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Nonlocality of high-dimensional two-photon orbital angular momentum states

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We propose an interferometric method to investigate the nonlocality of high-dimensional two-photon orbital angular momentum states generated by spontaneous parametric down conversion. We incorporate two half-integer spiral phase plates and a variable-reflectivity output beam splitter into a Mach-Zehnder interferometer to build an orbital angular momentum analyzer. This setup enables testing the nonlocality of high-dimensional two-photon states by repeated use of the Clauser-Horne-Shimony-Holt inequality.

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I. INTRODUCTION

Entangled qubits play a key role in many applications of quantum information [1] and quantum cryptography [2]. An example of a qubit is the polarization state of a photon. More generally, a qudit is a quantum system whose state lies in a d-dimensional Hilbert space. The higher dimensionality implies a greater potential for applications in quantum information processing and this explains the continuously growing interest in methods for creating entangled qudits.

Among these methods, spontaneous parametric down conversion (SPDC) appears to be the most reliable one for creating entangled photon pairs [3]. Recently, several techniques have been used to create entangled qudits from down-converted photons. For example, conservation of orbital angular momentum (OAM) in SPDC has been used to create entangled states with $d=3$ [4,5], and a time binning method was employed to realize states with $d=11$ [6]. Recently, spatial degrees of freedom in SPDC [7] have been exploited to demonstrate entanglement for the cases $d=4,8$ [8] and $d=6$ [9].

It is well known that *useful* high-dimensional entanglement can be witnessed by violation of Bell-type inequalities [10], which also furnish a test of nonlocality for a quantum system. However, tests of d-dimensional inequalities for bipartite quantum systems require the use of at least $2d$ detectors, which becomes exceedingly difficult (if not impossible) for large d.

In a previous paper [11] we proposed an experiment to show the entanglement of high-dimensional two-photon OAM states, with *two* detectors only. This scheme indeed allows us to verify the existence of high-dimensional nonseparability, as demonstrated by our subsequent experimental results [12]. In Ref. [11] we went on to use a two-dimensional Bell inequality to check the nonlocality of our OAM-entangled photons. In the meantime we have realized that this implicitly assumes dichotomic variables, a condition that was not fulfilled by the scheme proposed in Ref. [11].

In the present paper, we propose an experimental scheme to explicitly test the nonlocality (namely, the *useful* entanglement) of very-high-dimensional two-photon OAM states ($d\sim\infty$), by using just four detectors. The advantages of our method with respect to those using $2d$ detectors are obvious for $d>2$. Additionally, we stress that the scheme we propose is designed to realize dichotomic observables. The idea is first to project the infinite-dimensional two-photon state onto several different four-dimensional subspaces (in order to select different four-dimensional two-photon states), and then to apply the Clauser-Horne-Shimony-Holt (CHSH) inequality [13] to each selected state. It is not obvious *a priori* whether such a scheme will work or not. In fact several legitimate questions can be raised: (i) Does this dimensional reduction spoil the entanglement of the two-photon state? (ii) Do selected four-dimensional states maximally violate the CHSH inequality? (iii) Are distinct four-dimensional subspaces equivalent? In the rest of this paper we will address these questions.

II. THE PROPOSED EXPERIMENT

As shown in Fig. 1, a nonlinear crystal yields OAM-entangled photon pairs, and the two photons (say a and b) are fed into two balanced Mach-Zehnder interferometers which are shown in detail in Fig. 2. Each Mach-Zehnder MZ_x ($x=a,b$) is made of a 50-50 input beam splitter (BS)

