Large Time Behaviour of Neutral Delay Systems

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A meus pais, Lourival e Helena, e à Marília, com amor.

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Preface

The objective of this thesis is to study the large time behaviour of solutions of linear autonomous and periodic functional differential equations (FDE) of neutral type. The emphasis is on the explicit computation (also symbolically) of the large time behaviour by using spectral projections. Applications also include nonlinear equations.

In Chapter 1, we study FDE by analyzing a class of renewal equations (Volterra and Volterra-Stieltjes integro-differential equations) that can be associated with FDE. We present the large time behaviour of solutions of renewal equations, provided that the "neutral part" satisfies a (reasonable) additional hypothesis. The chapter is written for a general audience and plays the role of an overview of the key results which are presented in this thesis. Since explicit computations are harder to obtain directly for renewal equations, explicit formulas are postponed to later chapters. Renewal equations return in Chapter 8 and in Appendix A.

In Chapter 2, we study the basic spectral theory for autonomous and periodic FDE. We present the decomposition of the state space $\mathcal{C}$ into a direct sum of finite dimensional invariant subspaces — the generalized eigenspaces $\mathcal{M}_\lambda$ corresponding to the eigenvalues λ of the characteristic equation — and the complementary space. We derive strong estimates for the solutions restricted to these spaces. Therefore, projecting on the subspaces $\mathcal{M}_\lambda$ plays an important role in order to obtain the large time behaviour of solutions of FDE.

In Chapter 3, we study the spectral projection (from $\mathcal{C}$ to $\mathcal{M}_\lambda$) and present various methods to compute the projections explicitly. First we compute the spectral projections, in the autonomous case, for simple dominant eigenvalues using duality using the Hale bilinear form. Then we compute spectral projections using Dunford calculus, and we like to advertise this method, since it yields an algorithmic way to compute the spectral projections for eigenvalues of arbitrary order, in a simpler way compared with the method using duality. In addition, we can extend this later method to compute spectral projections for classes of periodic equations as well. The algorithmic nature of the method allows us to implement the computation of the spectral projection method for autonomous equations as a Maple library package, called `FDESpectralProj`. We present the usage of the library `FDESpectralProj` in Chapter 4, and its source code is provided in Appendix B.

In Chapter 5, we present a number of tests to decide whether a root of a given characteristic equations is simple and dominant.

In Chapter 6, we state our main results, that is, the large time behaviour for autonomous and periodic FDE and we present more applications of the basic results developed so far. In particular, we provide the analysis of the large time behaviour of a class of neutral FDE in $\mathbb{C}^2$, and we do explicit computations (also symbolically) for a particular case where the dominant eigenvalue of the infinitesimal generator of the solution semigroup is a dominant zero of sixth order of the corresponding characteristic equation.

In Chapter 7, we continue with some other applications of spectral projections in the context of computing invariant manifolds. As examples we discuss some nonlinear FDE where the linearization yields linear equations with 0 as the dominant eigenvalue of the characteristic equation. We present in detail the (symbolical) computation of the large time behaviour and the ordinary differential equations that give the flow on the center manifold in a neighbourhood of the origin.

In Chapter 8, we study a size-structured cell cycle model which has a neutral "nature". We use the renewal equations and similar techniques as in Chapter 1 to obtain the large time behaviour for the solution of the model, even when there is no dominant eigenvalue (or finite set of dominant eigenvalues) as in previous chapters, but when the large time behaviour is governed by the solution restricted to an infinite dimensional subspace.

We close the thesis with Appendices A and B. In Appendix A, we provide a justification for a bilinear form first introduced by Hale. With respect to this bilinear form, we obtain as the dual space of $\mathcal{C} \stackrel{\text{def}}{=} \mathcal{C}([-r,0],\mathbb{C}^n)$ the space $\mathcal{C}' \stackrel{\text{def}}{=} \mathcal{C}([0,r],\mathbb{C}^{n*})$, *i.e.*, the space of continuous functions from $[0,r]$ with values in the space of row n-vectors with complex numbers as entries. Finally, in Appendix B, we present the source code of the Maple library `FDESpectralProj`.

Chapter 1

Introduction

In this chapter we introduce the class of differential equations called *functional differential equations*, using renewal equations as the main tool. We introduce renewal equations and recall some basic results. Later, with help of Laplace transform techniques, we provide a characterization for the large time behaviour of solutions of linear functional differential equations.

1.1 Notation and definitions

We denote by $\mathbb{C}^n$ the set of *column* n-vectors with complex-valued entries, and $\mathbb{R}$ and $\mathbb{C}$ denote the fields of real and complex numbers, as usual. One can identify a *row* n-vector γ with entries in $\mathbb{C}$ with a functional in $\mathbb{C}^n$ given by $v \mapsto \gamma v$. In order to emphasize this, we denote the set of row vectors with complex entries as $\mathbb{C}^{n*}$. We identify the $n \times n$ matrices with complex-valued entries with the linear operators in $\mathbb{C}^n$ and denote this space by $\mathbb{C}^{n\times n}$.

Let $\mathcal{C} \stackrel{\text{def}}{=} \mathcal{C}\big([-r,0],\mathbb{C}^n\big)$ denote the Banach space of continuous functions from $[-r,0]$ $(r>0)$ with values in $\mathbb{C}^n$ endowed with the supremum norm. From the Riesz Representation Theorem (see for instance Rudin [49] or Royden [48]) it follows that every bounded linear mapping $L : \mathcal{C} \to \mathbb{C}^n$ can be represented by

$$L\varphi = \int_0^r d\eta(\theta)\varphi(-\theta), \tag{1.1}$$

where η is a function of bounded variation on $[0,r]$ normalized so that $\eta(0) = 0$ and η is continuous from the right in $(0,r)$ with values in the matrix space $\mathbb{C}^{n\times n}$. This set of functions is denoted by $NBV\big([0,r],\mathbb{C}^{n\times n}\big)$. We can trivially extend $\eta \in NBV([0,r],\cdot)$ in $\mathbb{R}$ by $\eta(\theta) = 0$ if $\theta < 0$ and $\eta(\theta) = \eta(r)$ if $\theta > r$. This will be done without further notification. In (1.1), the notation $d\eta$ before the integrand φ emphasizes that η is a matrix and φ is a column vector, and therefore the integral is column vector-valued. The variable between parenthesis after the "differential part" is the variable of integration. Sometimes, in order to avoid confusion, we denote the variable of integration as a index of the *"d"*, as in $\int d_\theta[\eta(t+\theta)]f(\theta)$.

The Riesz Representation Theorem yields that every bounded linear functional $f : \mathcal{C} \to \mathbb{C}$ can be written as

$$f(\varphi) = \int_0^r d\psi(\theta)\varphi(-\theta) \stackrel{\text{def}}{=} \langle \psi, \varphi \rangle,$$

where $\psi \in NBV\big([0,r], \mathbb{C}^{n*}\big)$, defined analogously as the space of NBV functions with values in $\mathbb{C}^{n*}$. Therefore we can represent the dual space of the space $\mathcal{C}$ by $\mathcal{C}^* = NBV\big([0,r], \mathbb{C}^{n*}\big)$.

As usual in the theory of Delay Equations, for a function x from $[-r,\infty)$ to some Banach space X, we define $x_t : [-r,0] \to X$ by $x_t(\theta) = x(t+\theta)$, $-r \leqslant \theta \leqslant 0$ and $t \geqslant 0$. We define $\mathcal{C}' \stackrel{\text{def}}{=} \mathcal{C}\big([0,r], \mathbb{C}^{n*}\big)$, which will be used as the state space for the transposed equation, see Section A.3. For a function $y : (-\infty, r] \to X$, we define $y^s \in X$ by

$$y^s(\xi) = y(-s+\xi), \qquad 0 \leqslant \xi \leqslant r, \quad s \geqslant 0. \tag{1.2}$$

Therefore $y^s \in \mathcal{C}'$.

Derivatives of a function f will be also denoted by Df, where D is viewed as a linear operator. The "dot" notation $\dot{f}$ will also be used. We denote the (classical) adjoint of a matrix A, *i.e.*, the matrix of cofactors of A, by $\operatorname{adj} A$, and the transpose of A is denoted by A^{T}. By $\widehat{g}(\cdot)$ and $\widehat{d\eta}(\cdot)$ we denote respectively the Laplace transform of a function g and the Laplace-Stieltjes transform of a measure $d\eta$. See Section 1.3.4. We denote by $\beth(\gamma)$ the line in the complex plane $\{z \in \mathbb{C} : \operatorname{Re} z = \gamma\}$ and we define $\int_{\beth(\gamma)} \cdots dz$ to denote the principal value integral $\lim_{\omega\to\infty} \int_{\gamma-i\omega}^{\gamma+i\omega} \cdots dz$.

1.2 Initial value problem for FDE

An initial value problem for a linear autonomous Functional Differential Equation (FDE) is given by the following relation

$$\begin{cases} \frac{d}{dt} M x_t = L x_t, & t \geqslant 0, \\ x_0 = \varphi, & \varphi \in \mathcal{C}, \end{cases} \tag{1.3}$$

where $L, M : \mathcal{C} \to \mathbb{C}^n$ are linear continuous, given respectively by

$$L\varphi = \int_0^r d\eta(\theta)\varphi(-\theta), \qquad M\varphi = \varphi(0) - \int_0^r d\mu(\theta)\varphi(-\theta), \tag{1.4}$$

where $\eta, \mu \in NBV\big([0,r], \mathbb{C}^{n\times n}\big)$ and μ is continuous at zero. See Hale & Verduyn Lunel [27] for a detailed introduction to these equations. As an example, we observe that the differential equation

$$\dot{x}(t) + C\dot{x}(t-1) = Ax(t) + Bx(t-1), \qquad t \geqslant 0,$$

where A, B and C are $n \times n$-matrices, can be written in the form (1.3) by taking $r = 1$, $\mu(\theta) = 0$ if $\theta < 1$, $\mu(\theta) = -C$ for $\theta \geqslant 1$ (i.e., a "jump" of size $-C$ at $\theta = 1$), and $\eta(\theta) = 0$ for $\theta \leqslant 0$, $\eta(\theta) = A$ for $0 < \theta < 1$ and $\eta(\theta) = A + B$ for $\theta \geqslant 1$ (the variation of η in (1.3) is concentrated at $\theta = 0$ and $\theta = 1$, where the "jumps" are, respectively, A and B.)

Differential equations of the form (1.3) with $\mu = 0$

$$\dot{x}(t) = Lx_t, \qquad x_0 = \varphi$$

are known as *retarded differential equations* or *delay differential equations*. The theory for these equations is well developed (see the books by Hale and Verduyn Lunel [27] and Diekmann *et al.* [15] for a comprehensive introduction) and form a basis for the theory of Neutral equations that we study here.

If for every $\varphi \in \mathcal{C}$ we have existence and uniqueness of continuous solutions of (1.3) for t in some interval $[0, t_0)$, then we can define a semigroup $T(t) : \mathcal{C} \to \mathcal{C}$ by

$$T(t)\varphi = x_t, \qquad t \in [0, t_0)$$

where x is the solution of (1.3). It is easy to see that T is indeed a strongly continuous semigroup, that is, $T(0) = I$, $T(t+s) = T(t)T(s)$ and $\lim_{t\downarrow 0} T(t)\varphi = \varphi$. The growth bound of the semigroup ensures that the solutions are defined for $t \in [0, \infty)$. The semigroup T is known as the *solution semigroup*.

We take M in the form as in (1.4) with μ continuous at zero as a sufficient condition in order to have well-poseness of the solution semigroup, that is, for the existence and uniqueness of solution through every state. See Hale & Verduyn Lunel [27] in the page 255 for the concept of *atomic at zero*. To derive necessary conditions for the well-poseness is still an open problem. See for instance Ito, Kappel & Turi [33], where the authors present examples of well-posed and non-well-posed linear neutral equations with an M operator which is non-atomic at 0. See also Burns, Herdman & Turi [8].

FDE (1.3) has been studied on a variety of state spaces. O'Connor & Tarn [44, 45] studied a class of neutral equations on the Sobolev space $W_2^1([-r,0];\mathbb{R}^n)$ and the controllability and stabilization of solutions. Verduyn Lunel & Yakubovich [58] consider the FDE (1.3) on the space $\mathbb{C}^n \times L^2([-r,0], \mathbb{C}^n)$. Burns, Herdman & Stech [7] and Salamon [51] presented neutral systems with $\mathcal{C}$, $R^n \times L^p$ and $W^{1,p}$ as state spaces.

1.3 Renewal equations

For functional differential equations, it is well known that the adjoint of a semigroup associated with the solution of a delay equation is not of the same type as the original one. The interpretation of the adjoint semigroup in terms of the underlying system equation was first given by Burns & Herdman [6] for Volterra integro-differential equations. These authors did show that the adjoint semigroup is associated with the transposed equation via an alternative state space concept which is due to Miller [42]. Further results in this direction can be found, for example, in Diekmann [9, 10], Diekmann, Gyllenberg & Thieme [12]. In the book by Diekmann *et al.* [15] it was shown that a delay equation can be written into the form

$$x = k * x + f, \tag{1.5}$$

where k is a function of bounded variation and f, the forcing function, is continuous. In order to develop similar results for FDE's like (1.3), we study *renewal equations*, or alternatively, *Volterra-Stieltjes convolution integral equations* (of the second kind), and convolution between measures and functions. Our main references here are the book by Salamon [51] and Gripenberg, Londen & Staffans [21]. We formulate results about existence, uniqueness, continuous dependence and representation of L^p-solutions to the Volterra-Stieltjes integral equation

$$z(t) = \int_0^t d\alpha(s) z(t-s) + f(s), \tag{1.6}$$

or alternatively,

$$z = d\alpha * z + f, \tag{1.7}$$

where $\alpha \in NBV\big([0,R), \mathbb{C}^{n\times n}\big)$ represents the Borel measure $d\alpha$ which is called *kernel* of the renewal equation (1.7) and $f \in L^p\big([0,R], \mathbb{C}^n\big)$ is called the *forcing function.*

1.3.1 Convolution product of functions and Borel measures

In this section we recall some results from Salamon [51] with small modifications. We focus our efforts in NBV functions. For more general results, covering a wide class of function spaces, we refer to the book by Gripenberg, Londen & Staffans [21].

In the following remark, items 2–6 are found in Remark 1.1.1 in Salamon [51].

Remark 1.1.

1. The function spaces $NBV\big([0,R), \mathbb{C}^{n\times n}\big)$ and $\mathcal{C}\big([0,R], \mathbb{C}^n\big)$ "are" subspaces of $L^p\big([0,R], \mathbb{C}^n\big)$ for $1 \leqslant p \leqslant \infty$. For $R > r$, we can embed $NBV\big([0,r], \mathbb{C}^{n\times n}\big)$ into $NBV\big([0,R), \mathbb{C}^{n\times n}\big)$ by extending $\psi \in NBV\big([0,r], \mathbb{C}^{n\times n}\big)$ to $[0,R)$, defining $\psi(t) = \psi(r)$ for $t \geqslant r$.

2. Let $1 \leqslant p \leqslant \infty$ and q such that $1/p + 1/q = 1$, $f \in L^p([0,R],\mathbb{C}^n)$ and $g \in L^q([0,R],\mathbb{C}^n)$. Then the function
$$g * f(t) = \int_0^t g(s)f(t-s)ds = \int_0^t g(t-s)f(s)ds, \qquad 0 \leqslant t \leqslant l,$$
is continuous. For $n = 1$ (scalar case) we have that $g * f(t) = f * g(t)$.

3. Every $\alpha \in NBV([0,R),\mathbb{C}^{n\times n})$ represents a Borel measure on $\mathbb{C}^n$ with no mass outside $[0,R)$. This measure will be denoted by $d\alpha$.

   Let $\alpha \in NBV([0,R),\mathbb{C}^{n\times n})$ and $f \in L^p([0,R],\mathbb{C}^n)$ for $1 \leqslant p \leqslant \infty$. Then $d\alpha * f \in L^p([0,R],\mathbb{C}^n)$ can be defined by the explicit expression
$$d\alpha * f(t) = \int_0^t d\alpha(s)f(t-s)ds \tag{1.8}$$
for almost $t \in [0,R]$. If f is continuous, then (1.8) can be understood as Stieltjes integral.

   By the Riesz Representation Theorem, $NBV([0,R),\mathbb{C}^{n\times n})$ is (isometrically isomorphic to) the dual space of
$$\mathcal{C}_R \stackrel{\text{def}}{=} \{g \in \mathcal{C}([0,R],\mathbb{C}^n) \;:\; g(0) = 0\}$$
where the pairing is given by $d\alpha^* * g(R)$ for $\alpha \in NBV([0,R),\mathbb{C}^{n\times n})$ and $g \in \mathcal{C}_R$, and α^* denotes the adjoint of α in $\mathbb{C}^{n\times n}$, that is, $\alpha^* = \bar{\alpha}^{\mathrm{T}}$.

4. The operator $f \mapsto d\alpha * f$ maps $\mathcal{C}_R$ into itself. The operator $f \mapsto d\alpha * f$ maps $NBV([0,R),\mathbb{C}^n)$ into itself. However, $\mathcal{C}([0,R],\mathbb{C}^n)$ is *not* mapped into itself by $f \mapsto d\alpha * f$.

   We show here that $f \mapsto d\alpha * f$ maps $\mathcal{C}_R$ into itself. It is known that "translation" is a continuous operation in the space of continuous functions with compact domain, that is, if f is continuous in an interval $[a,b]$, and f is extended as $f(t) = f(a)$ for $t < a$ and $f(t) = f(b)$ for $t > b$, then $|f(t) - f(t+\delta)| \to 0$ uniformly as $\delta \to 0$. Let f to be continuous on $[0,R]$ with $f(0) = 0$ and define $h(t) \stackrel{\text{def}}{=} d\alpha * f(t)$. Then we have that far $\delta > 0$,
$$\begin{aligned} h(t+\delta) - h(t) &= \int_0^{t+\delta} d\alpha(s)f(t+\delta-s) - \int_0^t d\alpha(s)f(t-s) \\ &= \underbrace{\int_0^t d\alpha(s)\big(f(t+\delta-s) - f(t-s)\big)}_{\stackrel{\text{def}}{=} I_1(\delta)} + \underbrace{\int_t^{t+\delta} d\alpha(s)f(t+\delta-s)}_{\stackrel{\text{def}}{=} I_2(\delta)}. \end{aligned}$$

When $\delta \downarrow 0$, we have that $f(t+\delta-s)-f(t-s) \to 0$ and hence $I_1(\delta) \to 0$. We observe that in $I_2(\delta)$, f is integrated over $(0,\delta]$. Therefore, since $f(0)=0$, we have that $I_2(\delta) \to 0$. Then, $h(t+\delta)-h(t) \to 0$ as $\delta \downarrow 0$. It is immediate that $h(0)=0$. Therefore $f \mapsto d\alpha * f$ maps $\mathcal{C}_R$ into itself.

Now suppose that $f \mapsto d\alpha * f$ maps $\mathcal{C}([0,R],\mathbb{C}^n)$ into itself. Then for every $f \in \mathcal{C}([0,R],\mathbb{C}^n)$ and $\alpha \in NBV$, we have that $d\alpha * f$ is continuous. Since g, defined by $g(\cdot) \overset{\text{def}}{=} f(\cdot) - f(0)$, belongs to $\mathcal{C}_R$, then $d\alpha * g$ is continuous (and $d\alpha * g(0) = 0$). This would imply that

$$d\alpha * f - d\alpha * g = d\alpha * (f-g) = d\alpha * (f(0)) = \alpha \cdot f(0)$$

is continuous, but there are plenty of $\alpha \in NBV$ such that α is not continuous. This is a contradiction.

5. The following inequality holds for every $f \in L^p\big([0,R],\mathbb{C}^n\big)$

$$\|d\alpha * f\|_p \leqslant \operatorname{Var}_{[0,R)} \alpha \cdot \|f\|_p \tag{1.9}$$

(see Hewitt & Ross [31], Theorem 20.12)

6. For $\alpha, \beta \in NBV\big([0,R),\mathbb{C}^{n\times n}\big)$, the convolution $d\alpha * d\beta$ is a Borel measure, and for $g \in \mathcal{C}\big([0,R],\mathbb{C}^n\big)$, $g * [d\alpha * d\beta]$ is defined by

$$\int_0^l g(\theta)[d\alpha * d\beta](\theta) = \int_0^l \Big[\int_0^{l-t} g(t+s)d\alpha(s)\Big] d\beta(t).$$

Moreover, $d\alpha * \beta \in NBV\big([0,R),\mathbb{C}^{n\times n}\big)$, $d\alpha * \beta = \alpha * d\beta$ and

$$d[\alpha * d\beta] = d\alpha * d\beta. \tag{1.10}$$

Note that the Borel measure $d\alpha$ can be interpreted as the distributional derivative of $\alpha \in NBV\big([0,R),\mathbb{C}^{n\times n}\big)$. In this sense (1.10) follows from the fact that in order to differentiate a convolution product of distributions, it suffices to differentiate one of the factors.

Remark 1.2. We cannot expect existence and uniqueness for solutions of (1.6) for arbitrary α. For example, if $\alpha = \rho \in NBV\big([0,R),\mathbb{C}^{n\times n}\big)$ given by $\rho(0)=0$ and $\rho(t) = I$ for $t>0$, where I is the identity on $\mathbb{C}^{n\times n}$, we have that $d\rho * z = z$ and then (1.6) is equivalent to $f(t) \equiv 0$. To exclude this situation, we assume that

$$1 \notin \sigma(\alpha_0), \qquad \alpha_0 = \lim_{t\downarrow 0} \alpha(t). \tag{1.11}$$

Note that $\lim_{t\downarrow 0} \operatorname{Var}_{[0,t]}(\alpha - \alpha_0\rho) = 0$ always holds.

A central result in this section is stated below. For a proof see Theorem 1.1.2 in Salamon [51].

Theorem 1.3. *Let $\alpha \in NBV\big([0,R),\mathbb{C}^{n\times n}\big)$ be such that Condition (1.11) is satisfied. Then the following statements hold.*

1. *For every $f \in \mathcal{C}\big([0,R],\mathbb{C}^n\big)$ with $f(0)=0$ there exists a unique solution z of (1.6) in $\mathcal{C}\big([0,R],\mathbb{C}^n\big)$, that is, $z = d\alpha * z + f$, satisfying $z(0)=0$. Here z depends continuously on f with respect to the sup-norm;*
2. *For every $f \in NBV\big([0,R),\mathbb{C}^n\big)$, there exists a unique solution z of (1.6) belonging to $NBV\big([0,R),\mathbb{C}^n\big)$, depending continuously on f with respect to the NBV-norm.*

Given $\alpha \in NBV\big([0,R),\mathbb{C}^n\big)$ and $f \in L^p\big([0,R],\mathbb{C}^n\big)$, there are two equivalent ways to solve the Volterra-Stieltjes equation (1.6): either by using the fundamental solution or by using the resolvent kernel. For sake of completeness we present both, but we prefer the resolvent kernel approach.

1.3.2 The fundamental solution

Definition 1.4. Let $\alpha \in NBV\big([0,R),\mathbb{C}^{n\times n}\big)$ satisfying Condition (1.11). The unique solution ξ of

$$\xi = d\alpha * \xi + \rho \tag{1.12}$$

($\rho(0)=0$, $\rho(t)=I$ for $t>0$) is said to be the *fundamental solution* of (1.6).

The existence and uniqueness of the fundamental solution follow from Theorem 1.3. In fact, the fundamental solution also solves the equation

$$\xi = \xi * d\alpha + \rho. \tag{1.13}$$

Indeed, suppose that ζ is a solution of (1.13). Then

$$\begin{aligned}\zeta &= \zeta * d\rho = \zeta * [\xi - d\alpha * \xi] \\ &= d[\zeta - \zeta * d\alpha] * \xi = d\rho * \xi = \xi.\end{aligned}$$

It follows that ξ^{T} is the fundamental solution of the transposed renewal equation

$$z = d\alpha^{\mathrm{T}} * z + f.$$

The following result is obtained by direct computations using convolutions and the definition of the fundamental solution. See Theorem 1.1.4 of Salamon [51] for a rigorous proof of the next lemma.

Lemma 1.5. *Let $\alpha \in NBV\big([0,R),\mathbb{C}^{n\times n}\big)$ satisfy Condition (1.11) and let ξ be the fundamental solution of (1.7). Then, for $f \in L^p\big([0,R],\mathbb{C}^n\big)$ with $1 \leqslant p \leqslant \infty$, there exists a unique solution $z \in L^p\big([0,R],\mathbb{C}^n\big)$ of (1.6), that is, $z = d\alpha * z + f$. This solution is given by*

$$z = d\xi * f \tag{1.14}$$

and the solution depends continuously on f with respect to the L^p norm.

1.3.3 The resolvent kernel

Definition 1.6. Let $\alpha \in NBV\big([0,R),\mathbb{C}^{n\times n}\big)$ satisfy Condition (1.11). The unique solution ζ of

$$\zeta = \zeta * d\alpha + \alpha \tag{1.15}$$

is said to be the *resolvent kernel* of (1.6).

Similarly as for the fundamental solution, a computation shows that the resolvent kernel also solves

$$\zeta = d\alpha * \zeta + \alpha \tag{1.16}$$

and therefore, ζ^{T} is the resolvent kernel of the transposed renewal equation

$$z = d\alpha^{\mathrm{T}} * z + f.$$

Lemma 1.7. *Let $\alpha \in NBV\big([0,R),\mathbb{C}^{n\times n}\big)$ satisfy Condition (1.11) and let ζ be the resolvent kernel of (1.7). Then, for $f \in L^p\big([0,R],\mathbb{C}^n\big)$ with $1 \leqslant p \leqslant \infty$, there exists a unique solution $z \in L^p\big([0,R],\mathbb{C}^n\big)$ of (1.6), that is, $z = d\alpha * z + f$. This solution is given by*

$$z = d\zeta * f + f \tag{1.17}$$

and depends continuously on f with respect to the L^p norm. Furthermore, if $f \in \mathcal{C}\big([0,R],\mathbb{C}^n\big)$ and $f(0) = 0$, then the solution z satisfies $z \in \mathcal{C}\big([0,R],\mathbb{C}^n\big)$ and $z(0) = 0$ as well. If $f \in NBV\big([0,R),\mathbb{C}^n\big)$, then the solution z satisfies $z \in NBV\big([0,R),\mathbb{C}^n\big)$ too.

Proof. From (1.6), we have

$$z = d\alpha * z + f.$$

By applying the convolution by $d\zeta$ in both sides and using the definition of resolvent kernel (1.15), we arrive that

$$\begin{aligned} d\zeta * z &= d\zeta * (d\alpha * z) + d\zeta * f \\ &= d(\zeta * d\alpha) * z + d\zeta * f \\ &= d(\zeta - \alpha) * z + d\zeta * f \\ &= d\zeta * z - d\alpha * z + d\zeta * f. \end{aligned}$$

Therefore

$$d\zeta * f = d\alpha * z. \tag{1.18}$$

Substituting (1.18) into (1.6), we find (1.14). The continuous dependence of z with respect to f follows from the explicit solution (1.14) and the estimate (1.9). The final statements follow from item 4 of Remark 1.1. ☐

Theorem 1.8. *Let $\alpha : [0,\infty) \to \mathbb{C}^{n\times n}$ satisfying Condition (1.11) such that $\alpha(0) = 0$ and α is locally of bounded variation, that is, for all $R \geqslant 0$, $\operatorname{Var}_{[0,R)} \alpha < \infty$. Then there exists a unique $\zeta : [0,\infty) \to \mathbb{C}^{n\times n}$ satisfying $\zeta(0) = 0$ and locally of bounded variation such that*

$$\zeta(t) = [\zeta * d\alpha](t) + \alpha(t), \qquad t \geqslant 0. \tag{1.19}$$

Proof. For β belonging to $NBV\big([0,R),\mathbb{C}^{n\times n}\big)$ we have that $\beta(t) = \beta(R-)$ for $t \geqslant R$. For $R > 0$, define $\alpha_R \in NBV\big([0,R),\mathbb{C}^{n\times n}\big)$ by $\alpha_R(t) = \alpha(t)$ for $0 \leqslant t < R$ and $\alpha_R(t) = \alpha(R-)$ for $t \geqslant R$. Denote by $\zeta_R \in NBV\big([0,R),\mathbb{C}^{n\times n}\big)$ the resolvent kernel of renewal equation (1.7), where α has been replaced by α_R, as in Definition 1.6. Therefore

$$\zeta_R(t) = \int_0^t d\alpha_R(s)\zeta_R(t-s) + \alpha_R(t), \qquad 0 \leqslant t \leqslant R. \tag{1.20}$$

Let $0 < R_2 < R_1$. Then $\alpha_{R_2}(t) = \alpha_{R_1}(t)$ for $0 \leqslant t < R_2$. We claim that

$$\zeta_{R_1}(t) = \zeta_{R_2}(t), \qquad 0 \leqslant t < R_2. \tag{1.21}$$

To prove this, define $\tilde\zeta_{R_1} \in NBV\big([0,R_2),\mathbb{C}^{n\times n}\big)$ by $\tilde\zeta_{R_1}(t) = \zeta_{R_1}(t)$ for $0 \leqslant t < R_2$ and $\tilde\zeta_{R_1}(t) = \zeta_{R_1}(R_2-)$ for $t \geqslant R_2$. Then, for $0 \leqslant t < R_2$,

$$\begin{aligned}\tilde\zeta_{R_1}(t) = \zeta_{R_1}(t) &= \int_0^t d\alpha_{R_1}(s)\zeta_{R_1}(t-s) + \alpha_{R_1}(t)\\ &= \int_0^t d\alpha_{R_2}(s)\tilde\zeta_{R_1}(t-s) + \alpha_{R_2}(t).\end{aligned}$$

For $t = R_2$,

$$\begin{aligned}\tilde\zeta_{R_1}(R_2) &= \lim_{t\uparrow R_2} \tilde\zeta_{R_1}(t)\\ &= \lim_{t\uparrow R_2} \int_0^t d\alpha_{R_1}(s)\tilde\zeta_{R_1}(t-s) + \alpha_{R_1}(t)\\ &= \lim_{t\uparrow R_2} \int_{[0,t]} d\alpha_{R_2}(s)\tilde\zeta_{R_1}(t-s) + \alpha_{R_2}(t).\end{aligned}$$

Using that $s \mapsto \tilde\zeta_{R_1}(t-s)$ is *left* continuous for $0 \leqslant s < t$, we conclude that

$$\tilde\zeta_{R_1}(R_2) = \int_{[0,R_2)} d\alpha_{R_2}(s)\tilde\zeta_{R_1}(R_2-s-) + \alpha_{R_2}(R_2-).$$

Observing that $d\alpha_{R_2}$ has no mass outside $[0,R_2)$ and using the definition of α_{R_2}, gives

$$\begin{aligned}\tilde{\zeta}_{R_1}(R_2) &= \int_{[0,R_2]} d\alpha_{R_2}(s)\tilde{\zeta}_{R_1}(R_2-s)+\alpha_{R_2}(R_2)\\ &= \int_0^{R_2} d\alpha_{R_2}(s)\tilde{\zeta}_{R_1}(R_2-s)+\alpha_{R_2}(R_2).\end{aligned}$$

So $\tilde{\zeta}_{R_1}$ solves (1.20) for $R=R_2$. Therefore, $\zeta_{R_2}=\tilde{\zeta}_{R_1}$ by uniqueness of the resolvent kernel. This proves the claim (1.21).

Define $\zeta(0)=0$ and, for $t>0$, $\zeta(t)=\zeta_{t+1}(t)$. By construction, ζ satisfies (1.19), is of bounded variation on finite intervals, and its uniqueness is inherited from ζ_{t+1} for all $t>0$. □

Corollary 1.9. *Let $\alpha:[0,\infty)\to\mathbb{C}^{n\times n}$ satisfying Condition (1.11) such that $\alpha(0)=0$ and α is locally of bounded variation. For all functions $f:[0,\infty)\to\mathbb{C}^n$ satisfying one of the following properties*

i. The restriction of f to the intervals $[0,R]$ belongs to $L^p\big([0,R],\mathbb{C}^n\big)$, for $R>0$ and $1\leqslant p\leqslant\infty$;

ii. The restriction of f to the intervals $[0,R]$ belongs to $NBV\big([0,R),\mathbb{C}^{n\times n}\big)$, for $R>0$;

iii. The restriction of f to the intervals $[0,R]$ belongs to $\mathcal{C}\big([0,R],\mathbb{C}^{n\times n}\big)$, for $R>0$ and $f(0)=0$.

Then there exists a unique solution z of (1.6), that is,

$$z(t)=[d\alpha * z](t)+f(t),$$

defined almost everywhere (according to Lebesgue measure) such that z has the same properties as f. Furthermore, z is given explicitly by the formula

$$z(t)=[d\zeta * f](t)+f(t),\qquad t\geqslant 0. \tag{1.22}$$

Proof. The existence and uniqueness of the solution of (1.6) for $f:[0,\infty)\to\mathbb{C}^n$, where f belongs to the the classes described in the statements now follow from Lemma 1.7, extending the conclusions to functions defined on $[0,\infty)$. The resolvent kernel is obtained from Theorem 1.8. □

1.3.4 The Laplace-Stieltjes transform

Renewal equations are more easily studied by using Laplace transform techniques, since convolutions of functions and measures are transformed into a simple product of complex functions.

If g is a locally integrable function with domain of definition $\mathbb{R}_+$ and σ_0-exponentially bounded, *i.e.*, there exists $C > 0$ such that

$$|g(t)| \leqslant Ce^{\sigma_0 t},$$

then the *Laplace transform* of g, denoted by $\widehat{g}$, is defined by

$$\widehat{g}(z) \stackrel{\text{def}}{=} \int_0^\infty e^{-zt} g(t)dt. \tag{1.23}$$

which is defined (at least) for those $z \in \mathbb{C}$ such that $\operatorname{Re} z > \sigma_0$. Analogously, for the measure on $[0, r]$ represented by a *NBV* function α we define the *Laplace-Stieltjes transform* of α by

$$\widehat{da}(z) \stackrel{\text{def}}{=} \int_0^\infty e^{-zt} d\alpha(t). \tag{1.24}$$

We have that $\widehat{d\alpha}$ is an entire function (because the interval of integration reduces to $[0, r]$). We collect some properties of Laplace-Stieltjes transforms from the book by Widder [60].

There is uniqueness of Laplace(-Stieltjes) transformations, that is, if α and β are *NBV* functions and $\widehat{d\alpha} = \widehat{d\beta}$, then $\alpha = \beta$, and if f and g are σ_0-exponentially bounded functions with domain in $\mathbb{R}_+$, then $\widehat{f}(z) = \widehat{g}(z)$ for z in a half plane implies that $f(t) = g(t)$ for almost all $t \in \mathbb{R}_+$.

One can derive the following lemma by carrying out some simple computations. The result also holds in the vector-valued case.

Lemma 1.10. *Let f and g be σ_0-exponentially bounded functions and let α be a NBV function. For z such that $\operatorname{Re} z \geqslant \sigma_0$ we have that*

$$\widehat{f * g}(z) = \widehat{f}(z)\widehat{g}(z), \qquad \widehat{f * d\alpha}(z) = \widehat{f}(z)\widehat{d\alpha}(z).$$

For a function f with values in $\mathbb{C}^n$, defined and of bounded variation on $\mathbb{R}_+$ and constant on $[r, \infty)$ (if in addition f is right-continuous in $(0, \infty)$ and $f(0) = 0$ then f is *NBV*, but we do not require this here) integration by parts leads to the identity

$$\widehat{f}(z) = \frac{1}{z}\left(f(0) + \int_0^r e^{-zt} df(t)\right) = \frac{1}{z}\big(f(0) + \widehat{df}(z)\big). \tag{1.25}$$

We present a result from standard Laplace transform literature (see for instance Widder [60]) on the inversion formula.

Lemma 1.11. *Let g be a σ_0-exponentially bounded function that is of bounded variation on bounded intervals. Then for $\gamma > \sigma_0$ and for $t > 0$ we have the inversion formula*

$$\frac{1}{2}\big(g(t+) + g(t-)\big) = \lim_{\omega \to \infty} \frac{1}{2\pi i} \int_{\gamma - i\omega}^{\gamma + i\omega} e^{zt} \widehat{g}(z) dz, \tag{1.26}$$

whereas for $t = 0$ we have

$$\frac{1}{2}\, g(0+) = \lim_{\omega\to\infty} \frac{1}{2\pi i} \int_{\gamma-i\omega}^{\gamma+i\omega} \widehat{g}(z)dz, \tag{1.27}$$

To facilitate the formulation of results like (1.26), we introduce some notation. We define $\daleth(\gamma)$ to denote the line $\{z \in \mathbb{C} : \operatorname{Re} z = \gamma\}$ and we define $\int_{\daleth(\gamma)} \cdots dz$ to denote the principal value integral $\lim_{\omega\to\infty} \int_{\gamma-i\omega}^{\gamma+i\omega} \cdots dz$.

1.4 Representation of FDE as a renewal equation

In this section we study the equivalence between functional differential equations and a class of renewal equations. The initial condition of FDE (1.3) is converted into a forcing function of renewal equation (1.7). Later in this chapter, using Laplace transform techniques within the renewal equation obtained from FDE (1.3), we will be able to provide the large time behaviour of solutions for FDE (1.3).

Lemma 1.12. *Equation (1.3) is equivalent to the following renewal equation*

$$x(t) = \int_0^t \big[d\mu(\theta) + \eta(\theta)d\theta\big]x(t-\theta) + F\varphi(t), \qquad t \geqslant 0 \tag{1.28}$$

where $F : \mathcal{C} \to L^\infty$ is defined by

$$F\varphi(t) = M\varphi + \int_t^r d\mu(\theta)\varphi(t-\theta) + \int_0^t \Big[\int_s^r d\eta(\theta)\varphi(s-\theta)\Big]ds, \quad t \geqslant 0, \tag{1.29}$$

maps the initial condition $\varphi \in \mathcal{C}$ into the corresponding forcing function of the renewal equation (1.28). For $\varphi \in \mathcal{C}$ one has that $F\varphi(\cdot)$ is constant on $[r, \infty)$, $F\varphi(0) = \varphi(0)$ and $F\varphi + \mu \cdot \varphi(0)$ is continuous.

