys. Rep. 336, 1 (2000).
[4] D. B. Gutman and Y. Gefen, cond-mat/0201007 (unpublished).
[5] B. Yurke and G. P. Kochanski, Phys. Rev. B 41, 8184 (1990).
[6] C. Presilla, R. Onofrio, and M. F. Bocko, Phys. Rev. B 45, 3735 (1992).
[7] N. F. Schwabe, A. N. Cleland, M. C. Cross, and M. L. Roukes, Phys. Rev. B 52, 12 911 (1995).
[8] A. V. Shytov, L. S. Levitov, and C. W. J. Beenakker, Phys.

Rev Lett **88**, 228303 (2002)

[9] M L Roukes, Phys World **14**, 25 (2001), cond-mat/0008187 (unpublished)

[10] Yu V Nazarov, Ann Phys (Leipzig) **8**, 507 (1999)

[11] K E Nagaev, Phys Rev B **66**, 075334 (2002)

[12] V M Kontorovich, Usp Fiz Nauk **142**, 265 (1984) [Sov Phys Usp **27**, 134 (1984)]

[13] V B Fiks, Zh Éksp Teor Fiz **75**, 137 (1978) [Sov Phys JETP **48**, 68 (1978)]

[14] Yu V Nazarov and M Kindermann, cond-mat/0107133 (unpublished)

[15] G Meyer and N M Amer, Appl Phys Lett **53**, 1045 (1988)

[16] M M J Treacy, T W Ebbesen, and J M Gibson, Nature (London) **381**, 678 (1996)

[17] A N Cleland and M L Roukes, Appl Phys Lett **69**, 2653 (1996)

[18] B A Muzykantskii and D E Khmelnitskii, Phys Rev B **50**, 3982 (1994)

[19] C W J Beenakker and H Schomerus, Phys Rev Lett **86**, 700 (2001)

[20] M Buttiker, Phys Rev Lett **65**, 2901 (1990)

[21] A Tajic, M Kindermann, and C W J Beenakker, cond-mat/0206306, Phys Rev B (to be published)

[22] L D Landau and E M Lifshitz, *Theory of Elasticity* (Pergamon, Oxford, 1959), Sec 18

[23] J Rammer and H Smith, Rev Mod Phys **58**, 323 (1986)

[24] Yu V Nazarov, in *Quantum Dynamics of Submicron Structures*, Vol 291 of *NATO Advanced Study Institute, Series E*, edited by H A Cerdeira, B Kramer, and G Schon (Kluwer, Dordrecht, 1995)

[25] Y Zhang and M P Blencowe, J Appl Phys **91**, 4249 (2002)