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ELSEVIER Stochastic Processes and their Applications 115 (2005) 1209–1232

stochastic processes and their applications

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Bad configurations for random walk in random scenery and related subshifts

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Received 17 August 2004; received in revised form 25 February 2005; accepted 25 February 2005
Available online 7 April 2005

Abstract

In this paper we consider an arbitrary irreducible random walk on $\mathbb{Z}^d$, $d \geq 1$, with i.i.d. increments, together with an arbitrary i.i.d. random scenery. Walk and scenery are assumed to be independent. Random walk in random scenery (RWRS) is the random process where time is indexed by $\mathbb{Z}$, and at each unit of time both the step taken by the walk and the scenery value at the site that is visited are registered. Bad configurations for RWRS are the discontinuity points of the conditional probability distribution for the configuration at the origin of time given the configuration at all other times. We show that the set of bad configurations is non-empty. We give a complete description of this set and compute its probability under the random scenery measure. Depending on the type of random walk, this probability may be zero or positive. For simple symmetric random walk we get three different types of behavior depending on whether $d = 1, 2$, $d = 3, 4$ or $d \geq 5$. Our classification is actually valid for a class of subshifts having a certain determinative property, which we call specifiable, of which RWRS is an example. We also consider bad configurations w.r.t. a finite time interval (replacing the origin) and obtain an almost complete generalization of our results. Remarkably, this extension turns out to be somewhat delicate.

MSC: primary 2000; 60G10; 82B20

Keywords: Random walk in random scenery; Conditional probability distributions; Bad configurations

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doi:10.1016/j.spa.2005.03.001

1. Introduction

1.1. Motivation

An important area in statistical physics concerns itself with the behavior of Gibbs measures under various types of transformations. In the past 20 years many examples have been studied in detail, showing that under (typically simple) transformations the Gibbs property may be preserved, lost or recovered. These examples include spin systems under renormalization, spins systems under stochastic dynamics, disordered spin systems, the Fortuin–Kasteleyn random cluster model, the fuzzy Potts model, hidden Markov models, g-function systems, Hamiltonian dynamics and chaotic dynamics. The history and recent developments of this research area are highlighted in the proceedings of a workshop held at EURANDOM in December 2003, organized by van Enter et al. [2], to appear as a special issue of Markov Processes and Related Fields. For an overview and for references, we refer the reader to that volume.

The present paper is a contribution to the above area. We consider the random process that is obtained by looking at a random scenery on $\mathbb{Z}^d$ along the path of a random walk on $\mathbb{Z}^d$. This random process, which is called random walk in random scenery (RWRS), can be viewed as a random transformation of the random scenery induced by the random walk. The random scenery is assumed to be i.i.d. and the random walk is assumed to have i.i.d. increments and to be independent of the random scenery. Under these assumptions we will show that RWRS is not Gibbs, i.e., the conditional probabilities for RWRS inside any finite time interval given the configuration outside are not uniformly positive and not everywhere continuous. We will give a *complete description* of the set of discontinuity points, which turns out to be non-empty. Moreover, we will compute the probability of this set under the random scenery measure. This probability may be zero or positive depending on the type of random walk.

1.2. Random walk in random scenery

We begin by defining the random process that will be the object of our study.

Fix an integer $d \geqslant 1$. Let $X = (X_n)_{n\in\mathbb{Z}}$ be a sequence of i.i.d. random variables taking values in a *finite* set $F \subset \mathbb{Z}^d$ according to a common distribution m having full support on F. Let $S = (S_n)_{n\in\mathbb{Z}}$ be the corresponding two-sided *random walk* on $\mathbb{Z}^d$, defined by

$$S_0 = 0 \quad \text{and} \quad S_n - S_{n-1} = X_n,\ n \in \mathbb{Z},$$

i.e., X_n is the step at time n and S_n is the position at time n. To make S into an *irreducible* random walk, we will assume that F generates $\mathbb{Z}^d$, i.e., for all $x \in \mathbb{Z}^d$ there exist $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in F$ such that $x_1 + \cdots + x_n = x$.

Let $C = (C_z)_{z\in\mathbb{Z}^d}$ be a field of i.i.d. random variables taking values in a *finite* set G with $|G| \geqslant 2$ according to a common distribution having full support on G. Denote the joint distribution of C (which is a product measure on $G^{\mathbb{Z}^d}$) by μ. We will refer to

G as the set of scenery values and to C as the *random scenery*, i.e., C_x is the scenery value at site x.

Let

$$Y = (Y_n)_{n\in\mathbb{Z}} \quad \text{with} \;\; Y_n = (C \circ S)_n = C_{S_n}$$

be the sequence of scenery values observed along the random walk. The joint process

$$Z = (Z_n)_{n\in\mathbb{Z}} \quad \text{with} \;\; Z_n = (X_n, Y_n)$$

is called the RWRS[1] associated with m and μ.

Let $H = F \times G$. The range of Z, which we denote by Ω, is the set of *compatible configurations*; in short

$$\Omega = \{z \in H^{\mathbb{Z}}\colon\; z = (x, y = c \circ s(x)) \text{ for some } x \in F^{\mathbb{Z}},\; c \in G^{\mathbb{Z}^d}\}$$

with $s(x)$ the walk associated with x. Observe that Ω is shift-invariant, is closed in the product topology and is a proper subshift of $H^{\mathbb{Z}}$. Let $\mathbb{P}$ denote the probability distribution of Z on Ω. From now on we will consider the random sequences X, Y and Z as being defined on the common sample space Ω. By our assumptions on m and μ, the cylinder set $\{Z = \omega \text{ on } I\} = \{Z_n = \omega_n \text{ for } n \in I\}$ has positive $\mathbb{P}$-measure for all $\omega \in \Omega$ and all finite $I \subset \mathbb{Z}$.

The main question that we will address in this paper is the following: Does there exist a version $V(\cdot \mid \eta)$ of the conditional probability distribution

$$\mathbb{P}(Z_0 \in \cdot \mid Z = \eta \text{ on } \mathbb{Z}\backslash\{0\}), \quad \eta \in \Omega,$$

such that the map $\eta \mapsto V(\cdot \mid \eta)$ is *everywhere/almost everywhere/not almost everywhere continuous* on Ω? The same question will be addressed for

$$\mathbb{P}(X_0 \in \cdot \mid Z = \eta \text{ on } \mathbb{Z}\backslash\{0\}), \quad \eta \in \Omega,$$
$$\mathbb{P}(Y_0 \in \cdot \mid Z = \eta \text{ on } \mathbb{Z}\backslash\{0\}), \quad \eta \in \Omega.$$

In a forthcoming paper we will look at

$$\mathbb{P}(Y_0 \in \cdot \mid Y = \zeta \text{ on } \mathbb{Z}\backslash\{0\}), \quad \zeta \in G^{\mathbb{Z}}.$$

It turns out that this conditional probability distribution has a behavior that is very different from the one for Z. Indeed, Y is the projection of RWRS where the steps of the random walk are not registered. Consequently, Y has as its support the full shift $G^{\mathbb{Z}}$. For our results on Z it is essential that Ω, the support of Z, is a proper subshift.

1.3. Bad configurations and discontinuity points for subshifts

In this section we view Ω as a subshift (a shift-invariant and closed subset) of an arbitrary product space $H^{\mathbb{Z}}$, with H a finite set, and we view the compatible configurations of RWRS as a specific example. For $\omega \in \Omega$, define $Z_k(\omega) = \omega_k$, and

[1] In ergodic theory Z is referred to as the T, T^{-1}-process. The interest in this process originally came from the fact that it was conjectured to be a simple and natural example of a K-automorphism that is not Bernoulli. Kalikow [4] showed that this is indeed the case for $d = 1$ and simple random walk. This result was extended by den Hollander and Steif [3] to essentially arbitrary recurrent random walk.

let $\mathbb{P}$ be a translation invariant probability measure on Ω that assigns positive measure to all cylinder sets. We view the conditional probability distribution $\mathbb{P}(Z_0 \in \cdot \mid (Z_n)_{n\neq 0})$ as a map from Ω_0 to $\mathscr{P}(H)$, where

$$\Omega_0 = \{\eta \in H^{\mathbb{Z}\setminus\{0\}}\colon \text{ there is an } \omega \in \Omega \text{ such that } \omega = \eta \text{ on } \mathbb{Z}\setminus\{0\}\}$$

is the set of *extendable configurations* and $\mathscr{P}(H)$ is the set of probability measures on H (as opposed to a map from Ω to $\mathscr{P}(H)$).

Our question about continuity of conditional probabilities will be formulated in terms of the so-called *bad configurations*. We use three different notions of badness for a configuration: (a) bad for Z_0, (b) bad for a Z_0-measurable random variable U, (c) bad for a set $A \subseteq H$. In what follows we will always identify a random variable that is measurable with respect to Z_0 with a function on H. For $n \in \mathbb{N}$, write $\Lambda_n = \mathbb{Z} \cap [-n, n]$.

Definition 1.1. Let Ω and $\mathbb{P}$ be as above.

(a) A configuration $\eta \in \Omega_0$ is said to be a *bad configuration for* Z_0 if there is an $\varepsilon > 0$ such that for all $n \in \mathbb{N}$ there are $m \geqslant n$ with $m \in \mathbb{N}$ and $\delta \in \Omega_0$ with $\delta = \eta$ on $\Lambda_n\setminus\{0\}$ such that

$$\|\mathbb{P}(Z_0 \in \cdot \mid Z = \eta \text{ on } \Lambda_m\setminus\{0\}) - \mathbb{P}(Z_0 \in \cdot \mid Z = \delta \text{ on } \Lambda_m\setminus\{0\})\| \geqslant \varepsilon,$$

where $\|\ \|$ denotes total variation norm on $\mathscr{P}(H)$.

(b) Let U be a random variable that is measurable with respect to Z_0. A configuration $\eta \in \Omega_0$ is said to be a *bad configuration for* U if there is an $\varepsilon > 0$ such that for all $n \in \mathbb{N}$ there are $m \geqslant n$ with $m \in \mathbb{N}$ and $\delta \in \Omega_0$ with $\delta = \eta$ on $\Lambda_n\setminus\{0\}$ such that

$$\|\mathbb{P}(U \in \cdot \mid Z = \eta \text{ on } \Lambda_m\setminus\{0\}) - \mathbb{P}(U \in \cdot \mid Z = \delta \text{ on } \Lambda_m\setminus\{0\})\| \geqslant \varepsilon.$$

(c) Let $A \subseteq H$. A configuration $\eta \in \Omega_0$ is said to be a *bad configuration for* A if there is an $\varepsilon > 0$ such that for all $n \in \mathbb{N}$ there are $m \geqslant n$ with $m \in \mathbb{N}$ and $\delta \in \Omega_0$ with $\delta = \eta$ on $\Lambda_n\setminus\{0\}$ such that

$$|\mathbb{P}(Z_0 \in A \mid Z = \eta \text{ on } \Lambda_m\setminus\{0\}) - \mathbb{P}(Z_0 \in A \mid Z = \delta \text{ on } \Lambda_m\setminus\{0\})| \geqslant \varepsilon.$$

In words, for (a), no matter how large n is, by tampering with the configuration inside $\Lambda_m\setminus\Lambda_n$ for some large $m \geqslant n$, the conditional distribution of Z_0 can be non-trivially affected; for (b), the distribution of U can be non-trivially affected; for (c), the probability that Z_0 falls in A can be non-trivially affected.

