Adiabatic Quantization of Andreev Quantum Billiard Levels

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We identify the time T between Andreev reflections as a classical adiabatic invariant in a ballistic chaotic cavity (Lyapunov exponent λ), coupled to a superconductor by an N-mode constriction. Quantization of the adiabatically invariant torus in phase space gives a discrete set of periods T_n, which in turn generate a ladder of excited states $\varepsilon_{nm} = (m + 1/2)\pi\hbar/T_n$. The largest quantized period is the Ehrenfest time $T_0 = \lambda^{-1} \ln N$. Projection of the invariant torus onto the coordinate plane shows that the wave functions inside the cavity are squeezed to a transverse dimension $W/\sqrt{N}$, much below the width W of the constriction.

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The notion that quantized energy levels may be associated with classical adiabatic invariants goes back to Ehrenfest and the birth of quantum mechanics [1]. It was successful in providing a semiclassical quantization scheme for special integrable dynamical systems but failed to describe the generic nonintegrable case. Adiabatic invariants play an interesting but minor role in the quantization of chaotic systems [2,3].

Since the existence of an adiabatic invariant is the exception rather than the rule, the emergence of a new one quite often teaches us something useful about the system. An example from condensed matter physics is the quantum Hall effect, in which the semiclassical theory is based on two adiabatic invariants: the flux through a cyclotron orbit and the flux enclosed by the orbit center as it slowly drifts along an equipotential [4]. The strong magnetic field suppresses chaotic dynamics in a smooth potential landscape, rendering the motion quasi-integrable.

Some time ago it was realized that Andreev reflection has a similar effect on the chaotic motion in an electron billiard coupled to a superconductor [5]. An electron trajectory is retraced by the hole that is produced upon absorption of a Cooper pair by the superconductor. At the Fermi energy E_F the dynamics of the hole is precisely the time reverse of the electron dynamics, so that the motion is strictly periodic. The period from electron to hole and back to electron is twice the time T between Andreev reflections. For finite excitation energy ε the electron (at energy $E_F + \varepsilon$) and the hole (at energy $E_F - \varepsilon$) follow slightly different trajectories, so the orbit does not quite close and drifts around in phase space. This drift has been studied in a variety of contexts [5–9] but not in connection with adiabatic invariants and the associated quantization conditions. It is the purpose of this Letter to make that connection and point out a striking physical consequence: The wave functions of Andreev levels fill the cavity in a highly nonuniform "squeezed" way, which has no counterpart in normal state chaotic or regular billiards. In particular, the squeezing is distinct from periodic orbit scarring [10] and entirely different from the random superposition of plane waves expected for a fully chaotic billiard [11].

Adiabatic quantization breaks down near the excitation gap, and we will argue that random-matrix theory [12] can be used to quantize the lowest-lying excitations above the gap. This will lead us to a formula for the gap that crosses over from the Thouless energy to the inverse Ehrenfest time as the number of modes in the point contact is increased.

To illustrate the problem we represent in Figs. 1 and 2 the quasiperiodic motion in a particular Andreev billiard. (It is similar to a Sinai billiard but has a smooth potential V in the interior to favor adiabaticity.) Figure 1 shows a trajectory in real space while Fig. 2 is a section of phase space at the interface with the superconductor ($y = 0$). The tangential component p_x of the electron momentum is plotted as a function of the coordinate x along the interface. Each point in this Poincaré map corresponds to one collision of an electron with the interface. (The collisions of holes are not plotted.) The electron is retroreflected as a hole with the same p_x. At $\varepsilon = 0$ the component p_y is also the same, and so the hole retraces the path of the electron (the hole velocity being opposite to its momentum). At nonzero ε the retroreflection occurs with a slight change in p_y, because of the difference 2ε in the kinetic energy of electrons and holes. The resulting slow drift of the periodic trajectory traces out a contour in the surface of section. The adiabatic invariant is the function of x, p_x that is constant on the contour. We have found numerically that the drift follows *isochronous* contours C_T of constant time $T(x, p_x)$ between Andreev reflections [13]. Let us now demonstrate analytically that T is an adiabatic invariant.