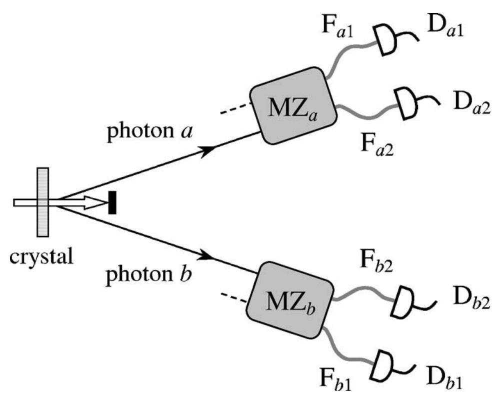

FIG. 1. Schematic of the proposed experimental setup. The boxes MZ_a and MZ_b represent the Mach-Zehnder interferometers in the path of the photon a and b, respectively. The thick gray lines $\mathrm{F}_{xi}(x=a,b;\ i=1,2)$, represent the single-mode optical fibers. Each of them is coupled with a detector D_{xi}. Further details are given in the text.

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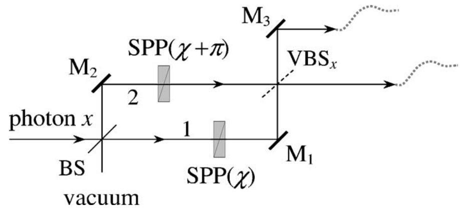

FIG. 2. Detailed scheme of the OAM analyzer in the path of the photon $x=a,b$. BS denotes a 50-50 beam splitter, and VBS_x a variable-reflectivity beam splitter. The two channels "1" and "2" of the interferometer are indicated. With SPP we denoted the two spiral phase plates ($\chi=\alpha,\beta$), and $\mathrm{M}_1,\mathrm{M}_2,\mathrm{M}_3$ represent three ordinary mirrors. The role of M_3 is to ensure that the photon undergoes an even number of reflections (thus maintaining the spatial symmetry of the input wave function), whichever path it takes.

and a variable-reflectivity output beam splitter (VBS_x). We denote with t_x and r_x the transmission and reflection coefficients of each VBS_x and assume $t_x=\cos\theta_x$, $r_x=\mathrm{i}\sin\theta_x$, where $x=a,b$ and $\theta_x\in[0,2\pi)$. The role of the VBS in such a scheme is that of a "channel selector" which can change the relative weight of the two arms of the interferometer. Such a VBS can be easily realized, for example, by exploiting the polarization degrees of freedom of the SPDC photons. Type I crystals emit photon pairs with a well-defined linear polarization [14]. Then, the combination of a half-wave plate before the Mach-Zehnder and a polarizing beam splitter as output BS of the same interferometer realizes the desired VBS. Another possibility is to use a Fabry-Pérot étalon whose mirror separation can be varied, to realize a so-called "Lorentzian beam splitter" [15], which acts as a VBS.

In channel 1 of interferometer MZ_a there is a spiral phase plate (SPP) [16] with step index $\mathcal{L}$ oriented at α (see Fig. 3), while in channel 2 there is a SPP with the same step index but oriented at $\alpha+\pi$. When the step index is half-integer, that is when $\mathcal{L}=\ell+1/2$, $(\ell=1,2,\ldots)$, these two antiparallel geometrical orientations (α and $\alpha+\pi$) define, in combination with single-mode fibers (see below), two orthogonal spatial modes [11]. Similarly, in channel 1 of interferometer MZ_b there is a spiral phase plate (SPP) with negative step index $-\mathcal{L}$ oriented at β, while in channel 2 there is a SPP with the same step index but oriented at $\beta+\pi$.

Each output port of the interferometers is coupled to a

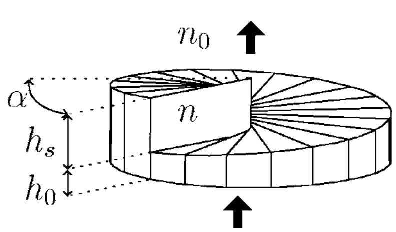

FIG. 3. Schematic drawing of a spiral phase plate (SPP) with a step index (= phase shift per unit angle) $\mathcal{L}=h_s(n-n_0)/\lambda$, where h_s is the step height, n and n_0 are the refractive indices of the SPP and the surrounding medium, respectively, and λ is the wavelength of the incident light. In this paper we assume $\mathcal{L}=\ell+1/2,\{\ell=1,2,\ldots\}$. The orientation angle α is indicated.