Note that (a) is (b) with $U = Z_0$, and that (c) is (b) with $U = 1_A$. Note that η is bad for Z_0 if and only if it is bad for some $A \subseteq H$, and that η is bad for U if and only if it is a bad configuration for some U-measurable subset of H. We write $B(U)$ for the set of bad configurations for U, and $B(A)$ for the set of bad configurations for A.

The relationship between bad configurations and discontinuity points is given by the following theorem, which will be proved in Section 2.

Theorem 1.2. (i) *Fix* $A \subseteq H$ *and let* $W(A \mid \eta)$ *be any version of the conditional probability* $\mathbb{P}(Z_0 \in A \mid Z = \eta$ *on* $\mathbb{Z}\backslash\{0\})$, *viewed as a map from* Ω_0 *to* $[0,1]$. *Then* $B(A)$ *is contained in the set of discontinuity points for the map* $\eta \mapsto W(A \mid \eta)$.

(ii) *Fix* $A \subseteq H$. *There is a version* $W(A \mid \eta)$ *of the conditional probability* $\mathbb{P}(Z_0 \in A \mid Z = \eta$ *on* $\mathbb{Z}\backslash\{0\})$ *such that* $B(A)$ *is equal to the set of discontinuity points for the map* $\eta \mapsto W(A \mid \eta)$.

(iii) *Analogous properties hold for the other two notions of bad configuration.*

Let $\mathbb{P}_0$ be the probability measure on Ω_0 induced by $\mathbb{P}$. Given a Z_0-measurable random variable U, the question whether there exists an everywhere/almost everywhere/not almost everywhere continuous version of the conditional probability distribution of U given $(Z_n)_{n \neq 0}$ translates into the question whether $B(U) = \emptyset$, $\mathbb{P}_0(B(U)) = 0$ or $\mathbb{P}_0(B(U)) > 0$.

1.4. Bad configurations for RWRS

In the context of RWRS, typical choices for U are X_0, Y_0 and Z_0. The following theorem will be proved in Section 5.

Theorem 1.3. *Assume that* m *and* μ *satisfy the conditions in Section* 1.2.

(i) $B(X_0)$, $B(Y_0)$ *and* $B(Z_0)$ *are non-empty.*

(ii) *For* $d = 1, 2$ *and* $\sum_{x \in F} x m(x) = 0$,

$$\mathbb{P}_0(B(X_0)) = \mathbb{P}_0(B(Y_0)) = \mathbb{P}_0(B(Z_0)) = 0.$$

(iii) *For* $d = 3, 4$ *and* $\sum_{x \in F} x m(x) = 0$,

$$\mathbb{P}_0(B(X_0)) = 0,$$
$$0 < \mathbb{P}(S_n \neq 0 \text{ for all } n \neq 0) = \mathbb{P}_0(B(Y_0)) = \mathbb{P}_0(B(Z_0)) < 1.$$

(iv) *For* $d \geqslant 5$ *and* $\sum_{x \in F} x m(x) = 0$ *or* $d \geqslant 1$ *and* $\sum_{x \in F} x m(x) \neq 0$,

$$0 < \mathbb{P}_0(B(X_0)) < 1,$$
$$0 < \mathbb{P}(S_n \neq 0 \text{ for all } n \neq 0) < \mathbb{P}_0(B(Y_0)) < \mathbb{P}_0(B(Z_0)) < 1.$$

The proof of Theorem 1.3 is based on a complete description of the sets $B(X_0)$, $B(Y_0)$ and $B(Z_0)$, obtained in Section 4.

So far, we have looked at how the conditional distribution at a *single time point* depends on the configuration elsewhere. It is quite natural to also ask how the conditional distribution in a *finite time interval* depends on the configuration elsewhere. Therefore, let Λ be a finite interval in $\mathbb{Z}$, and define the set of Λ-*extendable configurations*, in analogy with the case $\Lambda = \{0\}$, as

$$\Omega_\Lambda = \{\eta \in H^{\mathbb{Z}\backslash\Lambda}\colon \text{ there is an } \omega \in \Omega \text{ such that } \omega = \eta \text{ on } \mathbb{Z}\backslash\Lambda\}.$$

Given a probability measure on Ω, we can, in a way that is analogous to Definition 1.1, define a configuration in Ω_Λ to be bad for $Z_\Lambda = (Z_n)_{n \in \Lambda}$, bad for a

Z_Λ-measurable random variable U, and bad for a set $A \subseteq H^\Lambda$. These obvious formulations are left to the reader. A version of Theorem 1.2 again holds. In Section 7 we will obtain a full generalization of Theorem 1.3.

We finally mention that there are random processes with full support for which all configurations are bad. The following unpublished example is due to Rob van den Berg. Let $(X_n)_{n\in\mathbb{Z}}$ be i.i.d. $\{0,1\}$-valued random variables with $\mathbb{P}(X_n = 1) = p \in (0,1)$, $p \neq \frac{1}{2}$. For $n \in \mathbb{Z}$, let $Y_n = 1_{\{X_n \neq X_{n+1}\}}$. Then $Y = (Y_n)_{n\in\mathbb{Z}}$ is a stationary random process with full support, called a 2-block factor in symbolic dynamics. It is easy to show that, for reasons of parity, every configuration in $\{0,1\}^{\mathbb{Z}\setminus\{0\}}$ is bad for Y_0.

1.5. Outline

The outline of the rest of this paper is as follows. In Section 2 we prove Theorem 1.2. In Section 3 we look at arbitrary subshifts and give a complete classification of the bad configurations for those subshifts that have a certain determinative property, which we call ($\{0\}$)-specifiable. In Section 4 we show that RWRS has this property. In Section 5 we prove Theorem 1.3. In Sections 6 and 7 we move on to studying bad configurations for finite intervals Λ. As we will see, this extension is somewhat delicate. Indeed, since our main motivating example of RWRS is not Λ-specifiable when $|\Lambda| > 1$, we introduce another property of subshifts, which we call weakly Λ-specifiable, and study the bad configurations. In Section 7 we show that RWRS has this property when $|\Lambda| \geqslant 1$ and generalize Theorem 1.3.

Remark. In the present paper, although our subshifts are indexed by $\mathbb{Z}$, most of our results hold equally well for subshifts indexed by $\mathbb{Z}^d$. In addition, all our results for RWRS go through if the i.i.d. assumption on the random scenery is replaced by (translation invariance and) the weaker *uniform finite energy* property, i.e.,

$$\min_{c\in G} \operatorname{essinf} \mu(C_0 = c \mid (C_z)_{z\neq 0}) > 0.$$

2. Proof of Theorem 1.2

Proof. We give the proofs of (i) and (ii); the proof of (iii) is similar.

(i) Fix $A \subseteq H$ and any version $W(A \mid \eta)$ of the conditional probability $\mathbb{P}(Z_0 \in A \mid Z = \eta \text{ on } \mathbb{Z}\setminus\{0\})$, viewed as a map from Ω_0 to $[0,1]$. Suppose that $\eta \in \Omega_0$ is a continuity point of this map. Then

$$\lim_{n\to\infty} \sup_{\substack{\xi,\zeta\in\Omega_0 \\ \xi=\zeta=\eta \text{ on } \Lambda_n\setminus\{0\}}} |W(A \mid \xi) - W(A \mid \zeta)| = 0.$$

Fix $\varepsilon > 0$ and let n be so large that the supremum in the expression above is $\leqslant \varepsilon$. Let $m \geqslant n$ and $\delta \in \Omega_0$ be such that $\delta = \eta$ on $\Lambda_n\setminus\{0\}$. Abbreviating $\mathbb{P}_0(\cdot \mid \eta_{\Lambda_m}) =$

$\mathbb{P}_0(\cdot \mid Z = \eta \text{ on } \Lambda_m \backslash \{0\})$, we obtain

$$\begin{aligned}
&|\mathbb{P}(Z_0 \in A \mid Z = \eta \text{ on } \Lambda_m \backslash \{0\}) - \mathbb{P}(Z_0 \in A \mid Z = \delta \text{ on } \Lambda_m \backslash \{0\})| \\
&\quad = \left| \int_{\Omega_0} \mathrm{d}\,\mathbb{P}_0(\zeta \mid \eta_{\Lambda_m}) W(A \mid \zeta) - \int_{\Omega_0} \mathrm{d}\mathbb{P}_0(\xi \mid \delta_{\Lambda_m}) W(A \mid \xi) \right| \\
&\quad \leqslant \int_{\Omega_0} \int_{\Omega_0} \mathrm{d}\,\mathbb{P}_0(\zeta \mid \eta_{\Lambda_m})\, \mathrm{d}\mathbb{P}_0(\xi \mid \delta_{\Lambda_m}) |W(A \mid \zeta) - W(A \mid \xi)| \\
&\quad \leqslant \varepsilon.
\end{aligned}$$

Hence $\eta \notin B(A)$. (See also Maes et al. [6], Proposition 4.2.)

(ii) Fix $A \subseteq H$, and for $\eta \in \Omega_0$ define

$$W(A \mid \eta) = \liminf_{n \to \infty} w_n(A \mid \eta),$$

where

$$w_n(A \mid \eta) = \mathbb{P}(Z_0 \in A \mid Z = \eta \text{ on } \Lambda_n \backslash \{0\}), \quad n \in \mathbb{N}.$$

The martingale convergence theorem guarantees that $W(A \mid \eta)$ is a version of the conditional probability $\mathbb{P}(Z_0 \in A \mid Z = \eta \text{ on } \mathbb{Z} \backslash \{0\})$.