We consider the Poincaré map $C_T \to C(\varepsilon, T)$ at energy ε. If $\varepsilon = 0$ the Poincaré map is the identity, so $C(0, T) = C_T$. For adiabatic invariance we need to prove that $\lim_{\varepsilon \to 0} dC/d\varepsilon = 0$, so that the difference between

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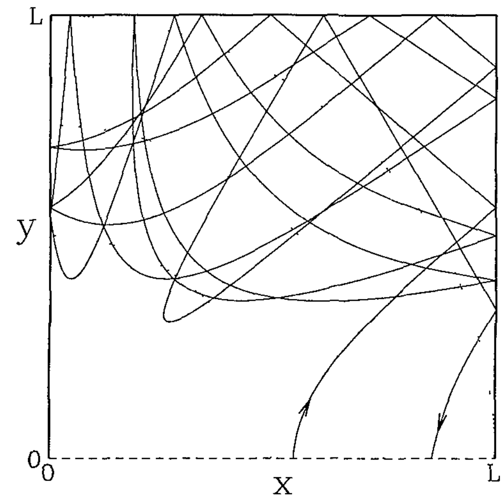

FIG. 1. Classical trajectory in an Andreev billiard. Particles in a two-dimensional electron gas are deflected by the potential $V = [1 - (r/L)^2]V_0$ for $r < L$, $V = 0$ for $r > L$. (The dotted circles are equipotentials.) There is specular reflection at the boundaries with an insulator (thick solid lines) and Andreev reflection at the boundary with a superconductor (dashed line). The trajectory follows the motion between two Andreev reflections of an electron near the Fermi energy $E_F = 0.84V_0$. The Andreev reflected hole retraces this trajectory in the opposite direction.

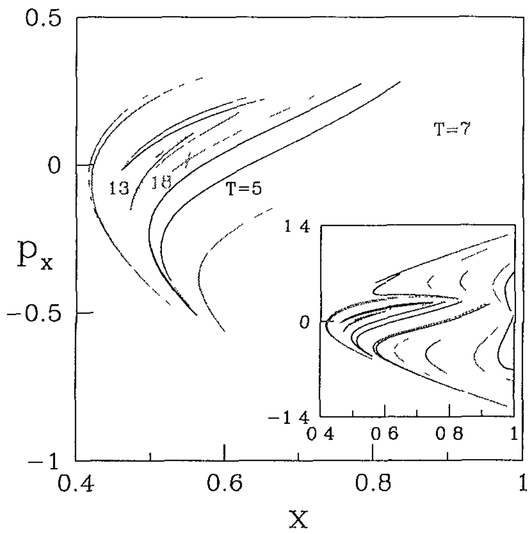

FIG. 2 (color online). Poincaré map for the Andreev billiard of Fig. 1. Each dot represents a starting point of an electron trajectory, at position x (in units of L) along the interface $y = 0$ and with tangential momentum p_x (in units of $\sqrt{mV_0}$). The inset shows the full surface of the section, while the main plot is an enlargement of the central region. The drifting quasiperiodic motion follows contours of constant time T between Andreev reflections. The cross marks the starting point of the trajectory shown in the previous figure, having $T = 18$ (in units of $\sqrt{mL^2/V_0}$).

$C(\varepsilon, T)$ and C_T is of higher order than ε [14]. Since the contour $C(\varepsilon, T)$ can be locally represented by a function $p_x(x, \varepsilon)$, we need to prove that $\lim_{\varepsilon\to 0} \partial p_x(x, \varepsilon)/\partial\varepsilon = 0$.