single-mode fiber which sustains the Laguerre-Gaussian mode $\mathrm{LG}_{p=0}^{l=0}$. When a photon in the arbitrary state $|\xi\rangle$ is coupled to such a single-mode fiber, the fiber projects the input state of the photon on the Laguerre-Gaussian state $|l=0,p=0\rangle\equiv|0,0\rangle$ with probability $|\langle 0,0|\xi\rangle|^2$. The output port of each fiber is coupled with a single-photon detector. Finally, each photon propagates from the crystal to the single-mode fibers through a suitable system of lenses (not shown in Fig. 1), which images the twin photons from the crystal to the SPPs, and from the SPPs to the input facets of the fibers. In this way, free-space propagation effects reduce to an azimuthal-independent longitudinal phase factor.

III. THE MACH-ZEHNDER INTERFEROMETER

Each photon enters the Mach-Zehnder interferometer through a single input port, say "port 1." The quantum state of the down-converted photon pair at the entrance of both interferometers, can be written as [17]

$$|\Psi^{\mathrm{in}}\rangle \propto \int \mathrm{d}^2\mathbf{x}\Lambda_P(r)\hat{a}_1^\dagger(\mathbf{x})\hat{b}_1^\dagger(\mathbf{x})|0\rangle \tag{1}$$

where $\Lambda_P(r)\equiv\mathrm{LG}_0^0(r,w)$ describes the transverse profile of the pump beam, $r=|\mathbf{x}|$, and w is the pump beam waist. The entangled photons cross both Mach-Zehnders, and are eventually detected. After a lengthy but straightforward calculation, it is possible to show [18] that the probability $P_{ij}(\theta_a,\theta_b)$ that the detector D_{ai} fires in coincidence with the detector D_{bj} is given by

$$P_{ij}(\theta_a,\theta_b) \propto |\langle 0,0|\langle 0,0|\hat{U}_i(\alpha,\theta_a)\otimes\hat{U}_j^\dagger(\beta,\theta_b)|\Psi^{\mathrm{in}}\rangle|^2, \tag{2}$$

where

$$\hat{U}_i(\chi,\theta_x)=\sum_{j=1}^{2}R_{ij}(\theta_x)\hat{S}(\chi_j), \qquad \begin{cases} i=1,2, \\ x=a,b, \end{cases} \tag{3}$$

is the operator representing the propagation of a photon through the channel "i" of MZ_x, and $\hat{S}(\chi_j)$ is the quantum-mechanical operator representing a half-integer SPP oriented at angle χ_j [18], where $\chi_j=\chi+(j-1)\pi$, with $\chi=\alpha,\beta$. Finally, we introduced

$$R(\theta_x)=\begin{pmatrix}\cos\theta_x & -\sin\theta_x \\ \sin\theta_x & \cos\theta_x\end{pmatrix}, \quad (x=a,b), \tag{4}$$

as the orthogonal matrix representing the VBS_x. Explicit expressions for $P_{ij}(\theta_a,\theta_b)$ are given in [18]. For our present purpose it is important to note that $P_{ij}(\theta_a,\theta_b)$ satisfies the no-signaling conditions

$$\sum_{j=1}^{2}P_{ij}(\theta_a,\theta_b)=P_i(\theta_a), \quad \sum_{i=1}^{2}P_{ij}(\theta_a,\theta_b)=P_j(\theta_b), \tag{5}$$

expected for a bipartite, 2×2 dimensional system. From Eqs. (2) and (3) it follows that when a coincidence detection