Proof. Recall equation (1.3)

$$\tfrac{d}{dt} M x_t = L x_t, \qquad t \geqslant 0$$

with $x_0 = \varphi \in \mathcal{C}$. We can integrate the system (1.3) to obtain

$$Mx_t - M\varphi = \int_0^t \Big[\int_0^r d\eta(\theta)x(s-\theta)\Big]ds. \tag{1.30}$$

Next we split off the part that explicitly depends in initial data. For any $\xi \in NBV\big([0, r], \mathbb{C}^{n\times n}\big)$ and for $0 < t \leqslant r$ (not for $t = 0$ because ξ is not right-continuous at 0), we have

$$\begin{aligned}\int_0^r d\xi(\theta)x(t-\theta) &= \int_0^t d\xi(\theta)x(t-\theta) + \int_t^r d\xi(\theta)x(t-\theta)\\ &= \int_0^t d\xi(\theta)x(t-\theta) + \int_t^r d\xi(\theta)\varphi(t-\theta)\end{aligned} \tag{1.31}$$

(since for $t \leqslant \theta \leqslant r$, $t-\theta \leqslant 0$ and $x(t-\theta) = \varphi(t-\theta)$.) Now, for $t > r$, using that for $\theta \notin [0, r]$, the variation of ξ vanishes on $[r, t]$ and we see that

$$\int_t^r d\xi(\theta)\varphi(t-\theta) = 0,$$
$$\int_0^t d\xi(\theta)x(t-\theta) = \int_0^r d\xi(\theta)x(t-\theta).$$

Therefore, for all $t > 0$ and for all $\xi \in NBV\big([0,r], \mathbb{C}^{n\times n}\big)$ we can write

$$\int_0^r d\xi(\theta)x(t-\theta) = \int_0^t d\xi(\theta)x(t-\theta) + \int_t^r d\xi(\theta)\varphi(t-\theta). \tag{1.32}$$

Returning to (1.30) and applying (1.32), for $t > 0$, we arrive at

$$\begin{aligned}
x(t) &= \int_0^r d\mu(\theta)x(t-\theta) + M\varphi + \int_0^t \Big[\int_0^r d\eta(\theta)x(s-\theta)\Big]ds \\
&= \int_0^t d\mu(\theta)x(t-\theta) + \int_t^r d\mu(\theta)\varphi(t-\theta) + M\varphi \\
&\qquad + \int_0^t \Big[\int_0^s d\eta(\theta)x(s-\theta) + \int_s^r d\eta(\theta)\varphi(s-\theta)\Big]ds \\
&= \int_0^t d\mu(\theta)x(t-\theta) + \int_0^t \Big[\int_0^s d\eta(\theta)x(s-\theta)\Big]ds \\
&\qquad + M\varphi + \int_t^r d\mu(\theta)\varphi(t-\theta) + \int_0^t \Big[\int_s^r d\eta(\theta)\varphi(s-\theta)\Big]ds.
\end{aligned}$$

Applying Fubini's theorem in the second integral and defining $F\varphi(t)$ as in (1.29), show that

$$\begin{aligned}
x(t) &= \int_0^t d\mu(\theta)x(t-\theta) + \int_0^t \Big[\int_0^s d\eta(\theta)x(s-\theta)\Big]ds + F\varphi(t) \\
&= \int_0^t d\mu(\theta)x(t-\theta) + \int_0^t d\eta(\theta)\Big[\int_\theta^t x(s-\theta)ds\Big] + F\varphi(t) \\
&= \int_0^t d\mu(\theta)x(t-\theta) + \int_0^t d\eta(\theta)\Big[\int_0^{t-\theta} x(s)ds\Big] + F\varphi(t) \\
&= \int_0^t d\mu(\theta)x(t-\theta) + \cancelto{0}{\eta(\theta)\Big[\int_0^{t-\theta} x(s)ds\Big]\Big|_{\theta=0}^{t}} + \int_0^t \eta(\theta)x(t-\theta)d\theta + F\varphi(t) \\
&= \int_0^t d\mu(\theta)x(t-\theta) + \int_0^t \eta(\theta)x(t-\theta)d\theta + F\varphi(t) \\
&= \int_0^t \big[d\mu(\theta) + \eta(\theta)d\theta\big]x(t-\theta) + F\varphi(t),
\end{aligned}$$

where $F\varphi(t)$, for $t > 0$, given in formula (1.29), is the part that explicitly contains the initial data. However, since $F\varphi(0) = \varphi(0)$ and μ is continuous at $\theta = 0$, equality (1.28) holds for $t = 0$ too. An easy evaluation shows that $F\varphi(t)$ is constant for $t \geqslant r$.

To prove the statement about the continuity of $F\varphi + \mu \cdot \varphi(0)$, we observe that the only term that is not continuous in t in the right hand side of the definition of F, in equation (1.29), is the second one. We extend $\varphi : [-r, 0]$ on $[-r, \infty)$ by defining $\varphi(\theta) = \varphi(0)$ for $\theta \geqslant 0$ and we observe that "shifting in time" is a continuous operation on the space of continuous functions. So the map

$$\begin{aligned} t \mapsto \int_0^r d\mu(\theta)\varphi_t(-\theta) &= \int_0^t d\mu(\theta)\varphi(t-\theta) + \int_t^r d\mu(\theta)\varphi(t-\theta) \\ &= \int_0^t d\mu(\theta)\varphi(0) + \int_t^r d\mu(\theta)\varphi(t-\theta) \\ &= \mu(t) \cdot \varphi(0) + \int_t^r d\mu(\theta)\varphi(t-\theta) \end{aligned}$$

is continuous too. Then the second term in the right hand side of the definition of F in (1.29) is

$$\int_t^r d\mu(\theta)\varphi(t-\theta) = \int_0^r d\mu(\theta)\varphi_t(-\theta) - \mu(t) \cdot \varphi(0).$$

It follows that $F\varphi + \mu \cdot \varphi(0)$ is continuous. □

From Lemma 1.12 it follows that equation (1.28) can be written in the form

$$x = dk * x + F\varphi, \tag{1.33}$$

where k is given by

$$k(\theta) = \mu(\theta) + \int_0^t \eta(\theta)d\theta. \tag{1.34}$$

We use Theorem 1.8 with $\alpha = k$, given in Formula (1.34), to obtain the (unique) resolvent kernel ζ that satisfies

$$\zeta = \zeta * dk + k = d\zeta * k + k$$

on $[0, \infty)$. By applying the convolution with the measure $d\zeta$ from the left in both sides of (1.33), we have that

$$\begin{aligned} d\zeta * x &= d\zeta * (dk * x) + d\zeta * F\varphi \\ &= d(\zeta * dk) * x + d\zeta * F\varphi \\ &= d(\zeta - k) * x + d\zeta * F\varphi \\ &= d\zeta * x - dk * x + d\zeta * F\varphi, \end{aligned}$$

Therefore $dk * x = d\zeta * F\varphi$, and we use it together with (1.33) to get the explicit representation for the solution x of (1.33) on $[0, \infty)$

$$x = d\zeta * F\varphi + F\varphi. \tag{1.35}$$

Alternatively, we can express FDE (1.3) as a renewal equation of the type discussed in Corollary 1.9, that is, both solution and forcing function are continuous and vanish at 0. This is done by transforming renewal equation (1.28) into

$$x - \varphi(0) = dk * [x - \varphi(0)] + F_0(\varphi) \tag{1.36}$$

with $F_0(\varphi)$ given by

$$F_0(\varphi) \stackrel{\text{def}}{=} F\varphi - \varphi(0) + k \cdot \varphi(0) \tag{1.37}$$

as in the following sequence of equivalent equations

$$\begin{aligned} x &= dk * x + F\varphi \\ x - \varphi(0) &= dk * x + F\varphi - \varphi(0) \\ x - \varphi(0) &= dk * [x - \varphi(0)] + F\varphi - \varphi(0) + dk * \varphi(0) \\ x - \varphi(0) &= dk * [x - \varphi(0)] + \underbrace{F\varphi - \varphi(0) + k \cdot \varphi(0)}_{\stackrel{\text{def}}{=} F_0(\varphi)} \end{aligned}$$

From the properties of $F\varphi$ from Lemma 1.12, we have that $F_0(\varphi)(\cdot)$ is continuous on $[0, \infty)$ and $F_0(\varphi)(0) = 0$. Using the explicit form of the solution, given by Corollary 1.9, we have

$$x - \varphi(0) = d\zeta * F_0(\varphi) + F_0(\varphi) \tag{1.38}$$

as an alternative representation of FDE (1.3) as a renewal equation in terms of the space of $\mathbb{C}^n$-valued continuous functions on $[0, \infty)$ which vanish at zero.

1.5 The space of forcing functions

The representation of FDE as renewal equations, as in (1.33) and (1.35) allows us to describe the space of forcing functions.

We have seen in item 4 of Remark 1.1 that for $\alpha \in NBV\big([0, r], \mathbb{C}^{n\times n}\big)$, the maps $f \mapsto d\alpha * f$ and $f \mapsto d\alpha * f$ map the space $\{f \in \mathcal{C}([0, r], \mathbb{C}^n) : f(0) = 0\}$ into itself, but the space $\mathcal{C}$ is not necessarily mapped into itself by these maps. This implies (cf. Theorem 1.3) that the one-to-one correspondence between the forcing function and the solution of a renewal equation is "closed" in the space $\{f \in \mathcal{C}([0, r], \mathbb{C}^n) : f(0) = 0\}$. We have that the same statements hold for the space $\{f \in \mathcal{C}([0, \infty), \mathbb{C}^n) : f(0) = 0\}$. In contrast, for retarded FDE, *i.e.*, when $\mu \equiv 0$, we find in the literature (see, for instance, Hale & Verduyn Lunel [27]) that $\{f \in \mathcal{C}([0,\infty),\mathbb{C}^n) : f(t) = f(r) \text{ for } t \geqslant r\}$ is a common choice for the space of forcing functions of the representation of this class of equations as renewal equations. The situation is simpler since it is not necessary to employ Laplace-Stieltjes transforms and convolutions between measures and functions, but it suffices the use of "standard" Laplace transforms and convolutions between functions.

After these considerations we choose to defined as the space $\mathcal{F}$ of forcing functions as follows.

Definition 1.13. We define the space $\mathcal{F}$ of forcing functions f of renewal equation

$$x = dk * x + f \tag{1.39}$$

by the set of $\mathbb{C}^n$-valued functions f on $[0,\infty)$ such that there exists a function $g \in \mathcal{C}([0,\infty),\mathbb{C}^n)$ such that

$$f = g - \mu \cdot g(0)$$

and $f(t) = f(r)$ for $t \geqslant r$. Obviously, for each $\varphi \in \mathcal{C}$, we have that $F\varphi \in \mathcal{F}$.

If $f \in \mathcal{F}$, then g is determined uniquely. This can be seen by observing that $f(0) = g(0)$, since $\mu(0) = 0$. Then

$$g = f + \mu \cdot f(0).$$

We have that $\mathcal{F}$ is a linear space and we can define a norm $\|\cdot\|_{\mathcal{F}}$ on $\mathcal{F}$ by

$$\|f\|_{\mathcal{F}} = \|g\|,$$

where $\|g\|$ is the sup-norm of g, such that the space is complete. A computation similar to the one presented to justify (1.36) shows that for each $f \in \mathcal{F}$, there is a unique solution x defined on $[0,\infty)$ of the renewal equation (1.39) such that x is continuous on $[0,\infty)$ and $x(0) = f(0) = g(0)$, for that $g \in \mathcal{C}([0,\infty),\mathbb{C}^n)$ such that $f = g + \mu \cdot g(0)$.

In the case of retarded FDE, we have that the space defined here $\mathcal{F}$ coincides with the set $\{f \in \mathcal{C}([0,\infty),\mathbb{C}^n) : f(t) = f(r) \text{ for } t \geqslant r\}$, which is the space of forcing functions used by Hale & Verduyn Lunel [27] for retarded FDE.

1.6 Solving FDE using Laplace transformation

In Section 1.4 we have seen that the renewal equation

$$x = dk * x + f, \tag{1.40}$$

where f is a *NBV* function, is related to FDE (1.3) by Lemma 1.12 and equations (1.33) and (1.36). The solution is given by

$$x = d\zeta * f + f, \tag{1.41}$$

where $d\zeta$ is the resolvent kernel of dk. (Moreover, ζ is a *NBV* function.) From Corollary 1.9 it follows that x is locally of bounded variation. From the representation of the solution (1.41), it is easy to see that x is exponentially bounded. Therefore we can apply Laplace transformation to both sides of (1.40) to obtain, for $\operatorname{Re} z$ sufficiently large, the algebraic equation

$$\widehat{x} = \widehat{dk}\widehat{x} + \widehat{f} \tag{1.42}$$

which can be solved, using (1.25), in order to obtain an explicit expression for $\widehat{x}$

$$\begin{aligned} \widehat{x}(z) &= (I - \widehat{dk}(z))^{-1}\widehat{f}(z) \\ &= \big(zI - z\widehat{dk}(z)\big)^{-1} z\widehat{f}(z) \\ &= \Delta(z)^{-1}\big(f(0) + \widehat{df}(z)\big), \end{aligned} \tag{1.43}$$

where $\Delta(z)$ is defined by (cf. equation (2.4))

$$\begin{aligned} \Delta(z) &= z(I - \widehat{dk}(z)) \\ &= \big(zI - z\widehat{d\mu}(z) - z\widehat{\eta}(z)\big) = \big(zI - z\widehat{d\mu}(z) - \widehat{d\eta}(z)\big) \\ &= z\Big[I - \int_0^r d\mu(t)e^{-zt}\Big] - \int_0^r d\eta(t)e^{-zt}. \end{aligned} \tag{1.44}$$

Since both $z \mapsto \Delta(z)$ and $z \mapsto \int_0^r e^{-zt} df(z)$ are entire functions, the right hand side of (1.43) is a meromorphic function with, possibly, poles at the roots of the *characteristic equation*

$$\det \Delta(z) = 0. \tag{1.45}$$

We show that there is a right half plane free of roots of $\det \Delta(z)$.

Lemma 1.14. *The roots of (1.45) are located in a left half plane $\{z \in \mathbb{C} : \operatorname{Re} z < \gamma_0\}$ for some $\gamma_0 \in \mathbb{R}$.*

Proof. First we show that, as $\operatorname{Re} z \to \infty$

$$\int_0^r d\mu(\theta)e^{-z\theta} \to 0 \qquad \text{and} \qquad \int_0^r d\eta(\theta)e^{-z\theta} \to \eta(0-). \tag{1.46}$$

Let $\epsilon > 0$ be given. Since μ is continuous at zero, there exists $\delta > 0$ such that the variation of μ on $[0, \delta]$ is less then $\epsilon/2$. Let $N > 0$ such that $e^{-\delta N} \operatorname{Var}_{[0,r]} \mu < \epsilon/2$.

Then, for z such that $\operatorname{Re} z > N$ we estimate

$$\begin{aligned}\left\|\int_0^r d\mu(\theta)e^{-z\theta}\right\| &= \left\|\int_0^\delta d\mu(\theta)e^{-z\theta} + \int_\delta^r d\mu(\theta)e^{-z\theta}\right\| \\ &\leqslant \left\|\int_0^\delta d\mu(\theta)e^{-z\theta}\right\| + \left\|\int_\delta^r d\mu(\theta)e^{-z\theta}\right\| \\ &\leqslant \max_{\theta\in[0,\delta]} |e^{-z\theta}| \operatorname{Var}_{[0,\delta]} \mu + \max_{\theta\in[\delta,r]} |e^{-z\theta}| \operatorname{Var}_{[\delta,r]} \mu \\ &\leqslant \operatorname{Var}_{[0,\delta]} \mu + e^{-\delta N} \operatorname{Var}_{[\delta,r]} \mu < \frac{\epsilon}{2} + \frac{\epsilon}{2} < \epsilon.\end{aligned}$$

We can conclude (1.46) arguing similarly for $\bar{\eta} \stackrel{\text{def}}{=} \eta - \eta(0-)$, which is continuous at zero. Now we observe that when $\operatorname{Re} z$ is sufficiently large, $\Delta(z) = z\big(I - \int_0^r d\mu(\theta)e^{-z\theta}\big) - \int_0^r d\eta(\theta)e^{-z\theta}$ is close to $zI - \eta(0-)$ and consequently it is nonsingular. □

We can invert the representation of the Laplace transform of the solution to obtain a characterization for the solution.

Theorem 1.15. *Let k be given by $k(\theta) = \mu(\theta) + \int_0^\theta \eta(s)ds$, where μ and η are NBV functions and μ is continuous at $\theta = 0$. For $f : \mathbb{R}_+ \to \mathbb{C}^n$ continuous, of bounded variation and constant on $[r, \infty)$, the solution x of the renewal equation*

$$x = dk * x + f \tag{1.47}$$

admits for $t > 0$ the representation

$$x(t) = \frac{1}{2\pi i}\int_{\daleth(\gamma)} e^{zt}\Delta(z)^{-1}\left(f(0) + \int_0^r e^{-z\theta}df(\theta)\right)dz \tag{1.48}$$

for γ sufficiently large.

Proof. By Lemma 1.14 we have that $\Delta(z)$ is non-singular for $\operatorname{Re} z$ sufficiently large. Since x is continuous, locally of bounded variation and exponentially bounded, we obtain (1.48) for γ sufficiently large by combining (1.26) with (1.43). □

From representation (1.48) for the solution of (1.47), we can obtain the large time behaviour of solutions, but first we need estimates on quantities related to $\Delta(z)$.

1.7 Estimates for $\Delta(z)$ and related quantities

From representation (1.48) for the solution of (1.47), we can obtain the large time behaviour of solutions, but some preparations are needed. This section contains several results concerning location of the roots and estimates on the characteristic matrix $\Delta(z)$, given in (1.44), its determinant and its inverse.

Set

$$\Delta(z) = z\Delta_0(z) - \int_0^r d\eta(\theta)e^{-z\theta} \tag{1.49}$$

where

$$\Delta_0(z) = I - \int_0^r d\mu(\theta)e^{-z\theta}. \tag{1.50}$$

It is possible that there are infinitely many roots of the characteristic equation (1.45) in a vertical strip. For example, the characteristic equation of the scalar functional differential equation

$$\dot{x}(t) - \dot{x}(t-1) = 0$$

is given by

$$\Delta(z) = z(1 - e^{-zt})$$

which has as all its roots of the form $2k\pi i$ with $k \in \mathbb{Z}$. This phenomenon is an added complication for neutral equations which is *not* observed in retarded equations (i.e., when $\mu \equiv 0$). See for instance Theorem I.4.4 of Diekmann *et al.* [15].

In order to control the behaviour of $|\Delta(z)|$ as $|z| \to \infty$, we make the following assumption on the kernel μ:

(J) The entries μ_{ij} of μ have a jump before they become constant, that is, there exists t_{ij} with $\mu_{ij}(t_{ij}-) \neq \mu_{ij}(t_{ij}+)$ and $\mu_{ij}(t_{ij}+) = \mu_{ij}(t)$ for $t > t_{ij}$.

For example, μ can be a step function. In that case

$$\Delta_0(z) = I - \sum_{j=1}^{\infty} e^{-zr_j} A_j$$

and $\det \Delta_0(\cdot)$ is an almost-periodic function. The jump condition (J) is more general and implies that $\det \Delta_0$ is asymptotically almost-periodic. We define

$$\mathbb{C}_{\gamma_1,\gamma_2} = \{z \in \mathbb{C} : \gamma_1 < \operatorname{Re} z < \gamma_2\}.$$

Lemma 1.16. *If μ satisfies (J), then the zeros of $\det \Delta_0(z)$ are locate in a finite strip $\mathbb{C}_{\alpha_0,\omega_0}$. For z in the strip $\mathbb{C}_{\omega_0,\infty}$, there are positive constants m and M such that*

$$m|e^{-zr}| \leqslant |\det \Delta_0(z)| \leqslant M|e^{zr}| \tag{1.51}$$

For any $\epsilon > 0$, for a suitable choice of m, estimate (1.51) holds for $z \in \mathbb{C}_{\alpha_0,\omega_0}$ outside circles of radius ϵ centered in the zeros of $\det \Delta_0(\cdot)$.

Proof. For any function $\alpha \in NBV\big([0,r], \mathbb{C}^{n\times n}\big)$ we see that

$$\int_0^r e^{-z\theta} d\alpha(\theta) = \int_0^\infty e^{-z\theta} d\alpha(\theta) = \widehat{d\alpha}(z)$$

Since the determinant of Δ_0 is a product of $\widehat{d\mu_{ij}}(z)$, that equals to the Laplace-Stieltjes transform of convolutions of $d\mu_{ij}$, there is a *NBV* function $\tilde{\mu}$ such that

$$\det \Delta_0(z) = \int_0^r e^{-zs} d\tilde{\mu}(s).$$

If μ satisfies (J), then $\tilde{\mu}$ has jumps at 0 and r. An application of Verduyn Lunel [55], Theorem 4.6, yields the lemma. $\square$

The next lemma describes the location of the zeros of $\det \Delta(\cdot)$ in the finite vertical strip where the roots of $\det \Delta_0(\cdot)$ are located.

Theorem 1.17. *Suppose that μ satisfies (J) and let $\mathbb{C}_{\alpha_0,\omega_0}$ as in Lemma 1.16. For any $\delta > 0$, there exists K such that for any zero ζ_0 of $\det \Delta_0(\cdot)$ with $|\zeta_0| > K$, there is a zero ζ of $\det \Delta(\cdot)$ with $|\zeta_0 - \zeta| < \delta/2$. Furthermore, there exist positive constants m and M such that*

$$m \leqslant |\det\big(\tfrac{1}{z}\Delta(z)\big)| \leqslant M \tag{1.52}$$

for $z \in \mathbb{C}_{\alpha_0,\omega_0}$ with $|z| > K$ and outside circles of radius ϵ centered in the zeros of $\det \Delta(\cdot)$.

Proof. Let $K_1 = \max\{1, \inf\{|z| : z \in \mathbb{C}_{\alpha_0,\omega_0}\}\}$. Then, when $|z| > K_1$, $\det \Delta(z) = 0$ if and only if $\det(\frac{1}{z}\Delta(z)) = 0$, and

$$\frac{1}{z}\Delta(z) = \Delta_0(z) + \frac{1}{z}\int_0^r d\eta(\theta) e^{-z\theta}.$$

From Lemma 1.16, there exists m_1 such that

$$|\det \Delta_0(z)| \geqslant m_1 |e^{-zr}| > m_1 \max\{e^{-\alpha_0 r}, e^{-\omega_0 r}\} \stackrel{\text{def}}{=} C_1$$

for z outside (and consequently over) circles of radius δ centered in the roots of $\det \Delta_0(z)$. We can estimate

$$\frac{1}{z}\bigg\|\int_0^r d\eta(\theta) e^{-z\theta}\bigg\| < \int_0^r d|\eta|(\theta)|e^{-z\theta}| \leqslant \frac{r}{z} \operatorname{Var}_{[0,r]} \eta \max\{e^{-\alpha_0 - r}, e^{-\omega_0}\} \stackrel{\text{def}}{=} \frac{C_2}{z},$$

where $d|\eta|$ denotes the total variation measure of $d\eta$. Therefore, since the determinant is continuous, it follows that there exists $K_2 > K_1$ such that for $|z| > K_2$

$$|\det(\tfrac{1}{z}\Delta(z)) - \det \Delta_0(z)| < \frac{C_1}{2}$$

Now Rouché Theorem (see for instance Rudin [49]) implies that $\det(\frac{1}{z}\Delta(\cdot))$ and $\det\Delta_0(\cdot)$ have the same number of roots inside circles of radius δ centered in the roots of $\det\Delta_0(\cdot)$. By applying the triangular inequality, it is possible to choose m and M to satisfy estimates (1.52). $\square$

Lemma 1.18. *For any $f \in NBV([0,r],\mathbb{C})$ and any constants C_0 and C_1 satisfying $0 < C_0 \leqslant C_1$, the estimate*

$$|\widehat{df}(z)| < \frac{|z|}{C_0}\operatorname{Var}_{[0,r]} f \tag{1.53}$$

holds for $z \in \mathbb{C}$ such that $|z| > C_0|e^{-zr}|$ and $|z| > C$.

Proof. For $0 \leqslant \theta \leqslant r$

$$|e^{-zt}| = e^{-t\operatorname{Re} z} \leqslant \max\{1, e^{-r\operatorname{Re} z}\}.$$

Hence

$$|\widehat{df}(z)| = \left|\int_0^r e^{-z\theta} df(\theta)\right| \leqslant \max\{1, e^{-r\operatorname{Re} z}\}\operatorname{Var}_{[0,r]} f < \frac{|z|}{C_0}\operatorname{Var}_{[0,r]} f$$

provided that $|z| > C_0|e^{-zr}|$ and $|z| > C \geqslant C_0 > 0$. $\square$

Lemma 1.19. *For* $\det\Delta(z)$ *we have the representation*

$$\det\Delta(z) = \det\Delta_0(z)z^n + \sum_{m=0}^{n-1}\big(\widehat{d\xi_m}(z)\big)z^m \tag{1.54}$$

where $d\xi_m$ are certain convolutions among μ_{ij} and η_{ij} for some indexes i and j.

Proof. First, we observe that if A and B are two $n\times n$-matrices, then $\det(zA+wB)$ is a n^{th} order polynomial in z and w. Setting $w = 0$ we see that the coefficient with the term z^n is $\det A$.

The matrix $\Delta(z)$ has as entries elements the sum of z or 0 (depending if entry is in the diagonal) with elements of form $z\widehat{\mu}_{ij}(z)$ and $\widehat{\eta}_{ij}$, where μ_{ij} and η_{ij} are the entries of the matrix-valued functions μ and η. Therefore the determinant of $\Delta(z)$ consists of sums of products of such elements. Grouping elements with the same power of z, it turns out that the coefficients are sums of convolutions among μ_{ij} and η_{ij} for some indexes i and j. Since $\Delta(z)$ has the form (1.49), we conclude representation (1.54) for the determinant of $\Delta(z)$. $\square$

Definition 1.20. Let a_M be defined by

$$a_M = \inf\{\lambda \in \mathbb{R} : \#_{\Delta_0}(\lambda) < \infty\} \tag{1.55}$$

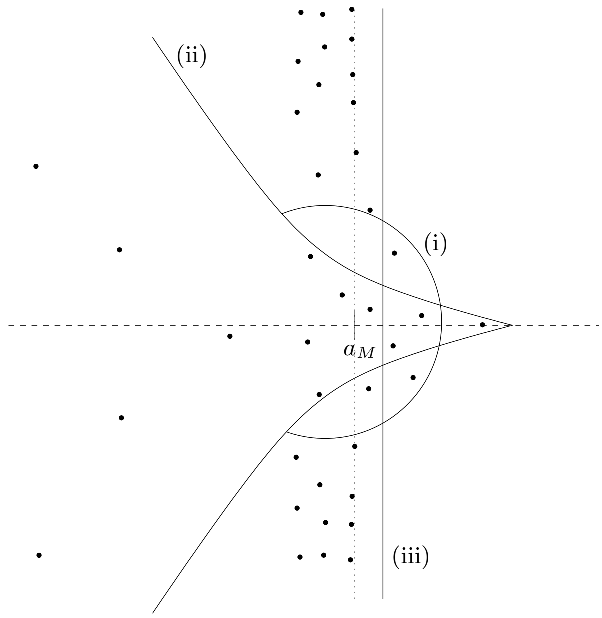

Figure 1.1: Distribution of zeros of $\det\Delta(z)$ in the complex plane; the region in the right, bounded by the curves (i) $|z| = C$, (ii) $|z| = C_0|e^{-zr}|$ and (iii) $\operatorname{Re} z = a_M + \epsilon$ (see Theorem 1.21), is free of zeros and estimate (1.56) holds. There is a vertical strip $\mathbb{C}_{\alpha_0,\omega_0}$ where all zeros of $\det\Delta_0(z)$ are, and in this region, for $|z|$ sufficiently large, the zeros of $\det\Delta(z)$ are close of the zeros of $\det\Delta_0(z)$ in a one-to-one correspondence.

⋆ ⋆ ⋆

where $\#_{\Delta_0}(\lambda)$ is the number of zeros of $\det\Delta_0(z)$ on $\mathbb{C}_{\lambda,\infty}$, that is, for each $\epsilon > 0$, there is a finite number of zeros ζ of $\det\Delta_0(z)$ such that $\operatorname{Re}\zeta > a_M + \epsilon$. An application of Theorem 1.17 (using also the fact that if all roots of $\det\Delta_0(z)$ are in $\mathbb{C}_{\alpha_0,\omega_0}$, then the same holds for on $\mathbb{C}_{\alpha_0,\omega}$ for any $\omega > \omega_0$) implies that we can substitute $\#_{\Delta_0}$ by $\#_{\Delta}$ in (1.55), where $\#_{\Delta}(\lambda)$ is defined as the number of zeros of $\det\Delta(z)$ on $\mathbb{C}_{\lambda,\infty}$.

Theorem 1.21. *For any $\epsilon > 0$, there exist positive constants q, C_0 and C such that*

$$|\det\Delta(z)| \geqslant q|z|^n \tag{1.56}$$

for those $z \in \mathbb{C}$ for which $|z| > C_0|e^{-zr}|$, $|z| > C$ and $\operatorname{Re} z > a_M + \epsilon$.

Proof. From Theorem 1.17, it follows that for there is no w with $\operatorname{Re} w > a_M + \epsilon$ and $|w|$ sufficiently large such that $\det\Delta(w) = 0$. From Lemma 1.16, there is q

such that $\det \Delta_0(z) \geqslant 2q$ for $z \in \mathbb{C}$ satisfying $\operatorname{Re} z > a_M + \epsilon$. From Lemma 1.19, we deduce that

$$|\det \Delta(z)| \geqslant \left| |\det \Delta_0(z)||z|^n - \sum_{m=0}^{n-1} \left|\widehat{d\xi}_m(z)\right||z|^m \right|$$

So, Lemma 1.18 yields, with $\varrho(C) = C^{-1} \sum_{m=0}^{n-1} \frac{\operatorname{Var} \xi_m}{C_0}$, that

$$|\det \Delta(z)| \geqslant |z|^n(\det \Delta_0(z) - \varrho(C)) > |z|^n(2q - \varrho(C)) > q|z|^n$$

if we choose C large enough such that $\varrho(C) < q$. □

The following lemma and its corollary will be useful later when computing contour integrals.

Lemma 1.22. *Let $f \in NBV\big([0,r], \mathbb{C}^{n\times n}\big)$. Then*

$$\lim_{\omega\to\pm\infty} e^{(s+i\omega)t}\Delta(s+i\omega)^{-1}\left(f(0) + \int_0^r e^{-(s+i\omega)\theta} df(\theta)\right) = 0 \tag{1.57}$$

uniformly for $a_M < \gamma_1 \leqslant s \leqslant \gamma_2$ and t on compact sets.

Proof. Given $\gamma_1 > a_M$, let $\epsilon \overset{\text{def}}{=} (\gamma_1 - a_M)/2$. Theorem 1.21 yields the existence of a positive constant q such that for $|z|$ sufficiently large we have

$$|\det \Delta(z)| \geqslant q|z|^n.$$

Let $\operatorname{adj} \Delta(z)$ denote the matrix of cofactors of $\Delta(z)$, that is, the elements of $\operatorname{adj} \Delta(z)$ are the $(n-1)\times(n-1)$ subdeterminants of $\Delta(z)$. Lemma 1.54 tells us that

$$|(\operatorname{adj} \Delta(z))_{ij}| \leqslant K|z|^{n-1}$$

for some constant K_0 and $z = s \pm \omega i$ and $\gamma_1 \leqslant s \leqslant \gamma_2$ and ω large enough. Since

$$\Delta(z)^{-1} = \frac{1}{\det \Delta(z)} \operatorname{adj} \Delta(z),$$

we obtain that

$$\left|(\Delta(z)^{-1})_{ij}\right| \leqslant \frac{K_0}{q}\frac{1}{|z|}$$

for z as before. Both e^{zt} and $\int_0^r e^{-z\theta} df(\theta)$ are uniformly bounded for z as before and t in a given compact set. Hence there exists a constant K_1 such that

$$\left| e^{zt}\Delta(z)^{-1}\left(f(0) + \int_0^r e^{-z\theta} df(\theta)\right)\right| \leqslant \frac{K_1}{|z|}$$

for those values of z and t with ω sufficiently large. □

Corollary 1.23. *Let $\beth_N(\gamma_1,\gamma_2)$, $N \in \mathbb{R}$ and $a_M < \gamma_1 \leqslant \gamma_2$, to be the contour in the complex plane defined by the (horizontal) straight segment connecting $N + i\gamma_1$ to $N + i\gamma_2$. Then, for $t > 0$,*

$$\lim_{N\to\pm\infty} \int_{\beth_N(\gamma_1,\gamma_2)} e^{zt}\Delta(z)^{-1}\Big(f(0) + \int_0^r e^{-z\theta} df(\theta)\Big)dz = 0.$$

1.8 Asymptotic behaviour for $t \to \infty$

In this section we obtain the asymptotic behaviour of the solution of the renewal equation

$$x = dk * x + f,$$

where $k(\theta) = \mu(\theta) + \int_0^\theta \eta(s)ds$ and μ satisfies the condition (J) (see page 27). Results in Section 1.7 and the representation for the solution of renewal equations, given in Theorem 1.15, provide the necessary ingredients for us to present the large time behaviour of solutions.

From the inversion formula (1.26) it follows that the value of the complex integral in (1.48) is independent of the choice of $\gamma > \gamma_0$ for some γ_0 sufficiently large. We shall prove this directly in order to demonstrate how to compute complex line integrals that will be used repeatedly in the sequel. We shall denote

$$Q(z) = \Delta(z)^{-1}\Big(f(0) + \int_0^r e^{-z\theta} df(\theta)\Big). \tag{1.58}$$

We define $\daleth_N(\gamma_1,\gamma_2)$ to be the closed positively-oriented contour in the complex plane consisting of four straight line segments through the vertexes $\gamma_1 - iN$, $\gamma_1 + iN$, $\gamma_2 - iN$ and $\gamma_2 + iN$. Since Q is analytic in the half-plane $\operatorname{Re} z > \sigma_0$, the Cauchy Theorem (see Rudin [49]) tells us that

$$\frac{1}{2\pi i}\int_{\daleth_N(\gamma_1,\gamma_2)} e^{zt}Q(z)dz = 0 \tag{1.59}$$

for $\sigma_0 < \gamma_1 < \gamma_2$. By taking the limit when $N \to \infty$ and using the limit in Corollary 1.23, for $a_M < \gamma_1 \leqslant \gamma_2$, we have

$$\lim_{N\to\pm\infty} \int_{\gamma_1+iN}^{\gamma_2+iN} e^{zt}Q(z)dz = 0, \tag{1.60}$$

and hence we can draw the conclusion that

$$\frac{1}{2\pi i}\int_{\beth(\gamma_1)} e^{zt}Q(z)dz = \frac{1}{2\pi i}\int_{\beth(\gamma_2)} e^{zt}Q(z)dz. \tag{1.61}$$

This discussion shows that we can take γ_0 as the infimum of those γ such that the half-plane $\operatorname{Re} z > \gamma$ is free of roots of $\det\Delta(z)$. However, estimate (1.60) is valid for $a_M < \gamma_1 \leqslant \gamma_2$ and obviously $\gamma_0 \geqslant a_M$. Thus we can move the vertical line $\operatorname{Re} z = \gamma_1$ of integration in (1.61) for $a_M < \gamma_1 < \gamma_0$, but the result of the contour integral (1.59), according to Cauchy Theorem, for γ_1 such that $\operatorname{Re} z = \gamma_1$ is free of zeros of $\det \Delta(z)$, would become

$$\frac{1}{2\pi i} \int_{\neg_N(\gamma_1,\gamma_2)} e^{zt} Q(z) dz = \sum_{j=1}^{m} \operatorname*{Res}_{z=\lambda_m} e^{zt} Q(z),$$

where λ_j, $1 \leqslant j \leqslant m$ with $\lambda_j \neq \lambda_l$ if $j \neq l$, are all the roots of $\det \Delta(z)$ such that $\operatorname{Re} z > \gamma_1$. Therefore,

$$\frac{1}{2\pi i} \int_{\daleth(\gamma)} e^{zt} Q(z) dz = \sum_{j=1}^{m} \operatorname*{Res}_{z=\lambda_m} Q(z) + \frac{1}{2\pi i} \int_{\daleth(\gamma_1)} e^{zt} Q(z) dz \tag{1.62}$$

for $\gamma > \gamma_0$. This discussion motivates the next couple of results. There Theorem 1.27, the main result of this chapter, concerns the large time behaviour for the solutions of equation (1.47).

The next lemma is a classical result. A proof can be found in Hewitt & Stromberg [32], Theorem 21.39.

Lemma 1.24 (Riemann-Lebesgue). *If f belongs to $L^1(\mathbb{R}_+)$ then*

$$\lim_{\omega \to \pm\infty} \left| \int_0^\infty e^{i\omega t} f(t) dt \right| = 0.$$

Lemma 1.25. *For $\gamma > a_M$, we have that*

$$\frac{1}{2\pi i} \int_{\daleth(\gamma)} e^{zt} Q(z) dz = o(e^{\gamma t}) \qquad \text{for } t \to \infty. \tag{1.63}$$

Proof. We have to prove that

$$\lim_{t \to \infty} \left(\lim_{N \to \infty} \int_{-N}^{N} e^{it\omega} Q(\gamma + i\omega) d\omega \right) = 0. \tag{1.64}$$

In the proof of Lemma 1.22 we showed that, for $|\omega|$ sufficiently large,

$$|Q(\gamma + i\omega)| \leqslant \frac{K_1}{|\gamma + i\omega|}$$

but this does not guarantee that $Q(\gamma + i\omega)$ is a L^1-function in ω (in other words, the integral above does not necessarily converge absolutely) and, hence, we cannot apply the Riemann-Lebesgue Lemma 1.24 directly. For every fixed N, however, the Riemann-Lebesgue Lemma tells us that

$$\lim_{t \to \infty} \int_{-N}^{N} e^{it\omega} Q(\gamma + i\omega) d\omega = 0. \tag{1.65}$$

So, if we prove that the limits $t \to \infty$ and $N \to \infty$ are interchangeable, at least for some terms of the integral (1.65), we obtain the desired conclusion for those terms. Therefore it suffices to show that the convergence for $N \to \infty$ is uniform for $t \geqslant t_0$ for some fixed value t_0.

For $f \in NBV([0,r],\mathbb{C})$ the integral

$$\int_0^r e^{-z\theta} df(\theta)$$

is uniformly bounded for $z = \gamma + i\omega$ with $\omega \in \mathbb{R}$. Since

$$\Delta(z) = z\Delta_0(z) - \int_0^r e^{-z\theta} d\eta(\theta),$$

it follows that for $z = \gamma + i\omega$ (cf. Lemma 1.19)

$$\det \Delta(z) = z^n \det \Delta_0(z) + O(|\omega|^{n-1}) \quad \text{as } |\omega| \to \infty$$

and

$$\operatorname{adj} \Delta(z) = z^{n-1} \operatorname{adj} \Delta_0(z) + O(|\omega|^{n-2}) \quad \text{as } |\omega| \to \infty.$$

Hence, for such z

$$Q(z) = \frac{1}{z}\Delta_0(z)^{-1}\left(f(0) + \int_0^r e^{-z\theta} df(\theta)\right) + O(|\omega|^{-2}). \tag{1.66}$$

For the $O(|\omega|^{-2})$ term we can estimate $|e^{it\omega}|$ by one and uniformity in t follows (or, in other words, the contribution of this term to the integral converges absolutely). So we can concentrate on the first term.

We observe that an application of Lemma 1.16 yields that for fixed $\gamma > a_M$, there exists a $\epsilon_\gamma > 0$ such that

$$|\det \Delta_0(z)| > \epsilon_\gamma \tag{1.67}$$

for $z \in \mathbb{C}$ such that $\operatorname{Re} z \geqslant \gamma$, since $\Delta_0(z) \to I$ uniformly as $\operatorname{Re} z \to \infty$, what implies that $\det \Delta_0(z) \to 1$ as $\operatorname{Re} z \to \infty$ uniformly, so we obtain a lower bound in a vertical strip $\mathbb{C}_{[N,\infty)}$ for N sufficiently large, and from Lemma 1.16 we obtain lower bounds for the finite vertical strip $\mathbb{C}_{[\gamma,N]}$.