In order to prove this, it is convenient to decompose the map $C_T \to C(\varepsilon, T)$ into three separate stages, starting out as an electron (from C_T to C_+), followed by Andreev reflection ($C_+ \to C_-$), and then concluded as a hole [from C_- to $C(\varepsilon, T)$]. Andreev reflection introduces a discontinuity in p_y but leaves p_x unchanged, so $C_+ = C_-$. The flow in phase space as electron (+) or hole (−) at energy ε is described by the action $S_\pm(\mathbf{q}, \varepsilon)$, such that $\mathbf{p}^\pm(\mathbf{q}, \varepsilon) = \partial S_\pm/\partial\mathbf{q}$ gives the local dependence of (electron or hole) momentum $\mathbf{p} = (p_x, p_y)$ on position $\mathbf{q} = (x, y)$. The derivative $\partial S_\pm/\partial\varepsilon = t_\pm(\mathbf{q}, \varepsilon)$ is the time elapsed since the previous Andreev reflection. Since by construction $t_\pm(x, y = 0, \varepsilon = 0) = T$ is independent of the position x of the end of the trajectory, we find that $\lim_{\varepsilon\to 0} \partial p_x^\pm(x, y = 0, \varepsilon)/\partial\varepsilon = 0$, completing the proof.

The drift $(\delta x, \delta p_x)$ of a point in the Poincaré map is perpendicular to the vector $(\partial T/\partial x, \partial T/\partial p_x)$. Using also that the map is area preserving, it follows that

$$(\delta x, \delta p_x) = \varepsilon f(T)(\partial T/\partial p_x, -\partial T/\partial x) + O(\varepsilon^2), \quad (1)$$

with a prefactor $f(T)$ that is the same along the entire contour.

The adiabatic invariance of isochronous contours may alternatively be obtained from the adiabatic invariance of the action integral I over the quasiperiodic motion from electron to hole and back to electron:

$$I = \oint p dq = \varepsilon \oint \frac{dq}{q} = 2\varepsilon T. \quad (2)$$

Since ε is a constant of the motion, adiabatic invariance of I implies adiabatic invariance of the time T between Andreev reflections. This is the way in which adiabatic invariance is usually proven in textbooks. Our proof explicitly takes into account the fact that phase space in the Andreev billiard consists of two sheets, joined in the constriction at the interface with the superconductor, with a discontinuity in the action on going from one sheet to the other.

The contours of large T enclose a very small area. This will play a crucial role when we quantize the billiard, so let us estimate the area. It is convenient for this estimate to measure p_x and x in units of the Fermi momentum p_F and width W of the constriction to the superconductor. The highly elongated shape evident in Fig. 2 is a consequence of the exponential divergence in time of nearby trajectories, characteristic of chaotic dynamics. The rate of divergence is the Lyapunov exponent λ. (We consider a fully chaotic phase space.) Since the Hamiltonian flow is area preserving, a stretching $\ell_+(t) = \ell_+(0)e^{\lambda t}$ of the dimension in one direction needs to be compensated by a squeezing $\ell_-(t) = \ell_-(0)e^{-\lambda t}$ of the dimension in the other direction. The area $A \simeq \ell_+\ell_-$ is then time independent. Initially, $\ell_\pm(0) < 1$. The constriction at the superconductor acts as a bottleneck, enforcing $\ell_\pm(T) < 1$. These two inequalities imply $\ell_+(t) < e^{\lambda(t-T)}$, $\ell_- < e^{-\lambda t}$. The enclosed area, therefore, has the upper bound

$$A_{\max} \simeq p_F W e^{-\lambda T} \simeq \hbar N e^{-\lambda T}, \tag{3}$$

where $N \simeq p_F W/\hbar \gg 1$ is the number of channels in the point contact.

We now continue with the quantization. The two invariants ε and T define a two-dimensional torus in the four-dimensional phase space. Quantization of this adiabatically invariant torus proceeds following Einstein-Brillouin-Keller [3], by quantizing the area

$$\oint p\,dq = 2\pi\hbar(m + \nu/4), \qquad m = 0, 1, 2, \ldots \tag{4}$$

enclosed by each of the two topologically independent contours on the torus. Equation (4) ensures that the wave functions are single valued. (See Ref. [15] for a derivation in a two-sheeted phase space.) The integer ν counts the number of caustics (Maslov index) and in our case should also include the number of Andreev reflections.