The main ingredient of the proof that $W(A \mid \eta)$ is continuous at configurations outside $B(A)$ is the fact that $(w_n(A \mid \eta))_{n \in \mathbb{N}}$ is a Cauchy sequence when $\eta \notin B(A)$. To see the latter, fix $\eta \notin B(A)$ and $\varepsilon > 0$. Then, by the definition of $B(A)$, we can fix an $n \in \mathbb{N}$ such that for all $m \geqslant n$ and $\delta \in \Omega_0$ with $\delta = \eta$ on $\Lambda_n \backslash \{0\}$,

$$|w_m(A \mid \eta) - w_m(A \mid \delta)| \leqslant \varepsilon.$$

Hence, for all $m \geqslant n$,

$$\begin{aligned}
|w_n(A \mid \eta) - w_m(A \mid \eta)| &= \left| \int_{\Omega_0} \mathrm{d}\,\mathbb{P}_0(\delta \mid \eta_{\Lambda_n}) w_m(A \mid \delta) - w_m(A \mid \eta) \right| \\
&\leqslant \int_{\Omega_0} \mathrm{d}\mathbb{P}_0(\delta \mid \eta_{\Lambda_n}) |w_m(A \mid \delta) - w_m(A \mid \eta)| \\
&\leqslant \varepsilon,
\end{aligned}$$

where we adopted the notation $\mathbb{P}_0(\cdot \mid \eta_{\Lambda_n})$ from the proof of part (i).

To prove continuity outside $B(A)$, fix $\varepsilon > 0$, $\eta \notin B(A)$ and choose n as above. Then for all $\delta \in \Omega_0$ with $\delta = \eta$ on $\Lambda_n \backslash \{0\}$,

$$\begin{aligned}
|W(A \mid \eta) - W(A \mid \delta)| &= |\liminf_{m \to \infty} w_m(A \mid \eta) - \liminf_{m \to \infty} w_m(A \mid \delta)| \\
&= |\lim_{m \to \infty} w_m(A \mid \eta) - \liminf_{m \to \infty} w_m(A \mid \delta)| \\
&= |\limsup_{m \to \infty} \{w_m(A \mid \eta) - w_m(A \mid \delta)\}| \\
&\leqslant \limsup_{m \to \infty} |w_m(A \mid \eta) - w_m(A \mid \delta)| \\
&\leqslant \varepsilon,
\end{aligned}$$

by the above inequality. □

3. Identification of bad configurations for subshifts

In this section we work with the more general setup of Section 1.3, where Ω is an arbitrary subshift of $H^{\mathbb{Z}}$ and $\mathbb{P}$ is a translation invariant probability measure on Ω. For Ω satisfying a certain determinative property (see Definition 3.3 below) we will explicitly describe the set of bad configurations for a Z_0-measurable random variable U (Theorem 3.6 below). This description will be *purely topological* and will not depend on $\mathbb{P}$.

3.1. Insertion and specifiable

Definition 3.1. For $\eta \in \Omega_0$, define

$$\text{insert}(\eta) = \{a \in H\colon\ \omega \text{ defined by } \omega_0 = a \text{ and } \omega = \eta \text{ on } \mathbb{Z}\backslash\{0\} \text{ is in } \Omega\}.$$

In words, insert(η) consists of those elements in H that can be inserted in η at time 0 to give a configuration in Ω. The following lemma states that if an element of H cannot be inserted in η, then it cannot be inserted in any configuration that agrees with η on a sufficiently large interval around 0.

Lemma 3.2. *Let* $\eta \in \Omega_0$. *Then there is an* $n \in \mathbb{N}$ *such that* $\text{insert}(\delta) \subseteq \text{insert}(\eta)$ *for all* $\delta \in \Omega_0$ *with* $\delta = \eta$ *on* $\Lambda_n\backslash\{0\}$.

Proof. Suppose that for all $n \in \mathbb{N}$, there is a $\delta^n \in \Omega_0$ with $\delta^n = \eta$ on $\Lambda_n\backslash\{0\}$ such that $a^n \in \text{insert}(\delta^n)$ for some $a^n \notin \text{insert}(\eta)$. Then, since H is finite, we can find an $a \notin \text{insert}(\eta)$ and a subsequence $(n_k)_{k\in\mathbb{N}}$ such that $a \in \text{insert}(\delta^{n_k})$ for all $k \in \mathbb{N}$. Define $(\omega^k)_{k\in\mathbb{N}}$ with $\omega^k \in \Omega$ by putting $\omega^k_0 = a$ and$\omega^k = \delta^{n_k}$ on $\mathbb{Z}\backslash\{0\}$. Then, clearly, $\lim_{k\to\infty}\omega^k = \omega$ with $\omega_0 = a$ and $\omega = \eta$ on $\mathbb{Z}\backslash\{0\}$. Since Ω is closed, it follows that $\omega \in \Omega$, and hence that $a \in \text{insert}(\eta)$, which is a contradiction. $\square$

Our key property of subshifts is the following.

Definition 3.3. A subshift Ω is *specifiable* if for all $\eta \in \Omega_0$, $a \in \text{insert}(\eta)$ and $n \in \mathbb{N}$, there is a $\delta \in \Omega_0$ such that $\delta = \eta$ on $\Lambda_n\backslash\{0\}$ and $\text{insert}(\delta) = \{a\}$.

In words, Ω is specifiable if the following holds. Let η be an extendable configuration for which *more than one* element of H can be inserted at time 0. Let a be any of these elements. Then, given an arbitrarily large interval around 0, we can tamper with η outside this interval such that a is the *only* element of H that can be inserted in the new configuration.

3.2. Bad configurations

Lemmas 3.4 and 3.5 below give an expression for $B(A)$, the set of bad configurations for $A \subseteq H$, by means of two inclusions, the second of which requires Ω to be specifiable. Although $B(A)$ depends on $\mathbb{P}$, the inclusions involve a set that depends on Ω only.

Lemma 3.4. *For every* $A \subseteq H$,

$$B(A) \subseteq \{\eta \in \Omega_0\text{: there are } a \in A \text{ and } b \notin A \text{ such that } a, b \in \text{insert}(\eta)\}.$$

Proof. Suppose that $\text{insert}(\eta) \subseteq A$ (resp. $\subseteq A^c$). By Lemma 3.2, there is an $n \in \mathbb{N}$ such that $\text{insert}(\delta) \subseteq A$ (resp. $\subseteq A^c$) for all $\delta \in \Omega_0$ with $\delta = \eta$ on Λ_n. Hence $\mathbb{P}(Z_0 \in A \mid Z = \delta \text{ on } \Lambda_m) = 1$ (resp. $= 0$) for all $m \geqslant n$ and for all $\delta \in \Omega_0$ with $\delta = \eta$ on Λ_n. Therefore η is not a bad configuration for A. □

Lemma 3.5. *Let* Ω *be a specifiable subshift. Then for every* $A \subseteq H$,

$$B(A) \supseteq \{\eta \in \Omega_0\text{: there are } a \in A \text{ and } b \notin A \text{ such that } a, b \in \text{insert}(\eta)\}.$$

Proof. The claim is trivial for $A = \emptyset, H$. Therefore assume that $A \neq \emptyset, H$. Let $\eta \in \Omega_0$, $a \in A$ and $b \notin A$ be such that $a, b \in \text{insert}(\eta)$. Fix $n \in \mathbb{N}$. Since Ω is specifiable, there are δ_a and δ_b in Ω_0 with $\delta_a = \delta_b = \eta$ on $\Lambda_n \backslash \{0\}$ such that $\text{insert}(\delta_a) = \{a\}$ and $\text{insert}(\delta_b) = \{b\}$. By Lemma 3.2, there is an $m \geqslant n$ such that if $\zeta = \delta_a$ on $\Lambda_m \backslash \{0\}$, then $\text{insert}(\zeta) = \{a\}$, while if $\zeta = \delta_b$ on $\Lambda_m \backslash \{0\}$, then $\text{insert}(\zeta) = \{b\}$. Hence,

$$\mathbb{P}(Z_0 \in A \mid Z = \delta_a \text{ on } \Lambda_m \backslash \{0\}) = 1,$$
$$\mathbb{P}(Z_0 \in A \mid Z = \delta_b \text{ on } \Lambda_m \backslash \{0\}) = 0.$$

The latter imply that either

$$|\mathbb{P}(Z_0 \in A \mid Z = \eta \text{ on } \Lambda_m \backslash \{0\}) - \mathbb{P}(Z_0 \in A \mid Z = \delta_a \text{ on } \Lambda_m \backslash \{0\})| \geqslant \tfrac{1}{2}$$

or

$$|\mathbb{P}(Z_0 \in A \mid Z = \eta \text{ on } \Lambda_m \backslash \{0\}) - \mathbb{P}(Z_0 \in A \mid Z = \delta_b \text{ on } \Lambda_m \backslash \{0\})| \geqslant \tfrac{1}{2}.$$

Hence η is bad for A with $\varepsilon = \frac{1}{2}$. □

Let U be a Z_0-measurable random variable. Then U in a natural way gives us a partition π_U of H and a σ-algebra σ_U on H. The following identification of $B(U)$, the set of bad configurations for U, follows from Lemmas 3.4 and 3.5.

Theorem 3.6. *Let* Ω *be a specifiable subshift and let* U *be a* Z_0*-measurable random variable. Then*

$$B(U) = \{\eta \in \Omega_0\text{: } |\{U(a)\text{: } a \in \text{insert}(\eta)\}| > 1\}.$$

In particular,

$$B(Z_0) = \{\eta \in \Omega_0\text{: } |\text{insert}(\eta)| > 1\}.$$

Proof. By definition,

$$B(U) = \bigcup_{A \in \sigma_U} B(A).$$

Since Ω is specifiable, it follows from Lemmas 3.4 and 3.5 that

$$B(U) = \{\eta \in \Omega_0\colon \text{ there are } A \in \sigma_U \text{ and } a \in A,\ b \notin A \text{ such that } a, b \in \text{insert}(\eta)\}. \qquad \square$$

In words, $B(U)$ is the set of configurations for which we can insert elements from different partition elements of π_U.

To close this section, we give an example of a subshift Ω that is not specifiable and a translation invariant probability measure $\mathbb{P}$ on Ω for which the containment in Lemma 3.5 fails (the reverse containment holds by Lemma 3.4). Let Ω be the subshift of $\{0,1,2\}^{\mathbb{Z}}$ consisting of those configurations in which 0 is followed by 0 or 1, 1 is followed by 1 or 2, and 2 is followed by 2 or 0 (this is an example of a so-called Markov shift). It is obvious that Ω is not specifiable. Let $\mathbb{P}$ be the unique stationary probability measure on Ω corresponding to the Markov chain on $\{0,1,2\}$ that with probability $\frac{1}{2}$ stands still and with probability $\frac{1}{2}$ increases by 1 (mod 3). For any $A \neq \emptyset, \{0,1,2\}$, trivially $B(A) = \emptyset$, but the right-hand set in Lemma 3.5 is non-empty.