We claim that $\Delta_0(z)^{-1}$ is the Laplace-Stieltjes transform of some NBV function, *i. e.*, there exists a $\zeta \in NBV([0,r],\mathbb{C})$ such that $\Delta_0(z)^{-1} = \widehat{d\zeta}(z)$. To show this, first observe that $|\det \Delta_0(z)|$ is bounded away from zero, implying that $\Delta_0(z)$ is non-singular for z on the line $\operatorname{Re} z = \gamma$. Now consider the resolvent $d\rho$ of the measure $-d\mu$. We have the following sequence of equivalent equalities (for

$\operatorname{Re} z > a_M$)

$$\begin{gathered}
\rho + (-d\mu) * \rho = -\mu \\
d\rho + (-d\mu) * d\rho = d\rho + d\rho * (-\mu) = -d\mu \\
\widehat{d\rho}(z) - \widehat{d\mu}(z)\widehat{d\rho}(z) = -\widehat{d\mu}(z) \\
[I - \widehat{d\mu}(z)]\widehat{d\rho}(z) = -\widehat{d\mu}(z) \\
I + [I - \widehat{d\mu}(z)]\widehat{d\rho}(z) = I - \widehat{d\mu}(z) I - \widehat{d\rho}(z) = [I - \widehat{d\mu}(z)]^{-1} = \Delta_0(z)^{-1}
\end{gathered}$$

(Only in the last step we used that $I - \widehat{d\mu}(z) = \Delta_0(z)$ is non-singular.) Then, we have the formula

$$\Delta_0(z)^{-1} = I - \widehat{d\rho}(z)$$

and the claim follows. After these considerations, we can write

$$\Delta_0(z)^{-1}\big(f(0) + \int_0^r e^{-z\theta} df(\theta)\big)$$

as the Laplace-Stieltjes transform of a NBV function ζ. We also point out that

$$\begin{aligned}
&\lim_{\tau\to\infty}\lim_{N\to\infty}\int_{-N}^{N} \frac{e^{i\omega\tau}}{\gamma + i\omega} d\omega \\
&\quad = \lim_{\tau\to\infty}\lim_{N\to\infty}\left[\frac{e^{i\omega\tau}}{\tau i}\,\frac{1}{\psi + i\omega}\Big|_{\omega=-N}^{N} - \frac{1}{\tau}\int_{-N}^{N}\frac{e^{i\omega\tau}}{(\gamma + i\omega)^2} d\omega\right] \\
&\quad = \lim_{\tau\to\infty}\left[\cancelto{0}{\frac{e^{i\omega\tau}}{\tau i}\,\frac{1}{\psi + i\omega}\Big|_{\omega=-N}^{N}} - \frac{1}{\tau}\cancelto{\text{some value bounded uniformly in } \tau}{\int_{-N}^{N}\frac{e^{i\omega\tau}}{(\gamma + i\omega)^2} d\omega}\right] \\
&\quad = 0.
\end{aligned}$$

Finally, we compute the limit of the integral given in (1.64) of the first (remaining) term of $Q(z)$ in (1.66) to obtain

$$\begin{aligned}
&\lim_{t\to\infty}\lim_{N\to\infty}\frac{e^{i\omega t}}{\gamma + i\omega}\Delta_0(\gamma + i\omega)^{-1}\left(f(0) + \int_0^r e^{-(\gamma+i\omega)\theta} df(\theta)\right) \\
&\quad = \lim_{t\to\infty}\lim_{N\to\infty}\int_{-N}^{N}\frac{e^{i\omega t}}{\gamma + i\omega}\int_0^r e^{-(\gamma+i\omega)\theta} d\zeta(\theta) d\omega \\
&\quad = \lim_{t\to\infty}\lim_{N\to\infty}\int_0^r e^{\gamma\theta}\Big(\int_{-N}^{N}\frac{e^{i\omega(t-\theta)}}{\gamma + i\omega} d\omega\Big) \\
&\quad = \int_0^r e^{\gamma\theta}\Big(\lim_{t\to\infty}\lim_{N\to\infty}\int_{-N}^{N}\frac{e^{i\omega(t-\theta)}}{\gamma + i\omega} d\omega\Big) \\
&\quad = 0
\end{aligned}$$

and the proof is complete. $\square$

Lemma 1.26. *Let λ be a zero of* $\det \Delta(z)$ *of order m. Then*

$$\operatorname*{Res}_{z=\lambda} e^{zt} Q(z) = p(t) e^{\lambda t}, \tag{1.68}$$

where p is a $\mathbb{C}^n$-valued polynomial in t of degree less than or equal to $m-1$.

Proof. In a neighborhood of $z = \lambda$, we have the series expansions

$$\begin{aligned} \Delta(z)^{-1} &= \frac{1}{\det \Delta(z)} \operatorname{adj} \Delta(z) = \sum_{k=-m}^{\infty} (z-\lambda)^k A_k, \\ f(0) + \int_0^r e^{-z\theta} df(\theta) &= \sum_{k=0}^{\infty} (z-\lambda)^k v_k, \\ e^{zt} = e^{\lambda t} e^{(z-\lambda)t} &= e^{\lambda t} \sum_{k=0}^{\infty} \frac{t^k}{k!} (z-\lambda)^k. \end{aligned}$$

Since the residue in $z = \lambda$ equals the coefficient of the $(z-\lambda)^{-1}$-term of the Laurent expansion of $e^{zt}Q(z)$ in a neighborhood, a multiplication of the above series expansions yields the desired result. □

We are ready to formulate the main result of this chapter.

Theorem 1.27. *Let x to be a solution of the FDE given by (1.3)–(1.4), where μ satisfies condition (J)*[1]*, corresponding to a initial function φ. For $\gamma > a_M$ such that* $\det \Delta(z) \neq 0$ *for z on the line* $\operatorname{Re} z = \gamma$ *we have the asymptotic expansion*

$$x(t) = \sum_{j=1}^{l} p_j(t) e^{\lambda_j t} + o(e^{\gamma t}) \qquad \textit{for } t \to \infty, \tag{1.69}$$

where $\lambda_1, \ldots, \lambda_l$ are the finite many zeros of $\det \Delta(\lambda)$ *with real part exceeding γ and where $p_j(t)$ are $\mathbb{C}^n$-valued polynomials in t with degree $\leqslant m_j - 1$, where m_j is the multiplicity of λ_j as zero of* $\det \Delta(z)$.

Proof. The idea for the proof is given in the discussion prior Lemma 1.24. For σ sufficiently large, we obtain from Theorem 1.15 that

$$x(t) = \frac{1}{2\pi i} \int_{\beth(\sigma)} e^{zt} Q(z) dz$$

From equation (1.62) we have that

$$\frac{1}{2\pi i} \int_{\beth(\sigma)} e^{zt} Q(z) dz = \sum_{j=1}^{m} \operatorname*{Res}_{z=\lambda_m} e^{zt} Q(z) + \frac{1}{2\pi i} \int_{\beth(\gamma)} e^{zt} Q(z) dz.$$

[1] The jump condition (J) can be found in the page 27.

From Lemma 1.25 we have that

$$\frac{1}{2\pi i}\int_{\daleth(\gamma)} e^{zt}Q(z)dz = o(e^{\gamma t}).$$

From Lemma 1.26, we get the polynomial form for the residue

$$\operatorname*{Res}_{z=\lambda_j} e^{zt}Q(z) = p_j(t)e^{\lambda_j t}.$$

The combination of these results completes the proof. □

1.9 Comments

The approach taken was motivated by the first chapter of the book by Diekmann *et al.* [15]. The jump condition (J)[2] appeared in the book by Hale & Verduyn Lunel [27] which also contains results on non-autonomous neutral systems, qualitative changes in the properties of solutions when there is perturbation on the delays, and the concept of stable M operator. A classical article by Henry [30] discusses the difference equation

$$x(t) = \sum_{k=1}^{\infty} A_k x(t - w_k), \qquad t \geqslant 0, \tag{1.70}$$

where $0 < w_k \leqslant r$, $\sum A_k < \infty$ and $\sum_{w_k \leqslant \epsilon} |A_k| \to 0$ as $\epsilon \to 0_+$. The characteristic equation associated to (1.70) is given by

$$\det H(z) = 0 \tag{1.71}$$

where

$$H(z) = I - \sum_{k=1}^{\infty} A_k e^{-zw_k}.$$

It is shown that for fixed $\alpha < \beta$ there is a number N such that for any $t \in \mathbb{R}$ there are no more than N zeros of (1.71) in the set

$$\{z \in \mathbb{C} : \alpha \leqslant \operatorname{Re} z \leqslant \beta,\ t \leqslant \operatorname{Im} z \leqslant t+1\}.$$

This implies that if $\det H(z)$ is bounded from zero on the lines $\operatorname{Re} z = \alpha$ and $\operatorname{Re} z = \beta$, there is a sequence on rectangular contours C_j having the vertical sides on the lines $\operatorname{Re} z = \alpha$ and $\operatorname{Re} z = \beta$ and vertical sides on the lines $\operatorname{Im} z = l_j$ for some l_j with $j \leqslant l_j < j+1$ such that $\det H(z)$ is uniformly bounded from zero in these contours. With this result it is possible to sharpen conclusions of

[2] The jump condition (J) is found in page 27.

Theorem 1.27 for $\gamma \geqslant a_M - \epsilon$ for some $\epsilon > 0$. Note that the asymptotic expansion (1.69) becomes a countable sum.

In Definition 1.13, actually there was no reason, in the realm of renewal equations, to impose the condition that $f \in \mathcal{F}$ must be constant on $[t, \infty)$ in order to ensure a continuous solution of the renewal equation. However, in the context of the representation of FDE as renewal equation, there is no initial condition $\varphi \in \mathcal{C}$ such that $F\varphi$ is not constant in $[r, \infty)$.

Consider the characteristic equation

$$\det \Delta = 0 \tag{1.72}$$

where $\Delta(z)$ and $\Delta_0(z)$ are given by (1.49)–(1.50). In Definition 1.55, a_M is presented as the infimum of λ such that the number of roots of the characteristic equation (1.72) in the right-plane $\operatorname{Re} z > \lambda$ is finite. Define $\tilde{a}_M$ as

$$\tilde{a}_M = \sup \left\{ \operatorname{Re} z : \det \Delta_0(z) = 0 \right\}.$$

Obviously $a_M \leqslant \tilde{a}_M$. It is known that $a_M = \tilde{a}_M$ for the FDE given by (1.3)–(1.4) with the additional hypothesis that the singular part of μ is sufficiently small. However, as far as this author knows, it is an open problem whether there exist examples of FDE where $a_M < \tilde{a}_M$. This problem is equivalent to the existence of examples of difference equations like

$$x(t) = \int_{-r}^{0} d\mu(\tau) x(\tau)$$

with (non-empty) finite dimensional unstable manifold.

Chapter 2

Spectral theory of neutral systems

In this chapter we study properties of autonomous and periodic functional differential equations that can be obtained by looking at the spectrum of the generator of the solution semigroup. The spectral decomposition of $\mathcal{C}$ into invariant subspaces with estimates and a variation-of-constants formula are the main goals in this chapter.

2.1 Spectral theory for autonomous FDE

It is convenient to view the linear autonomous Functional Differential Equation

$$\begin{cases} \frac{d}{dt}Mx_t = Lx_t, & t \geqslant 0, \\ x_0 = \varphi, & \varphi \in \mathcal{C}, \end{cases} \tag{2.1}$$

where $L, M : \mathcal{C} \to \mathbb{C}^n$ are linear continuous, as an evolutionary system describing the evolution of the state x_t in the Banach space $\mathcal{C}$. More precisely, L and M are given by

$$L\varphi = \int_0^r d\eta(\theta)\varphi(-\theta), \qquad M\varphi = \varphi(0) - \int_0^r d\mu(\theta)\varphi(-\theta), \tag{2.2}$$

where $\eta, \mu \in NBV\big([0,r], \mathcal{L}(\mathbb{C}^n)\big)$ and μ is continuous at zero (to ensure well-poseness of the solution semigroup; see Chapter 1). In order to do so, we associate with (2.1) a semigroup of solution operators in $\mathcal{C}$. The semigroup is strongly continuous and given by translation along the solution of (2.1)

$$T(t)\varphi = x_t(\cdot;\varphi),$$

where $x(\cdot;\varphi)$ denotes the solution of (2.1). See Hale & Verduyn Lunel [27] for further details and more information. The infinitesimal generator A of the semigroup $T(t)$ is given by

$$\begin{cases} \mathcal{D}(A) = \{\varphi \in \mathcal{C} \mid \frac{d\varphi}{d\theta} \in \mathcal{C},\ M\frac{d\varphi}{d\theta} = L\varphi\} \\ \quad A\varphi = \frac{d\varphi}{d\theta}. \end{cases} \tag{2.3}$$

Let $\lambda \in \sigma(A)$ be an eigenvalue of A. The kernel $\mathcal{N}\big(\lambda I - A\big)$ is called the eigenspace at λ and its dimension d_λ, the *geometric multiplicity.* The generalized eigenspace $\mathcal{M}_\lambda$ is the smallest closed subspace that contains all $\mathcal{N}\big((\lambda I - A)^j\big)$, $j = 1, 2, \dots$ and its dimension m_λ is called the *algebraic multiplicity.* It is known that there is a close connection between the spectral properties of the infinitesimal generator A and the characteristic matrix $\Delta(z)$, associated with (1.3), given by

$$\Delta(z) = z\Big[I - \int_0^r d\mu(t)e^{-zt}\Big] - \int_0^r d\eta(t)e^{-zt}. \tag{2.4}$$

See Diekmann *et al.* [15] and Kaashoek & Verduyn Lunel [35]. In particular, the geometric multiplicity d_λ equals the dimension of the null space of $\Delta(z)$ at λ and the algebraic multiplicity m_λ is equal to the multiplicity of $z = \lambda$ as a zero of $\det\Delta(z)$. Furthermore, the generalized eigenspace at λ is given by

$$\mathcal{M}_\lambda = \mathcal{N}\big((\lambda I - A)^{k_\lambda}\big), \tag{2.5}$$

where k_λ is the order of $z = \lambda$ as a pole of $\Delta(z)^{-1}$. Using the matrix of cofactors $\operatorname{adj}\Delta(z)$ of $\Delta(z)$, we have the representation

$$\Delta(z)^{-1} = \frac{1}{\det\Delta(z)}\ \operatorname{adj}\Delta(z). \tag{2.6}$$

From representation (3.9), we immediately derive that the spectrum of A consists of point spectrum only, and is given by the zero set of an entire function

$$\sigma(A) = \big\{\lambda \in \mathbb{C} \ \big|\ \det\Delta(\lambda) = 0\big\}.$$

The zero set of the function $\det\Delta(\lambda)$ is contained in a left half plane $\{z \mid \operatorname{Re} z < \gamma\}$ in the complex plane. For retarded equations (*i.e.*, $M\varphi = \varphi(0)$), the function $\det\Delta(\lambda)$ has finitely many zeros in strips of the form $S_{\alpha,\beta} = \{z \mid \alpha < \operatorname{Re} z < \beta\}$, where $\alpha, \beta \in \mathbb{R}$. However, in general, for neutral functional differential equations, $\det\Delta(z)$ can have infinitely many zeros in $S_{\alpha,\beta}$. See Section 1.7 for a more precise description of the location of the roots of $\det\Delta(\lambda)$.

An eigenvalue λ of A is called *simple* if $m_\lambda = 1$. So simple eigenvalues of A correspond to the simple roots of the characteristic equation

$$\det\Delta(\lambda) = 0.$$

For $k_\lambda = 1$, in particular if λ is simple, it is known that

$$\mathcal{M}_\lambda = \{\theta \mapsto e^{\lambda\theta} v \mid \theta \in [-r, 0], v \in \mathcal{N}(\Delta(\lambda))\}. \tag{2.7}$$

We refer to Chapter 7 of Hale & Verduyn Lunel [27]. In Kaashoek & Verduyn Lunel [35] and Section IV.3 of Diekmann *et al.* [15] a systematic procedure has been developed to construct a canonical basis for $\mathcal{M}_\lambda$ using Jordan chains for generic $\lambda \in \sigma(A)$. For the transposed system (A.9), we have similar notions.

Let $y(\cdot) \in \mathbb{C}^{n*}$ be a solution of Equation (A.9) on the interval $(-\infty, r]$. Similarly as before, we can write (A.9) as an evolutionary system for y^s for $s \geqslant 0$, in the Banach space $\mathcal{C}'$. In order to do so, we associate, by translation along the solution, a $\mathcal{C}_0$-semigroup $T^{\mathrm{T}}(s)$ with Equation (A.9), the *transposed semigroup*, defined by

$$T^{\mathrm{T}}(s)\psi = y^s(\cdot;\psi), \qquad s \geqslant 0. \tag{2.8}$$

The infinitesimal generator A^{T} associated with $T^{\mathrm{T}}(t)$ is given by (see Lemma 1.4 of Chapter 7 and Lemma 2.3 of Chapter 9 in Hale & Verduyn Lunel [27])

$$\begin{cases} \mathcal{D}(A^{\mathrm{T}}) = \{\psi \in \mathcal{C}' \mid \frac{d\psi}{d\xi} \in \mathcal{C}',\ M'\frac{d\psi}{d\xi} = -L'\psi\} \\ \quad A^{\mathrm{T}}\psi = \frac{d\psi}{d\xi}. \end{cases} \tag{2.9}$$

The spectra of A and A^{T} coincide. If we define

$$\mathcal{M}_\lambda(A^{\mathrm{T}}) = \mathcal{N}((\lambda I - A^{\mathrm{T}})^{k_\lambda}),$$

and if $k_\lambda = 1$, then

$$\mathcal{M}_\lambda(A^{\mathrm{T}}) = \{\theta \mapsto e^{-\lambda\theta} v \mid\ 0 \leqslant \theta \leqslant r,\ v \in \mathbb{C}^{n*},\ v\Delta(\lambda) = 0\}. \tag{2.10}$$

2.2 Spectral decomposition of $\mathcal{C}$

We denote by φ_λ the row m_λ-vector $\{\varphi_1, \dots, \varphi_{m_\lambda}\}$, where $\varphi_1, \dots, \varphi_{m_\lambda}$ form a basis of eigenvectors and generalized eigenvectors of A at λ. Let $\psi_1, \dots, \psi_{m_\lambda}$ be a basis of eigenvectors and generalized eigenvectors of A^{T} at λ. Define the column m_λ-vector Ψ_λ by $\mathrm{col}\{\psi_1, \dots, \psi_{m_\lambda}\}$ and let $(\Psi_\lambda, \varphi_\lambda) = ((\psi_i, \varphi_j))$, $i, j = 1, 2, \dots, m_\lambda$. The matrix $(\Psi_\lambda, \varphi_\lambda)$ is nonsingular and thus can be normalized to be the identity. The decomposition of $\mathcal{C}$ can be written explicitly as

$$\varphi = P_\lambda\varphi + (I - P_\lambda)\varphi,$$

where $P_\lambda\varphi \in \mathcal{M}_\lambda$ and $(I - P_\lambda)\varphi \in \mathcal{Q}_\lambda$ and

$$\begin{aligned} \mathcal{C} &= \mathcal{M}_\lambda \oplus \mathcal{Q}_\lambda \\ \mathcal{M}_\lambda &= \{\varphi \in \mathcal{C} : \varphi = \varphi_\lambda b \text{ for some } m_\lambda\text{-vector } b\}, \\ \mathcal{Q}_\lambda &= \{\varphi \in \mathcal{C} : (\Psi_\lambda, \varphi) = 0\}. \end{aligned}$$

The spaces $\mathcal{M}_\lambda$ and $\mathcal{Q}_\lambda$ are closed subspaces that are invariant under $T(t)$.

We finish this section with exponential estimates on the complementary subspace $\mathcal{Q}_{\lambda_d}$ when λ_d is simple and a *dominant eigenvalue of* A, that is, there exists a $\epsilon > 0$ such that if λ is another eigenvalue of A, then $\operatorname{Re}\lambda < \operatorname{Re}\lambda_d - \epsilon$. The next lemma shows the importance of computing the projections P_λ explicitly.

Lemma 2.1. *Suppose that λ_d is a dominant eigenvalue of A. For $\delta > 0$ sufficiently small there exists a positive constant $K = K(\delta)$ such that*

$$\|T(t)(I - P_\lambda)\varphi\| \leqslant K e^{(\operatorname{Re}\lambda_d - \delta)t}\|\varphi\|, \qquad t \geqslant 0. \tag{2.11}$$

Proof. From the fact that λ_d is dominant, it follows that we can choose $\delta > 0$ sufficiently small such that

$$\sigma(A \mid \mathcal{Q}_{\lambda_d}) \subset \{z \in \mathbb{C} \mid \operatorname{Re} z < \operatorname{Re}\lambda_d - 2\delta\}.$$

Therefore, the lemma follows from the spectral mapping theorem for retarded functional differential equations (see Theorem IV.2.16 of Diekmann *et al.* [15]) or from the spectral mapping theorem for neutral equations (see Corollary 9.4.1 of Hale & Verduyn Lunel [27]). $\square$

2.3 Spectral theory for periodic FDE

We begin to recall some of the basic theory for linear periodic delay equations that we use in this section. Consider the scalar periodic differential difference equation

$$\begin{aligned} \frac{dx}{dt}(t) &= a(t)x(t) + \sum_{j=1}^{m} b_j(t)x(t - r_j), \quad t \geqslant s, \\ x_s &= \varphi, \qquad \varphi \in \mathcal{C}, \end{aligned} \tag{2.12}$$

where the coefficients $a(\cdot)$ and $b_j(\cdot)$, for $1 \leqslant j \leqslant m$, are real continuous periodic functions with period ω and the delays $r_j = j\omega$ are multiples of the period ω.

To emphasize the dependence of the solution $x(t)$ of (2.12) with respect to the initial condition $x_s = \varphi$, we write $x(t) = x(t; s, \varphi)$. The evolutionary system associated with (2.12) is again given by translation along the solution

$$T(t,s)\varphi = x_t(s,\varphi), \tag{2.13}$$

where $x_t(s,\varphi)(\theta) = x(t+\theta; s, \varphi)$ for $-m\omega \leqslant \theta \leqslant 0$. The periodicity of the coefficients of (2.12) implies that

$$T(t+\omega, s+\omega) = T(t,s), \qquad t \geqslant s.$$

This together with the semigroup property $T(t,\tau)T(\tau,s) = T(t,s)$, $t \geqslant \tau \geqslant s$, yields to

$$T(t+\omega, s) = T(t,s)T(s+\omega, s) \qquad \text{for } t \geqslant s. \tag{2.14}$$

The periodicity property (2.14) allows us to define the monodromy map or period map $\Pi(s) : \mathcal{C} \to \mathcal{C}$ associated with (2.12) as follows

$$\Pi(s)\varphi = T(s+\omega, s)\varphi, \qquad \varphi \in \mathcal{C}. \tag{2.15}$$

From the general theory for functional differential equations (see Diekmann *et al.* [15] and Hale & Verduyn Lunel [27]), it follows that $\Pi(s)$ is a compact operator, *i.e.*, $\Pi(s)$ is a bounded operator with the property that the closure of the image of the unit ball in $\mathcal{C}$ is compact. Hence, the spectrum $\sigma(\Pi(s))$ of $\Pi(s)$ is at most countable with the only possible accumulation point being zero. If $\mu \neq 0$ belongs to $\sigma(\Pi(s))$, then μ is in the point spectrum of Π, *i.e.*, there exists $\varphi \in \mathcal{C}$, $\varphi \neq 0$, such that $\Pi(s)\varphi = \mu\varphi$. If μ belongs to the nonzero point spectrum of $\Pi(s)$, then μ is called a *characteristic multiplier* of (2.12) and λ for which $\mu = e^{\lambda\omega}$ (unique up to sums with multiples of $2\pi i/\omega$) is called a *characteristic exponent* of (2.12). The characteristic multipliers are in fact independent of s and the *generalized eigenspace* $\mathcal{M}_\mu(s)$ of $\Pi(s)$ at μ is defined to be $\mathcal{N}\big(\mu I - \Pi(s))^{k_\lambda}\big)$, where k_λ is the smallest positive integer such that

$$\mathcal{M}_\mu(s) = \mathcal{N}\big((\mu I - \Pi(s))^{k_\lambda}\big) = \mathcal{N}\big((\mu I - \Pi(s))^{k_\lambda+1}\big).$$

Since $\Pi(s)$ is compact, there are two closed subspaces $\mathcal{M}_\mu(s)$ and $\mathcal{Q}_\mu(s)$ of $\mathcal{C}$ such that the following properties hold:

(i) $\mathcal{C} = \mathcal{M}_\mu(s) \oplus \mathcal{Q}_\mu(s)$;

(ii) $m_\mu = \dim \mathcal{M}_\mu(s) < \infty$;

(iii) $\mathcal{M}_\mu(s)$ and $\mathcal{Q}_\mu$ are $\Pi(s)$-invariant;

(iv) $\sigma(\Pi(s)|\mathcal{M}_\mu(s)) = \{\mu\}$ and $\sigma(\Pi(s)|\mathcal{Q}_\mu(s)) = \sigma(\Pi(s)) \setminus \{\mu\}$.

Let $\phi_1(s), \dots, \phi_{m_\mu}(s)$ be a basis of eigenvectors and generalized eigenvectors of $\Pi(s)$ at μ. Define the row m_λ-vector $\Phi(s) = \{\phi_1(s), \dots, \phi_{m_\lambda}(s)\}$. Since $\mathcal{M}_\lambda(s)$ is invariant under $\Pi(s)$, there exists a $m_\lambda \times m_\lambda$ matrix $N(s)$ such that

$$\Pi(s)\Phi(s) = \Phi(s)N(s)$$

and Property (iv) implies that the only eigenvalue of $N(s)$ is $\mu \neq 0$. Therefore, there is a $m_\mu \times m_\mu$ matrix B_s such that $B_s = 1/\omega \log N(s)$.

Define the vector $P(t)$ with elements in $\mathcal{C}$ by

$$P(t) = T(t,0)\Phi\, e^{-Bt},$$

where $\Phi = \Phi(0)$, $N = N(0)$ and $B = B_0$. Then, for $t \geqslant 0$,

$$\begin{aligned} P(t+\omega) &= T(t+\omega,0)\Phi\, e^{-B(t+\omega)} = T(t,0)T(\omega,0)\Phi\, e^{-B\omega}e^{-Bt} \\ &= T(t,0)\Pi(0)\Phi\, e^{-B\omega}e^{-Bt} = T(t,0)\Phi\, Ne^{-B\omega}e^{-Bt} \\ &= P(t). \end{aligned}$$

Since $P(t)$ can be extended periodically for $t \in \mathbb{R}$, we conclude that

$$T(t,0)\Phi = P(t)e^{Bt} \qquad \text{for} \quad t \in \mathbb{R}.$$

As for the case $s = 0$, one can define $T(t,s)\Phi(s)$ for all $t \in \mathbb{R}$ and for any real number τ and $\mu \in \sigma(\Pi(s)) \setminus \{0\}$, we have

$$\begin{aligned}\Pi(\tau)T(\tau,s)\Phi(s) &= T(\tau+\omega,\tau)T(\tau,s)\Phi(s) = T(\tau+\omega,s)\Phi(s)\\ &= T(\tau,s)T(s+\omega,s)\Phi(s) = T(\tau,s)\Phi(s)N(s).\end{aligned}$$

Therefore,

$$[\mu I - \Pi(\tau)]T(\tau,s)\Phi(s) = T(\tau,s)\Phi(s)(\mu I - N(s)),$$

and it follows that $\mu \in \sigma(\Pi(\tau))$. Thus the dimension of $\mathcal{M}_\mu(\tau)$ is at least as large as the dimension of $\mathcal{M}_\mu(s)$. Since one can reverse the role of s and τ, we obtain that the characteristic multipliers of (2.12) are independent of the starting time and if $\Phi(s)$ is a basis for $\mathcal{M}_\mu(s)$, then $T(t,s)\Phi(s)$ is a basis for $\mathcal{M}_\mu(t)$ for any $t \in \mathbb{R}$. In particular, the subspaces $\mathcal{M}_\mu(s)$ and $\mathcal{M}_\mu(t)$ are diffeomorphic for all s and t. Note that if $\mu = e^{\lambda\omega}$ is simple, *i.e.* $m_\mu = 1$, then $N(s)$ is identical to μ, $\Phi \stackrel{\text{def}}{=} \Phi(0) = \{\phi_1\}$ and $P(t)$ assumes the simpler form

$$e^{\lambda t}P(t) = T(t,0)\phi_1. \tag{2.16}$$

The spectral projection onto $\mathcal{M}_\mu(s)$ along $\mathcal{Q}_\mu(s)$ can again be represented by a Dunford integral (see Gohberg, Goldberg & Kaashoek [20])

$$P_\mu(s) = \frac{1}{2\pi i}\int_{\Gamma_\mu}(zI - \Pi(s))^{-1}\,dz, \tag{2.17}$$

where Γ_μ is a small circle such that μ is the only singularity of $(zI - \Pi(s))^{-1}$ inside Γ_μ. *N.b.:* As far as the author knows, the spectral projection P_λ into $\mathcal{M}_\mu$ can only be computed using Dunford calculus.

We finish this section presenting the asymptotic behaviour of solutions of periodic delay equations, similar to approach used for autonomous equations. We can relate the solutions, corresponding to initial data $\varphi \in \mathcal{M}_\mu(s)$ to an arbitrary solution $x_t(s;\varphi)$ by an exponential bound for the remainder. The next couple of results in this section provide a new approach and generalize earlier results in Zhang [63] and Philos [46].

Theorem 2.2. *Let μ_j, $j = 1,2,\ldots$, denote the nonzero eigenvalues of the monodromy operator $\Pi(s)$ ordered by decreasing modulus and let $\varphi \in \mathcal{C}$. If γ is an arbitrary real number, then there are positive constants ϵ and N such that for $t \geqslant s$*

$$\Big\| x_t(s;\varphi) - \sum_{|\mu_n| \geqslant e^{\gamma\omega}} P_{\mu_n}(s)x_t(s;\varphi)\Big\| \leqslant Ne^{(\gamma-\epsilon)(t-s)}\|\varphi\|. \tag{2.18}$$

Proof. Let $k = k(\gamma)$ be the integer such that $|\mu_k| \geqslant e^{\gamma\omega}$ and $|\mu_{k+1}| < e^{\gamma\omega}$. Set $\Sigma = \{\mu_1, \mu_2, \ldots, \mu_k\}$ and define

$$\mathcal{M}_\Sigma(s) = \bigoplus_{\mu\in\Sigma} \mathcal{M}_\mu(s) \quad \text{and} \quad \mathcal{Q}_\Sigma(s) = \bigcap_{\mu\in\Sigma} \mathcal{Q}_\mu(s).$$

Then, one has $\mathcal{C} = \mathcal{M}_\Sigma \oplus \mathcal{Q}_\Sigma$. To prove the exponential estimate, set

$$R_k(s)\varphi = \varphi - \sum_{j=1}^{k} P_{\mu_j}(s)\varphi. \tag{2.19}$$

It follows that $R_k(s)\varphi \in \mathcal{Q}_\Sigma(s)$ and $T(t,s)R_k(s)\varphi \in \mathcal{Q}_\Sigma(t)$ for all $\varphi \in \mathcal{C}$. Furthermore, there exists a constant N_0 such that $\|R_k(s)\varphi\| \leqslant N_0\|\varphi\|$. If ϵ is such that $e^{(\gamma-2\epsilon)\omega} = |\mu_{k+1}|$, then the spectral radius of

$$\widehat{\Pi}(s) \stackrel{\text{def}}{=} \Pi(s)\big|_{\mathcal{Q}_\Sigma(s)}$$

is $e^{(\gamma-2\epsilon)\omega}$. Therefore, $\lim_{n\to\infty} \|\widehat{\Pi}(s)^n\|^{1/n} = e^{(\gamma-2\epsilon)\omega}$ and this implies that for some $m > 0$

$$\|\widehat{\Pi}(s)^m R_k(s)\varphi\| \leqslant e^{(\gamma-\epsilon)m\omega}\|R_k(s)\varphi\|.$$

Since $\widehat{T}(\tau,s) \stackrel{\text{def}}{=} T(\tau,s)|_{\mathcal{Q}_\Sigma(s)}$, $s \leqslant \tau \leqslant s+\omega$ is pointwise bounded, the uniform boundedness principle yields a constant N_1 such that $\|\widehat{T}(\tau,s)\| \leqslant N_1$ for $s \leqslant \tau \leqslant s+\omega$. Set $N_2 = N_1 \max_{j=1,\ldots,m-1} \|\widehat{\Pi}(s)^j\|$ and let $t \geqslant s$ be given. If k_t is the largest integer so that $s + k_t m\omega \leqslant t$, then

$$\|\widehat{T}(t,s)R_k(s)\varphi\| \leqslant N_2\|\widehat{\Pi}(s)^m\|^{k_t}\|R_k(s)\varphi\| \leqslant Ne^{(\gamma-\epsilon)(t-s)}\|\varphi\|,$$

where $N = N_0N_2$. This proves the exponential estimate for the remainder term. □

Theorem 2.3. *Suppose that $\mu_d = e^{\lambda_d\omega}$ is a simple dominant eigenvalue of $\Pi(s)$. If $P(t)$ is as in (2.16), then there are positive constants N and ϵ so that the large time behaviour of the solution $x(t;0,\varphi)$ is given by*

$$\|e^{-\lambda_d t}x_t(0,\varphi) - c(\varphi)P(t)\| \leqslant Ne^{-\epsilon t}\|\varphi\|, \qquad t \geqslant 0, \tag{2.20}$$

where $c(\varphi)$ is defined so that $P_{\mu_d}(0)\varphi = c(\varphi)\varphi_1$.

Proof. If $\mu = e^{\lambda\omega}$ is a simple eigenvalue of Π, it follows from Equation (2.16) together with the property that $T(t,s)$ maps $\mathcal{M}_{\mu_d}(s)$ into $\mathcal{M}_{\mu_d}(t)$ diffeomorphically, that

$$\begin{aligned} P_{\mu_d}(t)x_t(0,\varphi) &= P_{\mu_d}(t)T(t,0)\varphi \\ &= T(t,0)P_{\mu_d}(0)\varphi \\ &= c(\varphi)e^{\lambda_d t}P(t). \end{aligned}$$

Therefore the result follows from Theorem 2.2. □

Chapter 3

The spectral projection

In this chapter we study *spectral projections* and we are interested in efficient methods to explicitly compute them. We observed in Section 2.2 that, for a set of roots of the characteristic equation $\Lambda = \{\lambda_1, \ldots, \lambda_n\}$, the state space $\mathcal{C}$ can be decomposed in a direct sum of the finite-dimensional $\mathcal{M}_\Lambda = \mathcal{M}_{\lambda_1} \oplus \cdots \oplus \mathcal{M}_{\lambda_n}$ and the complementary space, and these spaces are invariant under the solution semigroup. Hence the spectral projection can be used to restrict the flow to a finite dimensional subspaces. Banks & Manitius [4], Burns, Herdman & Stech [7] and Verduyn Lunel [56] studied the convergence of the series expansion of the solution operator in terms of the spectral projections into the subspaces $\mathcal{M}_\lambda$. In Section 2.2 we have presented strong estimates on the complementary space. Later, in Chapter 6, we obtain explicit formulas for the large time behaviour of solutions. Therefore, it is important to being able to compute spectral projections.

The first approach used to compute spectral projections uses duality, with help of the bilinear form introduced by Hale. Although it looks simpler theoretically, it lacks flexibility, because it applies only for simple eigenvalues on autonomous FDE. The second approach is using the Dunford calculus to compute the spectral projections. This technique allows us to provide an explicit formula for the spectral projection on arbitrary order eigenvalues of the generator of the solution semigroup for autonomous FDE, with extensions for a class of periodic equations. Due to the algorithmic nature of the results presented in this chapter, we implemented our method to compute spectral projections as a library for the computer algebra system Maple. Our program easily computes spectral projections for "concrete" equations, a process which is often lengthy and difficult to do by hand. The source code for the library is presented in Appendix B.

3.1 Computing spectral projections using duality

For a simple eigenvalue λ of an operator A, the spectral projection onto the eigenspace $\mathcal{M}_\lambda$ can be explicitly given by

$$P_\lambda \varphi = \langle \varphi_\lambda^*, \varphi \rangle \, \varphi_\lambda,$$

where φ_λ is an eigenvector at λ for A, φ_λ^* is an eigenvector at λ for A^* and $\langle \varphi_\lambda^*, \varphi_\lambda \rangle = 1$ (here $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $\mathcal{C}$ and the dual space $\mathcal{C}^*$). Before we can compute the projection explicitly, we need some more definitions in order to avoid the adjoint operator A^* and the duality pairing between $\mathcal{C}$ and its dual space $\mathcal{C}^*$. It turns out that using a specific bilinear form, given in Lemma A.4, it is possible to view the transposed equation (A.9) as the 'adjoint' equation. This approach is developed in Hale & Verduyn Lunel [27] and here we present the functional analytic foundation for this approach.

The following lemma provides us with an explicit formula for the spectral projection onto the eigenspace $\mathcal{M}_\lambda$ corresponding to a simple eigenvalue λ_d.

Lemma 3.1. *Let A be given by (2.3). If λ is a simple eigenvalue of A, then the spectral projection P_λ onto $\mathcal{M}_\lambda(A)$ along $\mathcal{R}\big((\lambda I - A)^{k_\lambda}\big)$ can be written explicitly as follows*

$$P_\lambda \varphi = e^{\lambda \cdot} \Big[\frac{d}{dz} \det \Delta(\lambda)\Big]^{-1} \operatorname{adj} \Delta(\lambda) \Big(M\varphi + \int_{-r}^{0} d_\tau [\lambda \mu(\tau) + \eta(\tau)] \int_0^{-\tau} e^{-\lambda \sigma} \varphi(\sigma + \tau) \, d\sigma \Big), \tag{3.1}$$

where adj $\Delta(\lambda)$ *denotes the matrix of cofactors of* $\Delta(\lambda)$.