The first contour follows the quasiperiodic orbit of Eq. (2), leading to

$$\varepsilon T = (m + \tfrac{1}{2})\pi\hbar, \qquad m = 0, 1, 2, \ldots \tag{5}$$

The quantization condition (5) is sufficient to determine the smoothed density of states $\rho(\varepsilon)$, using the classical probability distribution $P(T) \propto \exp(-TN\delta/h)$ [16] for the time between Andreev reflections. (We denote by δ the level spacing in the isolated billiard.) The density of states

$$\rho(\varepsilon) = N \int_0^\infty dT P(T) \sum_{m=0}^{\infty} \delta\left[\varepsilon - \left(m + \frac{1}{2}\right)\pi\hbar/T\right] \tag{6}$$

has no gap but vanishes smoothly $\propto \exp(-N\delta/4\varepsilon)$ at energies below the Thouless energy $N\delta$. This "Bohr-Sommerfeld approximation" [12] has been quite successful [17–19], but it gives no information on the location of individual energy level—nor can it be used to determine the wave functions.

To find these we need a second quantization condition, which is provided by the area $\oint_T p_x dx$ enclosed by the contours of constant $T(x, p_x)$,

$$\oint_T p_x dx = 2\pi\hbar(n + \nu/4), \qquad n = 0, 1, 2, \ldots \tag{7}$$

Equation (7) amounts to a quantization of the period T, which together with Eq. (5) leads to a quantization of ε. For each T_n there is a ladder of Andreev levels $\varepsilon_{nm} = (m + \frac{1}{2})\pi\hbar/T_n$.

While the classical T can become arbitrarily large, the quantized T_n has a cutoff. The cutoff follows from the maximal area (3) enclosed by an isochronous contour. Since Eq. (7) requires $A_{\max} > 2\pi\hbar$, we find that the longest quantized period is $T_0 = \lambda^{-1}[\ln N + \mathcal{O}(1)]$. The lowest Andreev level associated with an adiabatically invariant torus is therefore

$$\varepsilon_{00} = \frac{\pi\hbar}{2T_0} = \frac{\pi\hbar\lambda}{2\ln N}. \tag{8}$$

The time scale $T_0 \propto |\ln\hbar|$ represents the Ehrenfest time of the Andreev billiard, which sets the scale for the excitation gap in the semiclassical limit [20–22].

We now turn from the energy levels to the wave functions. The wave function has electron and hole components $\psi_\pm(x, y)$, corresponding to the two sheets of phase space. By projecting the invariant torus in a single sheet onto the x-y plane we obtain the support of the electron or hole wave function. This is shown in Fig. 3, for the same billiard presented in the previous figures. The curves are streamlines that follow the motion of individual electrons, all sharing the same time T between Andreev reflections. (A single one of these trajectories was shown in Fig. 1.)

Together the streamlines form a flux tube that represents the support of ψ_+. The width δW of the flux tube is of order W at the constriction but becomes much smaller in the interior of the billiard. Since $\delta W/W < \ell_+ + \ell_- < e^{\lambda(t-T)} + e^{-\lambda t}$ (with $0 < t < T$), we conclude that the flux tube is squeezed down to a width

$$\delta W_{\min} \simeq W e^{-\lambda T/2}. \tag{9}$$

The flux tube for the level ε_{00} has a minimal width $\delta W_{\min} \simeq W/\sqrt{N}$. Particle conservation implies that $|\psi_+|^2 \propto 1/\delta W$, so that the squeezing of the flux tube is