(i,j) occurs, the input state $|\Psi^{\text{in}}\rangle$ is projected onto the state $|u_i(\alpha,\theta_a)\rangle|\bar{u}_j(\beta,\theta_b)\rangle$, where

$$|u_i(\alpha,\theta_a)\rangle = \hat{U}_i^\dagger(\alpha,\theta_a)|0,0\rangle \equiv \sum_{j=1}^{2} R_{ij}(\theta_a)|S(\alpha_j)\rangle, \qquad (6)$$

and $|S(\alpha_j)\rangle \equiv \hat{S}^\dagger(\alpha_j)|0,0\rangle$. In a similar manner we define $|\bar{u}_j(\beta,\theta_b)\rangle = \hat{U}_j(\beta,\theta_b)|0,0\rangle$ and $|\bar{S}(\beta_j)\rangle \equiv \hat{S}(\beta_j)|0,0\rangle$. From the orthogonality relations [19]

$$\langle S(\alpha_i)|S(\alpha_j)\rangle = \delta_{ij} = \langle \bar{S}(\beta_i)|\bar{S}(\beta_j)\rangle, \qquad (7)$$

it follows that $\{|S(\alpha)\rangle, |S(\alpha+\pi)\rangle\}$ and $\{|\bar{S}(\beta)\rangle, |\bar{S}(\beta+\pi)\rangle\}$ form an orthogonal two-dimensional basis for the photons a and b, respectively. Equations (4) and (6) show that the state $|u_i(\alpha,\theta_a)\rangle|\bar{u}_j(\beta,\theta_b)\rangle$ onto which the initial state $|\Psi^{\text{in}}\rangle$ is projected, remains confined to the four-dimensional two-photon subspace spanned by the basis $\{|S(\alpha_i)\rangle \otimes |\bar{S}(\beta_j)\rangle\}$, $(i,j=1,2)$ when the VBS's "angles" θ_a and θ_b are varied. Moreover, we can see that, e.g., the basis $\{|S(\alpha)\rangle, |S(\alpha+\pi)\rangle\}$ defines a dichotomic subspace, as the basis $\{|H\rangle, |V\rangle\}$ does in polarization space. It is clear then that, when we choose a pair (α,β) of SPP's orientations, we uniquely fix a four-dimensional two-photon subspace.

IV. ADDRESSING THE QUANTUM NONLOCALITY

At this point we know how to calculate the coincidence probabilities $P_{ij}(\theta_a,\theta_b)$ from the state $|\Psi^{\text{out}}\rangle$ at the output of both interferometers. However, to proceed further and test the quantum nonlocality of the input state $|\Psi^{\text{in}}\rangle$, we have to specify our scenario more precisely. We have two parties, say Alice and Bob, who share the two-photon entangled state $|\Psi^{\text{in}}\rangle$ given in Eq. (1). Each one of the two entangled photons belongs to an (in principle) infinite-dimensional Hilbert space. Alice and Bob each have a measuring apparatus: M_a and M_b respectively. Each apparatus M_x $(x=a,b)$ consists of a two-channel Mach-Zehnder interferometer MZ_x, with a parameter θ_x at the experimenter's disposal, followed by two (one per channel $i=1,2$) single-mode fibers F_{xi}. The output ports $i=1,2$ of each M_x are monitored by two detectors D_{x1} and D_{x2} respectively. We stress that in this scenario the SPP rotation angles α and β are *not* experimental "knobs" that are changed during an experiment. Different pairs $\{\alpha,\beta\}$ define *different* experiments which use the *same* initial two-photon entangled state $|\Psi^{\text{in}}\rangle$. In analogy with the polarization case, Alice can choose between two different measurements, say A and A', corresponding to two different choices for the varying-beam-splitter "angles" θ_a and θ_a', respectively. Similarly, Bob can choose between B and B', corresponding to θ_b and θ_b', respectively. Each time Alice and Bob perform a measurement, M_x $(x=a,b)$ gives the string $\{x_1,x_2\}$, where $x_i=1$ when the detector D_{xi} fires and $x_i=0$ when it does not. So, we have two parties (Alice and Bob), two measurements (θ_x and θ_x') per party, and two possible outcomes ({1,0} and {0,1}) per measurement for each party. This situation is usually indicated as a $d\times N_a\times N_b = 2\times 2\times 2$ Bell scenario. For this case, as is well known [20], the most important test of nonlocality is the CHSH inequality [21]