Proof. Let ψ_λ and φ_λ be a basis for $\mathcal{M}_\lambda(A^{\mathrm{T}})$ and $\mathcal{M}_\lambda(A)$, respectively, such that $(\psi_\lambda, \varphi_\lambda) = 1$. The spectral projection is given by

$$P_\lambda \varphi = \varphi_\lambda (\psi_\lambda, \varphi). \tag{3.2}$$

Since $z = \lambda$ is a simple zero of $\det \Delta(z)$, we know from (2.7) and (2.10) that

$$\begin{aligned} \psi_\lambda(s) &= e^{-\lambda s} d_\lambda, \qquad 0 \leqslant s \leqslant r, \qquad d_\lambda \Delta(\lambda) = 0, \\ \varphi_\lambda(\theta) &= e^{\lambda \theta} c_\lambda, \qquad -r \leqslant \theta \leqslant 0, \qquad \Delta(\lambda) c_\lambda = 0. \end{aligned}$$

So, using the bilinear form (A.19),

$$(\psi_\lambda, \varphi) = d_\lambda \Big[\varphi(0) - \int_{-r}^{0} d_s \Big(\int_{-r}^{s} e^{-\lambda(s-\theta)} \, d\mu(\theta) \Big) \varphi(s) + \int_{-r}^{0} \int_{-r}^{s} e^{-\lambda(s-\theta)} \, d\eta(\theta) \varphi(s) ds \Big]. \tag{3.3}$$

Therefore

$$
\begin{aligned}
(\psi_\lambda, \varphi_\lambda) &= d_\lambda \Big[c_\lambda - \int_{-r}^{0} d_s \Big(\int_{-r}^{s} e^{-\lambda(s-\theta)}\, d\mu(\theta) \Big) e^{\lambda s} c_\lambda \\
&\qquad\qquad + \int_{-r}^{0} \int_{-r}^{s} e^{-\lambda(s-\theta)}\, d\eta(\theta)\, e^{\lambda s} c_\lambda \, ds \Big] \\
&= d_\lambda \Big[I - \int_{-r}^{0} e^{\lambda\theta} d\mu(\theta) \\
&\qquad + \lambda \int_{-r}^{0} \int_{-r}^{s} e^{\lambda\theta}\, d\mu(\theta)\, ds + \int_{-r}^{0} \int_{-r}^{s} e^{\lambda\theta}\, d\eta(\theta)\, ds \Big] c_\lambda .
\end{aligned}
$$

Since

$$
\int_{-r}^{0} \int_{-r}^{s} e^{\lambda\theta}\, d\eta(\theta)\, ds = - \int_{-r}^{0} \theta e^{\lambda\theta}\, d\eta(\theta)
$$

and

$$
\lambda \int_{-r}^{0} \int_{-r}^{s} e^{\lambda\theta}\, d\mu(\theta)\, ds = -\lambda \int_{-r}^{0} \theta e^{\lambda\theta}\, d\mu(\theta),
$$

it follows that

$$
(\psi_\lambda, \varphi_\lambda) = d_\lambda \Big[I - \int_{-r}^{0} e^{\lambda\theta} d\mu(\theta) - \lambda \int_{-r}^{0} \theta e^{\lambda\theta}\, d\mu(\theta) - \int_{-r}^{0} \theta e^{\lambda\theta}\, d\eta(\theta) \Big] c_\lambda .
$$

This together with the definition of $\Delta(z)$ in (2.4) and the normalization condition, $(\psi_\lambda, \varphi_\lambda) = 1$, yields

$$
d_\lambda \Big[\frac{d}{dz} \Delta(\lambda) \Big] c_\lambda = 1. \tag{3.4}
$$

Returning to representation (3.2) for P_λ and representation (3.3) for (ψ_λ, φ), we claim that, for a given vector v,

$$
c_\lambda d_\lambda v = \Big[\frac{d}{dz} \det \Delta(\lambda) \Big]^{-1} \operatorname{adj} \Delta(z)\, v. \tag{3.5}
$$

From the relation

$$
\Delta(z) \operatorname{adj} \Delta(z) = \det \Delta(z), \tag{3.6}
$$

it follows that for a given vector v, the vector c_λ, defined by $c_\lambda = \operatorname{adj} \Delta(\lambda) v$, satisfies $\Delta(\lambda) c_\lambda = 0$. Next we choose d_λ such that $d_\lambda \Delta(\lambda) = 0$ and such that

$$
d_\lambda \Big[\frac{d}{dz} \Delta(\lambda) \Big] \operatorname{adj} \Delta(\lambda) v = 1,
$$

so that (3.4) is also satisfied. Differentiating equation (3.6) with respect to z yields

$$
\frac{d}{dz} \Delta(z) \operatorname{adj} \Delta(z) + \Delta(z) \frac{d}{dz} \operatorname{adj} \Delta(z) = \frac{d}{dz} \det \Delta(z). \tag{3.7}
$$

If we multiply (3.7) with $z = \lambda$ from the left by d_λ and from the right by v, and use that $d_\lambda \Delta(\lambda) = 0$ and that (3.4) holds, then it follows that

$$d_\lambda v = \Big[\frac{d}{dz} \det \Delta(\lambda)\Big]^{-1}.$$

This proves (3.5) and completes the proof of the lemma. □

An important application of Lemma 3.1 arises in the situation when $\lambda = \lambda_d$ is a dominant eigenvalue of A.

3.2 Spectral projection via Dunford calculus

From standard spectral theory (see for instance Diekmann *et al.* [15], Gohberg, Goldberg & Kaashoek [20] and Yosida [62]), it follows that the spectral projection onto $\mathcal{M}_\lambda$ along $\mathcal{R}\big((\lambda I - A)^{k_\lambda}\big)$ can be represented by a Dunford integral

$$P_\lambda = \frac{1}{2\pi i} \int_{\Gamma_\lambda} (zI - A)^{-1}\, dz, \tag{3.8}$$

where Γ_λ is a small circle such that λ is the only singularity of $(zI - A)^{-1}$ inside Γ_λ. In order to compute the projection explicitly, we need an explicit formula for the resolvent of A.

Lemma 3.2. *If A is defined by (2.3), then the resolvent $(zI - A)^{-1}$ of A is given by*

$$(zI - A)^{-1}\varphi = -e^{z\cdot} \int_0^{\cdot} e^{-z\tau}\varphi(\tau)d\tau + e^{z\cdot}\Delta(z)^{-1}H(z,\varphi) \tag{3.9}$$

where $\Delta(z)$ is given by (2.4) and $H(z,\cdot) : \mathcal{C} \to \mathbb{C}^n$ is an operator given by

$$H(z,\varphi) = M\varphi + \int_0^r d_\theta\big[z\mu(\theta) + \eta(\theta)\big] \int_0^\theta e^{-z\tau}\varphi(\tau - \theta)d\tau. \tag{3.10}$$

Furthermore, if $\lambda \in \sigma(A)$, then the spectral projection P_λ onto $\mathcal{M}_\lambda$ along $\mathcal{R}\big((\lambda I - A)^{k_\lambda}\big)$ is given by

$$P_\lambda\varphi = \operatorname*{Res}_{z=\lambda}\big\{e^{z\cdot}\Delta(z)^{-1}H(z,\varphi)\big\}. \tag{3.11}$$

Proof. Let φ be fixed. If we define $\psi = (zI - A)^{-1}\varphi$, then $\psi \in \mathcal{D}\big(A\big)$ and $z\psi - A\psi = \varphi$. From the definition of A, it follows that ψ satisfies the ordinary differential equation

$$z\psi - \frac{d\psi}{d\theta} = \varphi \tag{3.12}$$

with boundary condition

$$M\frac{d\psi}{d\theta} = L\psi. \tag{3.13}$$

Equation (3.12) yields

$$\psi(\theta) = e^{z\theta}\Big[\psi(0) - \int_0^\theta e^{-z\tau}\varphi(\tau)d\tau\Big]. \tag{3.14}$$

Applying M on both sides of (3.12) and using (3.13), we obtain

$$\begin{aligned}
0 &= zM\psi - L\psi - M\varphi \\
&= z\left[\psi(0) - \int_0^r d\mu(\theta)\psi(-\theta)\right] - \int_0^r d\eta(\theta)\psi(-\theta) - M\varphi \\
&= z\psi(0) - z\int_0^r d\mu(\theta)\Big(e^{-z\theta}\Big[\psi(0) - \int_0^{-\theta} e^{-z\tau}\varphi(\tau)d\tau\Big]\Big) \\
&\qquad\qquad - \int_0^r d\eta(\theta)\Big(e^{-z\theta}\Big[\psi(0) - \int_0^{-\theta} e^{-z\tau}\varphi(\tau)d\tau\Big]\Big) - M\varphi \\
&= \Big[zI - z\int_0^r d\mu(\theta)e^{-z\theta} - \int_0^r d\eta(\theta)e^{-z\theta}\Big]\psi(0) \\
&\qquad\qquad - M\varphi + \int_0^r d_\theta\big[z\mu(\theta) + \eta(\theta)\big]\int_0^{-\theta} e^{-z(\theta+\tau)}\varphi(\tau)d\tau \\
&= \Big[zI - z\int_0^r d\mu(\theta)e^{-z\theta} - \int_0^r d\eta(\theta)e^{-z\theta}\Big]\psi(0) \\
&\qquad\qquad - M\varphi - \int_0^r d_\theta\big[z\mu(\theta) + \eta(\theta)\big]\int_0^{\theta} e^{-z\tau}\varphi(\tau-\theta)d\tau \\
&= \Delta(z)\psi(0) - H(z,\varphi)
\end{aligned}$$

where $H(z,\varphi)$ is given by formula (3.10). This allows us to solve for $\psi(0)$ and

$$\psi(0) = \Delta(z)^{-1}H(z,\varphi). \tag{3.15}$$

Substituting (3.15) into (3.14) yields (3.9). The formula for the projection in (3.8) is precisely the residue of the resolvent of A in $z=\lambda$, and since

$$z \mapsto e^{z\cdot}\int_\cdot^0 e^{-z\tau}\varphi(\tau)d\tau$$

is analytic, the representation for the resolvent in (3.9) yields formula (3.11). □

To illustrate the power of Dunford calculus, we give next a second simpler proof of Lemma 3.1.

Second proof of Lemma 3.1. If λ is a simple eigenvalue of A, then λ is a simple zero of $\det \Delta(z)$. In order to compute the residue on (3.11), it suffices to compute

$\operatorname{Res}_{z=\lambda} \Delta(z)^{-1}$ explicitly

$$\begin{aligned}\operatorname*{Res}_{z=\lambda} \Delta(z)^{-1} &= \lim_{z\to\lambda} (z-\lambda)[\det \Delta(z)]^{-1} \operatorname{adj} \Delta(z)\\ &= \lim_{z\to\lambda} \left[\frac{\det \Delta(z) - \det \Delta(\lambda)}{z-\lambda}\right]^{-1} \operatorname{adj} \Delta(z)\\ &= \left[\frac{d}{dz} \det \Delta(\lambda)\right]^{-1} \operatorname{adj} \Delta(\lambda).\end{aligned}$$

in order to arrive at (3.1). □

3.3 Formula for the spectral projection for eigenvalues of arbitrary order

Our goal is to provide a explicit formula for a pole of $\Delta(z)^{-1}$ of arbitrary order. First we define the scalar functions $\phi_{\lambda,n}$, for $\lambda \in \mathbb{C}$ and $n \in \mathbb{Z}_+$, by

$$t \mapsto \phi_{\lambda,n}(t) = \frac{t^n}{n!} e^{\lambda t}.$$

Lemma 3.3. *The functions $\phi_{\lambda,n}$, $n \in \mathbb{Z}_+$ satisfy the following properties:*

a. (Derivative)

$$\begin{aligned} D\phi_{\lambda,0}(t) &= \lambda\phi_{\lambda,0}(t),\\ D\phi_{\lambda,n}(t) &= \lambda\phi_{\lambda,n}(t) + \phi_{\lambda,n-1}(t), \qquad n \geqslant 1.\end{aligned}$$

b. (Primitive) For $\lambda \neq 0$,

$$\int \phi_{\lambda,n}(t)dt = \sum_{j=0}^{n} \frac{(-1)^{n-j}}{\lambda^{n+1-j}}\, \phi_{\lambda,j}(t).$$

c. (Evaluation of sums)

$$\phi_{\lambda,n}(t_0+\theta) = \sum_{j=0}^{n} \phi_{\lambda,n-j}(t_0)\phi_{\lambda,j}(\theta).$$

d. (Linear operators) For $v \in \mathbb{C}^n$ and $t \in \mathbb{R}$, we have that $(\phi_{\lambda,n})_t\, v \in \mathcal{C}$. Furthermore, for any linear operator B on $\mathcal{C}$ into any complex linear space, we have

$$B\big[(\phi_{\lambda,n})_t v\big] = \sum_{j=0}^{n} \phi_{\lambda,n-j}(t) B\big[(\phi_{\lambda,j})_0 v\big].$$

Proof. Item a is clear. Item b follows from integration by parts. For item c, note that

$$\begin{aligned}\phi_{\lambda,n}(t+\theta) &= \frac{1}{n!}(t+\theta)^n e^{\lambda(t+\theta)} = \frac{1}{n!}\sum_{j=0}^{n}\binom{n}{j} t^{n-j}\,\theta^j\,e^{\lambda t}\,e^{\lambda\theta} \\ &= \frac{1}{n!}\sum_{j=0}^{n}\frac{n!}{(n-j)!\,j!}\,t^{n-j}\,\theta^j\,e^{\lambda t}\,e^{\lambda\theta} = \sum_{j=0}^{n}\frac{t^{n-j}}{(n-j)!}e^{\lambda t}\frac{\theta^j}{j!}\,e^{\lambda\theta} \\ &= \sum_{j=0}^{n}\phi_{\lambda,n-j}(t)\phi_{\lambda,j}(\theta).\end{aligned}$$

It follows that for $t \in \mathbb{R}$ and $\theta \in [-r,0]$, we have

$$\big(\phi_{\lambda,n}\big)_t(\theta) = \phi_{\lambda,n}(t+\theta) = \sum_{j=0}^{n}\phi_{\lambda,n-j}(t)\phi_{\lambda,j}(\theta) = \sum_{j=0}^{n}\phi_{\lambda,n-j}(t)\big(\phi_{\lambda,j}\big)_0(\theta),$$

and therefore

$$\big(\phi_{\lambda,n}\big)_t = \sum_{j=0}^{n}\phi_{\lambda,n-j}(t)\;\big(\phi_{\lambda,j}\big)_0.$$

This shows d. □

Next we characterize the residue of the product of $\Delta(z)^{-1}$ and an analytic function, at a pole of $\Delta(z)^{-1}$ of arbitrary order.

Lemma 3.4. *Let $E : \mathbb{C} \to \mathcal{C}$ be Fréchet differentiable on $z = \lambda$. If λ is a pole of $\Delta^{-1}(z)$ order n, that is, if*

$$\Delta(z)^{-1} = \frac{K(z)}{(z-\lambda)^n}$$

with $K(z)$ differentiable on a neighbourhood of $z = \lambda$ and $K(\lambda) \neq 0$, then

$$\operatorname*{Res}_{z=\lambda}\Delta(z)^{-1}E(z) = \sum_{j=0}^{n-1}\frac{D^{n-1-j}K(\lambda)}{(n-1-j)!}\;\frac{D^jE(\lambda)}{j!}. \tag{3.16}$$

Proof. We write down the Laurent expansions of $K(z)$, $\Delta(z)^{-1}$ and $E(z)$

$$\begin{aligned}K(z) &= K_0 + K_1(z-\lambda) + K_2(z-\lambda)^2 + \cdots \\ \Delta^{-1}(z) &= (z-\lambda)^{-n}\Big(K_0 + K_1(z-\lambda) + K_2(z-\lambda)^2 + \cdots\Big) \\ E(z) &= E_0 + E_1(z-\lambda) + E_2(z-\lambda)^2 + \cdots\end{aligned}$$

where $K_i = D^i K(\lambda)/i!$ and $E_i = D^i E(\lambda)/i!$, $i \in \mathbb{N}$. Then the Laurent expansion for $\Delta^{-1}(z)E(z)$ is given by

$$\Delta^{-1}(z)E(z) = \sum_{k=0}^{\infty}\Big(\sum_{j=0}^{k} K_{k-j}E_j\Big)(z-\lambda)^{k-n}$$

and we obtain that the coefficient of the term $(z-\lambda)^{-1}$ is given by

$$\sum_{j=0}^{n-1} K_{n-1-j}E_j = \sum_{j=0}^{n-1} \frac{D^{n-1-j}K(\lambda)}{(n-1-j)!}\,\frac{D^j E(\lambda)}{j!}.$$

This yields the representation of the residue of $\Delta^{-1}(z)E(z)$ at $z = \lambda$, given in (3.16). □

Lemma 3.5. *Suppose that a_i, b_i and c_i, $i = 0, \ldots, n$ are elements of a ring and that a_i commutes with b_j for $i, j = 0, \ldots, n$. Then*

$$\sum_{i=0}^{n}\Big\{a_i \sum_{j=0}^{i} b_{i-j}c_j\Big\} = \sum_{j=0}^{n}\Big\{b_j \sum_{i=j}^{n} a_i c_{i-j}\Big\} = \sum_{j=0}^{n}\Big\{b_j \sum_{m=0}^{n-j} a_{m+j}c_m\Big\}. \tag{3.17}$$

Theorem 3.6. *Suppose that λ is a pole of $\Delta(z)^{-1}$ of order n, that is, there exists a regular function K such that $K(\lambda) \neq 0$ and $\Delta(z)^{-1}$ is given by*

$$\Delta(z)^{-1} = \frac{K(z)}{(z-\lambda)^n}. \tag{3.18}$$

The spectral projection P_λ of $\mathcal{C}$ into $\mathcal{M}_\lambda$ along $\mathcal{R}\big((\lambda I - A)^{k_\lambda}\big)$ is given by

$$P_\lambda\varphi = \sum_{j=0}^{n-1} q_j\,\theta^j e^{\lambda\theta}, \tag{3.19}$$

where $q_j = q_j(n, \lambda, \varphi)$ is given by

$$q_j(n,\lambda,\varphi) = \frac{1}{j!}\sum_{k=j}^{n-1} \frac{D^{n-1-k}K(\lambda)}{(n-1-k)!}\,\frac{D_1^{k-j}H(\lambda,\varphi)}{(k-j)!}. \tag{3.20}$$

Proof. From Equation (3.11) we have that

$$P_\lambda\varphi = \operatorname*{Res}_{z=\lambda}\big\{\Delta(z)^{-1}\big[e^{z\cdot}H(z,\varphi)\big]\big\}. \tag{3.21}$$

For the form of Δ^{-1} from the hypothesis, we can use Lemma 3.4 in order to compute explicitly the resolvent in Equation (3.21). We get that

$$\big(P_\lambda\varphi\big)(\theta) = \sum_{j=0}^{n-1} \frac{D^{n-1-j}K(\lambda)}{(n-1-j)!}\,\frac{1}{j!}\,\frac{d^j}{d\lambda^j}\big[e^{\lambda\theta}H(\lambda,\varphi)\big]. \tag{3.22}$$

We can further simplify Equation (3.22) by using Leibniz Formula, that is, the n^{th} derivative of the product of functions $f(t)$ and $g(t)$ is given by

$$\frac{d^n}{dt^n}\big[f(t)g(t)\big] = \sum_{k=0}^{n} \binom{n}{k} D^k f(t)\, D^{n-k} g(t),$$

and Lemma 3.5 in order to obtain

$$\begin{aligned}
\big(P_\lambda \varphi\big)(\theta) &= \sum_{j=0}^{n-1} \frac{D^{n-1-j}K(\lambda)}{(n-1-j)!}\,\frac{1}{j!}\,\frac{d^j}{d\lambda^j}\big[e^{\lambda\theta}H(\lambda,\varphi)\big] \\
&= \sum_{j=0}^{n-1} \frac{D^{n-1-j}K(\lambda)}{(n-1-j)!}\,\frac{1}{j!}\sum_{k=0}^{j}\binom{j}{k}\frac{d^{j-k}}{d\lambda^{j-k}}e^{\lambda\theta}\,\frac{d^k}{d\lambda^k}H(\lambda,\varphi) \\
&= \sum_{j=0}^{n-1} \frac{D^{n-1-j}K(\lambda)}{(n-1-j)!}\sum_{k=0}^{j}\frac{\theta^{j-k}}{(j-k)!}\,e^{\lambda\theta}\,\frac{D_1^k H(\lambda,\varphi)}{k!} \\
&= \sum_{j=0}^{n-1} \frac{D^{n-1-j}K(\lambda)}{(n-1-j)!}\sum_{k=0}^{j}\phi_{\lambda,j-k}(\theta)\,\frac{D_1^k H(\lambda,\varphi)}{k!} \\
&= \sum_{j=0}^{n-1} \phi_{\lambda,j}(\theta)\sum_{k=j}^{n-1}\frac{D^{n-1-k}K(\lambda)}{(n-1-k)!}\,\frac{D_1^{k-j}H(\lambda,\varphi)}{(k-j)!}.
\end{aligned}$$

This shows (3.19). □

In order to compute the spectral projection for eigenvalues of second or higher orders, we also need several derivatives of $H(z,\varphi)$, given in formula (3.10), with respect to the first variable.

Lemma 3.7. *For integer $n \geqslant 1$, the n^{th} Fréchet derivative of $H(\lambda,\varphi)$ with respect to the first variable is given by*

$$\begin{aligned}
D_1^n H(z,\varphi) &= (-1)^{n+1} n \int_0^r d\mu(\theta)\int_0^\theta \tau^{n-1}e^{-z\tau}\varphi(\tau-\theta)d\tau \\
&\quad + (-1)^n z \int_0^r d\mu(\theta)\int_0^\theta \tau^{n}e^{-z\tau}\varphi(\tau-\theta)d\tau \\
&\quad + (-1)^n \int_0^r d\eta(\theta)\int_0^\theta \tau^{n}e^{-z\tau}\varphi(\tau-\theta)d\tau. \qquad (3.23)
\end{aligned}$$

Proof. Let $\Lambda_n(z,\varphi)$ be the expression in the right-hand side of (3.23). We differentiate (3.10)

$$\begin{aligned}H(z,\varphi) = M\varphi + z\int_0^r d\mu(\theta)\int_0^\theta e^{-z\tau}\varphi(\tau-\theta)d\tau\\ + \int_0^r d\eta(\theta)\int_0^\theta e^{-z\tau}\varphi(\tau-\theta)d\tau,\end{aligned}$$

with respect to z and we obtain

$$\begin{aligned}D_1H(z,\varphi) &= \int_0^r d\mu(\theta)\int_0^\theta e^{-z\tau}\varphi(\tau-\theta)d\tau\\ &\quad - z\int_0^r d\mu(\theta)\int_0^\theta \tau e^{-z\tau}\varphi(\tau-\theta)d\tau\\ &\quad - \int_0^r d\eta(\theta)\int_0^\theta \tau e^{-z\tau}\varphi(\tau-\theta)d\tau\\ &= \Lambda_1(z,\varphi).\end{aligned}$$

It is easy to check that for $n \geqslant 1$, we have

$$\frac{d}{dz}\Lambda_n(z,\varphi) = \Lambda_{n+1}(z,\varphi).$$

By induction, we have that (3.23) holds. □

Example 3.8. Consider the retarded equation

$$\dot{x}(t) = Bx(t-1), \quad t \geqslant 0, \qquad x_0 = \varphi \in \mathcal{C}, \tag{3.24}$$

where $B \neq 0$ is an $n \times n$-matrix. The characteristic equation is given by

$$\Delta(z) = zI - Be^{-z}. \tag{3.25}$$

For every simple root of $\det\Delta(z)$, the spectral projection is given by

$$(P_\lambda\varphi)(\theta) = \Big[\frac{d}{dz}\det\Delta(\lambda)\Big]^{-1}\operatorname{adj}\Delta(\lambda)\Big(\varphi(0) + B\int_0^1 e^{-\lambda\tau}\varphi(\tau-1)d\tau\Big)e^{\lambda\theta}.$$

In the scalar case, a root λ of Δ is not simple if and only if

$$\begin{cases}\lambda - Be^{-\lambda} = 0,\\ 1 + Be^{-\lambda} = 0.\end{cases}$$

Therefore, if $B \neq -1/e$ or equivalently $\lambda = -1$ is not a root of Δ, then all roots of (3.25) are simple. So the spectral projections are given by

$$(P_\lambda \varphi)(\theta) = \frac{1}{1+\lambda}\Big(\varphi(0) + B\int_0^1 e^{-\lambda\tau}\varphi(\tau-1)d\tau\Big)e^{\lambda\theta}, \tag{3.26}$$

where λ satisfies $\lambda - Be^{-\lambda} = 0$. Furthermore, it follows from Corollary 3.12 of Verduyn Lunel [57] that

$$x_t(\varphi) = \sum_{j=0}^{\infty} P_{\lambda_j} T(t)\varphi = \sum_{j=0}^{\infty} T(t) P_{\lambda_j}\varphi, \quad t > 0,$$

where λ_j, $j = 0, 1, \ldots$, denote the roots of $\lambda - Be^{-\lambda} = 0$, ordered according to decreasing real part. Using (3.26) and the fact that $T(t)e^{\lambda_j \cdot} = e^{\lambda_j(t+\cdot)}$, we can now explicitly compute the solution of (3.24) with initial condition $x_0 = \varphi$

$$x(t;\varphi) = \sum_{j=0}^{\infty} \frac{1}{1+\lambda_j}\Big(\varphi(0) + B\int_0^1 e^{-\lambda_j\tau}\varphi(\tau-1)d\tau\Big)e^{\lambda_j t}, \qquad t > 0.$$

(Compare Theorem 6 in Wright [61].)

If $B = -1/e$, then all zeros of $\Delta(z)$ are simple except for $\lambda = -1$. For the simple zeros we can again use (3.26). For the double zero $\lambda = -1$, we have to use (3.11) to compute the projection onto the two dimensional space $\mathcal{M}_{-1}$ and P_{-1} is given by

$$\begin{aligned}(P_{-1}\varphi)(\theta) = \frac{2}{3}\left(\varphi(0) + \int_0^1 (3\tau - 1)\, e^{\tau-1}\varphi(\tau-1)\, d\tau\right) e^{-\theta} \\ + 2\left(\varphi(0) - \int_0^1 e^{\tau-1}\varphi(\tau-1)\, d\tau\right)\theta\, e^{-\theta}\end{aligned}$$

Since $T(t)\phi = \phi(t+\cdot)$, where $\phi(\theta) = \theta e^{-\theta}$, we can again give the solution explicitly

$$\begin{aligned}x(t;\varphi) = {} & \frac{2}{3}\left(\varphi(0) + \int_0^1 (3\tau - 1)\, e^{\tau-1}\varphi(\tau-1)\, d\tau\right) e^{-t} \\ & + 2\left(\varphi(0) - \int_0^1 e^{\tau-1}\varphi(\tau-1)\, d\tau\right) t\, e^{-t} \\ & + \sum_{j=1}^{\infty} \frac{1}{1+\lambda_j}\Big(\varphi(0) - \int_0^1 e^{-\lambda_j\tau-1}\varphi(\tau-1)d\tau\Big)e^{\lambda_j t}, \qquad t > 0,\end{aligned}$$

where λ_j, $j = 1, 2, \ldots$ are the zeros of Δ ordered according to decreasing real part. Actually $\lambda = -1$ is a dominant eigenvalue, that is, $\operatorname{Re}\lambda_j < -1 + \epsilon$ for some $\epsilon > 0$. For $\tilde{B} > -1/e$, it is easy to see that

$$\lambda - \tilde{B}e^{-\lambda} = 0 \tag{3.27}$$

has two negative roots, and both tend to -1 as $\tilde{B} \downarrow -1/e$. Applying Lemma 5.4 to (3.27) with $\tilde{B} > -1/e$, we get that equation (3.27) has a real dominant eigenvalue λ_d. By continuity of roots of (3.27) with respect to $\tilde{B}$, we conclude that all roots of (3.27), with $\tilde{B} = -1/e$, have real part smaller or equal to -1. We can use the same arguments as in the proof of Lemma 5.4 to show that $\lambda = -1$ is indeed a dominant root. We can derive the large time behaviour

$$\lim_{t\to\infty} t\, e^t x(t,\varphi) = 2\left(\varphi\,(0) - \int_0^1 e^{\tau-1}\varphi(\tau-1)\,d\tau\right).$$

Chapter 4

Computing spectral projections symbolically

Given that λ is a zero of $\det \Delta(z)$ (see equation (2.4)), the computation of the spectral projection P_λ from $\mathcal{C}$ to the generalized eigenspace $\mathcal{M}_\lambda$ using Dunford calculus, as done in Chapter 3, is algorithmic. We developed a library that computes the characteristic matrix, the spectral projection for FDE's with fixed discrete time delays and some related tasks, called FDESpectralProj, for the computer algebra system Maple [43]. The library could be extended to deal with more general FDE's. The source code of the library is presented in the technical report [18][1]. The most useful routines of the library are `CharMatrix`, `CharEquation`, `OrderOfZero`, `MatrixOrderOfZero`, `SpectralProjection` and `Vfunction`. Their usage is described on this chapter.

4.1 Generating the loadable library from source code

In order to generate the file that provides a library that Maple can load like other standard packages, one just needs to evaluate (in Maple) the source code, found in the file `FDESpectralProj.mpl`, obtained in Frasson [18]. Maple interprets `.mpl` files as "Maple input", a plain text file with Maple code in it. Open the file and evaluate the whole worksheet. Once the code is evaluated, a file named `FDESpectralProj.m` is generated. This file could be "installed", put together with the standard library `.m`-files or kept somewhere, and one should customize the variable `libname`, which specifies where Maple looks for libraries, to include the

[1] At the time of this publication, the source code of the library FDESpectralProj was available for download the URL of [18], `http://www.math.leidenuniv.nl/reports/2005-01.shtml`.

path[2] to the generated file.

```
> libname:= "/path/to/library", libname;
> with(FDESpectralProj);
```

The library FDESpectralProj can handle neutral FDE with discrete time delays, that is, FDE of the type

$$\frac{d}{dt}Mx_t = Lx_t, \qquad x_0 = \varphi, \tag{4.1}$$

with

$$M\varphi = A_1\varphi(-t_1) + \cdots + A_k\varphi(-t_k), \tag{4.2}$$
$$L\,\varphi = B_1\varphi(-s_1) + \cdots + B_l\varphi(-s_l), \tag{4.3}$$

with $0 \leqslant t_i \leqslant r$, $0 \leqslant s_j \leqslant r$, A_i and B_j are $n \times n$-matrices or scalars if $n = 1$, for $i = 1, \dots, k$ and $j = 1, \dots, l$. Although the library does not require, the theory works that at delay time $\theta = 0$, the coefficient of M should be the identity (for instance, $t_1 = 0$ and $A_1 = I$).

Equation (4.1) is represented in the library by four Maple lists (expressions separated by commas, enclosed by square brackets), that we call here respectively *Mmat*, *Ldel*, *Lmat* and *Ldel*, where

- *Mmat* is the list of A_i;
- *Mdel* is the list of t_i;
- *Lmat* is the list of B_j;
- *Ldel* is the list of s_j.

As an example, we consider a one dimension neutral equation

$$\frac{d}{dt}\big(x(t) - x(t-1) + x(t-2)\big) = \frac{1}{4}x(t) + \frac{1}{2}x(t-1/2) - \frac{3}{4}x(t-1). \tag{4.4}$$

Then, for (4.4) we have that M and L are given by

$$M\varphi = \varphi(0) - \varphi\big(-\tfrac{1}{2}\big) + \varphi(-1), \qquad L\varphi = \frac{1}{4}\varphi(0) + \frac{1}{2}\varphi\big(-\tfrac{1}{2}\big) - \frac{3}{4}\varphi(-1). \tag{4.5}$$

The FDE (4.4) can be decoded into the Maple variables *Mmat*, *Ldel*, *Lmat* and *Ldel* with the following code

```
> Mmat_scalar:= [1, -1, 1];         Mdel_scalar:= [0, 1, 2];
  Lmat_scalar:= [1/4, 1/2, -3/4];   Ldel_scalar:= [0, 1, 2];
```

[2]If the library is in the same directory of the worksheet, you can use "." in place of "/path/to/library" in the Maple expression.

We consider also the following example with matrices as coefficients

$$\frac{d}{dt}\left(x(t) + \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix} x(t-1)\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{4.6}$$

Equation (4.6) can be decoded into the Maple variables *Mmat*, *Ldel*, *Lmat* and *Ldel* by evaluating the following code

```
> Mmat_mat:= [IdentityMatrix(2), Matrix([[-1,0],[1,0]])];
  Mdel_mat:= [0,1];    Lmat_mat:= [];      Ldel_mat:= [];
```

$$Mmat\_mat := \left[\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix}\right], \qquad Mmat\_mat := [0,1],$$
$$Lmat\_mat := [\,], \qquad Ldel\_mat := [\,].$$

For more examples where coefficients are matrices, see sections 6.3 and 7.4.

4.2 Usage of the routines

CharMatrix and CharEquation

We can compute the characteristic matrix $\Delta(z)$ and characteristic equation of FDE (4.1) with the routines `CharMatrix` and `CharEquation`. Both routines require 5 parameters.

CharMatrix(*Mmat, Mdel, Lmat, Ldel, z*)

CharEquation(*Mmat, Mdel, Lmat, Ldel, z*)

The parameters *Mmat*, *Ldel*, *Lmat*, *Ldel* are the variables that define the FDE and z is the variable. In the scalar case, *CharEquation* returns the characteristic matrix, while in the matrix case, it returns its determinant.

As a example, with equation (4.4), we evaluate the following code into Maple

```
> chareq_scalar:= CharEquation(Mmat_scalar, Mdel_scalar,
                               Lmat_scalar, Ldel_scalar, z);
```

$$chareq\_scalar := \left(-\frac{1}{2} - z\right)e^{-z} + \left(\frac{3}{4} + z\right)e^{-2z} + z - \frac{1}{4}.$$

In the case of (4.6), we can compute the characteristic matrix and equation by

```
> charmat_mat:= CharMatrix(Mmat_mat,Mdel_mat,Lmat_mat,Ldel_mat,z);
```

$$charmat\_mat := \begin{bmatrix} z - z\,e^{-z} & 0 \\ z\,e^{-z} & z \end{bmatrix}.$$

```
> chareq_mat:= CharEquation(Mmat_mat,Mdel_mat,Lmat_mat,Ldel_mat,z);
```

$$chareq\_mat := -z^2\,e^{-z} + z^2.$$

OrderOfZero and MatrixOrderOfZero

Given a symbolic expression *expr* (a matrix or a scalar algebraic expression), a variable z and a point z_0, the routines `OrderOfZero` and `MatrixOrderOfZero` (respectively for scalar expressions and matrices) compute the order of $z = z_0$ as zero of *expr*. The order of a zero is the smallest exponent of z in the Taylor expansion of *expr* around z_0. Hence, if 0 is returned, z_0 is *not* a zero. If 1 is returned, then z_0 is a simple zero of *expr*. If 2 is returned then z_0 is a double zero of *expr*, and so on. If $expr = 0$, then $z = 0$ is a zero of infinite order, and ∞ is returned by these routines.

OrderOfZero(*expr, z,* z_0 [*, maxorder*=$\langle$*value*$\rangle$])
MatrixOrderOfZero(*expr, z,* z_0 [*, maxorder*=$\langle$*value*$\rangle$])

To avoid infinite loop, the orders are computed up to a maximum 100. To override, use the optional parameter *maxorder*.

As examples, we compute the order of $z = 0$ as zeros of the characteristic equations.

```
> OrderOfZero(chareq_scalar,z,0);
  OrderOfZero(chareq_mat,z,0);
  MatrixOrderOfZero(Adjoint(charmat_mat),z,0);
```

$$2, \qquad 3, \qquad 1.$$

We obtain that, in the scalar case, $m_0 = 2$, and in the matrix case, $m_0 = 3$ and $k_0 = 3 - 1 = 2$. For the quantities m_λ and k_λ, see Section 2.1.

SpectralProjection

The routine `SpectralProjection` is the goal of this library. It needs 7 parameters: the first four to describe the equation, that is, *Mmat*, *Ldel*, *Lmat* and *Ldel*, and then respectively the zero of the characteristic equation z_0, a function or a name φ and the variable of the spectral projection θ. This would compute $\big(P_{z_0}\varphi\big)(\theta)$.

SpectralProjection(*Mmat, Mdel, Lmat, Ldel,* z_0*,* φ*,* θ
[*, maxorderofzero*=$\langle$*value*$\rangle$*, intvar*=$\langle$*variable*$\rangle$*,*
force_order_chareq=$\langle$*value*$\rangle$*,*
force_order_adjcharmat=$\langle$*value*$\rangle$])

Since the characteristic computes the order of z_0 as zero of the characteristic equation and the adjoint of the characteristic matrix, the optional parameter *maxorderofzero* can be used to override the default maximum order 100. However, if one wants to force those orders of z_0 as zero of the characteristic equation and the adjoint of the characteristic matrix, (for example when not making symbolic computations, but numerical computations), once can use the optional parameters *force_order_chareq* and *force_order_adjcharmat*. Also, the variable for the integrals can be specified with the optional parameter *intvar* (t is the default). The value of *intvar* only affects the presentation of the spectral projection, since it does not change anything in the computation (as x in $\int f(x)dx$ does not play any hole).

In the next example, from equation (4.4), we express $P_0\varphi(\theta)$ from $\mathcal{C}\big([-1,0],\mathbb{C}\big)$ into the two-dimensional generalized eigenspace $\mathcal{M}_0$, spanned by the functions $\theta \mapsto 1$ and $\theta \mapsto \theta$.

```
sp_scalar:= SpectralProjection(Mmat_scalar, Mdel_scalar,
                               Lmat_scalar, Ldel_scalar,
                               0, phi, theta);
```

$$\begin{aligned} sp\_scalar := &-\frac{28}{3}\varphi(0)+\frac{28}{3}\varphi(-1)-\frac{28}{3}\varphi(-2)-\frac{2}{3}\int_0^1 \varphi(t-1)dt \\ &+3\int_0^2 \varphi(t-2)dt-2\int_0^1 t\,\varphi(t-1)dt+3\int_0^2 t\,\varphi(t-2)dt \\ &+\theta\left(4\varphi(0)-4\varphi(-1)+4\varphi(-2)+2\int_0^1 \varphi(t-1)dt-3\int_0^2 \varphi(t-2)dt\right)\end{aligned}$$

We can evaluate the spectral projection in for a particular φ binding it with command `eval`. In the next example, we evaluate the spectral projection for the particular $\varphi(\theta)=\sin(\pi\theta)$, in two different ways.

```
> eval(sp_scalar, phi=(theta -> sin(Pi*theta)));
```

or, if one has a expression stored in a variable,

```
> phisin := sin(Pi*theta):
> eval(sp_scalar, phi=(t -> eval(phisin,theta=t)));
```

$$-\frac{8}{3\pi}-\frac{4}{\pi}\theta$$

N.b.: Note that we could not use just `phi=phisin` in the expression above since that does *not* pass φ as a function in Maple.

```
> eval(sp_scalar, phi=(t -> a0 + a1*t));
```

$$a_0+a_1\theta$$

We see that $m_0, m_1 \in \mathcal{C}$, given by $m_0(\theta)=1$ and $m_1(\theta)=\theta$, form a basis for $\mathcal{M}_0$ with FDE (4.4).

```
> sp_mat:= SpectralProjection(Mmat_mat, Mdel_mat,
                              Lmat_mat, Ldel_mat,
                              0, phi, theta);
```

$$sp\_mat := \begin{bmatrix} \frac{1}{2}\varphi_1(0) - \frac{1}{2}\varphi_1(-1) + \int_0^1 \varphi_1(t-1)dt + \theta\big(\varphi_1(0) - \varphi_1(-1)\big) \\ \frac{1}{2}\varphi_1(0) + \frac{1}{2}\varphi_1(-1) + \varphi_2(0) - \int_0^1 \varphi_1(t-1)dt + \theta\big(-\varphi_1(0) + \varphi_1(-1)\big) \end{bmatrix}$$

We can evaluate *sp_mat* at a vector of polynomials. In order to work in Maple, we have to provide a vector of functions.

```
> eval(sp_mat,phi:=[t -> a0+a1*t, t -> b0+b1*t]);
```

$$\begin{bmatrix} a_0 + a_1\theta \\ b_0 - a_1\theta \end{bmatrix}$$

One can easily identify that the three-dimensional subspace $\mathcal{M}_0$ is spanned by

$$m_0(\theta) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad m_1(\theta) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad m_2(\theta) = \begin{bmatrix} \theta \\ -\theta \end{bmatrix}.$$

Other routines

Between other routines, used internally in the library, we like mention here the routines `DiscreteTimeOperator` and `diffH`.