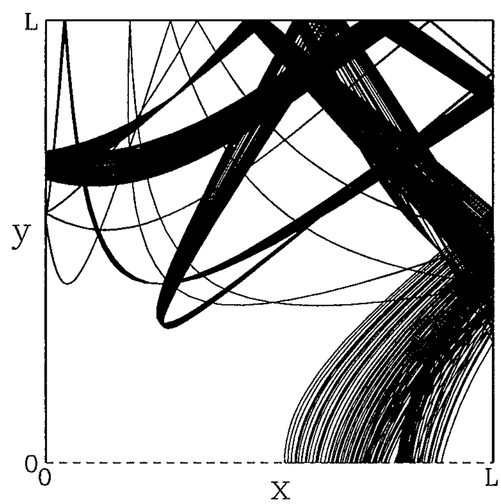

FIG. 3. Projection onto the x-y plane of the invariant torus with $T = 18$, representing the support of the electron component of the wave function. The flux tube has a large width near the superconductor, which is squeezed to an indistinguishably small value after a few collisions with the boundaries.

associated with an increase of the electron density by a factor of $\sqrt{N}$ as one moves away from the constriction

Let us examine the range of validity of adiabatic quantization The drift $\delta x, \delta p_x$ upon one iteration of the Poincare map should be small compared to W p_F We estimate

$$\frac{\delta x}{W} \simeq \frac{\delta p_x}{p_F} \simeq \frac{\varepsilon_{nm}}{\hbar \lambda N} e^{\lambda T_n} \simeq \left(m + \frac{1}{2}\right) \frac{e^{-\lambda(T_0 - T_n)}}{\lambda T_n} \tag{10}$$

For low-lying levels ($m \sim 1$) the dimensionless drift is $\ll 1$ for $T_n < T_0$ Even for $T_n = T_0$ one has $\delta x/W \simeq 1/\ln N \ll 1$

Semiclassical methods allow one to quantize only the trajectories with periods $T \leq T_0$ The part of phase space with longer periods can be quantized by random-matrix theory, according to which the excitation gap E_{gap} is the inverse of the mean time between Andreev reflections in that part of phase space [12,17]

$$E_{\text{gap}} = \gamma^{5/2} \hbar \frac{\int_{T_0}^{\infty} P(T) dT}{\int_{T_0}^{\infty} T P(T) dT} = \frac{\gamma^{5/2} \hbar}{T_0 + 2\pi\hbar/N\delta} \tag{11}$$

Here $\gamma = \frac{1}{2}(\sqrt{5} - 1)$ is the golden ratio This formula describes the crossover from $E_{\text{gap}} = \gamma^{5/2}\hbar/T_0 = \gamma^{5/2}\hbar\lambda/\ln N$ to $E_{\text{gap}} = \gamma^{5/2} N\delta/2\pi$ at $N \ln N \simeq \hbar\lambda/\delta$ It requires $\hbar\lambda/N\delta \gg 1$ (mean dwell time large compared to the Lyapunov time) The semiclassical (large-N) limit of Eq (11), $\lim_{N\to\infty} E_{\text{gap}} = 0\,30\,\hbar/T_0$ is a factor of 5 below the lowest adiabatic level, $\varepsilon_{00} = 1\,6\,\hbar/T_0$, so that indeed the energy range near the gap is not accessible by adiabatic quantization [23]

Up to now we considered two-dimensional Andreev billiards Adiabatic quantization may equally well be applied to three-dimensional systems, with the area enclosed by an isochronous contour as the second adiabatic invariant For a fully chaotic phase space with two Lyapunov exponents λ_1, λ_2, the longest quantized period is $T_0 = \frac{1}{2}(\lambda_1 + \lambda_2)^{-1} \ln N$ We expect interesting quantum size effects on the classical localization of Andreev levels discovered in Ref [7], which should be measurable in a thin metal film on a superconducting substrate