4. Identification of bad configurations for RWRS

The following lemma shows that the results of Section 3 apply to RWRS.

Lemma 4.1. *Let Ω be the subshift associated with RWRS. Then Ω is specifiable.*

Proof. Fix $\eta \in \Omega_0$, $(x,c) \in \text{insert}(\eta)$ and $n \in \mathbb{N}$. To prove that Ω is specifiable, we have to show that there is an $\omega \in \Omega$ such that $\omega = \eta$ on $\Lambda_n \backslash \{0\}$ and $\text{insert}(\omega_{\mathbb{Z}\backslash\{0\}}) = \{(x,c)\}$. We will achieve this by showing that the class of $\omega \in \Omega$ satisfying conditions (C1–C3) below have this property and that this class is non-empty. Before giving the mathematics, we describe the idea. After choosing ω to be (x,c) at time 0 and to agree with η on $\Lambda_n \backslash \{0\}$, we define ω elsewhere so that

(1) the random walk in positive time reaches the origin, a fixed site y far away, as well as all the sites nearby y,
(2) the scenery value revealed at y is different from that revealed at the sites nearby y,
(3) y is reached at some negative time.

In this way we can recover the scenery value seen at time 0 (since the walk comes back to 0 at some positive time) and we can recover the step at time 0 (since every choice for this step other than x yields an element outside Ω).

Let $\omega \in \Omega$ be such that

$$\text{(C1)}\colon\ \omega_0 = (x,c), \omega = \eta \text{ on } \Lambda_n \backslash \{0\} \text{ and } S_k(\omega) = 0 \text{ some } k > 0.$$

Let $y \in \mathbb{Z}^d$ be such that

$$(y + D) \cap R_{[-n,n]} = \emptyset,$$

where $D = \{x_1 - x_2\colon\ x_1, x_2 \in F\}$ and $R_{[-n,n]}$ is the set of sites the walk can possibly visit between times $-n$ and n (i.e., $\pm$ all the partial sums of $\leqslant n$ elements from F). Let $\omega \in \Omega$ be such that

(C2): $S_l(\omega) = y$ for some $l < -n$.

Let $c_1, c_2 \in G$ with $c_1 \neq c_2$. Let ω be such that for all $z \in y + D$ there is an integer $m_z = m_z(\omega) > n$ such that

$$\text{(C3):}\ \ S_{m_z}(\omega) = z \quad \text{and} \quad Y_{m_z}(\omega) = \begin{cases} c_1 & \text{if } z = y, \\ c_2 & \text{if } z \in y + D\backslash\{0\}. \end{cases}$$

It is easy to see that an $\omega \in \Omega$ satisfying conditions (C1–C3) exists (recall that the random walk is irreducible). Moreover, if δ is the restriction ω to $\mathbb{Z}\backslash\{0\}$, then $\text{insert}(\delta) = \{(x, c)\}$. Indeed, (C1) allows us to retrieve the scenery value c seen at time 0 while (C2–C3) allows us to retrieve the step x taken at time 0. $\square$

By Theorem 3.6 and Lemma 4.1, the respective sets of bad configurations for RWRS are given by:

Corollary 4.2.

$$B(X_0) = \{\eta \in \Omega_0\colon\ |\{x \in F\colon\ (x, c) \in \text{insert}(\eta)\ \textit{for some}\ c \in G\}| > 1\},$$
$$B(Y_0) = \{\eta \in \Omega_0\colon\ |\{c \in G\colon\ (x, c) \in \text{insert}(\eta)\ \textit{for some}\ x \in F\}| > 1\},$$
$$B(Z_0) = \{\eta \in \Omega_0\colon\ |\text{insert}(\eta)| > 1\}.$$

In words, the bad configurations for X_0, Y_0 and Z_0 are precisely those configurations for which *more than one* value can be inserted for the missing coordinate at time 0.

Note that

$$B(Z_0) = B(X_0) \cup B(Y_0).$$

5. Proof of Theorem 1.3

The proof is based on Lemmas 5.1–5.6 below.

5.1. Key lemmas

Lemma 5.1.

(i) $\mathbb{P}_0(B(Y_0)) \leqslant \mathbb{P}_0(B(Z_0)) = \mathbb{P}_0(B(X_0)) + \mathbb{P}_0(B(Y_0)\backslash B(X_0))$.
(ii) $\mathbb{P}_0(B(X_0)) \leqslant \mathbb{P}_0(B(Z_0)) = \mathbb{P}_0(B(Y_0)) + \mathbb{P}_0(B(X_0)\backslash B(Y_0))$.
(iii) $\mathbb{P}_0(B(Y_0)\backslash B(X_0)) \leqslant \mathbb{P}(S_n \neq 0\ \textit{for all}\ n \neq 0) \leqslant \mathbb{P}_0(B(Y_0))$.

Proof. (i–ii) These are immediate from the relation $B(Z_0) = B(X_0) \cup B(Y_0)$.

(iii) To prove the first inequality, let $\eta \in \Omega_0$ be good for X_0 but bad for Y_0, and let $\omega \in \Omega$ be a configuration such that $\omega = \eta$ on $\mathbb{Z}\backslash\{0\}$. Since η is good for X_0, we have by Corollary 4.2 that

$$|\{x \in F\colon\ (x,c) \in \text{insert}(\eta) \text{ for some } c \in G\}| = 1.$$

So, if $(x,c) \in \text{insert}(\eta)$ for some $c \in G$, then $x = X_0(\omega)$. Since η is bad for Y_0, we can find $c_1, c_2 \in G$ with $c_1 \neq c_2$ such that $(X_0(\omega), c_1), (X_0(\omega), c_2) \in \text{insert}(\eta)$. This implies that $S_n(\omega) \neq 0$ for all $n \neq 0$.

To prove the second inequality, let $\omega \in \Omega$ be such that $S_n(\omega) \neq 0$ for all $n \neq 0$, and let η be the restriction of ω to $\mathbb{Z}\backslash\{0\}$. Then $(X_0(\omega), c) \in \text{insert}(\eta)$ for all $c \in G$. Hence,

$$|\{c \in G\colon\ (x,c) \in \text{insert}(\eta) \text{ for some } x \in F\}| = |G| > 1,$$

and therefore η is bad for Y_0. □

Let $S_- = \{S_n\colon\ n < 0\}$ and $S_+ = \{S_n\colon\ n \geqslant 0\}$ denote the past, respectively, the future of the random walk. Define random sets $I_2 \subseteq I_1 \subseteq \mathbb{Z}^d$ by

$$I_1 = \{z \in S_-\colon\ (z+D) \cap S_+ \neq \emptyset\} \cup \{z \in S_+\colon\ (z+D) \cap S_- \neq \emptyset\},$$
$$I_2 = \{z \in S_-\colon\ (z+D) \subseteq S_+\},$$

where $D = \{x_1 - x_2\colon\ x_1, x_2 \in F\}$. Both these sets are measurable w.r.t. S.

Lemma 5.2. *Let $\rho = \max_{c \in G} \mu(C_0 = c)$. Then*

$$\mathbb{E}(\rho^{|I_1|}) \leqslant \mathbb{P}_0(B(X_0)) \leqslant \mathbb{E}((1 - \rho^{|D|-1}(1-\rho))^{|I_2|/|D|}),$$

where $\mathbb{E}$ denotes expectation w.r.t. $\mathbb{P}$.

Proof. To prove the first inequality, fix $c \in G$ with $\mu(C_0 = c) = \rho$ and $\omega \in \Omega$, and let η be the restriction of ω to $\mathbb{Z}\backslash\{0\}$. If all sites in $I_1(\omega)$ have scenery value c, then $(x,c) \in \text{insert}(\eta)$ for all $x \in F$. Indeed, $I_1(\omega)$ consists of those sites in the past (future) that lie in the D-neighborhood of the future (past). Changing the step at time 0 can only make two sites in $I_1(\omega) \cap S_+(\omega)$ and $I_1(\omega) \cap S_-(\omega)$ land on top of each other that are within the D-neighborhood of each other. Therefore, changing the step at time 0 can never lead to a conflict of scenery value. Since $|F| > 1$ (by the irreducibility of the random walk), the fact that $(x,c) \in \text{insert}(\eta)$ for all $x \in F$ implies, by Corollary 4.2, that η is a bad configuration for X_0. By the independence of the random walk and the random scenery and by the i.i.d. property of the random scenery, the conditional probability given the walk that all sites in $I_1(\omega)$ have scenery value c is equal to $\rho^{|I_1(\omega)|}$. From this, the first inequality follows.

To prove the second inequality, again fix $c \in G$ and $\omega \in \Omega$, and again let η be the restriction of ω to $\mathbb{Z}\backslash\{0\}$. Let $z \in I_2(\omega)$, and suppose that all sites in $z + D$ except z have scenery value c. Then $(x,c) \in \text{insert}(\eta)$ implies that $x = X_0(\omega)$ (since any change of step at time 0 changes the scenery value) and hence, by Corollary 4.2, that η is not bad for X_0. The probability that all sites in $z + D$ except z have scenery value c is equal to $\rho^{|D|-1}(1-\rho)$. Moreover, it is easy to see that there is a set $J(\omega) \subseteq I_2(\omega)$ such that $x \notin y + D$ for all $x, y \in J(\omega)$ and $|J(\omega)| \geqslant |I_2(\omega)|/|D|$. If η is bad for X_0, then for all $z \in J(\omega)$ it is not possible that all sites in $z + D$ except z have scenery value c. By the above estimate, the probability of this event is at most

$$(1-\rho^{|D|-1}(1-\rho))^{|J(\omega)|} \leqslant (1-\rho^{|D|-1}(1-\rho))^{|I_2(\omega)|/|D|}.$$

From this, the second inequality follows. □

Let $I = S_- \cap S_+$. Note that $I_1 \supseteq I \supseteq I_2$.

Lemma 5.3. $\mathbb{P}(|I_1| = \infty) = \mathbb{P}(|I| = \infty) = \mathbb{P}(|I_2| = \infty) \in \{0, 1\}$.

Proof. Since $\{|I_1| = \infty\}$, $\{|I| = \infty\}$ and $\{|I_2| = \infty\}$ are exchangeable events, they each have probability 0 or 1 by the Hewitt–Savage zero-one law (see e.g., Durrett [1, p. 174]). Since $I_1 \supseteq I \supseteq I_2$, we have

$$\mathbb{P}(|I_1| = \infty) \geqslant \mathbb{P}(|I| = \infty) \geqslant \mathbb{P}(|I_2| = \infty).$$

It follows from den Hollander and Steif [3], Lemma 3.2, that $\mathbb{P}(|I_2| = \infty) = 1$ whenever $\mathbb{P}(|I| = \infty) = 1$. Hence $\mathbb{P}(|I| = \infty) = \mathbb{P}(|I_2| = \infty)$.