DiscreteTimeOperator(*mat, del,* φ [, *dimension*=$\langle$*value*$\rangle$])

The routine `DiscreteTimeOperator` can compute the evaluation of an operator (like M or L) applied a function φ. In case of empty lists *mat* and *del*, in order to decide the format of the output, the optional parameter *dimension* can be used. Its value is either the name *scalar* or a nonnegative integer.

For the computation of the spectral projection as provided in Section 3.3, it is used a linear operator $H(z, \varphi)$ and its derivatives with respect to z (see Lemma 3.2, formula (3.10), Theorem 3.6 and Lemma 3.7).

diffH(*Mmat, Mdel, Lmat, Ldel, z,* φ*, t, n*)

The routine `diffH` computes $D_1^n H(z, \varphi)$, where $n \geqslant 0$ and t is the variable of the integrals involved, for a FDE given by lists *Mmat, Mdel, Lmat* and *Ldel*.

4.3 Comments

The library FDESpectralProj implements the routine `Vfunction`, that can be used to compute the function $V(\lambda)$, given by equation (5.3), introduced in Chapter 5, which is used to test sufficient conditions for a real root of a characteristic equation to be a simple dominant root. For the use of the routine `Vfunction`, see Section 5.2.

CHAPTER 5

Criterions on the dominance of roots of characteristic equations

In this chapter, we present tests to check if a real root of a typical characteristic equations is simple and dominant. More precisely, we study scalar characteristic equations corresponding to

$$\frac{d}{dt}\Big(x(t) + \sum_{l=1}^{m} c_l x(t - \sigma_l)\Big) = ax(t) + \sum_{j=1}^{k} b_j x(t - h_j). \tag{5.1}$$

where a, b_j $(j = 1, \dots, k)$, c_l $(l = 1, \dots, m)$ are real numbers, and h_j $(j = 1, \dots, k)$, σ_l $(l = 1, \dots, m)$ are positive real numbers. The characteristic equation associated to (5.1) is given by

$$\Delta(z) = z\Big(1 + \sum_{l=1}^{m} c_l e^{-z\sigma_l}\Big) - a - \sum_{j=1}^{k} b_j e^{-zh_j}, \tag{5.2}$$

For a FDE of the form (5.1) (or equivalently for a analytic function in the form of (5.2)), we introduce a function $V : \mathbb{R} \to \mathbb{R}$, defined by

$$V(\lambda) = \sum_{l=1}^{m} |c_l| \, (1 + |\lambda| \, \sigma_l) e^{-\lambda\sigma_l} + \sum_{j=1}^{k} |b_j| \, h_j e^{-\lambda h_j}, \qquad \lambda \in \mathbb{R}. \tag{5.3}$$

This function plays a role on estimates of the derivative of $\Delta(z)$ (compare [46, Eq. P(λ_0)]).

5.1 Tests to decide dominance of roots

The next lemma states a sufficient condition for a real root of (5.2) to be simple and dominant.

Lemma 5.1. *Suppose that there exists a real zero λ_0 of $\Delta(z)$. If*

$$V(\lambda_0) < 1, \tag{5.4}$$

then λ_0 is a real simple dominant zero of $\Delta(z)$.

Proof. Without loss of generality, we can assume that $\lambda_0 = 0$. We can see that by the change of variables

$$v(t) = e^{-\lambda_0 t} x(t) \tag{5.5}$$

on (5.1). Since zeros of the characteristic equation yield exponential solutions and vice-verse, then λ is a zero of $\Delta(z)$ if and only if $x(t) = e^{\lambda t}$ is a solution of (5.1), if and only if $v(t) = e^{(\lambda-\lambda_0)t}$ is solution of the equivalent to (5.1) with change of variables (5.5) applied, if and only if $\lambda - \lambda_0$ is a zero of the characteristic equation related to the differential equation (5.1) in the variable v. Therefore, we have

$$\Delta(0) = a + \sum_{j=1}^{k} b_j = 0$$

and condition (5.4) becomes

$$\sum_{l=1}^{m} |c_l| + \sum_{j=1}^{k} h_j \, |b_j| < 1. \tag{5.6}$$

The proof consists of four parts. First we prove that λ_0 is a simple zero of (6.14). Since

$$\frac{d}{dz}\Delta(0) = 1 + \sum_{l=1}^{m} c_l + \sum_{j=1}^{k} h_j b_j,$$

it follows from (5.6) that

$$\left|\frac{d}{dz}\Delta(0)\right| \geqslant 1 - \left|\sum_{l=1}^{m} c_l + \sum_{j=1}^{k} h_j b_j\right| > 0.$$

Thus $\lambda_0 = 0$ is a simple zero of $\Delta(z)$.

Now we prove that there are no other zeros on the imaginary axis. Suppose that for some $\nu \neq 0$, $z = i\nu$ is a zero of $\Delta(z)$. From the equations for the real and imaginary part of $\Delta(i\nu)$, it follows that

$$\begin{cases} \nu \sum_{l=1}^{m} c_l \sin(\nu\sigma_l) - a - \sum_{j=1}^{k} b_j \cos(\nu h_j) = 0, \\ \nu(1 + \sum_{l=1}^{m} c_l \cos(\nu\sigma_l)) + \sum_{j=1}^{k} b_j \sin(\nu h_j) = 0. \end{cases}$$

Since $\nu \neq 0$, we obtain from the equation for the imaginary part that

$$\Big(1+\sum_{l=1}^{m} c_l \cos(\nu\sigma_l)\Big) + \sum_{j=1}^{k} b_j \frac{\sin(\nu h_j)}{\nu} = 0. \tag{5.7}$$

We can estimate

$$\left|1+\sum_{l=1}^{m} c_l \cos(\nu\sigma_l)\right| \geqslant 1 - \left|\sum_{l=1}^{m} c_l \cos(\nu\sigma_l)\right| \geqslant 1 - \sum_{l=1}^{m} |c_l| > \sum_{j=1}^{k} h_j \left|b_j\right|. \tag{5.8}$$

On the other hand, using $|\sin(x)| \leqslant |x|$, we have

$$\left|\sum_{j=1}^{k} b_j \frac{\sin(\nu h_j)}{\nu}\right| \leqslant \sum_{j=1}^{k} h_j \left|b_j\right|.$$

Thus we obtain from (5.7) and (5.8) that

$$\sum_{j=1}^{k} h_j \left|b_j\right| < \left|1+\sum_{l=1}^{m} c_l \cos(\nu\sigma_l)\right| = \left|\sum_{j=1}^{k} b_j \frac{\sin(\nu h_j)}{\nu}\right| \leqslant \sum_{j=1}^{k} h_j \left|b_j\right|.$$

A contradiction to the assumption that $z = i\nu$, $\nu \neq 0$, is a zero of $\Delta(z)$. This proves that $\lambda = 0$ is the only zero on the imaginary axis.

Next we show that there are no zeros of $\Delta(z)$ with $\operatorname{Re} z > 0$. Suppose that $z = \alpha + i\nu$ satisfies $\Delta(\alpha + i\nu) = 0$. The equations for the real and imaginary part read

$$\begin{cases} \alpha\big(1+\sum_{l=1}^{m} c_l e^{-\alpha\sigma_l}\cos(\nu\sigma_l)\big) + \nu\sum_{l=1}^{m} c_l e^{-\alpha\sigma_l}\sin(\nu\sigma_l) \\ \qquad\qquad -a - \sum_{j=1}^{k} b_j e^{-\alpha h_j}\cos(\nu h_j) = 0, \\ -\alpha\sum_{l=1}^{m} c_l e^{-\alpha\sigma_l}\sin(\nu\sigma_l) + \nu\big(1+\sum_{l=1}^{m} c_l e^{-\alpha\sigma_l}\cos(\nu\sigma_l)\big) \\ \qquad\qquad + \sum_{j=1}^{k} b_j e^{-\alpha h_j}\sin(\nu h_j) = 0. \end{cases} \tag{5.9}$$

If $\nu = 0$ and $\alpha > 0$, then the first equation of (5.9) yields

$$\alpha\Big(1+\sum_{l=1}^{m} c_l e^{-\alpha\sigma_l}\Big) = a + \sum_{j=1}^{m} b_j e^{-\alpha h_j} = \sum_{j=1}^{m} b_j (e^{-\alpha h_j} - 1).$$

So

$$\left|1+\sum_{l=1}^{m} c_l e^{-\alpha\sigma_l}\right| = \left|\sum_{j=1}^{m} b_j \frac{e^{-\alpha h_j} - 1}{\alpha}\right| \leqslant \sum_{j=1}^{m} \left|b_j\right| h_j.$$

On the other hand, using (5.6), we find

$$\left|1+\sum_{l=1}^{m} c_l e^{-\alpha\sigma_l}\right| > \sum_{j=1}^{m} \left|b_j\right| h_j.$$

A contradiction to the assumption that $\nu = 0$. Thus we can assume that $\alpha > 0$ and $\nu > 0$. We claim that

$$\sum_{l=1}^{m} c_l e^{-\alpha\sigma_l} \sin(\nu\sigma_l) < 0. \tag{5.10}$$

Suppose first that the claim holds. Then it follows from the second equation of (5.9) that

$$1 + \sum_{l=1}^{m} c_l e^{-\alpha\sigma_l} \cos(\nu\sigma_l) < -\sum_{j=1}^{k} b_j e^{-\alpha h_j} \frac{\sin(\nu h_j)}{\alpha} \leqslant \sum_{j=1}^{k} h_j |b_j|.$$

On the other hand, using (5.6), we have

$$1 + \sum_{l=1}^{m} c_l e^{-\alpha\sigma_l} \cos(\nu\sigma_l) \geqslant 1 - \left|\sum_{l=1}^{m} c_l e^{-\alpha\sigma_l} \cos(\nu\sigma_l)\right| \geqslant 1 - \sum_{l=1}^{m} |c_l| > \sum_{j=1}^{k} h_j |b_j|.$$

A contradiction. Thus, if (5.10) holds, there are no zeros of $\Delta(z)$ with $\operatorname{Re} z > 0$.

To prove (5.10) (still assuming $\alpha > 0$), we first take the following combinations of (5.9). The combination α times the first plus ν times the second equation of (5.9) yields

$$\begin{aligned}(\alpha^2 + \nu^2)\Big(1 + \sum_{l=1}^{m} c_l e^{-\alpha\sigma_l} \cos(\nu\sigma_l)\Big)& \\ + \alpha \sum_{j=1}^{k} b_j \big(e^{-\alpha h_j} \cos(\nu h_j) - 1\big) + \nu \sum_{j=1}^{m} b_j e^{-\alpha h_j} \sin(\nu h_j) &= 0,\end{aligned} \tag{5.11}$$

and the combination ν times the first minus α times the second equation of (5.9) yields

$$\begin{aligned}(\alpha^2 + \nu^2) \sum_{l=1}^{m} c_l e^{-\alpha\sigma_l} \sin(\nu\sigma_l) + \nu \sum_{j=1}^{k} b_j (e^{-\alpha h_j} \cos(\nu h_j) - 1)& \\ - \alpha \sum_{j=1}^{m} b_j e^{-\alpha h_j} \sin(\nu h_j) &= 0.\end{aligned} \tag{5.12}$$

From the first equation of (5.9) and the fact that $1 + \sum_{l=1}^{m} c_l e^{-\alpha\sigma_l} \cos(\nu\sigma_l) > 0$, it follows that

$$\nu \sum_{l=1}^{m} c_l e^{-\alpha\sigma_l} \sin(\nu\sigma_l) \leqslant \sum_{j=1}^{k} b_j (e^{-\alpha h_j} \cos(\nu h_j) - 1). \tag{5.13}$$

From the second equation of (5.9) and the fact that $1+\sum_{l=1}^{m} c_l e^{-\alpha\sigma_l}\cos(\nu\sigma_l) > 0$, it follows that

$$\sum_{j=1}^{k} b_j e^{-\alpha h_j}\sin(\nu h_j) < \alpha \sum_{l=1}^{m} c_l e^{-\alpha\sigma_l}\sin(\nu\sigma_l). \tag{5.14}$$

From (5.11) and the fact that $1+\sum_{l=1}^{m} c_l e^{-\alpha\sigma_l}\cos(\nu\sigma_l) > 0$,

$$\alpha \sum_{j=1}^{k} b_j (e^{-\alpha h_j}\cos(\nu h_j) - 1) < -\nu \sum_{j=1}^{k} b_j e^{-\alpha h_j}\sin(\nu h_j). \tag{5.15}$$

From (5.13) and (5.15), it follows that

$$\alpha\nu \sum_{l=1}^{m} c_l e^{-\alpha\sigma_l}\sin(\nu\sigma_l) < -\nu \sum_{j=1}^{k} b_j e^{-\alpha h_j}\sin(\nu h_j)$$

and hence

$$\sum_{j=1}^{k} b_j e^{-\alpha h_j}\sin(\nu h_j) < -\alpha \sum_{l=1}^{m} c_l e^{-\alpha\sigma_l}\sin(\nu\sigma_l). \tag{5.16}$$

From (5.14) and (5.16), it now follows

$$\sum_{j=1}^{k} b_j e^{-\alpha h_j}\sin(\nu h_j) < 0$$

and hence, from (5.12),

$$(\alpha^2+\nu^2) \sum_{l=1}^{m} c_l e^{-\alpha\sigma_l}\sin(\nu\sigma_l) < -\nu \sum_{j=1}^{k} b_j (e^{-\alpha h_j}\cos(\nu h_j) - 1). \tag{5.17}$$

Finally, from (5.13) and (5.17), it follows that

$$\sum_{l=1}^{m} c_l e^{-\alpha\sigma_l}\sin(\nu\sigma_l) < 0$$

and this proves the claim (5.10).

Thus we have proved that $\lambda_0 = 0$ is the only zero of $\Delta(z)$ with $\operatorname{Re} z \geqslant 0$. To complete the proof of the lemma, it remains to prove that $\lambda_0 = 0$ is dominant.

Suppose to the contrary that, for every $\delta > 0$, there exists a zero of $\Delta(z)$ in the strip $-\delta < \operatorname{Re} z < 0$. Since $\Delta(z)$ is an entire function, its zeros cannot have a finite accumulation point. So it follows that $\Delta(z)$ has a sequence of zero's λ_n such that $\operatorname{Re}\lambda_n$ tends to zero and $|\operatorname{Im}\lambda_n|$ tends to infinity. From (5.6) it follows that for $\operatorname{Re}\lambda_n$ sufficiently small, there exists an $\epsilon > 0$ such that

$$\left|1 + \sum_{l=1}^{m} c_l e^{-\lambda_n \sigma_l}\right| \geqslant 1 - \sum_{l=1}^{m} |c_l|\, e^{-\operatorname{Re}\lambda_n \sigma_l} > \epsilon.$$

Thus

$$\epsilon\,|\lambda_n| < a + \sum_{j=1}^{k} |b_j|\, e^{-\operatorname{Re}\lambda_n h_j}. \tag{5.18}$$

As n tends to infinity the left hand side of (5.18) tends to infinity while the right hand side of (5.18) remains bounded. A contradiction to the assumption that, every $\delta > 0$, there exists a zero of $\Delta(z)$ in the strip $-\delta < \operatorname{Re} z < 0$. This completes the proof that $\lambda_0 = 0$ is dominant simple zero of $\Delta(z)$. □

Remark 5.2. Actually, condition (5.4) is sharp. This can be seen by considering the following example

$$\frac{d}{dt}\Big[x(t) - \frac{1}{2}x(t-1)\Big] = \frac{1}{2}x(t) - \frac{1}{2}x(t-1). \tag{5.19}$$

The characteristic equation associated with (5.19) is given by

$$z\Big(1 - \frac{1}{2}e^{-z}\Big) = \frac{1}{2} - \frac{1}{2}e^{-z}.$$

So $c_1 = -1/2$, $a = 1/2$, $b_1 = -1/2$, $\sigma_1 = 1$ and $h_1 = 1$. Thus $a + b_1 = 0$ and $\lambda_0 = 0$ is a real zero. Since $|c_1| + h_1|b_1| = 1$, condition (5.4) fails, and if we differentiate the characteristic equation, it follows that $\lambda_0 = 0$ is not a simple zero.

The next lemma gives sufficient conditions for the existence of a simple and dominant real root of (5.2).

Lemma 5.3. *If there exists γ such that*

$$\Delta(\gamma) < 0 \qquad \text{and} \qquad V(\gamma) \leqslant 1,$$

then there exists a unique real root λ_0 of Δ in (γ, ∞) such that $V(\lambda_0) < 1$. Therefore λ_0 is a simple real dominant root of (5.2).

Proof. First of all, we observe that V is positive and strictly decreasing. Indeed, since the functions $h_1(x) = e^{-x}$ and $h_2(x) = (1 + |x|)e^{-x}$ are positive and strictly decreasing for real x, we obtain that functions defined by linear combinations with positive coefficients of h_1 and h_2, with possible positive rescales of the variable x, are positive and strictly decreasing.

Computing the derivative of Δ for $\lambda > \gamma$, we obtain that

$$\begin{aligned}\frac{d}{d\lambda}\Delta(\lambda) &= 1 + \sum_{l=1}^{m} c_l e^{-\lambda\sigma_l} - \lambda \sum_{l=1}^{m} c_l \sigma_l e^{-\lambda\sigma_l} + \sum_{j=1}^{k} b_j h_j e^{-\lambda h_j} \\ &\geqslant 1 - \left(\sum_{l=1}^{m} |c_l| \left(1 + |\lambda| \, \sigma_l\right) e^{-\lambda\sigma_l} + \sum_{j=1}^{k} |b_j| \, h_j e^{-\lambda h_j} \right) \\ &= 1 - V(\lambda) > 0,\end{aligned}$$

since $V(\lambda) < V(\gamma) \leqslant 1$. Therefore $\Delta(\lambda)$ is strictly increasing for $\lambda > \gamma$. It is easy to see that

$$\lim_{\lambda\to\infty} \Delta(\lambda) = \infty.$$

This together with the fact that $\Delta(\gamma) < 0$ implies the existence of a unique real root λ_0 of Δ with $\lambda_0 > \gamma$. Since V is strictly decreasing, it follows that $V(\lambda_0) < 1$. The last assertion now follows from Lemma 5.1. $\square$

In a simpler case, we can improve the conclusions of Lemma 5.3 even further. The next lemma is used in Theorem 6.4 and improves the results in [16] and [46].

Lemma 5.4. *If $-e^{-1} < b\tau e^{-a\tau}$, then the equation*

$$\Delta(z) = z - a - be^{-\tau z} \tag{5.20}$$

has a simple real dominant zero $z = \lambda_d$.

Proof. Suppose first that $-e^{-1} < b\tau e^{-a\tau} < 0$. This implies that $b < 0$. If we define $\gamma = a - 1/\tau$, then

$$\Delta(\gamma) = \frac{1}{\tau} - be^{-a\tau-1} < 0,$$

and (recall equation (5.3))

$$V(\gamma) = |b| \, \tau e^{-a\tau+1} = -b\tau e^{-a\tau+1} < 1.$$

So the hypotheses of Lemma 5.3 are satisfied and the existence of the simple real dominant zero λ_d of Δ follows.

Before we continue with the case $b > 0$, we need to study how the location of the real roots of Δ depends on b. If λ is a real zero of Δ, then

$$\lambda = a + be^{-\tau\lambda}.$$

Observe that, if $b < 0$ then $\lambda < a$ and if $b > 0$, then $\lambda > a$.

Moreover, for $b > 0$, such real λ always exists, is simple and unique, because $\Delta(a) < 0$,

$$\lim_{z\to\infty} \Delta(z) = \infty$$

and for all z real

$$\frac{d}{dz}\Delta(z) = 1 + b\tau e^{-\tau z} > 0.$$

Denote this root by λ_d. Fix a and consider λ_d as a function of $b > 0$. If we differentiate $\Delta\big(\lambda_d(b)\big)$ with respect to b, we obtain

$$\frac{d\lambda_d}{db} = \frac{e^{-\tau\lambda_d}}{1 + b\tau e^{-\tau\lambda_d}} > 0.$$

Therefore $b \mapsto \lambda_d(b)$ is strictly increasing on $[0,\infty)$.

With these preliminaries, we can now show that in case $b > 0$, the root λ_d is also dominant. Suppose that $z = x+iy$ is another zero of Δ with x, y real numbers and $y > 0$. The equations for the real and imaginary parts for z are given by

$$x - a - b\cos(\tau y)e^{-\tau x} = 0, \tag{5.21}$$

$$y + be^{-\tau x}\sin(\tau y) = 0. \tag{5.22}$$

Define $\bar{b} = b\cos(\tau y)$. Equation (5.21) becomes

$$x - a - \bar{b}e^{-\tau x} = 0. \tag{5.23}$$

If $\bar{b} \leqslant 0$, then (5.23) and the arguments just given imply that $x \leqslant a < \lambda_d$. If $\bar{b} > 0$ then equation (5.22) and $y > 0$ imply that $\tau y \neq k\pi$ for all k integer and therefore $\bar{b} < b$, but this again implies that $x < \lambda_d$.

In order to show that there exists an $\epsilon > 0$ such that actually all roots with $z \neq \lambda_d \in \mathbb{C}$ satisfies $\operatorname{Re} z < \lambda_d - \epsilon$, we argue by contradiction and suppose that such ϵ does not exist. So there is a sequence z_n of zeros of Δ such that $\operatorname{Re} z_n \to \lambda_d$. Since Δ is an analytic function, its zeros are isolated, and therefore the only possibility is that $|\operatorname{Im} z_n| \to \infty$, but (5.22) shows that $\operatorname{Im} z_n$ is in fact bounded. A contradiction. Therefore also in the case $b > 0$, λ_d is a simple real dominant zero of Δ. This completes the proof of the lemma. □

Now we study the location of roots of the scalar neutral functional differential equation

$$\dot{x}(t) + a_1\dot{x}(t-1) = b_0 x(t) + b_1 x(t-1). \tag{5.24}$$

where $a_1 \neq 0$, b_0 and b_1 are real numbers. The characteristic equation associated with (5.24) is given by

$$\Delta(z) = z(1 + a_1 e^{-z}) - b_0 - b_1 e^{-z} = 0 \tag{5.25}$$

For this class of equation we can state

Theorem 5.5. *Suppose that λ_0 is a real root of $\Delta(z)$ in (5.25). If*

$$|a_1| < e^{\lambda_0}(1 - |b_0 - \lambda_0|), \tag{5.26}$$

then λ_0 is a simple dominant root of (5.24). Even when

$$|a_1| = e^{\lambda_0}(1 - |b_0 - \lambda_0|), \tag{5.27}$$

if in addition we have that $b_0 \neq \lambda_0$, then λ_0 is a dominant root, which is also simple if and only if

$$a_1 \neq -e^{\lambda_0}(1 - b_0 + \lambda_0). \tag{5.28}$$

Proof. First we transform (5.24) into a simples FDE in v by applying the change of variables $v(t) = e^{-\lambda t}x(t)$, suggested in the beginning of proof of Lemma 5.1

$$\dot v(t) + \underbrace{a_1 e^{-\lambda_0}}_{\stackrel{\text{def}}{=}\tilde a_1} \dot v(t-1) = \underbrace{(b_0 - \lambda_0)}_{\stackrel{\text{def}}{=}\tilde b_0} v(t) + \underbrace{(b_1 - a_1\lambda_0 e^{-\lambda_0})}_{\stackrel{\text{def}}{=}\tilde b_1} v(t-1)$$

which has as characteristic equation

$$\tilde\Delta(\tilde\lambda) = \tilde\lambda(1 + \tilde a_1 e^{-\tilde\lambda}) - \tilde b_0 - \tilde b_1 e^{-\tilde\lambda} = 0. \tag{5.29}$$

Since roots of $\Delta(z)$ correspond to exponential solutions of (5.24), it is easy to see that λ is root of $\Delta(z)$ if and only if $\lambda - \lambda_0$ is root of $\tilde\Delta(z)$. But $\tilde\Delta(0) = -\tilde b_0 - \tilde b_1 = 0$ what implies that $\tilde b_0 = -\tilde b_1$ (this also comes from the fact that λ_0 is a root of $\Delta(z)$.)

Using this fact, the result with the inequality (5.26) follows as a direct application of Lemma 5.1, where the condition $V(\lambda_0) < 1$ becomes $|\tilde a_1| < 1 - |\tilde b_0|$, which is equivalent to condition (5.26).

Now, suppose that (5.27) holds, that is, $|\tilde a_1| = 1 - |\tilde b_0|$ with $\tilde b_0 \neq 0$. Then $|\tilde a_1| < 1$ and necessarily $|\tilde b_0| \leqslant 1$.

We apply Theorem 1.21, where $\Delta_0(z) = 1 + \tilde a_1 e^{-\tilde\lambda}$ in case of characteristic equation (5.29), to obtain that $a_M = -\log(1/|\tilde a_1|) < 0$ (see Figure 1.1). Therefore there exists $\epsilon > 0$ such that there are there are finitely many roots of $\tilde\Delta(z)$ in the right-half plane $\{z \in \mathbb{C} : \operatorname{Re} z \geqslant -\epsilon\}$. Therefore, to prove that 0 is a dominant root of $\tilde\Delta(z)$ it suffices to show that there is no root λ of $\tilde\Delta(z)$ with $\operatorname{Re}\lambda > 0$ and that there is no root of $\tilde\Delta(z)$ in the imaginary axis other than 0.

Consider the perturbation of $\tilde\Delta(z)$ given by

$$f_a(z) = z(1 + ae^{-z}) - \tilde b_0 + \tilde b_0 e^{-z} = 0$$

with $|\tilde a_1|/2 < |a| < |\tilde a_1| = 1 - |\tilde b_0|$. We have that $f_a(0) = 0$. Applying again Lemma 5.1, we conclude that $z = 0$ is a simple dominant root of $f_a(z)$. By continuity of roots with respect to a when $a \to \tilde a_1$ and $|\tilde a_1| < |a| < |\tilde a_1|$, we conclude that there is no root λ of $\tilde\Delta(z)$ with $\operatorname{Re}\lambda > 0$.

Finally, suppose that for some $w > 0$ we have $\tilde{\Delta}(iw) = 0$. Taking real and imaginary parts on both sides, we obtain the system

$$\begin{aligned} w + a_1 w \cos w - \tilde{b}_0 \sin w &= 0 \\ a_1 w \sin w - \tilde{b}_0 + \tilde{b}_0 \cos w &= 0 \end{aligned} \tag{5.30}$$

Since $|\tilde{a}_1| = 1 - |\tilde{b}_0|$ there are four possibilities:

i) $\tilde{a}_1 = -1 - \tilde{b}_0$ if $\tilde{a}_1 < 0$ and $\tilde{b}_0 < 0$;

ii) $\tilde{a}_1 = -1 + \tilde{b}_0$ if $\tilde{a}_1 < 0$ and $\tilde{b}_0 > 0$;

iii) $\tilde{a}_1 = 1 + \tilde{b}_0$ if $\tilde{a}_1 > 0$ and $\tilde{b}_0 < 0$;

iv) $\tilde{a}_1 = 1 - \tilde{b}_0$ if $\tilde{a}_1 > 0$ and $\tilde{b}_0 > 0$.

In any of there cases, system (5.30) becomes a system in the variables w and $\tilde{b}_0$, linear in $\tilde{b}_0$. In the case i), system (5.30) is equivalent to

$$\underbrace{\begin{bmatrix} -w \sin w & -w \sin w - 1 + \cos w \\ w - w \cos w & -w \cos w - \sin w \end{bmatrix}}_{\overset{\text{def}}{=} M_i} \begin{bmatrix} 1 \\ \tilde{b}_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{5.31}$$

The matrix M_i is singular if and only if

$$\frac{\det M_i}{w} = w \sin w + 2 - 2 \cos w = 0. \tag{5.32}$$

If w is of form $w = k\pi$ with $k \in \mathbb{Z}$, $k \neq 0$, then necessarily k is even and (5.31) becomes

$$\begin{bmatrix} 0 & 0 \\ 0 & -w \end{bmatrix} \begin{bmatrix} 1 \\ \tilde{b}_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

what implies that $\tilde{b}_0 = 0$, but this contradicts our hypothesis. If $w \neq k\pi$ for all $k \in \mathbb{Z}$, then from (5.32) we obtain that $w \sin w = -2(1 - \cos w) \neq 0$ and (5.31) becomes

$$\begin{bmatrix} 2(1 - \cos w) & (1 - \cos w) \\ w(1 - \cos w) & -w \cos w - \sin w \end{bmatrix} \begin{bmatrix} 1 \\ \tilde{b}_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

and the only possible solution, if it exists at all, is $\tilde{b}_0 = -2$, which has modulus larger than 1, contradicting earlier discussions. Therefore, for case i), there are no $\lambda = iw$ root of $\tilde{\Delta}(z)$ with $w \neq 0$. We can argue in a similar way for the other cases. Finally,

$$\frac{d}{dz}\tilde{\Delta}(0) = 1 + \tilde{a}_1 - \tilde{b}_0$$

which is nonzero, implying that $z = 0$ is a simple root, if and only if condition (5.28) holds This completes the proof. □

5.2 Computing $V(\lambda)$ symbolically

The function V, given by equation (5.3), can be also computed symbolically, using for instance the routine `Vfunction` implemented in the Maple library FDESpectralProj, presented in Chapter 4. In order to use *Vfunction*, one has to "translate" the FDE (5.1) in Maple lists, like the variables *Mmat*, *Mdel*, *Lmat* and *Ldel* that are described in Chapter 4. The syntax of `Vfunction` is the following.

Vfunction(*Mmat, Mdel, Lmat, Ldel,* λ)

As an illustration, we consider the equation

$$\frac{d}{dt}\big(x(t)+x(t-1)-3x(t-2)\big)=ax(t)+2x(t-1)-\frac{1}{2}x(t-1)+x(t-3). \quad (5.33)$$

We observe that $V(\lambda)$ is independent from a, but a plays an important role in determining a real root of the characteristic equation. We have that the characteristic equation associated to (5.33) is given by

$$\Delta(z):=(-2+z)e^{-z}+\Big(\frac{1}{2}-3z\Big)e^{-2z}+z-a-e^{-3z} \quad (5.34)$$

If one likes to use the Maple library FDESpectralProj, presented in Chapter 4, one should evaluate in Maple a code similar to the following

```
> libname:= "/path/to/library", libname;
> with(FDESpectralProj);
> Mmat:= [1,1,-3];        Mdel:= [0,1,2];
  Lmat:= [a,2,-1/2,1];    Ldel:= [0,1,2,3];
> V:=Vfunction(Mmat,Mdel,Lmat,Ldel,lambda);
```

$$V:=(1+|\lambda|)e^{-\lambda}+3(1+2|\lambda|)e^{-2\lambda}+2e^{-\lambda}+e^{-2\lambda}+3e^{-3\lambda}. \quad (5.35)$$

We have that $V(\lambda)$ is always a positive decreasing function of λ such that $V(\lambda)\downarrow 0$ as $\lambda\to\infty$. Therefore, there exists λ_0 such that $V(\lambda)<1$ for all $\lambda>\lambda_0$. In our particular case, one can plot $V(\lambda)$, or solve $V(\lambda)=1$ numerically in order to discover that $\lambda_0\approx 1.978313308$. Therefore, for any $z_0>\lambda_0$, for the particular value of a given by

$$a=(-2+z_0)e^{-z_0}+\Big(\frac{1}{2}-3z_0\Big)e^{-2z_0}+z_0-e^{-3z_0}, \quad (5.36)$$

we have that z_0 is a root of $\Delta(z)=0$ and that $V(z_0)<1$. Therefore, the hypothesis of Lemma 5.1 are satisfied and we have that z_0 is a real simple dominant root of $\Delta(z)$. As an example, for the particular value of a given by

$$a=-\frac{11}{2}e^{-4}-e^{-6}+2\approx 1.897,$$

we have that $z_0=2>\lambda_0$ is a real simple dominant root of the characteristic equation $\Delta(z)$.

5.3 Comments

The contents of this chapter were presented in Frasson & Verduyn Lunel [19], with exception of Theorem 5.5, which is first presented here.

A remarkable feature of the function V is that it does not depend on the coefficient with the term $x(t)$, and this coefficient appears in the characteristic equation subtracted in a simple way, see the formula of $\Delta(z)$ on (5.2). We have that $V(\lambda)$ is a strictly decreasing function and that $\Delta(z)$ becomes strictly increasing for z sufficiently large (more precisely $\Delta(z) = z - a + o(z)$ as $z \to \infty$.) Therefore, for a sufficiently large, we have from Lemma 5.3 that there exists a real root λ_0 of the characteristic equation which is simple and dominant. This remark shows that the criteria presented here are useful.

Chapter 6

The large time behaviour

In Chapter 1, we considered the autonomous linear FDE

$$\begin{cases} \frac{d}{dt} M x_t = L x_t, & t \geqslant 0, \\ x_0 = \varphi, & \varphi \in \mathcal{C}, \end{cases} \tag{6.1}$$

where $L, M : \mathcal{C} \to \mathbb{C}^n$ are linear continuous, given respectively by

$$L\varphi = \int_0^r d\eta(\theta)\varphi(-\theta), \qquad M\varphi = \varphi(0) - \int_0^r d\mu(\theta)\varphi(-\theta), \tag{6.2}$$

where $\eta, \mu \in NBV\big([0, r], \mathcal{L}(\mathbb{C}^n)\big)$ and μ is continuous at zero. In order to make the analysis of the characteristic equations easier, we assume throughout this section that the coefficients of the functional differential equations are real-valued (the state space remains $\mathcal{C}([-r, 0], \mathbb{C}^n)$). We presented the solution semigroup $T(t) : \mathcal{C} \to \mathcal{C}$ as a strongly continuous semigroup that maps φ to x_t, where x is the solution of (6.1). We denote the infinitesimal generator of $T(t)$ as A. We showed that the large time behavior of the solutions of (6.1) is governed by polynomials times exponential functions which are the residues of some functions on zeros of the associated characteristic equation (see Theorem 1.27)

$$\Delta(z) = z\Big[I - \int_0^r d\mu(t)e^{-zt}\Big] - \int_0^r d\eta(t)e^{-zt}. \tag{6.3}$$

provided that μ satisfies the jump condition (J), that is, the entries μ_{ij} of μ have a jump before they become constant, that is, there exists t_{ij} with $\mu_{ij}(t_{ij}-) \neq \mu_{ij}(t_{ij}+)$ and $\mu_{ij}(t_{ij}+) = \mu_{ij}(t)$ for $t > t_{ij}$. We used the representation of (6.1) as a renewal equation, and the Laplace-Stieltjes transform in the analysis.

In Chapter 2, we decompose the state space according to the spectrum of the generator A, both for the autonomous case and for a class of periodic FDE. The spectrum of A consists only of point spectrum and is given by

$$\sigma(A) = \{\lambda \in \mathbb{C} : \det \Delta(z) = 0\}$$

where

$$\Delta(z) = z\Big[I - \int_0^r d\mu(t)e^{-zt}\Big] - \int_0^r d\eta(t)e^{-zt}. \tag{6.4}$$

Let $\lambda \in \sigma(A)$. See Section 2.1 for more on the relation between the spectral properties of the infinitesimal generator A and the characteristic matrix $\Delta(z)$, and the definition of the generalized eigenspace $\mathcal{M}_\lambda$. We denote by P_λ the projection from $\mathcal{C}$ onto $\mathcal{M}_\lambda$. Some estimates are shown for the restriction of the solution semigroup $T(t)$ to the $\mathcal{M}_\lambda$, which are invariant subspaces. See, for instance, Lemma 2.1 and Theorem 2.3. In Chapter 3 we gave explicit formulas for the computation of the spectral projection on the generalized eigenspace of any eigenvalue of the A.

In this chapter, we combine these ideas in the study of the large time behaviour of solutions of autonomous and periodic Functional Differential Equation. Note that there is no further hypothesis, like condition (J), used in Chapter 1, on the *NBV* function μ.

6.1 Large time behaviour for autonomous FDE

In this section we shall further investigate the case when $\det \Delta(z)$ has a dominant root $z = \lambda_d$.

Theorem 6.1. *Consider the FDE (6.1)–(6.2) and let $\Delta(z)$ be as in (6.4). Suppose that* $\det \Delta(z)$ *has a dominant simple zero λ_d,* i.e., *suppose that* $\det \Delta(\lambda_d) = 0$ *and there exists $\delta > 0$ such that if* $\det \Delta(\lambda) = 0$ *and $\lambda \neq \lambda_d$, then* $\operatorname{Re} \lambda < \operatorname{Re} \lambda_d - \delta$. *Then there exist positive numbers ϵ and N such that*

$$\|e^{-\lambda_d t}T(t)\varphi - P_{\lambda_d}\varphi\| \leqslant Ne^{-\epsilon t} \tag{6.5}$$

and

$$\begin{aligned} \lim_{t\to\infty} e^{-\lambda_d t}T(t)\varphi = e^{\lambda_d \cdot}\Big[\frac{d}{dz}\det \Delta(\lambda_d)\Big]^{-1} \operatorname{adj} \Delta(\lambda_d)\Big[M\varphi \\ + \int_{-r}^0 [\lambda_d d\mu(\tau) + d\eta(\tau)] \int_0^{-\tau} e^{-\lambda_d \sigma}\varphi(\sigma + \tau)\, d\sigma\Big]. \end{aligned} \tag{6.6}$$

Proof. From representation of the spectral projection P_λ on (3.8), it follows that P_λ and A commute, and therefore P_λ and $T(t)$ commute as well. The spectral decomposition with respect to λ_d yields

$$e^{-\lambda_d t}T(t)\varphi = e^{-\lambda_d t}T(t)P_{\lambda_d}\varphi + e^{-\lambda_d t}T(t)(I - P_{\lambda_d})\varphi.$$

From the exponential Lemma 2.1, it follows that there exist positive ϵ and N such that

$$\|e^{-\lambda_d t}T(t)(I - P_{\lambda_d})\varphi\| \leqslant Ne^{-\epsilon t}, \qquad t \geqslant 0.$$

The action of $T(t)$ restricted to a one-dimensional eigenspace $\mathcal{M}_{\lambda_d}$ is given by

$$e^{-\lambda_d t}T(t)P_{\lambda_d} = P_{\lambda_d}.$$

This shows (6.5) and using Lemma 3.1, we arrive at (6.6). □

If we evaluate (6.6) at $\theta = 0$ we obtain the following corollary.

Corollary 6.2. *Consider the FDE (6.1)–(6.2) and let $\Delta(z)$ be as in (6.4) and let λ_d to be a simple dominant zero of $\det \Delta(z)$. If $x(t) = x(t;\varphi)$ denotes the solution of (6.1) with initial data $x_0 = \varphi$, then the large time as a function of the initial data φ is given by*

$$\lim_{t\to\infty} e^{-\lambda_d t}x(t) = \Big[\frac{d}{dz}\det\Delta(\lambda_d)\Big]^{-1} \operatorname{adj}\Delta(\lambda_d)\Big[M\varphi + \int_{-r}^{0}[\lambda_d d\mu(\tau) + d\eta(\tau)]\int_{0}^{-\tau} e^{-\lambda_d\sigma}\varphi(\sigma+\tau)\,d\sigma\Big]. \tag{6.7}$$

The next theorem, that gives a result similar to the Corollary 6.2, but can be applied to real dominant eigenvalues that are not simple.