One important challenge for future research is to test the adiabatic quantization of Andreev levels numerically, by solving the Bogoliubov–de Gennes equation on a computer The characteristic signature of the adiabatic invariant that we have discovered, a narrow region of enhanced intensity in a chaotic region that is squeezed as one moves away from the superconductor, should be readily observable and distinguishable from other features that are unrelated to the presence of the superconductor, such as scars of unstable periodic orbits [10] Experimentally these regions might be observable using a scanning tunneling probe, which provides an energy and spatially resolved measurement of the electron density

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[1] P Ehrenfest, Ann Phys (Leipzig) **51**, 327 (1916)

[2] C C Martens, R L Waterland, and W P Reinhardt, J Chem Phys **90**, 2328 (1989)

[3] M C Gutzwiller, *Chaos in Classical and Quantum Mechanics* (Springer, Berlin, 1990)

[4] R E Prange, in *The Quantum Hall Effect*, edited by R E Prange and S M Girvin (Springer, New York, 1990)

[5] I Kosztin, D L Maslov, and P M Goldbart, Phys Rev Lett **75**, 1735 (1995)

[6] M Stone, Phys Rev B **54**, 13 222 (1996)

[7] A V Shytov, P A Lee, and L S Levitov, Phys Usp **41**, 207 (1998)

[8] I Adagideli and P M Goldbart, Phys Rev B **65**, 201306 (2002)

[9] J Wiersig, Phys Rev E **65**, 036221 (2002)

[10] E J Heller, Phys Rev Lett **53**, 1515 (1984)

[11] P W O'Connor, J Gehlen, and E Heller, Phys Rev Lett **58**, 1296 (1987)

[12] J A Melsen, P W Brouwer, K M Frahm, and C W J Beenakker, Europhys Lett **35**, 7 (1996)

[13] Isochronous contours are defined as $T(x, p_x) =$ const at $\varepsilon = 0$ We assume that the isochronous contours are closed This is true if the border $p_x = 0$ of the classically allowed region in the x, p_x section is itself an isochronous contour, which is the case if $\lim_{y\to 0} \partial V/\partial y \leq 0$ In this case the particle leaving the superconductor with infinitesimal p_x cannot penetrate into the billiard

[14] Adiabatic invariance is defined in the limit $\varepsilon \to 0$ and is therefore distinct from invariance in the sense of Kolmogorov-Arnold Moser (KAM), which would require a critical ε^* such that a contour is exactly invariant for $\varepsilon < \varepsilon^*$ Numerical evidence [5] suggests that the KAM theorem does not apply to a chaotic Andreev billiard

[15] K P Duncan and B L Gyorffy, Ann Phys (NY) **298**, 273 (2002)

[16] W Bauer and G F Bertsch, Phys Rev Lett, **65**, 2213 (1990)

[17] H Schomerus and C W J Beenakker, Phys Rev Lett **82**, 2951 (1999)

[18] W Ihra, M Leadbeater, J L Vega, and K Richter, Europhys J B **21**, 425 (2001)

[19] J Cserti, A Kormányos, Z Kaufmann, J Koltai, and C J Lambert, Phys Rev Lett **89**, 057001 (2002)

[20] A Lodder and Yu V Nazarov, Phys Rev B **58**, 5783 (1998)

[21] D Taras-Semchuk and A Altland, Phys Rev B **64**, 014512 (2001)

[22] I Adagideli and C W J Beenakker, Phys Rev Lett **89**, 237002 (2002)

[23] The density of states near the gap is obtained in the same way as Eq (11), with the result $\rho(\varepsilon) = c(\varepsilon - E_{\text{gap}})^{1/2} \times N_{\text{eff}}^{1/2} \delta_{\text{eff}}^{3/2}$, where $N_{\text{eff}} = N^{1 - N\delta/\hbar\lambda}$, $\delta_{\text{eff}}^{-1} = (\delta^{-1} + N \ln N/\hbar\lambda) N^{-N\delta/\hbar\lambda}$, and $c = 4(\pi/\sqrt{5})^{1/2}\gamma^{5/4}(9 + 4\sqrt{5})^{2/3} \approx 18$