For $v \in D$, let E_v be the event

$$E_v = \{|\{z \in S_-\colon\ z - v \in S_+\}| = \infty\}.$$

Suppose that $\mathbb{P}(|I_1| = \infty) = 1$. Then there is a $v \in D$ such that $\mathbb{P}(E_v) > 0$. Since the random walk is irreducible, we can find $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in F$ such that $v = x_1 + \cdots + x_n$. Let $p_n = \mathbb{P}(X_k = x_k \text{ for } 1 \leqslant k \leqslant n)$, and define

$$\begin{aligned} &E_v(x_1, \ldots, x_n) \\ &\quad = \{\omega \in \Omega\colon \text{ there is an } \omega' \in E_v \text{ such that } X_k(\omega) = X_k(\omega') \text{ for } k \leqslant 0, \\ &\qquad X_k(\omega) = x_k \text{ for } 1 \leqslant k \leqslant n,\ X_k(\omega) = X_{k-n}(\omega') \text{ for } k \geqslant n+1\}. \end{aligned}$$

We have

$$\mathbb{P}(|I| = \infty) \geqslant \mathbb{P}(E_v(x_1, \ldots, x_n)) = p_n \mathbb{P}(E_v) > 0,$$

and so $\mathbb{P}(|I| = \infty) = 1$. Hence $\mathbb{P}(|I_1| = \infty) = \mathbb{P}(|I| = \infty)$. □

Lemma 5.4. *If* $\mathbb{P}(|I_1| < \infty) = 1$, *then*

$$\mathbb{P}_0(B(X_0) \backslash B(Y_0)) > 0.$$

Proof. Fix $c \in G$ and define the following events:

$$\begin{aligned} E_1 &= \{S_n = 0 \text{ for some } n > 0\}, \\ E_2 &= \{Y_n = c \text{ for all } n \in \mathbb{Z} \text{ with } S_n = 0\}, \\ E_3 &= \{Y_n = c \text{ for all } n \in \mathbb{Z} \text{ with } S_n \in I_1\}. \end{aligned}$$

(Note that it is not necessary that $0 \in I_1$.) Then

$$\mathbb{P}_0(B(X_0) \backslash B(Y_0)) \geqslant \mathbb{P}(E_1 \cap E_2 \cap E_3).$$

Indeed, if $\omega \in E_1 \cap E_2$, then $Y_0(\omega) = c$ and hence, by Corollary 4.2, $\eta \notin B(Y_0)$, where η is the restriction of ω to $\mathbb{Z}\backslash\{0\}$. If $\omega \in E_3$, then $(x, c) \in \text{insert}(\eta)$ for all $x \in F$ and hence $\eta \in B(X_0)$.

Since E_1 is measurable with respect to S, and $\mathbb{P}(E_2 \cap E_3|S) \geqslant \rho^{|I_2|+1}$ a.s. with $\rho = \mu(C_0 = c) > 0$, we obtain

$$\mathbb{P}(E_1 \cap E_2 \cap E_3) = \mathbb{E}(1_{E_1}\mathbb{P}(E_2 \cap E_3|S)) \geqslant \mathbb{E}(1_{E_1}\rho^{|I_1|+1}).$$

Since $\mathbb{P}(E_1) > 0$ (by the irreducibility of the random walk) and $\mathbb{P}(\rho^{|I_1|+1} > 0) = 1$ (by the assumption that $\mathbb{P}(|I_1| < \infty) = 1$), we obtain that $\mathbb{P}(E_1 \cap E_2 \cap E_3) > 0$. $\square$

Lemma 5.5. $\mathbb{P}_0(B(Z_0)) < 1$.

Proof. Fix $c_1, c_2 \in G$ with $c_1 \neq c_2$. Let E_1 be the set of $\omega \in \Omega$ for which for all $z \in D$ there are $m_z > 0$ such that

$$S_{m_z}(\omega) = z \quad \text{and} \quad Y_{m_z}(\omega) = \begin{cases} c_1 & \text{if } z = 0, \\ c_2 & \text{if } z \in D\backslash\{0\}. \end{cases}$$

Let E_2 be the set of $\omega \in \Omega$ such that $S_n(\omega) = 0$ for some $n < 0$. Then

$$1 - \mathbb{P}_0(B(Z_0)) = \mathbb{P}_0(\Omega\backslash B(Z_0)) \geqslant \mathbb{P}(E_1 \cap E_2).$$

Indeed, if $\omega \in E_1 \cap E_2$, then $Z_0(\omega) = (Y_0(\omega), c_1)$ (i.e., only ω_0 is insertable at time 0) and hence $\eta \notin B(Z_0)$, where η is the restriction of ω to $\mathbb{Z}\backslash\{0\}$.

Since E_1 and E_2 are independent, and

$$\mathbb{P}(E_1) \geqslant \rho^{|D|}\mathbb{P}(\forall z \in D \exists m_z > 0\colon\ S_{m_z} = z)$$

with $\rho = \min\{\mu(C_0 = c_1), \mu(C_0 = c_2)\} > 0$, we obtain that $\mathbb{P}(E_1 \cap E_2) > 0$. $\square$

Lemma 5.6. *If* $\mathbb{P}(|I_1| < \infty) = 1$, *then*

$$\mathbb{P}(S_n \neq 0 \textit{ for all } n \neq 0) < \mathbb{P}_0(B(Y_0)).$$

Proof. Fix $c_1, c_2 \in G$ with $c_1 \neq c_2$. For $x, y \in F$ with $x \neq y$, define the event

$$\begin{aligned} E_1(x,y) = \{&(X_0, Y_0) = (x, c_1), \text{ and} \\ &Y_n = \begin{cases} c_1 & \text{if } S_n = k(x-y) \text{ for some } k \in \mathbb{Z}, \\ c_2 & \text{otherwise,} \end{cases} \\ &\text{for all } n \in \mathbb{Z} \text{ with } S_n \in I_1\}, \end{aligned}$$

and define

$$\begin{aligned} E_2 &= \{S_n \neq 0 \text{ for all } n > 0\}, \\ E_3 &= \{S_n = 0 \text{ for some } n < 0\}. \end{aligned}$$

Fix $x, y \in F$ with $x \neq y$ and $\omega \in E_1(x, y) \cap E_2$. We claim that ω' defined by

$$\omega'_n = \begin{cases} \omega_n & \text{if } n \neq 0, \\ (y, c_2) & \text{if } n = 0, \end{cases}$$

is an element of Ω. To prove this, we have to show that for all $m < n$,

$$Y_m(\omega') = Y_n(\omega') \text{ whenever } S_m(\omega') = S_n(\omega').$$

For $0 \leqslant m < n$, the claim holds because $S_n(\omega) = S_n(\omega')$ and $Y_n(\omega) = Y_n(\omega')$ for all $n > 0$. For $m < n < 0$, the claim holds because a change of the step at time 0 does not affect the self-intersection pattern of the past of the walk. For $m < 0 \leqslant n$, note that $S_m(\omega') = S_n(\omega')$ implies that $S_m(\omega) = S_n(\omega) - (x - y)$. Hence, $S_m(\omega)$ is a multiple of $x - y$ if and only if $S_n(\omega)$ is. Since $S_m(\omega), S_n(\omega) \in I_1(\omega)$ (because $\omega \in E_1(x, y)$), this in turn implies that $S_m(\omega), S_n(\omega)$ have the same color (either c_1 or c_2, depending on whether they are a multiple of $x - y$ or not). Thus

$$\{(x, c_1), (y, c_2)\} \subseteq \text{insert}(\eta),$$

where $\eta \in \Omega_0$ is the restriction of ω to $\mathbb{Z} \backslash \{0\}$, and hence $\eta \in B(Y_0)$ by Corollary 4.2.

The above shows that $B(Y_0) \supseteq E_1 \cap E_2$ with $E_1 = \bigcup_{x,y \in F, x \neq y} E_1(x, y)$. But $\{S_n \neq 0 \text{ for all } n \neq 0\} \subseteq \Omega \backslash E_3$, and so it follows that

$$B(Y_0) \backslash \{S_n \neq 0 \text{ for all } n \neq 0\} \supseteq E_1 \cap E_2 \cap E_3.$$

Trivially, $B(Y_0) \supseteq \{S_n \neq 0 \text{ for all } n \neq 0\}$, and hence

$$\mathbb{P}_0(B(Y_0)) - \mathbb{P}(S_n \neq 0 \text{ for all } n \neq 0) \geqslant \mathbb{P}(E_1 \cap E_2 \cap E_3).$$

Since E_2 and E_3 are measurable with respect to S, and $\mathbb{P}(E_1 | S) \geqslant \rho^{|I_1|+1}$ a.s. with $\rho = \min\{\mu(C_0 = c_1), \mu(C_0 = c_2)\}$, we obtain that $\mathbb{P}(E_1 \cap E_2 \cap E_3) > 0$, as before. □

5.2. Proof of Theorem 1.3

(i) Fix $\bar{x} \in F$, $\bar{c} \in G$ and define a configuration $\eta \in \Omega_0$ by $\eta_n = (\bar{x}, \bar{c})$ for all $n \in \mathbb{Z} \backslash \{0\}$. It is easily seen that the sets $\{(x, \bar{c})\colon x \in F\}$ and $\{(\bar{x}, c)\colon c \in G\}$ are both contained in insert(η). It follows from Corollary 4.2 that η is an element of $B(X_0)$, $B(Y_0)$ and $B(Z_0)$.

(ii) If $d = 1, 2$ and $\sum_{x \in F} x m(x) = 0$, then the random walk is recurrent. So, $\mathbb{P}(S_- = S_+ = \mathbb{Z}^d) = 1$, hence $\mathbb{P}(|I_2| = \infty) = 1$, and therefore the upper bound in Lemma 5.2 gives $\mathbb{P}_0(B(X_0)) = 0$. Consequently, Lemmas 5.1 (i,iii) yield $0 = \mathbb{P}(S_n \neq 0 \text{ for all } n \neq 0) = \mathbb{P}_0(B(Y_0)) = \mathbb{P}_0(B(Z_0))$.