Theorem 6.3. *Consider the FDE (6.1)–(6.2) and let $\Delta(z)$ be as in (6.4) and let λ_d to be a real dominant zero of $\det\Delta(z)$ of geometric multiplicity $n \geqslant 1$. If $x(t) = x(t;\varphi)$ denotes the solution of (6.1) with initial data $x_0 = \varphi$, then the large time as a function of the initial data φ is given by:*

1. *if $P_{\lambda_d}\varphi \neq 0$, then*
$$\lim_{t\to\infty}\frac{1}{t^m}e^{-\lambda_d t}x(t) = q_m(n,\lambda_d,\varphi). \tag{6.8}$$
where $m = \max\{j \in \{0,1,\dots,n-1\} : q_j(n,\lambda_d,\varphi) \neq 0\}$, with q_j given by formula (3.20) in Theorem 3.6.

2. *if $P_{\lambda_d}\varphi = 0$, then*
$$\lim_{t\to\infty} e^{-\lambda_d t}x(t) = 0. \tag{6.9}$$

Proof. We can use the same arguments as in the proof of Theorem 6.1, but now the action of $T(t)$ on the nth dimensional generalized eigenspace $\mathcal{M}_{\lambda_d}$ is given by

$$e^{-\lambda_d t}\big(T(t)P_{\lambda_d}\varphi\big)(0) = \sum_{j=0}^{n-1} q_j t^j, \tag{6.10}$$

where the "exponential times polynomial" form of P_{λ_d} is given by Theorem 3.6. Analyzing the large time behaviour of polynomial on (6.10) yields relations (6.8) and (6.9). □

6.2 Application for a scalar delay equation

Consider the scalar delay equation

$$\dot{x}(t) = ax(t) + bx(t-\tau), \quad t \geqslant 0, \tag{6.11}$$

with a and b real numbers. The characteristic equation of (6.11) is given by

$$\Delta(z) = z - a - be^{-\tau z}.$$

From Lemma 5.4, it follows that if $-e^{-1} < b\tau e^{-a\tau}$, then $\Delta(z)$ has a simple real dominant root and we can use Corollary 6.2 to compute the large time behaviour of the solutions of (6.11). The next theorem presents a new approach for the result in [16], where it was required that $-e^{-1} < b\tau e^{-a\tau} < e$, in order to get the same conclusion.

Theorem 6.4. *If $-e^{-1} < b\tau e^{-a\tau}$, then the long-time behaviour of the solution $x = x(\cdot;\varphi)$ of (6.11) with initial data $x_0 = \varphi$ is given by*

$$\lim_{t\to\infty} e^{-\lambda_d t}x(t) = \frac{1}{1 + b\tau e^{-\lambda_d \tau}}\Big[\varphi(0) + be^{-\lambda_d\tau}\int_{-\tau}^{0} e^{-\lambda_d s}\varphi(s)\,ds\Big], \tag{6.12}$$

where λ_d is the simple real dominant root of $z = a + be^{-\tau z}$.

Proof. Lemma 5.4 implies the existence of a simple real dominant zero λ_d of $\Delta(z)$. So if we apply Corollary 6.2 with $r = \tau$, $\mu = 0$ and η a sum of a point-mass at 0 of size a and at $-\tau$ of size b, Equation (6.12) follows from (6.7). □

The second application concerns the main result in [39]. Consider the scalar neutral equation

$$\frac{d}{dt}\Big[x(t) + \sum_{l=1}^{m} c_l x(t-\sigma_l)\Big] = ax(t) + \sum_{j=1}^{k} b_j x(t-h_j). \tag{6.13}$$

The characteristic equation associated with (6.13) is given by

$$\Delta(z) = z\Big(1 + \sum_{l=1}^{m} c_l e^{-z\sigma_l}\Big) - a - \sum_{j=1}^{k} b_j e^{-zh_j}. \tag{6.14}$$

If $\Delta(z)$ in (6.14) has a real zero λ_d that satisfies the condition

$$\sum_{l=1}^{m} |c_l|\,(1 + |\lambda_d|\,\sigma_l)e^{-\lambda_d\sigma_l} + \sum_{j=1}^{k} h_j\,|b_j|\,e^{-\lambda_d h_j} < 1, \tag{6.15}$$

then Lemma 5.1 implies that λ_d is a simple real dominant zero of (6.14).

Thus, if (6.15) holds, then we can again use Corollary 6.2 to compute the large time behaviour of the solutions of (6.13).

Theorem 6.5. *If λ_d is a real zero of (6.14) such that (6.15) holds, then the large time behaviour of the solution $x = x(\cdot;\varphi)$ of (6.13) with initial data $x_0 = \varphi$ is given by*

$$\lim_{t\to\infty} e^{-\lambda_d t}x(t;\varphi) = \frac{1}{H(\lambda_d)}\Big[\varphi(0) - \sum_{l=1}^{m} c_l\varphi(-\sigma_l)$$
$$+ \sum_{l=1}^{m}\lambda_d c_l\int_0^{-\sigma_l} e^{-\lambda_d\sigma}\varphi(\sigma+\sigma_l)\,d\sigma + \sum_{j=1}^{k} b_j\int_0^{-h_j} e^{-\lambda_d\sigma}\varphi(\sigma+h_j)\,d\sigma\Big], \quad (6.16)$$

where

$$H(\lambda_d) = 1 + \sum_{l=1}^{m} c_l e^{-\lambda_d\sigma_l} - \lambda_d\sum_{l=1}^{m} c_l\sigma_l e^{-\lambda_d\sigma_l} + \sum_{j=1}^{k} b_j h_j e^{-\lambda_d h_j}.$$

Proof. Let $r_1 = \max_{1\leqslant l\leqslant m}\sigma_l$, $r_2 = \max_{1\leqslant j\leqslant k} h_j$ and $r = \max\{r_1, r_2\}$. Let μ be a finite sum of point masses at $-\sigma_l$ of size c_l, $1 \leqslant l \leqslant m$, and let η be a finite sum of point masses at $-h_j$ of size b_j, $1 \leqslant j \leqslant k$. With these definitions we apply Corollary 6.2 in the scalar case. Since $H(\lambda_d)$ is the derivative of $\Delta(z)$ evaluated at $z = \lambda_d$, the theorem follows from (6.7). $\square$

Corollary 6.6. *If $a + \sum_{j=1}^{k} b_j = 0$ and $\sum_{j=1}^{k} h_j\,|b_j| < 1$, then the large time behaviour of the solution $x = x(\cdot;\varphi)$ of (6.13) with initial data $x_0 = \varphi$ is given by*

$$\lim_{t\to\infty} x(t) = \Big(1 + \sum_{l=1}^{m} c_l + \sum_{j=1}^{k} b_j h_j\Big)^{-1}\Big[\varphi(0) - \sum_{l=1}^{m} c_l\varphi(-\sigma_l)$$
$$+ \sum_{j=1}^{k} b_j\int_0^{-h_j}\varphi(\sigma+h_j)\,d\sigma\Big].$$

The examples in this subsection illustrate that computing the spectral projection on the dominant (finite dimensional) eigenspace yields an easy way to find explicit formulas for the large time behaviour of solutions.

6.3 Application for a two-dimensional neutral FDE

We consider the FDE of the form

$$\frac{d}{dt}\big(x(t) + Ax(t-1)\big) = Bx(t) + Cx(t-1), \quad t \geqslant 0 \quad (6.17)$$

with initial condition

$$x_0 = \varphi \in \mathcal{C}([-1,0], \mathbb{C}^2), \quad (6.18)$$

where $x(t) \in \mathbb{C}^2$ and A, B and C are 2×2-matrices with real entries. It is our aim to analyze the large time behaviour of solutions of (6.17) and we like to demonstrate how we can compute the large time behaviour symbolically using the Maple library `FDESpectralProj`, developed in Chapter 4.

The characteristic matrix corresponding to (6.17) is given by

$$\Delta(z) = z\Delta_0(z) - B - Ce^{-z} \tag{6.19}$$

with

$$\Delta_0(z) = I + Ae^{-z}. \tag{6.20}$$

We compute the characteristic equation corresponding to (6.19) and spectral projections with the help of the Maple library `FDESpectralProj`. In order to do so, we perform the following steps. First, we load the library by setting the internal Maple variable `libname` to include the path[1] to the library and then we load the package FDESpectralProj. For matrix manipulations we use also the (standard) library `linalg`.

```
> libname:= "/path/to/library", libname:
> with(FDESpectralProj):
> with(linalg):
```

Equation (6.17) can be written in form (1.3)

$$\tfrac{d}{dt}Mx_t = Lx_t, \qquad t \geqslant 0$$

by setting the operators $M, L : \mathcal{C}([-1,0], \mathbb{C}^2) \to \mathbb{C}^2$ as

$$M\varphi = I\varphi(0) + A\varphi(-1), \qquad L\varphi = B\varphi(0) + C\varphi(-1). \tag{6.21}$$

This representation is used to load FDE (6.17) into Maple according to the format presented by library.

```
> A:= Matrix([[a11,a12],[a21,a22]]);
  B:= Matrix([[b11,b12],[b21,b22]]);
  C:= Matrix([[c11,c12],[c21,c22]]);
```

$$A := \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \qquad B := \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \qquad C := \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$$

```
> Mmat:= [IdentityMatrix(2), A]:  Mdel:= [0, 1]:
  Lmat:= [B, C]:                  Ldel:= [0, 1]:
```

Next, we compute the characteristic equation using the routine `CharEquation`.

[1]If the library is in the same directory of the worksheet, you can use "." in place of "/path/to/library" in the Maple expression.

```
> chareq_original:= CharEquation(Mmat,Mdel,Lmat,Ldel,z):
```

The expansion of `chareq` yields the characteristic equation corresponding to (6.19) in terms of the coefficients of A, B and C. A manipulation of the expression stored in `chareq_original` yields that the characteristic equation is given by

$$\begin{aligned}\det \Delta(z) = {} & z^2 e^{-2z} \det A + z^2 e^{-z} \operatorname{tr} A + z^2 \\ & + z e^{-2z}\big(\det A + \det C - \det(A+C)\big) \\ & + z e^{-z}\big(\det A + \det B - \det(A+B) - \operatorname{tr} C\big) \\ & - z \operatorname{tr} B + e^{-2z} \det C \\ & + e^{-z}\big(\det(B+C) - \det B - \det C\big) \\ & + \det B = 0. \end{aligned} \tag{6.22}$$

One can see that (6.22) holds using Maple, evaluating the following code.

```
> AplusB:= MatrixAdd(A,B):
> AplusC:= MatrixAdd(A,C):
> BplusC:= MatrixAdd(C,B):
> simplify(chareq_original -
    (   z^2 * exp(-2*z) * det(A)
      + z^2 * exp(-z)   * trace(A)
      + z^2
      + z   * exp(-2*z) * (det(A) + det(C) - det(AplusC))
      + z   * exp(-z)   * (det(A) + det(B) - det(AplusB) - trace(C))
      - z   *             trace(B)
      +       exp(-2*z) * det(C)
      +       exp(-z)   * (det(BplusC) - det(B) - det(C))
      +                   det(B)));
```

$$0$$

We see that the characteristic equation in (6.22) depends on 9 parameters (determinants and traces of the matrices formed by sums of A, B and C), instead of the 12 coefficients of the matrices A, B and C. For the next manipulations, Maple needs variables instead of formulas. We are going to use "`detA`" as a variable to represent the formula "`det(A)`", "`traceA`" to represent "`trace(A)`" and so forth for the other formulas.

```
> chareq:=   z^2 * exp(-2*z) * detA
           + z^2 * exp(-z)   * traceA
           + z^2
           + z   * exp(-2*z) * (detA + detC - detAplusC)
           + z   * exp(-z)   * (detA + detB - detAplusB - traceC)
```

```
- z   *             traceB
+       exp(-2*z) * detC
+       exp(-z)   * (detBplusC - detB - detC)
+                   detB:
```

6.3.1 Relations between parameters and the order of $z = z_0$ as zero of the characteristic equation

We establish conditions on the quantities $\det A$, $\operatorname{tr} A$, $\det(A+B)$, $\det(A+C)$, $\det B$, $\operatorname{tr} B$, $\det(B+C)$ and $\det C$, to obtain when $z = z_0$ is a zero of $\det \Delta(z)$, up to eighth order. The parameter $\operatorname{tr} C$ does not have to satisfy any condition. Also, for each condition, we provide also the Maple code that yields the condition to be applied. To use this code, we assume that one has evaluated all the code from the start of Section 6.3 up to here.

Zero (of first order) of the characteristic equation

From now on, we assume that $z = z_0$ is a zero of characteristic equation (6.22). From (6.22), we obtain that $\det(B+C)$ must satisfy

$$\begin{aligned} e^{-z_0}\det(B+C) = &-z_0^2\big(e^{-2z_0}\det A + e^{-z_0}\operatorname{tr} A + 1\big) \\ &- z_0\big(e^{-2z_0}(\det A + \det C - \det(A+C)) + e^{-z_0}\det A \\ &\qquad + e^{-z_0}\det B - e^{-z_0}\det(A+B) - e^{-z_0}\operatorname{tr} C - \operatorname{tr} B\big) \\ &- e^{-2z_0}\det C - e^{-z_0}\det B + e^{-z_0}\det C - \det B. \end{aligned} \tag{6.23}$$

The expression can be obtained in Maple, evaluating the expression

```
> detBplusC:= solve(diffchareq[0],detBplusC);
```

We continue to specify conditions on the parameters to obtain a zero of the characteristic equation of higher order.

Zero of second order of the characteristic equation

We assume condition (6.23) on $\det(B+C)$ and we solve

$$\frac{d}{dz}\det\Delta(z)\,\Big|_{z=z_0} = 0$$

in terms of the parameter $\det(A+B)$. This yields that $\det(A+B)$ must satisfy the equation

$$\begin{aligned} e^{-z_0}\det(A+B) = {}& z_0^2(-e^{-2z_0}\det A+1) \\ &+ z_0\big(-e^{-2z_0}\det C + e^{-2z_0}\det A \\ &\qquad + e^{-2z_0}\det(A+C) - \operatorname{tr} B + 2e^{-z_0}\operatorname{tr} A + 2\big) \\ &- \operatorname{tr} B + \det B + e^{-2z_0}\det A + e^{-z_0}\det B \\ &\qquad - e^{-2z_0}\det(A+C) + e^{-z_0}\det A - e^{-z_0}\operatorname{tr} C. \end{aligned} \tag{6.24}$$

This expression follows from a simple manipulation of the formula obtained when the following code is evaluated in Maple

```
> detAplusB:= solve(diffchareq[1],detAplusB);
```

Zero of third order of the characteristic equation

Assuming conditions (6.23) and (6.24) on $\det(B+C)$ and $\det(A+B)$, we solve

$$\frac{d^2}{dz^2}\det\Delta(z)\,\Big|_{z=z_0} = 0$$

in terms of the remaining parameters. If $z_0 = 2$, we have that $\frac{d^2}{dz^2}\det\Delta(z)|_{z=z_0}$ is independent of the parameter $\det(A+C)$, and when $z=-2$ then $\frac{d^2}{dz^2}\det\Delta(z)|_{z=z_0}$ is independent of the parameter $\operatorname{tr} B$. In order to have $z = z_0$ as a zero of $\det\Delta(z)$ of order (at least) 2, we must have either one of the following conditions.

- If $z_0 \neq -2$, then

$$\begin{aligned} (z_0+2)\operatorname{tr} B = {}& z_0^2(e^{-2z_0}\det A+1) \\ &+ z_0(-e^{-2z_0}\det(A+C) + 4 + e^{-2z_0}\det C - 3e^{-2z_0}\det A) \\ &+ 2 + 2e^{-z_0}\operatorname{tr} A + \det B \\ &\qquad + 2e^{-2z_0}\det(A+C) - e^{-2z_0}\det C. \end{aligned} \tag{6.25}$$

- If $z_0 = -2$, then

$$\det B = 2 - 2e^2\operatorname{tr} A - 10e^4\det A + 3e^4\det C - 4e^4\det(A+C). \tag{6.26}$$

In Maple, the following code[2] assigns the condition.

```
> if z0 <> -2 then
    traceB:= solve(diffchareq[2],traceB);
  else
    detB:= solve(diffchareq[2],detB);
  end if;
```

[2]In Maple syntax, "<>" denotes "$\neq$".

Zero of fourth order of the characteristic equation

We take the following assumptions

- If $z_0 \neq -2$, we assume that the conditions (6.23), (6.24) and (6.25) hold for $\det(B+C)$, $\det(A+B)$ and $\operatorname{tr} B$, respectively.
- If $z_0 = -2$, we assume that the conditions (6.23), (6.24) and (6.26) hold for $\det(B+C)$, $\det(A+B)$ and $\det B$, respectively.

These assumptions imply that z_0 is a zero of the characteristic equation of order at least 3. We solve the expression $\frac{d^3}{dz^3}\det\Delta(z)|_{z=z_0}$ in $\det B$ (when $z_0 \neq -2$) and $\operatorname{tr} B$ (when $z_0 = -2$) to obtain conditions for a zero of fourth order.

- If $z_0 \neq -2$, then

$$\begin{aligned} e^{z_0}\det B = z_0^2(&3e^{-z_0}\det A + e^{z_0} + 2e^{-z_0}\det(A+C) - 2e^{-z_0}\det C) \\ &+ z_0(16e^{-z_0}\det A - 2e^{-z_0}\det C + 4e^{z_0} - 2\operatorname{tr} A) \\ &+ 6e^{z_0} + 7e^{-z_0}\det C - 12e^{-z_0}\det(A+C) \\ &\qquad - 6\operatorname{tr} A - 6e^{-z_0}\det A. \end{aligned} \tag{6.27}$$

- If $z_0 = -2$, then

$$\operatorname{tr} B = -2e^2\operatorname{tr} A - 27e^4\det A + 7e^4\det C - 9e^4\det(A+C). \tag{6.28}$$

The evaluation of the following code the in Maple applies the condition.

```
> if z0 <> -2 then
    detB:= solve(diffchareq[3],detB);
  else
    traceB:= solve(diffchareq[3],traceB);
  end if;
```

Zero of fifth order of the characteristic equation

We take the following assumptions

- If $z_0 \neq -2$, we assume that the conditions (6.23), (6.24), (6.25) and (6.27) hold for $\det(B+C)$, $\det(A+B)$, $\operatorname{tr} B$ and $\det B$, respectively.
- If $z_0 = -2$, we assume that the conditions (6.23), (6.24), (6.26) and (6.28) hold for $\det(B+C)$, $\det(A+B)$, $\det B$ and $\operatorname{tr} B$, respectively.

These assumptions imply that z_0 is a zero of the characteristic equation of order at least 4. We solve the expression $\frac{d^4}{dz^4}\det\Delta(z)|_{z=z_0}$ in $\det C$ (when $z_0 \neq -2$) or $\det(A+C)$ (when $z_0 = -2$) to obtain conditions for the fourth order.

- If $z_0 \neq 2$, then

$$\begin{aligned}\frac{2(z_0-2)}{e^{z_0}}\det C = -2z_0^2 e^{-z_0}\det A \\ + z_0(2e^{-z_0}\det(A+C) + 10e^{-z_0}\det A) \\ - 6e^{-z_0}\det(A+C) - \operatorname{tr} A - e^{z_0} - 7e^{-z_0}\det A. \quad (6.29)\end{aligned}$$

- If $z_0 = 2$, then

$$\det(A+C) = -\frac{e^4}{2} - \frac{e^2}{2}\operatorname{tr} A + \frac{5}{2}\det A. \quad (6.30)$$

Next we present the Maple code that applies the condition.

```
> if z0 <> 2 then
    detC:= simplify(solve(diffchareq[4],detC));
  else
    detAplusC:= simplify(solve(diffchareq[4],detAplusC));
  end if;
```

Zero of sixth order of the characteristic equation

We take the following assumptions

- If $z_0 = 2$, we assume that the conditions (6.23), (6.24), (6.25), (6.27) and (6.30) hold for $\det(B+C)$, $\det(A+B)$, $\operatorname{tr} B$, $\det B$ and $\det(A+C)$, respectively.
- If $z_0 = -2$, we assume that the conditions (6.23), (6.24), (6.26), (6.28) and (6.29) hold for $\det(B+C)$, $\det(A+B)$, $\det B$, $\operatorname{tr} B$ and $\det C$, respectively.
- If $z_0 \neq -2$ and $z_0 \neq 2$, we assume that the conditions (6.23), (6.24), (6.25), (6.27) and (6.29) hold for $\det(B+C)$, $\det(A+B)$, $\operatorname{tr} B$, $\det B$ and $\det C$, respectively.

These assumptions imply that z_0 is a zero of the characteristic equation of order at least 5. We solve the expression $\frac{d^5}{dz^5}\det\Delta(z)|_{z=z_0}$ in $\det(A+C)$ (when $z_0 \neq -2$) or $\det C$ (when $z_0 = -2$) to obtain conditions for the fifth order.

- If $z_0 \neq 2$, then

$$\begin{aligned}\det(A+C) = z_0^2\det A + z_0(-4\det A + e^{2z_0} + e^{z_0}\operatorname{tr} A) \\ + \frac{13}{2}\det A - \frac{3}{2}e^{z_0}\operatorname{tr} A - \frac{5}{2}e^{2z_0}. \quad (6.31)\end{aligned}$$

- If $z_0 = 2$, then
$$\det C = -e^2 \operatorname{tr} A + \frac{7}{2} \det A - \frac{3}{2} e^4. \tag{6.32}$$

The Maple code that applies the condition follows.

```
> if z0 <> 2 then
    detAplusC:= simplify(solve(diffchareq[5],detAplusC));
  else
    detC:= simplify(solve(diffchareq[5],detC));
  end if;
```

Zero of seventh or higher order of the characteristic equation

We take the following assumptions

- If $z_0 = 2$, we assume that the conditions (6.23), (6.24), (6.25), (6.27), (6.30) and (6.32) hold for $\det(B + C)$, $\det(A + B)$, $\operatorname{tr} B$, $\det B$, $\det(A + C)$ and $\det C$, respectively.
- If $z_0 = -2$, we assume that the conditions (6.23), (6.24), (6.26), (6.28), (6.29) and (6.31) hold for $\det(B+C)$, $\det(A+B)$, $\det B$, $\operatorname{tr} B$, $\det C$ and $\det(A+C)$, respectively.
- If $z_0 \neq -2$ and $z_0 \neq 2$, we assume that the conditions (6.23), (6.24), (6.25), (6.27), (6.29) and (6.31) hold for $\det(B + C)$, $\det(A + B)$, $\operatorname{tr} B$, $\det B$, $\det C$ and $\det(A + C)$, respectively.

Then, we have that

$$\frac{d^6}{dz^6} \det \Delta(z)|_{z=z_0} = 8 + 8e^{-2z_0} \det A + 2e^{-z_0} \operatorname{tr} A.$$

Hence, a necessary condition for z_0 to be a zero of $\det \Delta(z)$ of seventh order is that $\det A$ and $\operatorname{tr} A$ satisfy the (linear) relation

$$\operatorname{tr} A = -4e^{z_0}(1 + e^{-2z_0} \det A). \tag{6.33}$$

The Maple code that yields (6.33) is

```
> traceA:= simplify(solve(diffchareq[6],traceA));
```

Let us assume that (6.33) also holds. We have now that

$$\frac{d^7}{dz^7} \det \Delta(z)|_{z=z_0} = -64e^{-2z_0} \det A - 14e^{-z_0} \operatorname{tr} A - 48.$$

So, in order for z_0 to be a zero of eighth order of $\det\Delta(z)$, then we must also have that

$$\det A = e^{2z_0}, \qquad \operatorname{tr} A = -8e^{z_0}. \tag{6.34}$$

However, if we assume that conditions (6.34) also hold, when we have that

$$\frac{d^8}{dz^8}\det\Delta(z)|_{z=z_0} = 16,$$

and hence it is impossible for $\det\Delta(z)$ to have a zero of order ninth or higher.

6.3.2 Does $a_M < \operatorname{Re} z_0$ hold?

In Section 1.7, we already observed that, for any $\epsilon > 0$, there are at most finitely many roots of the characteristic equation (6.22) in the right-half plane $\{z \in \mathbb{C} : \operatorname{Re} z > a_M + \epsilon\}$ and infinitely many roots of the characteristic equation (6.22) in the left-half plane $\{z \in \mathbb{C} : \operatorname{Re} z < a_M - \epsilon\}$. We have that

$$a_M = \sup\{\operatorname{Re} z : \det\Delta_0(z) = 0\}. \tag{6.35}$$

with $\Delta_0(z)$ given by (6.20). After some simple computations, we have the following expression for $\det\Delta_0(z)$.

$$\det\Delta_0(z) = 1 + e^{-z}\operatorname{tr} A + e^{-2z}\det A. \tag{6.36}$$

The following lemma presents a criterion to decide whether a real number γ is larger than a_M, given the values of $\det A$ and $\operatorname{tr} A$.

Lemma 6.7. *Suppose that* $(\det A, \operatorname{tr} A) \neq (0,0)$. *For* $\gamma \in \mathbb{R}$, *we have that* $a_M < \gamma$ *if and only if the matrix* A *is such that the point in the plane of coordinates* $(\det A, \operatorname{tr} A)$ *is inside the triangle of vertexes* $(-e^{2\gamma}, 0)$, $(e^{2\gamma}, 2e^{\gamma})$ *and* $(e^{2\gamma}, -2e^{\gamma})$ *(See Figure 6.1 for a representation of the described region.)*

Proof. Let z_0 be a root of $\det\Delta_0(z) = 0$, with $\det\Delta_0(z)$ given in (6.36). We observe that $\det\Delta_0(z)$ is a polynomial of degree 2 in e^{-z}. Therefore e^{-z_0} can take at most two different values. Since the map $z \mapsto e^{-z}$ is $2\pi i$-periodic, it follows that there are one or two vertical lines in the complex plane with infinitely many roots of $\det\Delta_0(z)$, and all roots of $\det\Delta_0(z)$ are on those vertical lines.

Performing technical but simple analysis on the modulus of the roots of the polynomial

$$\alpha x^2 + \beta x + 1 = 0, \tag{6.37}$$

we obtain that, for any value $\sigma > 0$, the subset of the plane of those (α, β) such the polynomial in (6.37) has σ as the minimum of the modulus of its roots, is precisely the triangle of vertexes $\left(-\frac{1}{\sigma^2}, 0\right)$, $\left(\frac{1}{\sigma^2}, \frac{2}{\sigma}\right)$ and $\left(\frac{1}{\sigma^2}, -\frac{2}{\sigma}\right)$. It is easy to see that, when $(\alpha, \beta) \to (0,0)$, the modulus of the roots of (6.37) tend to ∞. We have that $\operatorname{Re} z < \gamma$ if and only if $|e^{-z}| = e^{\operatorname{Re} z} > e^{-\gamma}$. Therefore, taking $\alpha = \det A$, $\beta = \operatorname{tr} A$ and $\sigma = e^{-\gamma}$, the proof of the lemma follows. $\square$

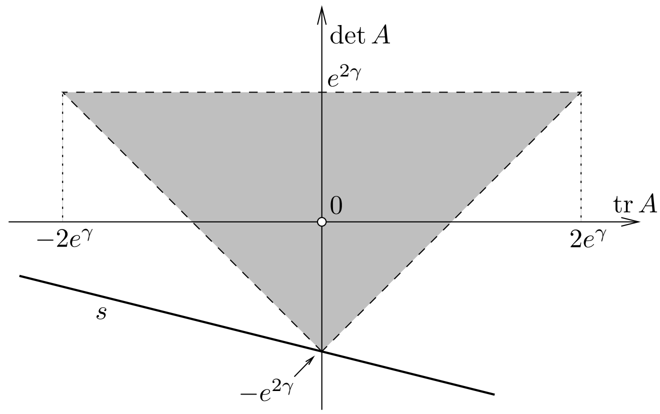

Figure 6.1: For a given $\gamma \in \mathbb{R}$, if the matrix A is such that the point of coordinates $(\det A, \operatorname{tr} A) \neq (0,0)$ is inside the region of the figure in gray, then one has that $a_M < \gamma$, where a_M is given in (6.35), with $\det \Delta_0(z)$ given in (6.36). The region is the triangle with vertexes $(-e^{2\gamma}, 0)$, $(e^{2\gamma}, 2e^{\gamma})$ and $(e^{2\gamma}, -2e^{\gamma})$. If $(\det A, \operatorname{tr} A)$ is over the border of the region, we have $a_M = \gamma$. When $(\det A, \operatorname{tr} A) = (0,0)$, we have that $a_M = -\infty$. The straight line marked with s represents the geometric place were the necessary condition (6.33) on z_0, satisfying $\gamma = \operatorname{Re} z_0$, to be a zero of $\det \Delta(z)$ seventh order.

⋆ ⋆ ⋆

When A is such that $\det A = 0$ and $\operatorname{tr} A = 0$, we have that $\det \Delta_0(z) \equiv 1$! We can take advantage of this situation to conclude that the location of the roots of the characteristic equation $\det \Delta(z)$ are similar to the location of the roots of the characteristic equation of a delay equation.

Lemma 6.8. *Suppose that* $\det A = 0$ *and* $\operatorname{tr} A = 0$*. Then, for any vertical strip* $\mathbb{C}_{\gamma_1,\gamma_2} = \{z \in \mathbb{C} : \gamma_1 < \operatorname{Re} z < \gamma_2\}$*, there are at most finitely many zeros of the characteristic equation (6.22). Hence, we have that* $a_M = -\infty$*.*

Proof. If $\det A = 0$ and $\operatorname{tr} A = 0$, then $\det \Delta(z)$, given in formula (6.22), can be reduced to

$$\begin{aligned}\det \Delta(z) = z^2 &+ ze^{-2z}\big(\det C - \det(A+C)\big)\\ &+ ze^{-z}\big(\det B - \det(A+B) - \operatorname{tr} C\big) - z \operatorname{tr} B\\ &+ e^{-2z}\det C + e^{-z}\big(\det(B+C) - \det B - \det C\big) + \det B. \end{aligned} \tag{6.38}$$

Define the functions $g_1(z)$ and $g_0(z)$, respectively, by

$$g_1(z) \stackrel{\text{def}}{=} e^{-2z}\big(\det C - \det(A+C)\big) + e^{-z}\big(\det B - \det(A+B) - \operatorname{tr} C\big) - \operatorname{tr} B, \tag{6.39}$$

$$g_0(z) \stackrel{\text{def}}{=} e^{-2z} \det C + e^{-z}\big(\det(B+C) - \det B - \det C\big) + \det B. \tag{6.40}$$

The functions g_1 and g_2 are analytic and we have that there exists $C > 0$ such that $|g_1(z)| < C$ and $|g_0(z)| < C$ for all $z \in \mathbb{C}_{\gamma_1,\gamma_2}$. Combining (6.38), (6.39) and (6.40), we obtain that

$$\det \Delta(z) = z^2 + g_1(z)z + g_0(z).$$

For $z \in \mathbb{C}_{\gamma_1,\gamma_2}$ with $|z| > C + 1$, we have that

$$\begin{aligned}
|\det \Delta(z)| &= |z^2 + g_1(z)z + g_0(z)| \\
&\geqslant |z^2| - |g_1(z)|\,|z| - |g_0(z)| \\
&\geqslant |z^2| - C|z| - C \\
&= (|z|+1)\Big(|z| - 1 + \frac{1}{|z|+1}\Big) - C(|z|+1) \\
&= (|z|+1)\Big(|z| - 1 + \frac{1}{|z|+1} - C\Big) \\
&> \frac{|z|+1}{|z|+1} = 1.
\end{aligned}$$

Therefore, we have that $\det \Delta(z) \neq 0$ for $z \in \mathbb{C}_{\gamma_1,\gamma_2}$ with $|z| > C + 1$. Since $\det \Delta(z)$ is an analytic function, its zeros are isolated and therefore there exist at most finitely many zeros of $\det \Delta(z)$ in the vertical strip $\mathbb{C}_{\gamma_1,\gamma_2}$. □

Remark 6.9. If A is a 2×2-matrix, then some simple computations show that the hypothesis $\det A = 0$ and $\operatorname{tr} A = 0$ are equivalent to the hypothesis $A^2 = 0$. It is known that the spectrum of the infinitesimal generator of the neutral equation

$$\frac{d}{dt}(x(t) + Ax(t-1)) = 0$$

is closely related with the spectral properties of the difference equation

$$x(t) = -Ax(t-1), \qquad t > 0. \tag{6.41}$$

The solution of the difference equation (6.41) can be build by the method of steps, *i.e.*,

$$x(t+2) = -Ax(t+1) = A^2 x(t) = 0.$$

It we start with $x_0 = \varphi$, then we have that $x(t) = 0$ for $t > 1$. Hence (6.41) has no exponential solutions. (All solutions are “small solutions”), and them the point spectrum of the infinitesimal generator is empty.

Corollary 6.10. *Let z_0 to be a zero of $\det \Delta(z)$. If $a_M < \operatorname{Re} z_0$, then the order of z_0 as zero of $\det \Delta(z)$ is at most six.*

Proof. The line in the $(\det A, \operatorname{tr} A)$-plane given by the necessary condition (6.33), on z_0 to be a zero of $\det \Delta(z)$ of seventh order, does not cross the interior of the region described in Lemma 6.7. See Figure 6.1. □

6.3.3 A concrete case

We consider FDE of the form (6.17) with the matrices A, B and C given by

$$A = \begin{bmatrix} \frac{1}{2}(1+\sqrt{7}) & 2 \\ -1 & \frac{1}{2}(1-\sqrt{7}) \end{bmatrix} \approx \begin{bmatrix} 1.823 & 2 \\ -1 & -0.823 \end{bmatrix}, \tag{6.42}$$

$$B = \frac{1}{4} \begin{bmatrix} 10+\sqrt{61}-\sqrt{7} & -4 \\ 23-\frac{1}{2}\sqrt{427} & 10-\sqrt{61}+\sqrt{7} \end{bmatrix} \approx \begin{bmatrix} 3.791 & -1 \\ 3.167 & 1.209 \end{bmatrix}, \tag{6.43}$$

$$C = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \approx \begin{bmatrix} 13.521 & 5.083 \\ -3.43 & -1.271 \end{bmatrix}, \tag{6.44}$$

where

$$\begin{aligned} c_{11} &= \frac{49}{8} + \frac{24797\sqrt{7}+3241\sqrt{61}+126\sqrt{71041}-3\sqrt{7}\sqrt{71041}\sqrt{61}}{14600}, \\ c_{12} &= \frac{24797}{3650} - \frac{3\sqrt{4333501}}{3650}, \\ c_{21} &= \frac{18467}{14600} - \frac{3241\sqrt{7}\sqrt{61}}{29200} - \frac{63\sqrt{7}\sqrt{71041}}{14600} + \frac{9\sqrt{61}\sqrt{71041}}{29200}, \\ c_{22} &= \frac{49}{8} - \frac{24797\sqrt{7}+3241\sqrt{61}+126\sqrt{71041}-3\sqrt{7}\sqrt{71041}\sqrt{61}}{14600}. \end{aligned} \tag{6.45}$$

We do the computations using the Maple library `FDESpectralProj`, described in Chapter 4. We input equation (6.17), with A, B and C are given in formulas (6.42)–(6.45) evaluating the following code in Maple.

```
> libname:= "/path/to/library", libname:
  with(FDESpectralProj):
> A:= Matrix([[1/2+1/2*sqrt(7), 2], [-1, 1/2-1/2*sqrt(7)]]):
  B:= Matrix([[5/2+1/4*sqrt(61)-1/4*sqrt(7), -1],
              [23/4-1/8*sqrt(427), 5/2-1/4*sqrt(61)+1/4*sqrt(7)]]):
  C:= Matrix([[49/8+(24797*sqrt(7)+3241*sqrt(61)+126*sqrt(71041)
                     - 3*sqrt(7)*sqrt(71041)*sqrt(61))/14600,
               24797/3650-3/3650*sqrt(4333501)],
              [18467/14600-3241/29200*sqrt(7)*sqrt(61)
                 -63/14600*sqrt(7)*sqrt(71041)
                 +9/29200*sqrt(61)*sqrt(71041),
               49/8-(24797*sqrt(7)+3241*sqrt(61)+126*sqrt(71041)
                     - 3*sqrt(7)*sqrt(71041)*sqrt(61))/14600]]):
  AplusB:= MatrixAdd(A,B):
  AplusC:= MatrixAdd(A,C):
  BplusC:= MatrixAdd(C,B):
```

```
Mmat:= [IdentityMatrix(2), A]:  Mdel:= [0, 1]:
Lmat:= [B, C]:                  Ldel:= [0, 1]:
```

Hence, we obtain that

$$\operatorname{tr} A = 1, \quad \det A = \frac{1}{2}, \quad \operatorname{tr} B = 5, \quad \det B = \frac{31}{4}, \quad \operatorname{tr} C = \frac{49}{4}, \quad \det C = \frac{1}{4},$$
$$\det(A+B) = 0, \quad \det(A+C) = -\frac{3}{4} \quad \text{and} \quad \det(B+C) = 0.$$

For $z_0 = 0$, we have that identities (6.23), (6.24), (6.25), (6.27), (6.29) and (6.31). Hence, z_0 is a zero of sixth order of the characteristic equation. Furthermore, we that that $(\det A, \operatorname{tr} A) = (\frac{1}{2}, 1)$ is inside the triangular region described in Lemma 6.7, and therefore we have that $a_M < 0$. Indeed, in this case we have that $a_M = -\ln\sqrt{2} \approx -0.34657$.

We obtain the characteristic equation associated to (6.17) evaluating

```
> chareq:= CharEquation(Mmat, Mdel, Lmat, Ldel, z);
```

The expansion of `chareq` yields

$$\det\Delta(z) = \left(\frac{1}{2}z^2 + \frac{3}{2}z + \frac{1}{4}\right)e^{-2z} + (z^2 - 4z - 8)\,e^{-z} + \frac{31}{4} + z^2 - 5z. \qquad (6.46)$$

We can also check that $z_0 = 0$ is a zero of (6.46) by evaluating

```
> OrderOfZero(chareq,z,0);
```

6.

In order to show that $z_0 = 0$ is a dominant root, we rewrite (6.46) in the form

$$\underbrace{z\left(\frac{1}{2}e^{-2z} + e^{-z} + 1\right)}_{g(z)} + \underbrace{e^{-2z} - 5e^{-z} - 6}_{h(z)} = \frac{3e^{-2z} + 12e^{-z} - 55}{4(z+1)}. \qquad (6.47)$$

Lets restrict our analysis to those z such that $\operatorname{Re} z \geqslant 0$. We have that $g(z) = f(e^{-z})$ where f is th polynomial $f(z) = \frac{z^2}{2} + z + 1$. The roots of f are $-1 \pm i$. If $\operatorname{Re} z \geqslant 0$, then $|e^{-z}| \leqslant 1$ and then e^{-z} is uniformly bounded away from the roots of f. We find that $|f(w)| > \frac{1}{3}$ if $|w| \geqslant 1$. Hence, $|g(z)| > \frac{1}{3}$ for $\operatorname{Re} z \geqslant 0$. For $\operatorname{Re} z \geqslant 0$, we have also that

$$|h(z)| = |e^{-2z} - 5e^{-z} - 6| \leqslant |e^{-2z}| + 5|e^{-z}| + 6 \leqslant 12.$$

We conclude that when $|z| \geqslant 40$ and $\operatorname{Re} z \geqslant 0$, we have that the left-hand side of (6.47) has modulus larger than 1. The analysis of the right-hand side of (6.47) yields

$$\left|\frac{3e^{-2z} + 12e^{-z} - 55}{4(z+1)}\right| \leqslant \left|\frac{70}{4(z+1)}\right| < \frac{1}{2}$$

when $|z| \geqslant 40$ and $\operatorname{Re} z \geqslant 0$. We have shown that there is no zero of the characteristic equation in the region $\{z \in \mathbb{C} : \operatorname{Re} z \geqslant 0 \text{ and } |z| \geqslant 40\}$.