$$S = |E(\theta_a,\theta_b) - E(\theta_a',\theta_b) + E(\theta_a,\theta_b') + E(\theta_a',\theta_b')| \leq 2, \quad (8)$$

where, in our notation, $E(\theta_a,\theta_b)$ is given by

$$\frac{P_{11}(\theta_a,\theta_b) - P_{12}(\theta_a,\theta_b) - P_{21}(\theta_a,\theta_b) + P_{22}(\theta_a,\theta_b)}{P_{11}(\theta_a,\theta_b) + P_{12}(\theta_a,\theta_b) + P_{21}(\theta_a,\theta_b) + P_{22}(\theta_a,\theta_b)}. \qquad (9)$$

We first choose as a special case a common orientation $\alpha=\beta$ for the SPPs for the two photons. It is then straightforward to show that $E(\theta_a,\theta_b)=\cos[2(\theta_a-\theta_b)]$ and, with the particular choice of varying-beam-splitter angles $\theta_a=0$, $\theta_a'=\pi/4$, $\theta_b=\pi/8$, $\theta_b'=3\pi/8$, we achieve the maximum violation $S=2\sqrt{2}$ of the CHSH inequality. This result is valid for *all* values of α. For this special case, we find thus the same result as one would achieve describing an experiment involving dichotomic variables, as in the case of polarization-entangled two-photon states. However, unlike the polarization case, here we have an additional parameter at our disposal, namely the SPP orientation angle α.

Next, we pass to the more general case $\alpha\neq\beta$. For this case we have to use numerical methods. We found, by numerical search, many pairs $\alpha\neq\beta$ which produce violation close to $2\sqrt{2}$. This result is quite interesting since it is a signature that the entanglement of the photon pair may survive this "dimensional reduction" even when different subspaces (viz, different degrees of freedom) are tested. Now, provided that the state vectors $\{|S(\chi)\rangle, |S(\chi+\pi)\rangle, |S(\chi')\rangle, |S(\chi'+\pi)\rangle, |S(\chi'')\rangle, |S(\chi''+\pi)\rangle, \ldots\}(\chi=\alpha,\beta)$ are chosen to be linearly independent, we can extend the CHSH test to the N pairs $\{(\alpha,\beta),(\alpha',\beta'),(\alpha'',\beta''),\ldots,(\alpha^{(N)},\beta^{(N)})\}$ defining N pairs of two-dimensional subspaces whose union defines a $2N\times 2N$ two-photon subspace. In this way we can demonstrate the nonlocal nature of the high-dimensional two-photon OAM-entangled states.

Let us compare our results with the questions (i–iii) posed in the Introduction. From an initial entangled ∞-dimensional state [Eq. (1)] we obtain entangled four-dimensional states; each dimensionally reduced state is maximally entangled; all four-dimensional subspaces are, in this sense, equivalent. All questions posed in the Introduction have thus been positively answered.

V. CONCLUSIONS

In this paper we proposed an experimental setup to investigate the nonlocality (viz, the degree of useful entanglement) of very high-dimensional two-photon OAM-entangled states, by using four detectors only. We use a pair of modified Mach-Zehnder interferometers as OAM analyzers. They reduce the effective dimensionality of the two-photon Hilbert space from ∞ to 4. This entanglement-preserving dimensional reduction permits us to check the nonlocality of the two-photon state with a $2\times 2\times 2$ inequality [20]. In this way we find the maximum violation $2\sqrt{2}$ of the CHSH inequality for any four-dimensional two-photon subspace we choose.

Moreover, because of the strict analogy between our four-dimensional two-photon sub-spaces and four-dimensional two-photon *polarization* space, other interesting experiments (e.g., teleportation of spatial degrees of freedom) can be implemented by using our scheme.

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