(iii) If $d = 3, 4$ and $\sum_{x \in F} x m(x) = 0$, then the random walk is transient. However, $\mathbb{P}(|I| = \infty) = 1$ (see Lawler [5, Section 3]), and therefore Lemma 5.3 gives $\mathbb{P}(|I_2| = \infty) = 1$. So, by the upper bound in Lemma 5.2, again $\mathbb{P}_0(B(X_0)) = 0$. Consequently, Lemmas 5.1(i,iii) yield $0 < \mathbb{P}(S_n \neq 0 \text{ for all } n \neq 0) = \mathbb{P}_0(B(Y_0)) = \mathbb{P}_0(B(Z_0)) < 1$.

(iv) If $d \geqslant 5$ and $\sum_{x \in F} x m(x) = 0$ or if $d \geqslant 1$ and $\sum_{x \in F} x m(x) \neq 0$, then $\mathbb{P}(|I| < \infty) = 1$ (see Lawler [5, Section 3]), and therefore Lemma 5.3 gives $\mathbb{P}(|I_1| < \infty) =$

$\mathbb{P}(|I_2|<\infty)=1$. So, Lemma 5.2 gives $0<\mathbb{P}_0(B(X_0))<1$. Consequently, Lemmas 5.1(i,iii) and 5.5 yield $0<\mathbb{P}(S_n\neq 0 \text{ for all } n\neq 0)\leqslant \mathbb{P}_0(B(Y_0))\leqslant \mathbb{P}_0(B(Z_0))<1$.The second inequality is strict by Lemma 5.6 and the third inequality is strict by Lemmas 5.1 (ii) and 5.4.

Remark. For the proof of Theorem 1.3, the i.i.d. property of the random scenery was used only in the proofs of Lemmas 5.2, 5.4, 5.5 and 5.6. It is easily checked that for all these lemmas the uniform finite energy condition actually suffices (recall the remarks made in Section 1.5).

6. Identification of bad configurations for subshifts for finite intervals Λ

In this section we deal with the situation where the time span on which we consider the conditional probabilities is not just a single point, but a finite interval $\Lambda\subseteq\mathbb{Z}$. Remarkably, the extension turns out to be somewhat delicate.

6.1. Insertion, Λ-specifiable, Λ-irreducible and weakly Λ-specifiable

We begin by extending the definition of being specifiable. Recall the definition of Ω_Λ in Section 1.4.

Definition 6.1. For $\eta\in\Omega_\Lambda$, define

$$\text{insert}_\Lambda(\eta)=\{\gamma\in H^\Lambda\colon\ \omega \text{ given by } \omega=\gamma \text{ on } \Lambda \text{ and } \omega=\eta \text{ on } \mathbb{Z}\backslash\Lambda \text{ is in } \Omega\}.$$

Definition 6.2. A subshift Ω is *Λ-specifiable* if for all $\eta\in\Omega_\Lambda$, $\gamma\in\text{insert}_\Lambda(\eta)$ and $n\in\mathbb{N}$, there is a $\delta\in\Omega_\Lambda$ such that $\delta=\eta$ on $\Lambda_n\backslash\Lambda$ and $\text{insert}_\Lambda(\delta)=\{\gamma\}$.

Clearly, specifiable in the sense of Definition 3.3 is $\{0\}$-specifiable. It is easily checked (we leave this to the reader) that the analogues of Lemmas 3.2, 3.4, 3.5 and Theorem 3.6 all extend when Λ is an arbitrary finite interval.

All this is fine. However, RWRS is not Λ-specifiable when $|\Lambda|\geqslant 2$. Indeed, it is never possible to read off from the configuration outside Λ in which *order* the steps are taken during the time interval Λ. At most it is possible to read off their *total sum*. Thus, it is never possible to bring $\text{insert}_\Lambda(\delta)$ down to a single configuration inside Λ when $|\Lambda|\geqslant 2$. To remedy this problem, we introduce a weaker property of subshifts (see Definition 6.4 below) that we believe is the key property for RWRS when $|\Lambda|\geqslant 2$.

To define this property, we need some more definitions.

Definition 6.3. Recall that $\Lambda_n=[-n,n]\cap\mathbb{Z}$ for $n\in\mathbb{N}$.

(a) Define the set of *Λ-irreducible configurations* as

$$I_\Lambda=\{\eta\in\Omega_\Lambda\colon \text{ there is an } n\geqslant 0 \text{ such that } \Lambda_n\backslash\Lambda\neq\emptyset \text{ and if } \delta=\eta \text{ on } \Lambda_n\backslash\Lambda, \text{ then } \text{insert}_\Lambda(\delta)=\text{insert}_\Lambda(\eta)\}.$$

(b) For U a Z_Λ-measurable random variable ($Z_\Lambda = (Z_n)_{n\in\Lambda}$), define the set of *$\Lambda$-irreducible configurations for U* as

$$I_\Lambda(U) = \{\eta \in \Omega_\Lambda\colon \text{ there is an } n \geqslant 0 \text{ such that } \Lambda_n \backslash \Lambda \neq \emptyset \text{ and} \\ \text{if } \delta = \eta \text{ on } \Lambda_n\backslash\Lambda, \text{ then} \\ \{U(\gamma)\colon\ \gamma \in \text{insert}_\Lambda(\delta)\} = \{U(\gamma)\colon\ \gamma \in \text{insert}_\Lambda(\eta)\}\}.$$

In words, I_Λ is the set of those configurations for which the possible insertions in Λ cannot be reduced by tampering with the configuration far outside Λ. (Note that, by the obvious analogue of Lemma 3.2, there is an $n \in \mathbb{N}$ with $\Lambda_n\backslash\Lambda \neq \emptyset$ such that $\text{insert}_\Lambda(\delta) \subseteq \text{insert}_\Lambda(\eta)$ for all $\delta \in \Omega_\Lambda$ with $\delta = \eta$ on $\Lambda_n\backslash\Lambda$.) Similarly for $I_\Lambda(U)$. Note that $I_\Lambda = I_\Lambda(Z_\Lambda)$.

The key property replacing Λ-specifiable reads:

Definition 6.4. A subshift Ω is *weakly Λ-specifiable* if for all $\eta \in \Omega_\Lambda$, $\gamma \in \text{insert}_\Lambda(\eta)$ and $n \in \mathbb{N}$ with $\Lambda_n\backslash\Lambda \neq \emptyset$, there is a $\delta \in I_\Lambda$ such that $\delta = \eta$ on $\Lambda_n\backslash\Lambda$ and $\gamma \in \text{insert}_\Lambda(\delta)$.

In words, being weakly Λ-specifiable guarantees that, by tampering with the configuration outside any annulus around Λ, the configuration can be made Λ-irreducible and can be made to contain a specified insert of η on Λ. In Section 7 we prove that RWRS is weakly Λ-specifiable for all finite intervals Λ.

Obviously, for all Λ, being Λ-specifiable implies being weakly Λ-specifiable. The converse is false even when $\Lambda = \{0\}$: the full shift is weakly $\{0\}$-specifiable but not $\{0\}$-specifiable.

6.2. *Bad configurations*

Recall that, by Lemmas 3.4, 3.5 and Theorem 3.6, if our subshift Ω is $\{0\}$-specifiable, then the bad configurations are identified purely topologically, i.e., they do not depend on the probability measure $\mathbb{P}$. As indicated above, this is also the case if Ω is Λ-specifiable. However, since the full shift is weakly Λ-specifiable, we should not expect in general that for subshifts satisfying this weaker property the bad configurations can still be described purely topologically. Rather it is clear that some conditions must now be placed on the probability measure $\mathbb{P}$. These conditions are formulated in:

Definition 6.5. (a) A probability measure $\mathbb{P}$ on a subshift Ω is *uniformly non-null on I_Λ* if there is a $c = c_\Lambda > 0$ such that for all $n \in \mathbb{N}$, $\eta \in I_\Lambda$ and $\gamma \in \text{insert}_\Lambda(\eta)$,

$$\mathbb{P}(Z = \gamma \text{ on } \Lambda \mid Z = \eta \text{ on } \Lambda_n\backslash\Lambda) \geqslant c.$$

(b) Let U be a Z_Λ-measurable random variable. A probability measure $\mathbb{P}$ on a subshift Ω is *finitarily Markov for U on $I_\Lambda(U)$* if for all $\eta \in I_\Lambda(U)$ there is an $n \in \mathbb{N}$ with $\Lambda_n\backslash\Lambda \neq \emptyset$ such that for all $m \geqslant n$ and for all $\delta \in I_\Lambda(U)$ with $\delta = \eta$ on $\Lambda_n\backslash\Lambda$,

$$\mathbb{P}(U \in \cdot \mid Z = \delta \text{ on } \Lambda_m\backslash\Lambda) = \mathbb{P}(U \in \cdot \mid Z = \eta \text{ on } \Lambda_n\backslash\Lambda).$$

While the latter property is technical, it is precisely the one satisfied by RWRS that will allow for a full identification of the bad configurations for X_Λ and Z_Λ. It is, roughly speaking, a two-sided version of a notion recently introduced by Morvai and Weiss [7,8], which they call finitarily Markov. We point out that if Ω is the full shift, then the property of being finitarily Markov for Z_Λ on I_Λ trivializes, in the sense that $\mathbb{P}$ must be i.i.d.

Lemmas 6.6 and 6.7 below identify the bad configurations for a Z_Λ-measurable random variable U in analogy with Lemmas 3.4 and 3.5.