Now, we analysis of the regular function $\det \Delta(z)$ on (6.46) in the region $\{z \in \mathbb{C} : \operatorname{Re} z \geqslant 0 \text{ and } |z| < 40\}$. We observe that the Taylor expansion of $\det \Delta(z)$ around $z = 0$ yields

$$\det \Delta(z) = \frac{14}{120} z^6 + o(|z|^6) \qquad \text{as } z \to 0.$$

Hence, $z = 0$ is a local minimum of the function $z \mapsto |\det \Delta(z)|$. Now, plots of $|\det \Delta(z)|$ outside a neighbourhood of $z = 0$ show that $|\det \Delta(z)|$ is bounded is bounded away from 0 outside such neighbourhood. Therefore, $z = 0$ is a dominant root of the characteristic equation (6.47).

Now, we compute the spectral projection. In Maple, we evaluate the following commands. First, we compute the spectral projection, storing it in the variable `sp`. Then we simplify the expression, grouping coefficients and simplifying them.

```
> sp:=SpectralProjection(Mmat,Mdel,Lmat,Ldel,0,phi,theta,intvar=t):
> sp1:=[collect(sp[1],[int,exp,z],simplify),
        collect(sp[2],[int,exp,z],simplify)]:
```

According to formula (3.19) in Theorem 3.6, the projection is a polynomial[3] in θ with vectors as coefficients. Since $z_0 = 0$ is a zero of sixth order of $\det \Delta(z)$, the degree of such polynomial is at most 5. However, the analysis of the adjoint of the characteristic matrix $\Delta(z)$ shows that $z = 0$ is not a zero of $\operatorname{adj} \Delta(z)$ ($k_0 = 6$, where k_λ was introduced in Section 2.1), and this implies that $(P_0\varphi)(\theta)$ is indeed a polynomial in θ of degree 5. The expansion of `sp1` conform this scenario. We would like to provide the formula for the projection, but of the formulas, with symbolical representation of the coefficients, with simplifications, easily occupy more than 7 pages. Then we are going to present the formulas with symbolic expressions of numbers[4] approximated as float numbers, with exception of the expression $s_5(\theta)$. The spectral projection $P_0 : \mathcal{C} \to \mathcal{C}$ is given by

$$P_0\varphi(\theta) = \sum_{j=0}^{5} l_j(\varphi) \begin{bmatrix} \theta^j \\ s_j(\theta) \end{bmatrix}, \tag{6.48}$$

where

$$s_j(\theta) = \frac{d^{5-j}}{d\theta^{5-j}} \frac{j!}{5!} s_5(\theta), \qquad j = 0, 1, 2, 3 \text{ and } 4, \tag{6.49}$$

[3] The spectral projection is a polynomial only when $z_0 = 0$. Actually the spectral projection is a sum of vectors times terms of the form $\theta^j e^{z_0 \theta}$ with $j = 0, \ldots, 5$. See Theorem 3.6.

[4] All numbers are sums of rational numbers with times combinations with products involving 1, $\sqrt{7}$, $\sqrt{61}$ and $\sqrt{71041}$, like $s_5(\theta)$.

$$
\begin{aligned}
s_5(\theta) = \quad & \theta^5\Big(-\frac{239\sqrt{71041}}{99408} - \frac{553\sqrt{61}}{5232} - \frac{23\sqrt{61}\sqrt{71041}}{99408} - \frac{8533}{5232} - \frac{\sqrt{7}}{4}\Big) \\
& + \theta^4\Big(-\frac{6535\sqrt{71041}}{570288} - \frac{64740325}{10835472} - \frac{2184745\sqrt{61}}{3611824} - \frac{275\sqrt{61}\sqrt{71041}}{190096}\Big) \\
& + \theta^3\Big(-\frac{105626466355}{5314799016} - \frac{4871980985\sqrt{71041}}{100981181304} \\
& \qquad - \frac{627603755\sqrt{61}\sqrt{71041}}{100981181304} - \frac{12505840585\sqrt{61}}{5314799016}\Big) \\
& + \theta^2\Big(-\frac{185357010381665}{3668982920712} - \frac{59546529065\sqrt{71041}}{407664768968} \\
& \qquad - \frac{68679737335\sqrt{61}\sqrt{71041}}{3668982920712} - \frac{7787618499005\sqrt{61}}{1222994306904}\Big) \\
& + \theta\Big(-\frac{82130881276618405\sqrt{61}}{7198544490436944} - \frac{4987512047173055\sqrt{61}\sqrt{71041}}{136772345318301936} \\
& \qquad - \frac{642584019900255655}{7198544490436944} - \frac{38976075861411605\sqrt{71041}}{136772345318301936}\Big) \\
& - \frac{15398492960799828 2945\sqrt{61}}{14908185639694911024} - \frac{12034765187215640082 31}{14908185639694911024} \\
& - \frac{406805091703140725 5\sqrt{71041}}{14908185639694911024} - \frac{520705889036848769\sqrt{61}\sqrt{71041}}{14908185639694911024}
\end{aligned} \tag{6.50}
$$

$$
\approx -4.2403\,\theta^5 - 16.765\,\theta^4 - 64.048\,\theta^3 - 178.15\,\theta^2 - 330.24\,\theta - 306.84,
$$

and

$$
\begin{aligned}
l_0(\varphi) \approx\; & 0.59797\,\varphi_1(0) + 1.3002\,\varphi_1(-1) - 0.21019\,\varphi_2(0) + 1.3688\,\varphi_2(-1) \\
& + 0.49668\,\Phi_{10} - 28.566\,\Phi_{11} + 79.648\,\Phi_{12} - 56.410\,\Phi_{13} \\
& + 1.813\,\Phi_{14} + 5.6423\,\Phi_{15} - 5.6447\,\Phi_{20} + 11.553\,\Phi_{21} + 8.598\,\Phi_{22} \\
& - 20.638\,\Phi_{23} + 4.293\,\Phi_{24} + 2.0888\,\Phi_{25},
\end{aligned} \tag{6.51}
$$

$$
\begin{aligned}
l_1(\varphi) \approx\; & 4.2277\,\varphi_1(0) + 8.3094\,\varphi_1(-1) - 0.60294\,\varphi_2(0) + 8.9513\,\varphi_2(-1) \\
& + 28.566\,\Phi_{10} - 159.30\,\Phi_{11} + 169.24\,\Phi_{12} - 7.257\,\Phi_{13} - 28.212\,\Phi_{14} \\
& - 11.553\,\Phi_{20} - 17.196\,\Phi_{21} + 61.913\,\Phi_{22} - 17.160\,\Phi_{23} - 10.443\,\Phi_{24},
\end{aligned} \tag{6.52}
$$

$$
\begin{aligned}
l_2(\varphi) \approx\; & 8.5743\,\varphi_1(0) + 15.332\,\varphi_1(-1) + 0.29797\,\varphi_2(0) + 16.902\,\varphi_2(-1) \\
& + 79.648\,\Phi_{10} - 169.24\,\Phi_{11} + 10.882\,\Phi_{12} + 56.423\,\Phi_{13} \\
& + 8.598\,\Phi_{20} - 61.913\,\Phi_{21} + 25.76\,\Phi_{22} + 20.888\,\Phi_{23},
\end{aligned} \tag{6.53}
$$

$$
l_3(\varphi) \approx 3.7251\,\varphi_1(0) + 11.753\,\varphi_1(-1) - 4.9633\,\varphi_2(0) + 11.535\,\varphi_2(-1)
$$

$$
\begin{aligned}
&\quad + 56.410\,\Phi_{10} - 7.257\,\Phi_{11} - 56.423\,\Phi_{12} \\
&\quad + 20.638\,\Phi_{20} - 17.160\,\Phi_{21} - 20.888\,\Phi_{22}, && (6.54)\\
l_4(\varphi) &\approx -2.2164\,\varphi_1(0) + 2.7455\,\varphi_1(-1) - 6.7856\,\varphi_2(0) + 1.1511\,\varphi_2(-1) \\
&\quad + 1.813\,\Phi_{10} + 28.212\,\Phi_{11} + 4.293\,\Phi_{20} + 10.443\,\Phi_{21}, && (6.55)\\
l_5(\varphi) &\approx 0.0266\,\varphi_1(0) - 1.7012\,\varphi_1(-1) + 1.7497\,\varphi_2(0) - 1.3866\,\varphi_2(-1) \\
&\quad - 5.6428\,\Phi_{10} - 2.0888\,\Phi_{20}. && (6.56)
\end{aligned}
$$

where φ_i is the ith component of $\varphi \in \mathcal{C} = \mathcal{C}([-1,0], \mathbb{C}^2)$ and

$$\Phi_{ij} = \int_0^1 t^j \varphi_i(\tau - 1)\, d\tau.$$

Setting $m_j(\theta) = [\theta^j \quad s_j(\theta)]^T$ for $j = 0, \ldots, 5$, we have that $\{m_j(\theta) : j = 0, \ldots, 5\}$ form a basis for $\mathcal{M}_0$. Furthermore, by Theorem 6.3 we have that, for an initial condition $\varphi \in \mathcal{C}$, the large time behaviour of the solution $x(t)$ of FDE given by (6.17), with A, B and C given respectively by (6.42), (6.43) and (6.44), is given as follows. Let $S = \big\{ j \in \{0, \ldots, 5\} : l_j(\varphi) \neq 0 \big\}$. Then

1. if $S \neq \varnothing$, let $n \overset{\text{def}}{=} \max S$. Then

$$\lim_{t \to \infty} \frac{1}{t^n} x(t) = l_n(\varphi) \begin{bmatrix} 1 \\ s_{5,5} \end{bmatrix} \tag{6.57}$$

   where

$$
\begin{aligned}
s_{5,5} &= -\frac{239\sqrt{71041} + 10507\sqrt{61} + 23\sqrt{61}\sqrt{71041} + 162127 + 24852\sqrt{7}}{99408} \\
&\approx -4.2403
\end{aligned}
$$

   is the coefficient of θ^5 in $s_5(\theta)$, given by (6.50).

2. if $S = \varnothing$, then

$$\lim_{t \to \infty} x(t) = 0. \tag{6.58}$$

6.4 Retarded autonomous FDE with positive L

Under certain conditions, the solution semigroup of a FDE is a positive operator, and the spectral bound is a strictly dominant eigenvalue of the generator of the semigroup. In the particular case that the FDE is a retarded differential equation of the form

$$\dot{x}(t) = L x_t$$

and L is a *positive operator*, that is, if for all $\varphi \in \mathcal{C}$ with $\varphi(\theta) \geqslant 0$ for $\theta \in [-r, 0]$ we have that all entries of $L\varphi$ are real and nonnegative. In Theorem 6.14 we will give the large time behaviour of solutions, using the positivity of the solution semigroup and Theorem 6.3. We start this section stating some results from the book by Arendt *et al.* [1]. For a proof of Lemmas 6.12 and 6.13, see Theorem 1.6 and Corollary 2.11 on [1].

Definition 6.11. An (unbounded) operator A on $\mathcal{C}([-r,0],\mathbb{C})$ is said to satisfy the *positive minimum principle* if for every $f \in \mathcal{D}\big(A\big)$ such that $f \geqslant 0$, for every $x \in [-r,0]$ such that $f(x) = 0$, then $(Af)(x) \geqslant 0$.

Lemma 6.12. *Let A be the generator of a strongly continuous semigroup. Then the semigroup is positive if and only if the generator A satisfies the positive minimum principle on Definition 6.11.*

Lemma 6.13. *Let $T(t)$ a positive semigroup. Assume that for some $t_0 > 0$ (hence for all $t > 0$) one has that $r_{\mathrm{ess}}(T(t_0)) < r(T(t_0))$, e. g., that $T(t_0)$ is compact and $r(T(t_0)) > 0$. Then $s(A)$ is a strictly dominant eigenvalue.*

Theorem 6.14. *Suppose that $M\varphi = \varphi(0)$ and let A be given by (2.3). If for every $\psi \in \mathcal{D}\big(A\big)$ with $\psi \geqslant 0$ and $\psi(0) = 0$ we have $L\psi \geqslant 0$, in particular if L is a positive operator, then the spectral bound $s(A)$ is a strictly dominant eigenvalue of A and, if we let n to be the geometric multiplicity of $s(A)$ as eigenvalue of A, the large time behaviour of solutions $x = x(\cdot,\varphi)$ of (6.1) is given by*

1. *if $P_{s(A)}\varphi \neq 0$, then*

$$\lim_{t\to\infty} \frac{1}{t^m} e^{-s(A)t} x(t) = q_m(n, s(A), \varphi). \tag{6.59}$$

   where $m = \max\{j \in \{0,1,\ldots,n-1\} : q_j(n,s(A),\varphi) \neq 0\}$, with q_j given by formula (3.20) in Theorem 3.6.

2. *if $P_{s(A)}\varphi = 0$, then*

$$\lim_{t\to\infty} e^{-s(A)t} x(t) = 0. \tag{6.60}$$

Proof. Let $\psi \in \mathcal{D}\big(A\big)$ such that $\psi(\theta) \geqslant 0$ for $\theta \in [-r,0]$. If $\psi(-r) = 0$ then $\dot{\psi}(-r) \geqslant 0$, otherwise there exist $\epsilon > 0$ such that $\psi(\theta) < 0$ for $\theta \in (-r,-r+\epsilon)$, which is a contradiction. It is easy to see that for all $\theta \in (-r,0)$ such that $\psi(\theta) = 0$ we have that $\dot{\psi}(\theta) = 0$ (recall that $A\psi = \dot{\psi}$). If $M\varphi = \varphi(0)$, then we have that $\dot{\psi}(0) = L\psi \geqslant 0$. Therefore, the operator A satisfies the positive minimum principle. From Lemma 6.12 it follows that the solution semigroup $T(t)$ is positive and from Lemma 6.13 we obtain that $s(A)$ is a strictly dominant eigenvalue of A. Estimates (6.59) and (6.60) follow from Theorem 6.3. □

Remark 6.15. Results similar to Theorem 6.14, using the argument of the positivity of the infinitesimal generator of the solution semigroup, are not possible when the measure $d\mu$, in the definition of the operator M (see equation (1.4)), has any point masses. We suppose that this is not true. Suppose that $d\mu$ has a point mass at some $a < 0$. It is possible to show the existence of a positive function $f \in \mathcal{C}$ with $\|f\|$ arbitrarily small, but with $\dot{f}(a) < 0$ and $|\dot{f}(a)|$ arbitrarily large. Therefore, we could take f such that Lf is arbitrarily small with $M\dot{f} < -C$ with $C > 0$.

6.5 Application for periodic equations

In this section, we use Theorem 2.3 to show that also for linear periodic equations, the large time behaviour can be given explicitly by computing a spectral projection onto the dominant eigenvalue of the monodromy operator.

Consider the following linear periodic delay equation

$$\dot{x}(t) = a(t)x(t) + \sum_{j=1}^{k} b_j(t)x(t-\tau_j), \quad t \geqslant s, \tag{6.61}$$

where $a(t+\omega) = a(t)$ and $b_j(t+\omega) = b_j(t)$ for $j = 1, 2\dots, k$. We assume the particular case where $\tau_j = j\omega$ (i.e., the delays are integer multiples of the period ω). The following lemma is clear.

Lemma 6.16. *If $y(t) = e^{-\int_0^t a(s)ds}x(t)$, then y satisfies*

$$\dot{y}(t) = \sum_{j=1}^{k} \hat{b}_j(t)y(t-\tau_j),$$

where $\hat{b}_j(t) = e^{-\int_{t-\tau_j}^{t} a(s)ds}b_j(t)$, $j = 1, 2, \dots, k$, are also ω-periodic.

So, it suffices to analyze the following system (recall $\tau_j = j\omega$)

$$\begin{aligned} \dot{x}(t) &= \sum_{j=1}^{k} b_j(t)x(t-\tau_j), \quad t \geqslant s, \\ x_s &= \varphi, \quad \varphi \in \mathcal{C}([-k\omega, 0], \mathbb{C}^n), \end{aligned} \tag{6.62}$$

where b_j are continuous real matrix-valued functions such that $b_j(t+\omega) = b_j(t)$.

Let $\Pi(s) : \mathcal{C} \to \mathcal{C}$ denote the monodromy operator $\Pi(s) = T(s+\omega, s)$, i.e.,

$$(\Pi(s)\varphi)(\theta) = x(\omega+\theta; s, \varphi), \quad -k\omega \leqslant \theta \leqslant 0.$$

Using the differential equation (6.62) and periodicity of b_j, we have the following representation for $\Pi(s)$

$$(\Pi(s)\varphi)(\theta) = \begin{cases} \varphi(0) + \sum_{j=1}^{k} \int_0^{\omega+\theta} b_j(\sigma+s)\varphi(\sigma-\tau_j)d\sigma, & -\omega \leqslant \theta \leqslant 0, \\ \varphi(\theta+\omega), & -k\omega \leqslant \theta \leqslant -\omega. \end{cases}$$

Since the large time behaviour of the solutions is independent of the starting time s, we can set $s = 0$ and define $\Pi = \Pi(0)$.

Lemma 6.17. *The resolvent $(zI - \Pi)^{-1}\varphi$ of the monodromy operator Π is given by*

$$(zI - \Pi)^{-1}\varphi(\theta) = \Omega^{\theta}_{-\omega}(1/z)[zI - \Omega^{0}_{-\omega}]^{-1}\big(\varphi(-\omega) + G_{\varphi,z}(0)\big) + G_{\varphi,z}(\theta), \quad (6.63)$$

for $-\omega \leqslant \theta \leqslant 0$ and

$$(zI - \Pi)^{-1}\varphi(\theta) = \frac{1}{z^m}(zI - \Pi)^{-1}\varphi(\theta + m\omega) + \sum_{j=0}^{m-1}\frac{\varphi(\theta - j\omega)}{z^{j+1}}, \quad (6.64)$$

for $-\omega \leqslant \theta + m\omega \leqslant 0$, $m = 1, \dots, k-1$, where

$$G_{\varphi,z}(\theta) = \frac{1}{z}\Bigg(\varphi(\theta) - \Omega^{\theta}_{-\omega}\big(\tfrac{1}{z}\big)\varphi(-\omega) + \sum_{l=1}^{k}\sum_{j=0}^{l-1}\int_{-\omega}^{\theta}\frac{1}{z^{l-j}}\Omega^{\theta}_{\sigma}\big(\tfrac{1}{z}\big)b_l(\sigma)\varphi(\sigma - j\omega)d\sigma\Bigg) \quad (6.65)$$

and $\Omega^t_s(1/z)$ denotes the fundamental matrix solution of the ordinary differential equation

$$\frac{d\psi}{dt}(t) = \sum_{l=1}^{k}\frac{1}{z^l}b_l(t)\psi(t).$$

Proof. If $(zI - \Pi)^{-1}\varphi = \psi$ (φ is given), then

$$\varphi = z\psi - \Pi\psi.$$

Suppose at first that φ is differentiable. We can derive the following system of equations

$$\varphi(\theta) = z\psi(\theta) - \psi(\theta + \omega), \qquad -k\omega \leqslant \theta \leqslant -\omega, \quad (6.66)$$

$$\left.\begin{aligned} \tfrac{d}{d\theta}\varphi(\theta) &= z\tfrac{d}{d\theta}\psi(\theta) - \textstyle\sum_{j=1}^{k} b_j(\theta)\psi(\theta - (j-1)\omega), \\ \varphi(-\omega) &= z\psi(-\omega) - \psi(0), \end{aligned}\right\} \quad -\omega \leqslant \theta \leqslant 0. \quad (6.67)$$

Using (6.66) inductively, for $l = 1, 2, \dots, k-1$ and $-\omega \leqslant \theta \leqslant 0$, we obtain

$$\psi(\theta - l\omega) = \frac{1}{z^l}\psi(\theta) + \sum_{j=1}^{l}\frac{1}{z^{l-j+1}}\varphi(\theta - j\omega). \quad (6.68)$$

Using (6.68), we can reduce the equation in (6.67) to an ordinary differential equation

$$
\begin{aligned}
\frac{d\varphi}{d\theta}(\theta) &= z\frac{d\psi}{d\theta}(\theta) - \sum_{l=1}^{k} b_l(\theta)\psi(\theta-(l-1)\omega) \\
&= z\frac{d\psi}{d\theta}(\theta) - b_1(\theta)\psi(\theta) - \sum_{l=2}^{k} b_l(\theta)\psi(\theta-(l-1)\omega) \\
&= z\frac{d\psi}{d\theta}(\theta) - b_1(\theta)\psi(\theta) - \sum_{l=2}^{k} b_l(\theta)\Big(\frac{1}{z^{l-1}}\psi(\theta) + \sum_{j=1}^{l-1}\frac{1}{z^{(l-1)-j+1}}\varphi(\theta-j\omega)\Big)
\end{aligned}
$$

and therefore

$$
\frac{d\psi}{d\theta}(\theta) - \sum_{l=1}^{k} b_l(\theta)\frac{1}{z^l}\psi(\theta) = \frac{1}{z}\Big[\frac{d\varphi}{d\theta}(\theta) + \sum_{l=2}^{k}\Big(b_l(\theta)\sum_{j=1}^{l-1}\frac{1}{z^{l-j}}\varphi(\theta-j\omega)\Big)\Big]. \tag{6.69}
$$

We first solve the homogeneous equation ($\varphi \equiv 0$), i.e.,

$$
\frac{d\psi}{d\theta}(\theta) = \sum_{l=1}^{k} b_l(\theta)\frac{1}{z^l}\psi(\theta), \tag{6.70}
$$

and let $\Omega_s^t(1/z)$ denote the fundamental matrix solution of (6.70) with $\Omega_s^s(1/z) = I$. Using the variation-of-constants formula and the fact that $\Omega_\sigma^t(1/z)$ is differentiable (use (6.70)), we can solve the differential equation (6.69)

$$
\begin{aligned}
\psi(\theta) &= \Omega_{-\omega}^{\theta}\big(\tfrac{1}{z}\big)\psi(-\omega) + \int_{-\omega}^{\theta}\Omega_\sigma^\theta\big(\tfrac{1}{z}\big)\frac{1}{z}\Big[\frac{d\varphi}{d\sigma}(\sigma) + \sum_{l=2}^{k}\Big(b_l(\sigma)\sum_{j=1}^{l-1}\frac{1}{z^{l-j}}\varphi(\sigma-j\omega)\Big)\Big]d\sigma \\
&= \Omega_{-\omega}^{\theta}\big(\tfrac{1}{z}\big)\psi(-\omega) + \frac{1}{z}\Big[\Omega_\sigma^\theta\big(\tfrac{1}{z}\big)\varphi(\sigma)\Big|_{\sigma=-\omega}^{\theta} + \int_{-\omega}^{\theta}\Omega_\sigma^\theta\big(\tfrac{1}{z}\big)\Big[\sum_{l=1}^{k}\frac{b_l(\sigma)}{z^l}\Big]\varphi(\sigma)d\sigma\Big] \\
&\qquad + \int_{-\omega}^{\theta}\Omega_\sigma^\theta\big(\tfrac{1}{z}\big)\frac{1}{z}\Big[\sum_{l=2}^{k}\Big(b_l(\sigma)\sum_{j=1}^{l-1}\frac{1}{z^{l-j}}\varphi(\sigma-j\omega)\Big)\Big]d\sigma \\
&= \Omega_{-\omega}^{\theta}\big(\tfrac{1}{z}\big)\psi(-\omega) + \frac{1}{z}\varphi(\theta) - \frac{1}{z}\Omega_{-\omega}^{\theta}\big(\tfrac{1}{z}\big)\varphi(-\omega) \\
&\qquad + \frac{1}{z}\int_{-\omega}^{\theta}\Omega_\sigma^\theta\big(\tfrac{1}{z}\big)\Big[\sum_{l=1}^{k}\Big(b_l(\sigma)\sum_{j=0}^{l-1}\frac{1}{z^{l-j}}\varphi(\sigma-j\omega)\Big)\Big]d\sigma \\
&= \Omega_{-\omega}^{\theta}\big(\tfrac{1}{z}\big)\psi(-\omega) + \frac{1}{z}\varphi(\theta) - \frac{1}{z}\Omega_{-\omega}^{\theta}\big(\tfrac{1}{z}\big)\varphi(-\omega) \\
&\qquad + \frac{1}{z}\sum_{l=1}^{k}\sum_{j=0}^{l-1}\int_{-\omega}^{\theta}\frac{1}{z^{l-j}}\Omega_\sigma^\theta\big(\tfrac{1}{z}\big)b_l(\sigma)\varphi(\sigma-j\omega)d\sigma
\end{aligned}
$$

and therefore

$$\psi(\theta) = \Omega^{\theta}_{-\omega}\big(\tfrac{1}{z}\big)\psi(-\omega) + G_{\varphi,z}(\theta), \tag{6.71}$$

where

$$G_{\varphi,z}(\theta) = \frac{1}{z}\left(\varphi(\theta) - \Omega^{\theta}_{-\omega}\big(\tfrac{1}{z}\big)\varphi(-\omega) + \sum_{l=1}^{k}\sum_{j=0}^{l-1}\int_{-\omega}^{\theta}\frac{1}{z^{l-j}}\Omega^{\theta}_{\sigma}\big(\tfrac{1}{z}\big)b_l(\sigma)\varphi(\sigma - j\omega)d\sigma\right).$$

We can solve $\psi(-\omega)$ from the boundary condition in (6.67)

$$\varphi(-\omega) = z\psi(-\omega) - \Omega^{0}_{-\omega}\big(\tfrac{1}{z}\big)\psi(-\omega) - G_{\varphi,z}(0). \tag{6.72}$$

Hence

$$\psi(-\omega) = \left[zI - \Omega^{0}_{-\omega}\big(\tfrac{1}{z}\big)\right]^{-1}\big(\varphi(-\omega) + G_{\varphi,z}(0)\big). \tag{6.73}$$

Equations (6.71) and (6.73) yield (6.63). To find ψ on $[-k\omega, -\omega]$, we use again relation (6.66) inductively to obtain, for $-(m+1)\omega \leqslant \theta \leqslant -m\omega$, $m = 1, \dots, k-1$

$$\psi(\theta) = \frac{\psi(\theta + m\omega)}{z^m} + \sum_{j=0}^{m-1}\frac{\varphi(\theta + j\omega)}{z^{j+1}}.$$

To finish the proof, note that the expression for $(zI - \Pi)^{-1}\varphi$ on Formula (6.63) is well defined for $\varphi \in \mathcal{C}$, and we can drop the assumption that φ is differentiable, using the fact that the set of differentiable functions on $\mathcal{C}$ is dense on $\mathcal{C}$. $\square$

The representation for the resolvent of Π yields important information about the spectral properties of the operator. For example, it follows that the nonzero spectrum of Π, $\sigma(\Pi) \setminus \{0\}$, consists of point spectrum only, given by

$$\sigma(\Pi) \setminus \{0\} = \left\{z \mid \det\big(zI - \Omega^{0}_{-\omega}(1/z)\big) = 0\right\}.$$

Furthermore, questions about completeness of the eigenvectors and generalized eigenvectors of Π (i.e., denseness of the Floquet solutions in $\mathcal{C}([-k\omega, 0], \mathbb{C}^n)$) can be answered using resolvent estimates. Furthermore, using the Dunford representation of the spectral projection P_μ of Π onto a generalized eigenspace $\mathcal{M}_\mu$, $\mu \in \sigma(\Pi) \setminus \{0\}$,

$$P_\mu = \operatorname*{Res}_{z=\mu}\,(zI - \Pi)^{-1},$$

we can explicitly compute the spectral projection of Π using residue calculus. In particular, if μ is a simple eigenvalue of Π, the spectral projection onto the one-dimensional eigenspace is given by

$$P_\mu\varphi = \lim_{z\to\mu}(z - \mu)(zI - \Pi)^{-1}\varphi.$$

Together with Lemma 6.17, this yields a formula for $P_\mu\varphi$ on $[-\omega,0]$, namely,

$$(P_\mu\varphi)(\theta)=\Omega^{\theta}_{-\omega}(1/\mu)\Big[\lim_{z\to\mu}(z-\mu)\big(zI-\Omega^{0}_{-\omega}(1/z)\big)^{-1}\Big]\big(\varphi(-\omega)+G_{\varphi,\mu}(0)\big), \quad (6.74)$$

where $G_{\varphi,z}$ is given by (6.65). The extension of $P_\mu\varphi$ to $[-k\omega,-\omega]$, found using (6.64), is given by

$$(P_\mu\varphi)(\theta)=\frac{1}{z^m}(P_\mu\varphi)(\theta+m\omega), \quad -\omega\leqslant\theta+m\omega\leqslant 0, \quad m=1,\dots,k-1. \quad (6.75)$$

So, if we can compute the residue of $[zI-\Omega^{\theta}_{-\omega}(1/z)]^{-1}$ at μ (which is a question about ordinary differential equations), we can find an explicit formula for P_μ. Like in the autonomous case, the existence of a simple Floquet multiplier μ_d that dominates the others (in the sense that $\mu\neq\mu_d\in\sigma(\Pi(s))$ implies $|\mu|<|\mu_d|$) allows us to give an explicit formula for the large time behaviour of solutions.

Corollary 6.18. *If $\mu_d\neq 0$ is a simple and dominant eigenvalue of Π, then the large time behaviour of the solution $x(\cdot;0,\varphi)$ of (6.61) is given by*

$$\lim_{t\to\infty}\Omega^{t}_{-\omega}\big(\tfrac{1}{\mu}\big)x(t;0,\varphi)=\operatorname*{Res}_{z=\mu}\big[zI-\Omega^{0}_{-\omega}\big(\tfrac{1}{z}\big)\big]^{-1}\big(\varphi(-\omega)+G_{\varphi,\mu}(0)\big) \quad (6.76)$$

where $\Omega^t_s\big(\frac{1}{z}\big)$ is as in Lemma 6.17.

In the scalar case, we can solve $\Omega^t_s(1/z)$, namely,

$$\Omega^t_s\big(\tfrac{1}{z}\big)=\exp\left(\sum_{j=1}^{k}z^{-j}\int_s^t b_j(\sigma)d\sigma\right). \quad (6.77)$$

The poles of resolvent $[z-\Omega^0_{-\omega}(1/z)]^{-1}$ are the zeros μ of the equation

$$\mu-\exp\left(\sum_{j=1}^{k}\mu^{-j}\int_{-\omega}^{0} b_j(\sigma)d\sigma\right)=0. \quad (6.78)$$

Define

$$B_j=\frac{1}{\omega}\int_{-\omega}^{0}b_j(t)\,dt.$$

For $\mu\in\sigma(\Pi)\setminus\{0\}$, we can write $\mu=e^{\lambda\omega}$ and (6.78) can be reduced to a simpler form

$$\lambda-\sum_{j=1}^{k}e^{-j\lambda\omega}B_j=\frac{2k(\lambda)\pi i}{\omega}. \quad (6.79)$$

If a Floquet exponent satisfies (6.79) for $k(\lambda) \neq 0$, then $\tilde{\lambda} = \lambda - 2k(\lambda)\pi i/\omega$ is also a Floquet exponent which satisfies (6.79) with $k(\tilde{\lambda}) = 0$. Since we are only interested in the real part of λ, it suffices to consider the equation

$$\Delta(\lambda) \stackrel{\text{def}}{=} \lambda - \sum_{j=1}^{k} e^{-j\lambda\omega} B_j = 0. \tag{6.80}$$

So, in the scalar case, we can again find the large time behaviour of the solutions of (6.61) when μ_d is a simple dominant eigenvalue of Π. See Chapter 5 for sufficient conditions for the existence of such a μ_d, specially Lemma 5.1 and Lemma 5.3.

Corollary 6.19. *Consider the scalar periodic equation given by (6.61). If $\mu_d = e^{\lambda_d \omega}$ is a simple real dominant eigenvalue of Π, then the large time behaviour of the solution $x(t;0,\varphi)$ is given by*

$$\begin{aligned}\lim_{t\to\infty} \Omega_0^t\big(\tfrac{1}{\mu_d}\big)x(t;0,\varphi) = \\ \Big[1 + \sum_{j=1}^{k} j\mu_d^{-j}\omega B_j\Big]^{-1}\Big(\varphi(0) + \sum_{l=1}^{k}\int_{-l\omega}^{0} \frac{1}{\mu_d^l}\Omega_\tau^0\big(\tfrac{1}{\mu_d}\big)b_l(\tau)\varphi(\tau)d\tau\Big),\end{aligned} \tag{6.81}$$

with $\Omega_s^t(1/\mu_d)$ given by (6.77).

Proof. We apply Corollary 6.18, but in the scalar case we can provide explicit formulas for the elements on the formula (6.74).

The fact that μ_d satisfies the equation $\mu_d = \Omega_{-\omega}^0(1/\mu_d)$ yields

$$\begin{aligned}\operatorname*{Res}_{z=\mu}\big[zI - \Omega_{-\omega}^0\big(\tfrac{1}{z}\big)\big]^{-1} &= \lim_{z\to\mu_d}(z-\mu_d)\big[z - \Omega_{-\omega}^0\big(\tfrac{1}{z}\big)\big]^{-1} \\ &= \Big[\frac{d}{dz}\big(z - \Omega_{-\omega}^0\big(\tfrac{1}{z}\big)\big)\Big|_{z=\mu_d}\Big]^{-1} \\ &= \Big[1 + \sum_{j=1}^{k} j\mu_d^{-j}\omega B_j\Big]^{-1}.\end{aligned}$$

Now, using that $b_j(t) = b_j(t+\omega)$, $\Omega_{s+\omega}^{t+\omega}(1/z) = \Omega_s^t(1/z)$ and $\mu_d = \Omega_{-\omega}^0(1/\mu_d)$, what implies $1/\mu_d^j = \Omega_{j\omega}^0(1/\mu_d)$, we obtain the identity

$$\begin{aligned}&\sum_{l=1}^{k}\sum_{j=0}^{l-1}\int_{-\omega}^{0}\frac{1}{\mu_d^{l-j}}\Omega_\sigma^0\big(\tfrac{1}{\mu_d}\big)b_l(\sigma)\varphi(\sigma - j\omega)d\sigma \\ &\quad= \sum_{l=1}^{k}\sum_{j=0}^{l-1}\int_{-\omega-j\omega}^{-j\omega}\frac{1}{\mu_d^{l-j}}\Omega_{\tau+j\omega}^0\big(\tfrac{1}{\mu_d}\big)b_l(\tau+j\omega)\varphi(\tau)d\tau\end{aligned}$$

$$= \sum_{l=1}^{k}\sum_{j=0}^{l-1}\int_{-\omega-j\omega}^{-j\omega}\frac{1}{\mu_d^l}\Omega_\sigma^0\big(\tfrac{1}{\mu_d}\big)b_l(\sigma)\varphi(\sigma)d\sigma$$
$$= \sum_{l=1}^{k}\int_{-l\omega}^{0}\frac{1}{\mu_d^l}\Omega_\sigma^0\big(\tfrac{1}{\mu_d}\big)b_l(\sigma)\varphi(\sigma)d\sigma$$

which we use to simplify

$$\begin{aligned}\varphi(-\omega)+G_{\varphi,\mu_d}(0) &= \varphi(-\omega)+\frac{1}{\mu_d}\varphi(0)-\frac{1}{\mu_d}\Omega_{-\omega}^0\big(\tfrac{1}{\mu_d}\big)\varphi(-\omega)\\ &\quad+\frac{1}{\mu_d}\sum_{l=1}^{k}\sum_{j=0}^{l-1}\int_{-\omega}^{0}\frac{1}{\mu_d^{l-j}}\Omega_\sigma^0\big(\tfrac{1}{\mu_d}\big)b_l(\sigma)\varphi(\sigma-j\omega)d\sigma\\ &= \frac{1}{\mu_d}\varphi(0)+\frac{1}{\mu_d}\sum_{l=1}^{k}\int_{-l\omega}^{0}\frac{1}{\mu_d^l}\Omega_\sigma^0\big(\tfrac{1}{\mu_d}\big)b_l(\sigma)\varphi(\sigma)d\sigma.\end{aligned}$$

Therefore P_{μ_d}, given by (6.74) and (6.75), takes the form

$$P_{\mu_d}\varphi(\theta)=\Omega_0^\theta\big(\tfrac{1}{\mu_d}\big)\Big[1+\sum_{j=1}^{k}j\mu_d^{-j}\omega B_j\Big]^{-1}\bigg(\varphi(0)+\sum_{l=1}^{k}\int_{-l\omega}^{0}\frac{1}{\mu_d^l}\Omega_\sigma^0\big(\tfrac{1}{\mu_d}\big)b_l(\sigma)\varphi(\sigma)d\sigma\bigg).$$

Note that from the representation for $\big(P_{\mu_d}\varphi\big)(\theta)$, it follows that $\mathcal{M}_{\mu_d}$ is spanned by

$$\xi(\theta)=\Omega_0^\theta\big(\tfrac{1}{\mu_d}\big),\quad -k\omega\leqslant\theta\leqslant 0.$$

Furthermore $P_\mu\varphi=c(\varphi)\xi$ with $c(\varphi)$ given by

$$c(\varphi)=\Big[1+\sum_{j=1}^{k}j\mu_d^{-j}\omega B_j\Big]^{-1}\bigg(\varphi(0)+\sum_{l=1}^{k}\int_{-l\omega}^{0}\frac{1}{\mu_d^l}\Omega_\sigma^0\big(\tfrac{1}{\mu_d}\big)b_l(\sigma)\varphi(\sigma)d\sigma\bigg).$$

A direct computation shows that

$$\big(T(t,0)\xi\big)(\theta)=\Omega_0^{t+\theta}\big(\tfrac{1}{\mu_d}\big).$$

and $P(t)$, defined in (2.16), is given by

$$P(t)(\theta)=e^{-\lambda_d(t+\theta)}\Omega_0^{t+\theta}\big(\tfrac{1}{\mu_d}\big). \tag{6.82}$$

Applying Theorem 2.3, evaluating the functions involved at $\theta=0$, and using the fact that $P(t)$ in (6.82) is invertible, we arrive at

$$\lim_{t\to\infty}\Omega_0^t\big(\tfrac{1}{\mu_d}\big)x(t;0,\varphi)=c(\varphi).$$

This proves formula (6.81) and completes the proof. □

Chapter 7

Dynamics on the center manifold

In this chapter, we continue with some other applications of spectral projections in the context of computing invariant manifolds. We discuss some nonlinear FDE where the linearization yields linear equations with 0 as the dominant eigenvalue of the characteristic equation. We present in detail the (symbolically) computation of the large time behaviour and the ordinary differential equations that give the flow on the center manifold in a neighbourhood of the the origin.