Lemma 6.6. *Assume that Ω is weakly Λ-specifiable and that $\mathbb{P}$ is a probability measure on Ω that is uniformly non-null on I_Λ. Let U be a Z_Λ-measurable random variable. Then*

$$B(U) \supseteq \Omega_\Lambda \backslash I_\Lambda(U).$$

Proof. Assume that $\eta \in \Omega_\Lambda \backslash I_\Lambda(U)$. Let $\varepsilon = c/2$, where c is the constant in Definition 6.5(a). Let n be sufficiently large so that $\Lambda_n \backslash \Lambda \neq \emptyset$ and $\mathrm{insert}_\Lambda(\delta) \subseteq \mathrm{insert}_\Lambda(\eta)$ for all $\delta \in \Omega_\Lambda$ with $\delta = \eta$ on $\Lambda_n \backslash \Lambda$, which is possible by the analogue of Lemma 3.2. The fact that η is not in $I_\Lambda(U)$ now implies the existence of a $\delta \in \Omega_\Lambda$ such that $\delta = \eta$ on $\Lambda_n \backslash \Lambda$ and

$$\{U(\gamma):\ \gamma \in \mathrm{insert}_\Lambda(\delta)\} \subsetneq \{U(\gamma):\ \gamma \in \mathrm{insert}_\Lambda(\eta)\}.$$

Take $\gamma' \in \mathrm{insert}_\Lambda(\eta)$ such that $U(\gamma) \neq U(\gamma')$ for all $\gamma \in \mathrm{insert}_\Lambda(\delta)$. Being weakly Λ-specifiable implies that there is a $\delta' \in I_\Lambda$ such that $\delta' = \eta$ on $\Lambda_n \backslash \Lambda$ and $\gamma' \in \mathrm{insert}_\Lambda(\delta')$. By the analogue of Lemma 3.2, there is an $m \geqslant n$ such that for all $\zeta \in \Omega_\Lambda$,

$$\begin{aligned} &\mathrm{insert}_\Lambda(\zeta) \subseteq \mathrm{insert}_\Lambda(\delta) \quad \text{whenever } \zeta = \delta \text{ on } \Lambda_m \backslash \Lambda, \\ &\mathrm{insert}_\Lambda(\zeta) = \mathrm{insert}_\Lambda(\delta') \quad \text{whenever } \zeta = \delta' \text{ on } \Lambda_m \backslash \Lambda. \end{aligned}$$

Hence

$$\mathbb{P}(U = U(\gamma') \mid Z = \delta \text{ on } \Lambda_m \backslash \Lambda) = 0$$

while, by the uniform non-null assumption,

$$\mathbb{P}(U = U(\gamma') \mid Z = \delta' \text{ on } \Lambda_m \backslash \Lambda) \geqslant c.$$

Therefore at least one of the latter two conditional probabilities must differ from

$$\mathbb{P}(U = U(\gamma') \mid Z = \eta \text{ on } \Lambda_m \backslash \Lambda)$$

by at least $c/2 = \varepsilon$. Consequently, $\eta \in B(U)$. □

Lemma 6.7. *Let $\mathbb{P}$ be a probability measure on a subshift Ω and U be a Z_Λ-measurable random variable. Assume that $\mathbb{P}$ is finitarily Markov for U on $I_\Lambda(U)$. Then*

$$B(U) \subseteq \Omega_\Lambda \backslash I_\Lambda(U).$$

Proof. Assume that $\eta \in I_\Lambda(U)$. Then there is an $n \in \mathbb{N}$ with $\Lambda_n \backslash \Lambda \neq \emptyset$ such that for all $\delta \in \Omega_\Lambda$ with $\delta = \eta$ on $\Lambda_n \backslash \Lambda$,

$$\{U(\gamma):\ \gamma \in \text{insert}_\Lambda(\delta)\} = \{U(\gamma):\ \gamma \in \text{insert}_\Lambda(\eta)\}.$$

Observe that any such δ is in $I_\Lambda(U)$. Using that $\mathbb{P}$ is finitarily Markov for U on $I_\Lambda(U)$, we obtain that for all $m \geqslant n$ and for all $\delta \in \Omega_\Lambda$ with $\delta = \eta$ on $\Lambda_n \backslash \Lambda$,

$$\mathbb{P}(U \in \cdot \mid Z = \delta \text{ on } \Lambda_m \backslash \Lambda) = \mathbb{P}(U \in \cdot \mid Z = \eta \text{ on } \Lambda_m \backslash \Lambda).$$

Hence $\eta \notin B(U)$. □

7. Identification of bad configurations for RWRS for finite intervals Λ

As in Section 6, we assume that $\Lambda \subsetneq \mathbb{Z}$ is a finite interval. For $\gamma = (x_n, y_n)_{n \in \Lambda} \in H^\Lambda$, we define $X(\gamma) = (X_n(\gamma))_{n \in \Lambda}$ and $Y(\gamma) = (Y_n(\gamma))_{n \in \Lambda}$ by putting $X_n(\gamma) = x_n$ and $Y_n(\gamma) = y_n$.

7.1. X_Λ and Z_Λ

The following lemma identifies the sets of irreducible configurations for X_Λ and Z_Λ. In Corollary 7.4 below we will see that the complements of these sets coincide with the sets of bad configurations for X_Λ and Z_Λ.

Lemma 7.1. (i)

$$I_\Lambda(X_\Lambda) = \left\{ \eta \in \Omega_\Lambda:\ \sum_{k \in \Lambda} X_k(\gamma) = \sum_{k \in \Lambda} X_k(\gamma')\ \forall \gamma, \gamma' \in \text{insert}_\Lambda(\eta) \right\}.$$

(ii)

$$I_\Lambda(Z_\Lambda) = I_\Lambda(X_\Lambda) \cap \{\eta \in \Omega_\Lambda:\ [\gamma, \gamma' \in \text{insert}_\Lambda(\eta), X(\gamma) = X(\gamma')] \Longrightarrow \gamma = \gamma'\}.$$

Proof. (i) Write $\mathscr{R}$ for the set in the right-hand side. To show that $I_\Lambda(X_\Lambda) \subseteq \mathscr{R}$, assume that $\eta \in \Omega_\Lambda$ is such that $\sum_{k \in \Lambda} X_k(\gamma) \neq \sum_{k \in \Lambda} X_k(\gamma')$ for some $\gamma, \gamma' \in \text{insert}_\Lambda(\eta)$. Fix $n \in \mathbb{N}$ with $\Lambda_n \backslash \Lambda \neq \emptyset$, and choose $\omega \in \Omega_\Lambda$ such that $\omega = \gamma$ on Λ and $\omega = \eta$ on $\Lambda_n \backslash \Lambda$. Analogously to the proof of Lemma 4.1, we define ω on $\mathbb{Z} \backslash \Lambda_n$ so that

(1) the random walk at some time $> n$ reaches a fixed site y far away from the origin, as well as all the sites nearby y (where far away and nearby depend on $|\Lambda|$),
(2) the scenery value revealed at y is different from that revealed at the sites nearby y,
(3) y is reached at some time $< -n$.

In this way, we get

$$\sum_{k \in \Lambda} X_k(\gamma) = \sum_{k \in \Lambda} X_k(\gamma'') \quad \forall \gamma'' \in \text{insert}_\Lambda(\delta)$$

for $\delta \in \Omega_\Lambda$ with $\delta = \omega$ on $\mathbb{Z} \backslash \Lambda$. Hence, $X(\gamma')$ is not in $\{X(\gamma''):\ \gamma'' \in \text{insert}_\Lambda(\delta)\}$, and therefore $\eta \notin I_\Lambda(X_\Lambda)$.

To show that $\mathscr{R} \subseteq I_\Lambda(X_\Lambda)$, let $\eta \in \Omega_\Lambda \backslash I_\Lambda(X_\Lambda)$. Fix $n \in \mathbb{N}$ with $\Lambda_n \backslash \Lambda \neq \emptyset$, and choose $\delta \in \Omega_\Lambda$ such that $\delta = \eta$ on $\Lambda_n \backslash \Lambda$ and $\text{insert}_\Lambda(\delta) \subseteq \text{insert}_\Lambda(\eta)$. Since $\eta \notin I_\Lambda(X_\Lambda)$, we have

$$\{X(\gamma):\ \gamma \in \text{insert}_\Lambda(\delta)\} \subsetneq \{X(\gamma):\ \gamma \in \text{insert}_\Lambda(\eta)\}.$$

Pick $\gamma \in \text{insert}_\Lambda(\eta)$ such that $X(\gamma) \neq X(\gamma'')$ for all $\gamma'' \in \text{insert}_\Lambda(\delta)$. Then $\sum_{k \in \Lambda} X_k(\gamma) \neq \sum_{k \in \Lambda} X_k(\gamma'')$ for all $\gamma'' \in \text{insert}_\Lambda(\delta)$. Since $\text{insert}_\Lambda(\delta) \subseteq \text{insert}_\Lambda(\eta)$, this shows that $\eta \notin \mathscr{R}$.

(ii) Write $\mathscr{R}$ for the set in the right-hand side. To show that $I_\Lambda(Z_\Lambda) \subseteq \mathscr{R}$, we argue as follows. From the definition of Λ-irreducibility it is clear that $I_\Lambda(Z_\Lambda) \subseteq I_\Lambda(X_\Lambda)$. Hence, assume that $\eta \in \Omega_\Lambda$ is such that $X(\gamma) = X(\gamma')$ for some $\gamma, \gamma' \in \text{insert}_\Lambda(\eta)$ with $\gamma \neq \gamma'$. Fix $n \in \mathbb{N}$ with $\Lambda_n \backslash \Lambda \neq \emptyset$, and choose $\omega \in \Omega$ such that $\omega = \gamma$ on Λ and $\omega = \eta$ on $\Lambda_n \backslash \Lambda$. Define ω on $\mathbb{Z} \backslash \Lambda_n$ such that for all $k \in \Lambda$ there is an $l \notin \Lambda$ with $S_k(\omega) = S_l(\omega)$. If $\delta \in \Omega_\Lambda$ is given by $\delta = \omega$ on $\mathbb{Z} \backslash \Lambda$, then $\gamma' \notin \text{insert}_\Lambda(\delta)$. Hence $\eta \notin I_\Lambda(Z_\Lambda)$.

To show that $\mathscr{R} \subseteq I_\Lambda(Z_\Lambda)$, let $\eta \in \Omega_\Lambda \backslash I_\Lambda(Z_\Lambda)$. Fix $n \in \mathbb{N}$ with $\Lambda_n \backslash \Lambda \neq \emptyset$, and choose $\delta \in \Omega_\Lambda$ such that $\delta = \eta$ on $\Lambda_n \backslash \Lambda$ and $\text{insert}_\Lambda(\delta) \subseteq \text{insert}_\Lambda(\eta)$. Since $\eta \notin I_\Lambda(Z_\Lambda)$, we have in fact that $\text{insert}_\Lambda(\delta) \subsetneq \text{insert}_\Lambda(\eta)$. Let $\gamma \in \text{insert}_\Lambda(\eta) \backslash \text{insert}_\Lambda(\delta)$. There are now two possibilities:

(1) $X(\gamma) \neq X(\gamma'')$ for all $\gamma'' \in \text{insert}_\Lambda(\delta)$. This implies that $\eta \notin I_\Lambda(X_\Lambda)$.
(2) $X(\gamma) = X(\gamma'')$ for some $\gamma'' \in \text{insert}_\Lambda(\delta)$. Such a γ'' cannot be equal to γ, and hence $\eta \notin \mathscr{R}$. □

We next show that RWRS fits into the framework of Section 6.

Lemma 7.2. *Let Ω be the subshift associated with RWRS. Then Ω is weakly Λ-specifiable.*

Proof. The proof is similar to that of Lemmas 4.1 and 7.1. The details are left to the reader. □

Lemma 7.3. *Let Ω be the subshift associated with RWRS and $\mathbb{P}$ the corresponding probability measure on Ω.*

(i) *$\mathbb{P}$ is uniformly non-null on I_Λ.*
(ii) *$\mathbb{P}$ is finitarily Markov for X_Λ on $I_\Lambda(X_\Lambda)$ and for Z_Λ on $I_\Lambda(Z_\Lambda)$.*

Proof. (i) Recall Definitions 6.3(a) and 6.5(a). By the i.i.d. property of the walk and the scenery, and the fact that the steps and the scenery values are drawn from finite sets, each possible insertion on Λ has a probability $\geqslant c_\Lambda > 0$. For Λ-irreducible configurations, the set of possible insertions on Λ is independent of the configuration on $\Lambda_n \backslash \Lambda$ for some n large enough, and non-empty for the configuration on $\mathbb{Z} \backslash \Lambda$.

(ii) Recall Definitions 6.3(b) and 6.5(b). By the i.i.d. property of the walk and the scenery, once the set of possible insertions on Λ is determined by the configuration on $\Lambda_n \backslash \Lambda$ for some n large enough, the probability distribution for the possible insertions on Λ no longer depends on the configuration on $\mathbb{Z} \backslash \Lambda_n$. □

Corollary 7.4. *$B(X_\Lambda) = \Omega_\Lambda \backslash I_\Lambda(X_\Lambda)$ and $B(Z_\Lambda) = \Omega_\Lambda \backslash I_\Lambda(Z_\Lambda)$.*

Proof. This is a direct consequence of Lemmas 6.6, 6.7, 7.2 and 7.3. □

The above results complete our analysis for X_Λ and Z_Λ. The situation for Y_Λ is different and more delicate.

7.2. Y_Λ

The following lemma shows the relation between the respective sets of irreducible and bad configurations.

Lemma 7.5.

(i) $I_\Lambda(Z_\Lambda) \subseteq I_\Lambda(X_\Lambda) \cap I_\Lambda(Y_\Lambda)$.
(ii) $B(Z_\Lambda) = B(X_\Lambda) \cup B(Y_\Lambda)$.

Proof. (i) This is immediate from the definition of Λ-irreducibility. In Fig. 1, a configuration is given that is irreducible for X_Λ and Y_Λ, but not for Z_Λ. Hence the inclusion may be strict.

(ii) It is immediate from the definition of bad configuration that $B(Z_\Lambda) \supseteq B(X_\Lambda) \cup B(Y_\Lambda)$. To prove the reverse inclusion, it suffices to show that if

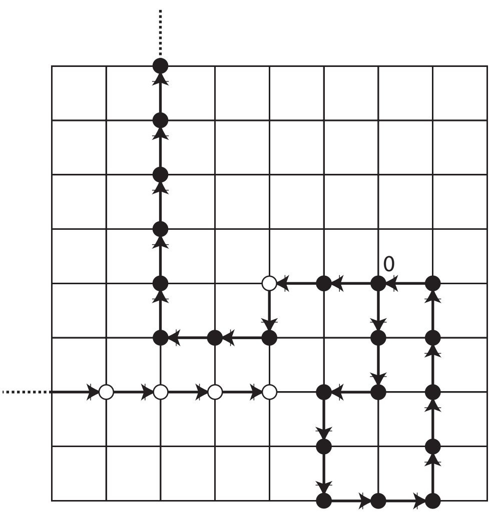

Fig. 1. The arrows and the colors represent an element η of Ω_Λ, where $\Lambda = \{-3, -2, -1, 0\}$, the set of possible steps is $F = \{\rightarrow, \uparrow, \leftarrow, \downarrow\}$ and the set of possible scenery values is $G = \{\circ, \bullet\}$. The picture corresponds to $\eta_n = (\rightarrow, \circ)$ for $n \in (-\infty, -4] \cap \mathbb{Z}$, $\eta_1 = (\downarrow, \bullet)$, $\eta_2 = (\downarrow, \bullet)$, $\eta_3 = (\leftarrow, \bullet), \ldots, \eta_{14} = (\leftarrow, \circ)$, $\eta_{15} = (\downarrow, \bullet)$, $\eta_{16} = (\leftarrow, \bullet)$, $\eta_{17} = (\leftarrow, \bullet)$, $\eta_n = (\uparrow, \bullet)$ for $n \in [18, \infty) \cap \mathbb{Z}$.

$\omega \in B(Z_\Lambda)\backslash B(X_\Lambda)$, then $\omega \in B(Y_\Lambda)$. This goes as follows. For $\omega \notin B(X_\Lambda)$, we have $\sum_{k\in\Lambda} X_k(\omega)$ determined by ω on $\mathbb{Z}\backslash\Lambda$, say s. If $\omega \in B(Z_\Lambda)$ also, then there are steps on Λ, with the prescribed sum s, for which the path on Λ visits a site z for which the scenery value is not determined. But the presence of such a z guarantees that $\omega \in B(Y_\Lambda)$ (in the same way as the example in Fig. 1 gives an element of $B(Y_\Lambda)$). □

In general, $\mathbb{P}$ is *not* finitarily Markov for Y_Λ on $I_\Lambda(Y_\Lambda)$. Lemma 6.6 tells us that

$$B(Y_\Lambda) \supseteq \Omega_\Lambda \backslash I_\Lambda(Y_\Lambda),$$

but *the reverse inequality fails in general*. Indeed, the configuration given in Fig. 1 is both bad and irreducible for Y_Λ.

Next we explain Fig. 1. Let $\eta \in \Omega_\Lambda$ denote the configuration that is drawn in the figure:

(1) To see that $\eta \in I_\Lambda(X_\Lambda)$, note that $\sum_{k\in\Lambda} X_k(\gamma)$ has to be located inside the diamond $\{(x,y) \in \mathbb{Z}^2\colon\ |x|+|y| \leqslant 4\}$ for all $\gamma \in \text{insert}_\Lambda(\eta)$. The only value of this sum that does not lead to a conflicting coloring of the sites is $(2,2)$, corresponding to $S_{-4} = (-2,-2)$, as drawn.

(2) To see that $\eta \in I_\Lambda(Y_\Lambda)$, note that for any $\delta \in \Omega_\Lambda$ that agrees with η on $\Lambda_{22}\backslash\Lambda$,

$$\{Y(\gamma)\colon\ \gamma \in \text{insert}_\Lambda(\delta)\} = \{(\bullet,\circ,\bullet,\bullet),(\bullet,\bullet,\bullet,\bullet)\}.$$

Indeed, there are six possible paths on Λ from $(-2,-2)$ to $(0,0)$, and along each of these walks the colors seen on Λ are the two sequences indicated, irrespective of the color of $(-1,-1)$.

(3) To see that $\eta \notin I_\Lambda(Z_\Lambda)$, note that it is possible to construct an $\omega \in \Omega$ such that $\omega = \eta$ on $\Lambda_{22}\backslash\Lambda$ and $S_n(\omega) = (-1,-1)$ for some $n \notin \Lambda$. For this ω the color of $(-1,-1)$ is determined.

(4) To see that $\eta \in B(Y_\Lambda)$, note that the color of $(-1,-1)$ may be determined by making the walk return to that site for the first time after an arbitrarily large time.

Remark. It is possible to give an expression for $I_\Lambda(Y_\Lambda)$ in the same spirit as the ones for $I_\Lambda(X_\Lambda)$ and $I_\Lambda(Z_\Lambda)$ in Lemma 7.1. However, this expression is complicated, and since its complement does not coincide with $B(Y_\Lambda)$ anyway, it is of less interest.

7.3. Generalization of Theorem 1.3

Using basically the same types of arguments as in Section 5, we obtain the following generalization of Theorem 1.3. The details are left to the reader.

Theorem 7.6. *Assume that m and μ satisfy the conditions in Section* 1.2.

(i) *$B(X_\Lambda)$, $B(Y_\Lambda)$ and $B(Z_\Lambda)$ are non-empty.*

(ii) *For* $d = 1, 2$ *and* $\sum_{x \in F} xm(x) = 0$,

$$\mathbb{P}_\Lambda(B(X_\Lambda)) = \mathbb{P}_\Lambda(B(Y_\Lambda)) = \mathbb{P}_\Lambda(B(Z_\Lambda)) = 0.$$

(iii) *For* $d = 3, 4$ *and* $\sum_{x \in F} xm(x) = 0$,

$$\mathbb{P}_\Lambda(B(X_\Lambda)) = 0,$$
$$0 < \mathbb{P}(\exists n \in \Lambda\colon\ S_m \neq S_n \, \forall m \notin \Lambda) \leqslant \mathbb{P}_\Lambda(B(Y_\Lambda)) = \mathbb{P}_\Lambda(B(Z_\Lambda)) < 1.$$

(iv) *For* $d \geqslant 5$ *and* $\sum_{x \in F} xm(x) = 0$ *or* $d \geqslant 1$ *and* $\sum_{x \in F} xm(x) \neq 0$,

$$0 < \mathbb{P}_\Lambda(B(X_\Lambda)) < 1,$$
$$0 < \mathbb{P}(\exists n \in \Lambda\colon\ S_m \neq S_n \, \forall m \notin \Lambda) < \mathbb{P}_\Lambda(B(Y_\Lambda)) < \mathbb{P}_\Lambda(B(Z_\Lambda)) < 1.$$

The fact that the $\leqslant$ in Theorem 7.6(iii) is an $=$ in Theorem 1.3(iii) is due to our lack of control of $B(Y_\Lambda)$.

Finally, define

$$B(W) = \bigcup_{|\Lambda| < \infty} B(W_\Lambda), \quad W = X, Y, Z,$$

i.e., the sets of configurations that are bad for some finite interval. By ergodicity, $\mathbb{P}(B(X)), P(B(Y)), \mathbb{P}(B(Z)) \in \{0, 1\}$. It follows from Theorem 7.6 that cases (ii–iv) correspond to

$$\mathbb{P}(B(X)) = \mathbb{P}(B(Y)) = \mathbb{P}(B(Z)) = 0,$$
$$\mathbb{P}(B(X)) = 0,\ \mathbb{P}(B(Y)) = \mathbb{P}(B(Z)) = 1,$$
$$\mathbb{P}(B(X)) = \mathbb{P}(B(Y)) = \mathbb{P}(B(Z)) = 1.$$

Acknowledgements

The authors are grateful to Frank Redig (Eindhoven) for discussions, to the Lorentz Center (Leiden) for hospitality, and to the European Science Foundation Scientific Program "Random Dynamics in Spatially Extended Systems" for travel support. The research of JES was supported by the Swedish National Science Research Council and through grant DMS-010384 of the American National Science Foundation. The research of PvdW took place within the framework of the Dutch–German Bilateral Research Group on "Mathematics of Random Spatial Models from Physics and Biology", which is jointly funded by the Netherlands Organization for Scientific Research and the Deutsche Forschungsgemeinschaft.

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