7.1 Perturbation theory

Consider the non-homogeneous delay differential equation

$$\dot{x}(t) = Lx_t + F(x_t), \tag{7.1}$$

where F is sufficiently smooth and satisfies $F(0) = 0$ and $DF(0) = 0$. The linearization of (7.1) reads

$$\dot{x}(t) = Lx_t. \tag{7.2}$$

Analyzing the infinitesimal generator A of the solution operator $T(t)$ of (7.2), we observe the inconvenience that A has the "differential equation" encoded in its domain

$$\mathcal{D}(A) = \{\varphi \in \mathcal{C} : \dot{\varphi} \in \mathcal{C}, \dot{\varphi}(0) = L\varphi\},$$

instead of its action

$$A\varphi = \dot{\varphi},$$

what makes perturbation theory particularly hard to be done in this framework, since a perturbation may change the domain of an operator instead of its action.

To avoid this difficulty, we adopt the perturbation theory developed in the book by Diekmann *et al.* [15], and we comment briefly their notation and ideas presented. Those authors show that the solution semigroup of a generic linear autonomous delay differential equation (7.2) can be studied as a perturbation of the solution semigroup of the "trivial" delay equation

$$\dot{x}(t) = 0.$$

This is done by writing the delay differential equation (7.2) in an abstract ordinary differential equation

$$\dot{u}(t) = \underbrace{(A_0^{\odot *} + B)}_{\stackrel{\text{def}}{=} A^{\odot *}} u(t) \tag{7.3}$$

where $A_0^{\odot *}$ is the infinitesimal generator of the an extension $T_0^{\odot *}(t) : \mathcal{C}^{\odot *} \to \mathcal{C}^{\odot *}$ of the solution semigroup $T_0(t)$ of the delay equation $\dot{x}(t) = 0$ to a "larger space" $\mathcal{C}^{\odot *} = \mathbb{C}^n \times L^\infty([-r, 0], \mathbb{C}^n)$, and B is a operator on $\mathcal{C}^{\odot *}$ with finite dimensional range. The space $\mathcal{C}$ can be naturally embed into $\mathcal{C}^{\odot *}$ by the mapping $\varphi \mapsto (\varphi(0), \varphi)$. We obtain a semigroup $T^{\odot *}$ which is an extension of T to $\mathcal{C}^{\odot *}$ and which has $A^{\odot *} \stackrel{\text{def}}{=} A_0^{\odot *} + B$ as its infinitesimal generator. The operators $A_0^{\odot *}$ and $A^{\odot *}$ have the same domain

$$\begin{aligned} \mathcal{D}\big(A^{\odot *}\big) &= \mathcal{D}\big(A_0^{\odot *}\big) \\ &= \big\{(\alpha, \varphi) \in \mathcal{C}^{\odot *} : \varphi \text{ is Lipschitz continuous and } \varphi(0) = \alpha\big\}, \end{aligned}$$

and the (original) differential equation is "encoded" in the action of the involved operators

$$\begin{aligned} A_0^{\odot *}(\alpha, \varphi) &= (0, \dot{\varphi}), \\ B(\alpha, \varphi) &= (L\varphi, 0). \end{aligned}$$

Formally, the "shift in time" and "extension according to FDE" are split into the operators $A_0^{\odot *}$ and B, respectively.

Moreover, for $f : \mathbb{R}_+ \to \mathcal{C}^{\odot *}$ norm continuous, let w to be the (at first) $\mathcal{C}^{\odot *}$-valued map defined as the following (weak*) integral

$$w(r, s, t) = \int_0^t T^{\odot *}(\tau) f(\tau) d\tau, \qquad 0 \leqslant r \leqslant s \leqslant t < \infty. \tag{7.4}$$

Then w is norm continuous and takes values[1] in $\mathcal{C}$. (See Lemma 2.1 on Section III.2 of [15]). Similar ideas can be applied to study other classes of perturbations of the solution semigroup, like non-linear and non-autonomous equations.

[1] Actually w takes values in another space $\mathcal{C}^{\odot\odot}$. However, we have that $\mathcal{C}^{\odot\odot} = \mathcal{C}$. See the $\odot$-*reflexivity* property in [15].

7.2 The center manifold

We consider the nonlinear autonomous functional differential equation (7.1). We are interested in analyze the behaviour of the nonlinear semiflow near a *nonhyperbolic* equilibrium; that is, we consider the situation where the generator A of the solution semigroup $T(t)$ of the linearization (7.2) of (7.1) at the equilibrium $0 \in \mathcal{C}$ has spectrum over the imaginary axis. We apply the ideas presented in Section 7.1 and in [15] in this analysis. We use the spectral decomposition, see Section 2.1, to write $\mathcal{C}$ as the direct sum

$$\mathcal{C} = \mathcal{C}_- \oplus \mathcal{C}_0 \oplus \mathcal{C}_+$$

where $\mathcal{C}_0$, $\mathcal{C}_+$ and $\mathcal{C}_-$ are the subspaces such that $\sigma(A|_{\mathcal{C}_0}) \subset i\mathbb{R}$, $\sigma(A|_{\mathcal{C}_+}) \subset \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$ and $\sigma(A|_{\mathcal{C}_-}) \subset \{z \in \mathbb{C} : \operatorname{Re} z < 0\}$. The subspace $\mathcal{C}_0$ is called *center subspace*, and $\mathcal{C}_- \oplus \mathcal{C}_+$ the *hyperbolic subspace*. The *center manifold* is the nonlinear analogue of the center subspace for FDE (7.1).

In Chapter IX of Diekmann *et al.* [15], the following integral functional equation is studied

$$u(t) = T(t-s)u(s) + \int_s^t T^{\odot *}(t-\tau)R(u(\tau))d\tau. \tag{7.5}$$

It is shown the existence of the center manifold in a sufficiently small neighbourhood of the origin, which is the image of a mapping $C : \mathcal{C}_0 \to \mathcal{C}$, with $C(0) = 0$, and which is tangent to the center space $\mathcal{C}_0$. In Section IX.8 of [15], it is shown that the flow in the center manifold, restricted to a sufficiently small neighbourhood around 0, is equivalent to the finite dimensional ODE

$$\dot{y} = Ay + P^{\odot *}_{\mathcal{C}_0} R(C(y)), \tag{7.6}$$

where $y \in \mathcal{C}_0$.

We can write (7.1) in the form (7.5), where T is taken as solution semigroup of the linearized equation (7.2), $P^{\odot *}_{\mathcal{C}_0}$ is the spectral projection from $\mathcal{C}$ to $\mathcal{C}_0$, that is

$$P^{\odot *}_{\mathcal{C}_0} = \sum_{\substack{\lambda \in \sigma(A) \\ \operatorname{Re} \lambda = 0}} P_\lambda,$$

and the nonlinearity R embeds F in formula (7.1) as a finite-dimensional bounded perturbation of the linear flow $T(t)$, and is defined as

$$R\varphi \stackrel{\text{def}}{=} r^{\odot *} F\varphi, \tag{7.7}$$

with $r^{\odot *} : \mathbb{C}^n \to \mathcal{C}^{\odot *} = \mathbb{C}^n \times L^\infty$, defined by

$$r^{\odot *} c = (c, 0). \tag{7.8}$$

7.3 The case of a double eigenvalue

In this section we present the study of the flow in the center manifold restricted to a neighbourhood of the origin. We carry out explicit computations to obtain the ordinary differential equation that describes the flow.

Consider the scalar FDE

$$\dot{x}(t) = \alpha x(t) + f(x(t-1)) \tag{7.9}$$

where $a \in \mathbb{R}$ and $f \in \mathcal{C}(\mathbb{R}, \mathbb{R})$ and $f(0) = 0$. Linearization around the zero equilibrium yields

$$\dot{x}(t) = \alpha x(t) + f'(0)x(t-1) \tag{7.10}$$

The characteristic matrix is

$$\Delta(z) = z - \alpha - f'(0)e^{-z}$$

which has a double root $z = \alpha - 1$ if $f'(0) = -e^{\alpha-1}$. We claim that this root is a double dominant eigenvalue. To see this, for any $\epsilon > 0$ arbitrarily small, we consider the characteristic equation

$$\Delta_\epsilon(z) = z - \alpha + e^{\alpha-1+\epsilon}e^{-z}.$$

We can apply Lemma 5.4 to conclude that $\Delta_\epsilon(z)$ has a real simple dominant root λ_ϵ. The continuity of the roots with respect to parameter ϵ implies that λ_ϵ tends to the unique real root of $\Delta(z) = \Delta_0(z)$, that is $\lambda_0 = \alpha - 1$. This implies that there exists no root λ of $\Delta(z)$ such that $\operatorname{Re}\lambda > \operatorname{Re}\lambda_0$. To prove the dominance, suppose that for $x > 0$, $\lambda = \alpha - 1 + ix$ is a root of $\Delta(z)$. We have

$$\begin{aligned} 0 &= |\Delta(\alpha - 1 + ix)| \\ &= |\alpha - 1 + ix - \alpha + e^{\alpha-1}e^{-\alpha+1-ix}| = |-1 + ix + e^{-ix}| \\ &\geqslant |-1 + ix| - |e^{ix}| > 0 \end{aligned}$$

which is a contradiction. Therefore $z = \alpha - 1$ is a real double dominant eigenvalue.

We are able to compute the spectral projection $P_{\alpha-1}$ on the generalized eigenspace $X_{\alpha-1}$, for instance, using the Maple library given in Appendix B and described in Section 4, running in Maple the code

```
> libname:= ".",libname;
> with(FDESpectralProj);
> df0:= -exp(alpha-1);
> Mmat:= [1]; Mdel:= [0]; Lmat:= [alpha,df0]; Ldel:= [0,1];
> sp:= simplify(
        SpectralProjection(Mmat, Mdel, Lmat, Ldel,
                                alpha-1, phi,theta, intvar=t));
```

On the first two lines, the library `FDESpectralProj` is loaded. Then the FDE (7.10) is defined, using the library notation. The code `Mmat:= [1]; Mdel:= [0]` defines M as

$$M\varphi = 1 \cdot \varphi(0),$$

and `Lmat:= [alpha,df0]; Ldel:= [0,1];` defines L as

$$L\varphi = \alpha\varphi(0) + f'(0)\varphi(-1),$$

where `alpha` decodes α and `df0` decodes $f'(0) \stackrel{\text{def}}{=} -e^{\alpha-1}$ in the previous line.

As result of `sp`, we obtain

$$\begin{aligned} P_{\alpha-1}\varphi(\theta) = e^{(\alpha-1)\theta}\Big(\frac{2}{3}\,\varphi(0) + \int_{-1}^{0}\Big(2t - \frac{2}{3}\Big)e^{-(\alpha-1)t}\varphi(t)\,dt\Big) \\ + \theta e^{(\alpha-1)\theta}\Big(2\varphi(0) - 2\int_{-1}^{0} e^{-(\alpha-1)t}\varphi(t)\,dt\Big) \end{aligned} \tag{7.11}$$

For $\alpha = 1$, the eigenvalue is zero, and on the imaginary axis. We investigate (7.9) in a neighbourhood of 0 in $\mathcal{C}$. Since all other roots lie in the left half plane, the center manifold is two dimensional and locally attracting. We compute the vector field on the generalized eigenspace X_0, describing the flow in the center manifold, tangent to X_0. From (7.11) with $\alpha = 1$ we get

$$P_0\varphi(\theta) = \frac{2}{3}\,\varphi(0) + \int_{-1}^{0}\Big(2t - \frac{2}{3}\Big)\varphi(t)dt + \theta\Big(2\varphi(0) - 2\int_{-1}^{0}\varphi(t)dt\Big). \tag{7.12}$$

We have that $\{\phi_0, \phi_1\}$, with $\phi_0(\theta) = 1$ and $\phi_1(\theta) = \theta$, is A basis for X_0.

This spectral projection, first defined in $\mathcal{C}$, can be extended to $\mathcal{C}^{\odot *} = \mathbb{C} \times L^\infty$ by a adaptation of the formula of P (compare Theorem IV.3.1 on p. 104 and Corollary IV.5.4 on p. 125 on Diekmann *et al.* [15]) and for $(c, \varphi) \in \mathbb{C} \times L^\infty$ we have

$$P_0^{\odot *}(c,\varphi)(\theta) = \frac{2}{3}\,c + \int_{-1}^{0}\Big(2t - \frac{2}{3}\Big)\varphi(t)dt + \theta\Big(2c - 2\int_{-1}^{0}\varphi(t)dt\Big). \tag{7.13}$$

Introducing small parameters

$$\begin{aligned} \alpha &= 1 + \epsilon, \\ f'(0) &= -1 + \delta, \end{aligned}$$

and writing

$$\begin{aligned} g(\varphi, \epsilon, \delta) &= \epsilon\varphi(0) + \delta\varphi(-1) + f(\varphi(-1)) - (\delta - 1)\varphi(-1) \\ &= \epsilon\varphi(0) + \delta\varphi(-1) + \frac{f''(0)}{2}\varphi(-1)^2 + \mathcal{O}(\|\varphi\|^3), \end{aligned}$$

we rewrite (7.9) as

$$\dot{x}(t) = x(t) - x(t-1) + g(x_t, \epsilon, \delta), \tag{7.14}$$

and g satisfies

$$g(\varphi, \epsilon, \delta) = \mathcal{O}(|\varphi| + |\epsilon| + |\delta|).$$

The abstract integral equation corresponding to (7.14), see formulas (7.5)–(7.8), can be written as

$$u(t) = T(t)u(0) + \int_0^t T^{\odot *}(t-\tau) r^{\odot *} g(u(\tau), \epsilon, \delta) d\tau, \tag{7.15}$$

where $T(t)$ is the semigroup of the linear part of (7.14) and $r^{\odot *}$ is given by (7.8) embeds g in formula (7.15) as a finite-dimensional bounded perturbation of the linear flow $T(t)$.

The center manifold is tangent to $\mathcal{M}_0$, that is, the center manifold is the locally the image of a map C from $\mathcal{M}_0$ to $\mathcal{C}$ such that $C(u, \epsilon, \delta) = u + \mathcal{O}(\|u\|^2)$. To obtain a description of the flow on the center manifold, in a neighbourhood of 0, we look to the $\mathcal{M}_0$-coordinate of $\mathcal{C}$ by projecting (7.15) and we get

$$\dot{u}(t) = Au(t) + P^{\odot *} r^{\odot *} g(C(u(t)), \epsilon, \delta). \tag{7.16}$$

The action of the infinitesimal generator A of $T(t)$ is differentiation, and we observe that $A\phi_0 = 0$ and $A\phi_1 = \phi_0$. We introduce coordinates into $\mathcal{M}_0$ identifying it with $\mathbb{C}^2$ by setting $u(t) = x(t)\phi_0 + y(t)\phi_1 \simeq (x(t), y(t))$. Then,

$$\begin{aligned}
&P_0^{\odot *} r^{\odot *} g(C(u(t)), \epsilon\delta) \\
&\quad = P_0^{\odot *}\big(g(C(u(t)), \epsilon\delta), 0\big) \\
&\quad = \Big(\frac{2}{3} + 2\theta\Big) g\big(u(t) + \mathcal{O}(\|u(t)\|^2), \epsilon\delta\big), 0\big) \\
&\quad = \Big(\frac{2}{3}, 2\Big)\Big(\epsilon u(t)(0) + \delta u(t)(-1) + \frac{f''(0)}{2} u(t)(-1)^2, \\
&\qquad\qquad + \mathcal{O}(\|u(t)\|^3 + (|\epsilon| + |\delta|)\|u(t)\|^2))\Big)
\end{aligned}$$

where we have used $u(t)(0) = x(t)$ and $u(t)(-1) = x(t) - y(t)$

$$\begin{aligned}
&\quad = \Big(\frac{2}{3}, 2\Big)\Big(\epsilon x + \delta(x-y) + \frac{f''(0)}{2}(x-y)^2 \\
&\qquad\qquad + \mathcal{O}((|x| + |y|)^3 + (|\epsilon| + |\delta|(|x| + |y|)^2))\Big) \\
&\quad \stackrel{\text{def}}{=} \Big(\frac{2}{3}, 2\Big) h(x, y, \epsilon, \delta).
\end{aligned}$$

Therefore, the dynamics in the center manifold is given by the system of ordinary differential equations

$$\begin{aligned}\dot{x} &= y + \frac{2}{3}h(x, y, \epsilon, \delta), \\ \dot{y} &= 2h(x, y, \epsilon, \delta).\end{aligned} \tag{7.17}$$

For further analysis on the dynamics of system (7.17), see section IX.10 of [15].

7.4 The case of a triple eigenvalue

In article [52], Krauskopf & Sieber consider the FDE

$$\dot{x}(t) = f\big(x(t), x(t-\tau), \lambda\big) \tag{7.18}$$

where $x \in \mathbb{R}^2$, and $f : \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^3 \to \mathbb{R}^2$ is given by

$$f(x, y, \lambda) = \begin{bmatrix} f_1(x, y, \lambda) \\ f_2(x, y, \lambda) \end{bmatrix} \tag{7.19}$$

where $x = (x_1, x_2)^{\mathrm{T}}$, $y = (y_1, y2)^{\mathrm{T}}$, $\lambda = (a, b, \epsilon)$ is a parameter variable, and

$$f_1(x, y, \lambda) = x_2, \tag{7.20}$$

$$f_2(x, y, \lambda) = \frac{-\frac{3}{8}\epsilon \sin(2x_1)x_2^2 + \sin(x_1) - \cos x_1(ay_1 + by_2)}{1 - \frac{3}{4}\epsilon \cos^2 x_1}. \tag{7.21}$$

The origin is always an equilibrium since f is odd, that is,

$$f(-x, -y, \lambda) = -f(x, y, \lambda).$$

Linearization of f with respect to x and y at origin gives

$$\dot{x}(t) = \partial_1 f(0, 0, \lambda)x(t) + \partial_2 f(0, 0, \lambda)x(t-\tau) \tag{7.22}$$

with

$$\partial_1 f(0, 0, \lambda) = \begin{bmatrix} 0 & 1 \\ \kappa & 0 \end{bmatrix}, \qquad \partial_2 f(0, 0, \lambda) = \begin{bmatrix} 0 & 0 \\ -\kappa a & -\kappa b \end{bmatrix}, \tag{7.23}$$

where $\kappa = 4/(4 - 3\epsilon)$. The FDE (7.22) can be written in the form (1.3) if $\mu \equiv 0$ and $\eta(\theta) = 0$ for $\theta \geqslant 0$,

$$\eta(\theta) = \begin{bmatrix} 0 & 1 \\ \kappa & 0 \end{bmatrix}$$

for $-1 < \theta < 0$, and

$$\eta(\theta) = \begin{bmatrix} 0 & 1 \\ \kappa(1 - a) & -\kappa b \end{bmatrix}$$

for $\theta \leqslant -1$.

The characteristic matrix $\Delta(z) = zI - \partial_1 f(0,0,\lambda) - \partial_2 f(0,0,\lambda)e^{-\tau z}$ is expressed in matrix notation by

$$\Delta(z) = \begin{bmatrix} z & -1 \\ -\kappa + a\kappa e^{-z\tau} & z + b\kappa e^{-z\tau} \end{bmatrix},$$

and

$$\det \Delta(z) = z^2 - \kappa\big(1 - e^{-\tau z}(a + zb)\big). \tag{7.24}$$

We see that

$$\begin{aligned} \det \Delta(0) &= \kappa(a-1), \\ D(\det \Delta)(0) &= \kappa(b - a\tau) \\ D^2(\det \Delta)(0) &= 2 - \kappa\tau(2b + a\tau), \end{aligned}$$

and then for the set of parameters $a = 1$, $b = \tau$ and $\epsilon = (4 - 2\tau^2)/3$ (or alternatively $\kappa = 2/\tau^2$), linearized FDE (7.22) becomes

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ \frac{2}{\tau^2} & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 \\ \frac{-2}{\tau^2} & \frac{-2}{\tau} \end{bmatrix} x(t - \tau), \tag{7.25}$$

has 0 as a triple zero of the characteristic equation $\det \Delta(z) = 0$. If $\det \Delta(i\omega) = 0$ for $w \in \mathbb{R}$ then the equations for the real and imaginary parts are

$$\begin{aligned} 1 &= \cos(\tau\omega)\Big(\frac{\omega^2}{\kappa} + 1\Big), \\ \tau\omega &= \sin(\tau\omega)\Big(\frac{\omega^2}{\kappa} + 1\Big). \end{aligned}$$

Squaring both equations and summing, one finds that ω must be zero. Therefore $z = 0$ is the only zero of $\det \Delta(z)$ over the imaginary axis.

For our choice of parameters, the characteristic equation (7.24) reads

$$z^2\tau^2 + 2z\tau e^{-z\tau} - 2 + 2e^{-z\tau} = 0. \tag{7.26}$$

We claim that the origin is a dominant eigenvalue. To show this, we rewrite (7.26) in the form

$$\frac{1}{z\tau + 1} = 2e^{-z\tau} + z\tau - 1 \tag{7.27}$$

For $\operatorname{Re} z \geqslant 0$, we have that $|\frac{1}{z\tau+1}| \leqslant 1$ and the equality is obtained only for $z = 0$. On the other hand, when $\operatorname{Re} z \geqslant 0$ and $|z\tau - 1| > 3$ we have that

$$|2e^{-z\tau} + z\tau - 1| \geqslant |z\tau - 1| - |2e^{-z\tau}| \geqslant |z\tau - 1| - 2 > 1.$$

Therefore, there are no roots of the characteristic function in the region when $\operatorname{Re} z \geqslant 0$ and $|z\tau - 1| > 3$. The analysis of the function $z \mapsto |z^2 + 2ze^{-z} - 2 + 2e^{-z}|$ shows that $z = 0$ is a local minimum and a plot shows that indeed there are no roots in the "small" region $\operatorname{Re} z \geqslant 0$ and $|z - 1| \leqslant 3$. As (7.18) is a delay equation, it is not possible that there is a sequence of roots λ_n of (7.26) with $\operatorname{Re} \lambda_n \uparrow 0$, since there are at most finitely many roots of the characteristic equation in any vertical strip. This shows that $z = 0$ is a dominant root of (7.26).

For parameters $(a, b, \kappa) = (1, \tau, 2/\tau^2)$, we compute the spectral projection using the computer algebra system Maple and the routines provided in Appendix B. We evaluate the following sequence of commands.

We load the Maple libraries `linalg` (for some matrix manipulations) and `FDESpectralProj`. The compiled code of the library `FDESpectralProj` is in the same directory of worksheet.

```
> with(linalg):
> libname:= ".", libname:
> with(FDESpectralProj);
```

[*CharEquation, CharMatrix, EvalDerivsFracFunction, MatrixEvalDerivsFracFunction, MaxOrderZero, OrderOfZero, SpectralProjection, SystemConsistent, diffH, init*]

We load the FDE (7.18)–(7.21)

```
> f:= Vector( [ x_2 ,
        (-3/8*epsilon*sin(2*x_1)*x_2^2 + sin(x_1)
              - cos(x_1)*(a*y_1 + b*y_2) )
        / (1 - 3/4* epsilon*(cos(x_1))^2) ] ):
```

$$f := \begin{bmatrix} x_2 \\ \dfrac{-\frac{3}{8}\epsilon \sin(2x_1)x_2^2 + \sin(x_1) - \cos(x_1)(ay_1 + by_2)}{1 - \frac{3}{4}\epsilon(\cos^2(x_1)} \end{bmatrix}$$

We define $A \stackrel{\text{def}}{=} \partial_1 f(0,0)$ and $B \stackrel{\text{def}}{=} \partial_2 f(0,0)$, replacing $1/(1 - 3\epsilon/4)$ by κ

```
> A:= subs( 1/(1-3/4*epsilon)=kappa,
            eval( Matrix( jacobian(f,[x_1,x_2]) ),
                  {x_1=0,x_2=0,y_1=0,y_2=0}));
  B:= subs( 1/(1-3/4*epsilon)=kappa,
            eval( Matrix( jacobian(f,[y_1,y_2]) ),
                  {x_1=0,x_2=0,y_1=0,y_2=0}));
```

$$A := \begin{bmatrix} 0 & 1 \\ \kappa & 0 \end{bmatrix} \qquad B := \begin{bmatrix} 0 & 0 \\ -a\kappa & -b\kappa \end{bmatrix}$$

and we use A and B to load the linearized FDE (7.25) in variables according do library `FDESpectralProj`

```
> Mmat:= [IdentityMatrix(2)];   Mdel:= [0];
  Lmat:= [A, B];                Ldel:= [0, tau]:
```

$$Mmat := \left[\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\right], \quad Mdel := [0],$$
$$Lmat := \left[\begin{bmatrix}0 & 1\\\kappa & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\-a\kappa & -b\kappa\end{bmatrix}\right], \quad Ldel := [0, \tau]$$

Define parameters $(a, b, \kappa) = (1, \tau, 2/\tau^2)$

```
> a:=1;   b:= tau;   kappa := 2/tau^2;
```

$$a := 1, \quad b := \tau, \quad \kappa := \frac{2}{\tau^2}$$

The library allows one to compute the characteristic equation (7.24) and the order of $z = 0$ as zero of the characteristic equation.

```
> ce := CharEquation(Mmat,Mdel,Lmat,Ldel,z);
```

$$ce := z^2 + ze^{-z\tau}b\kappa - \kappa + e^{-z\tau}a\kappa$$

```
> OrderOfZero(ce,z,0):
```

$$3$$

We compute the spectral projection. We use the routine `SpectralProjection` which returns the formula[2] for the spectral projection, and the result is stored in `sp1`. For evaluation of the spectral projection in specific function (see below), we use the expression in `sp1`.

```
> sp1:= SpectralProjection
          (Mmat, Mdel, Lmat, Ldel, 0, phi, theta, intvar=t):
```

We observe that the expansion of `sp1` yields a vector where the first and second components were polynomials in θ of degree 2 and 1, respectively. To improve the presentation, we substitute $\Phi_{jn} = \int_0^\tau t^n \varphi_j(t-\tau)\, dt$, storing this substitution in `sp2`. Finally we group terms to get a cleaner expression. The result is stored in `sp3`.

```
> sp2:=
   [subs({int(       phi[1](t-tau), t=0..tau ) = Phi10,
          int(  t  * phi[1](t-tau), t=0..tau ) = Phi11,
          int( t^2 * phi[1](t-tau), t=0..tau ) = Phi12,
```

[2]Note that we have to specify `intvar=t` to make modifications that involve t. Afterward we substitute integrals in (the global variable) t by symbols.

```
        int(       phi[2](t-tau), t=0..tau ) = Phi20,
        int(  t  * phi[2](t-tau), t=0..tau ) = Phi21,
        int( t^2 * phi[2](t-tau), t=0..tau ) = Phi22}, sp1[1]),
    subs({int(     phi[1](t-tau), t=0..tau ) = Phi10,
        int(  t  * phi[1](t-tau), t=0..tau ) = Phi11,
        int( t^2 * phi[1](t-tau), t=0..tau ) = Phi12,
        int(       phi[2](t-tau), t=0..tau ) = Phi20,
        int(  t  * phi[2](t-tau), t=0..tau ) = Phi21,
        int( t^2 * phi[2](t-tau), t=0..tau ) = Phi22}, sp1[2])]:

> sp3:=[ expand(coeff(sp2[1],theta,0),tau)
          + expand(coeff(sp2[1],theta,1),tau)*theta
          + expand(coeff(sp2[1],theta,2),tau)*theta^2,
        expand(coeff(sp2[2],theta,0),tau)
          + expand(coeff(sp2[2],theta,1),tau)*theta ]:
```

Expanding `sp3`, we have the expression for the spectral projection P_0 from $\mathcal{C}$ onto the generalized eigenspace $\mathcal{M}_0$

$$\begin{aligned} P_0\varphi(\theta) = \quad & \begin{bmatrix} 1 \\ 0 \end{bmatrix} \left(\frac{339}{320}\varphi_1(0) + \frac{39\tau}{640}\varphi_2(0) - \frac{39}{320}\frac{\Phi_{10}}{\tau} - \frac{39}{320}\Phi_{20} \right. \\ & \qquad \left. + \frac{9}{8}\frac{\Phi_{11}}{\tau^2} + \frac{9}{8}\frac{\Phi_{21}}{\tau} - \frac{3}{2}\frac{\Phi_{12}}{\tau^3} - \frac{3}{2}\frac{\Phi_{22}}{\tau^2} \right) \\ & + \begin{bmatrix} \theta \\ 1 \end{bmatrix} \left(-\frac{3}{8}\frac{\varphi_1(0)}{\tau} + \frac{9}{16}\varphi_2(0) - \frac{9}{8}\frac{\Phi_{10}}{\tau^2} - \frac{9}{8}\frac{\Phi_{20}}{\tau} + 3\frac{\Phi_{11}}{\tau^3} + 3\frac{\Phi_{21}}{\tau^2} \right) \\ & + \begin{bmatrix} \frac{\theta^2}{2} \\ \theta \end{bmatrix} \left(\frac{3}{\tau^2}\varphi_1(0) + \frac{3}{2\tau}\varphi_2(0) - \frac{3}{\tau^3}\Phi_{10} - \frac{3}{\tau^2}\Phi_{20} \right), \end{aligned} \tag{7.28}$$

where

$$\Phi_{jn} = \int_0^\tau t^n \varphi_j(t-\tau)\,dt, \qquad j \in \{1,2\}, \quad n \in \{0,1,2\} \tag{7.29}$$

and $\varphi_j(\theta)$ is the jth component of $\varphi(\theta) \in \mathbb{C}^2$. From (7.28) we see that the set $\{m_1, m_2, m_3\}$, where

$$m_1(\theta) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad m_2(\theta) = \begin{bmatrix} \theta \\ 1 \end{bmatrix}, \qquad m_3(\theta) = \begin{bmatrix} \frac{\theta^2}{2} \\ \theta \end{bmatrix},$$

form a basis for the three-dimensional generalized eigenspace $\mathcal{M}_0$. Our computations match the expression for P_0 and the basis of $\mathcal{M}_0$ (up to changes of variables in the integrals and linear combinations on the basis of $\mathcal{M}_0$), found in Krauskopf & Sieber [52], where computations were made for the particular value of $\tau = 1$, since a rescaling on time is always possible.

Remark 7.1. In order to illustrate that the symbolical computation of the spectral projection can be used in the evaluation of the spectral projection to arbitrary φ, we explain how to arrive to (7.28). We have observed that the expansion of `sp3` yields a vector where the first and second components were polynomials in θ of degree 2 and 1, respectively. Therefore we can compute $\mathcal{M}_0$ explicitly by trying φ as a vector of polynomials and solving $P_0\varphi = \varphi$ coefficient-wise

$$\varphi(\theta) = \begin{bmatrix} a_0 + a_1\theta + a_2\theta^2 \\ b_0 + b_1\theta \end{bmatrix}, \qquad P_0\varphi(\theta) = \begin{bmatrix} c_0 + c_1\theta + c_2\theta^2 \\ d_0 + d_1\theta \end{bmatrix}. \tag{7.30}$$

In the library Maple `FDESpectralProj`, the variable passed as "function" in the the routine "`SpectralProjection`" (`phi` in this case) should be a Maple function in the scalar case or a vector of Maple functions, in the matrix case. The following code we evaluate the spectral projection stored in `sp1` against the function φ given in (7.30), and we solve coefficient-wise $P_0\varphi = \varphi$.

```
> sp_polynom:= eval(sp1, phi=[t -> a0+a1*t+a2*t^2, t -> b0+b1*t]):

> c0, c1, c2 := seq( coeff(sp_polynom[1],theta,i), i=0..2 ):
> d0, d1     := seq( coeff(sp_polynom[2],theta,i), i=0..1 ):

> solve( {c0=a0,c1=a1,c2=a2,d0=b0,d1=b1}, {a0,a1,a2} );
```

$$\left\{ a_0 = a_0,\ a_1 = b_0,\ a_2 = \frac{b_1}{2} \right\}.$$

■

We are interested in determine (an equivalent flow to) the flow on the center manifold in a neighbourhood of $0 \in \mathcal{C}$. To do so, we apply results on Section 2.2 in order to decompose $\mathcal{C}$ as a direct sum of $\mathcal{M}_0$ and $\mathcal{Q}_0$ by $x = P_0x + (I - P_0)x$. We rewrite (7.18)–(7.21) as

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ \frac{2}{\tau^2} & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 \\ \frac{-2}{\tau^2} & \frac{-2}{\tau} \end{bmatrix} x(t-\tau) + \begin{bmatrix} 0 \\ g(x_t, \delta) \end{bmatrix}, \tag{7.31}$$

where $\delta \in \mathbb{R}^3$ satisfies $\|\delta\| \ll 1$ and $g : \mathcal{C} \times \mathbb{R}^3 \to \mathbb{C}$ is given by

$$g(\varphi, \delta) = f_2(\varphi(0), \varphi(-\tau), \lambda + \delta) - \frac{2}{\tau^2}\varphi_1(0) + \frac{-2}{\tau^2}\varphi_1(-\tau) + \frac{-2}{\tau}\varphi_2(-\tau), \tag{7.32}$$

with f_2 given in (7.21) and $\lambda = (a, b, \epsilon) \stackrel{\text{def}}{=} (1, \tau, (4-2\tau^2)/3)$. (The first component of the last term in the right hand side of (7.31) is zero because $f(x, y, \lambda)$, given in (7.19), has its first component f_1 linear in x and independent of y and λ.)

As we have done in Section 7.2, we write the abstract integral equation corresponding to (7.31) as

$$u(t) = T(t)u(0) + \int_0^t T^{\odot *}(t-\tau) r^{\odot *} \begin{bmatrix} 0 \\ g(u(\tau), \delta) \end{bmatrix} d\tau, \tag{7.33}$$

where $T(t)$ is the solution semigroup of the linear equation (7.25) and $r^{\odot *} : \mathbb{C}^2 \to \mathcal{C}^{\odot *} \sim \mathbb{C}^2 \times L^\infty([-\tau, 0], \mathbb{C}^2)$ is given by

$$r^{\odot *} c = (c, 0). \tag{7.34}$$

We observe again that the center manifold is tangent to the three-dimensional space $\mathcal{M}_0$, that is, the center manifold is the locally the image of a map C from $\mathcal{M}_0$ to $\mathcal{C}$ such that $C(u, \delta) = u + \mathcal{O}(\|u\|^2)$. To obtain a description of the flow on the center manifold, in a neighbourhood of 0, we look to the $\mathcal{M}_0$-coordinate of $\mathcal{C}$ by differentiating (7.33) and considering the $\mathcal{M}_0$-component of $\mathcal{C} = \mathcal{M}_0 \oplus \mathcal{Q}_0$ (see Section 2.2.) We observe "differentiation" (the action of the infinitesimal generator of the solution semigroup) always commutes with spectral projection, see formula (3.8). We obtain

$$\dot{u}(t) = Au(t) + P_0^{\odot *} r^{\odot *} \begin{bmatrix} 0 \\ C(g(u(t)), \delta) \end{bmatrix}. \tag{7.35}$$

where $P_0^{\odot *}$ is the extension of the spectral projection P_0, given by (7.28)–(7.29), at first defined in $\mathcal{C}$, to $\mathcal{C}^{\odot *}$. We have that $P^{\odot *}$ is given by

$$\begin{aligned} P_0^{\odot *}(c, \varphi) = \quad & m_1 \left[\frac{339}{320} c_1 + \frac{39\tau}{640} c_2 - \frac{39}{320} \frac{\Phi_{10}}{\tau} - \frac{39}{320} \Phi_{20} \right. \\ & \left. \qquad + \frac{9}{8} \frac{\Phi_{11}}{\tau^2} + \frac{9}{8} \frac{\Phi_{21}}{\tau} - \frac{3}{2} \frac{\Phi_{12}}{\tau^3} - \frac{3}{2} \frac{\Phi_{22}}{\tau^2} \right] \\ & + m_2 \left[-\frac{3}{8} \frac{c_1}{\tau} + \frac{9}{16} c_2 - \frac{9}{8} \frac{\Phi_{10}}{\tau^2} - \frac{9}{8} \frac{\Phi_{20}}{\tau} + 3 \frac{\Phi_{11}}{\tau^3} + 3 \frac{\Phi_{21}}{\tau^2} \right] \\ & + m_3 \left[3 \frac{c_1}{\tau^2} + \frac{3}{2} \frac{c_2}{\tau} - 3 \frac{\Phi_{10}}{\tau^3} - 3 \frac{\Phi_{20}}{\tau^2} \right] \end{aligned} \tag{7.36}$$

where c_j denotes the j^{th} component of c and Φ_{jn} is given in (7.29). From (7.36) we obtain that

$$\begin{aligned} P^{\odot *} r^{\odot *} \begin{bmatrix} 0 \\ g\big((x, y, w), \delta\big) \end{bmatrix} &= P^{\odot *} \left(\begin{bmatrix} 0 \\ g\big((x, y, w), \delta\big) \end{bmatrix}, 0 \right) \\ &= \left(\frac{39\tau}{640} m_1 + \frac{9}{16} m_2 + \frac{3}{4\tau} m_3 \right) g\big((x, y, w), \delta\big). \end{aligned} \tag{7.37}$$

Since the action of the infinitesimal generator of the solution semigroup is "differentiation" (on $\mathcal{C}$, not in the variable t) and in this particular case, it holds that $\dot{m}_3(\theta) = m_2(\theta)$, $\dot{m}_2(\theta) = m_1(\theta)$ and $\dot{m}_1(\theta) = 0$, we have that

$$A(x(t), y(t), w(t)) = (y(t), w(t), 0). \tag{7.38}$$

Therefore, from (7.35), (7.38) and (7.37), we obtain flow on the center manifold in a neighbourhood of 0 is equivalent to the flow in $\mathbb{C}^3$ given by the following system of ordinary differential equations

$$\begin{aligned}
\dot{x}(t) &= y(t) + \frac{39\tau}{640}\, g\big((x(t), y(t), w(t)), \delta\big),\\
\dot{y}(t) &= w(t) + \frac{9}{16}\, g\big((x(t), y(t), w(t)), \delta\big),\\
\dot{w}(t) &= \frac{3}{4\tau}\, g\big((x(t), y(t), w(t)), \delta\big).
\end{aligned}$$

Chapter 8

A cell cycle model

This chapter is based on results from a technical report by Diekmann *et al.* [11] and it is our aim to show the importance of being able to compute the spectral projections explicitly. We study a certain size structured cell cycle model, originally due to Bell & Anderson [5] and Sinko & Streifer [53]. The model was formulated as the following Kolmogorov functional[1] partial differential equation

$$\frac{\partial}{\partial t} n(t,x) = -\frac{\partial}{\partial x}\big(g(x)n(t,x)\big) - \mu(x)n(t,x) - b(x)n(t,x) + 4b(2x)n(t,2x) \quad (8.1)$$

where t denotes time, x denotes cell size and $n(t,\cdot)$ the cell size density function at time t. The coefficients g, μ and b have the following interpretation:

- g is the size-specific individual growth rate;
- μ is the size-specific per capita death rate;
- b is the size-specific probability per unit of time of division.

The last term of equation (8.1) reflects the assumption that a dividing cell produces two daughters, each of which has exactly half the size of the mother. The basic idea is to write (8.1) abstractly in the form

$$\frac{dn}{dt} = (A_0 + B)n \quad (8.2)$$

where the operator A_0 incorporates the first three terms at the right-hand side (the "local" terms) and the operator B corresponds to the non-local last term. The fact that A_0 generates a semigroup of operators can be checked by explicit integration. Next, perturbation theory is used (the Phillips-Dyson expansion) is

[1] Equation (8.1) is a functional equation because of the dependence of $\frac{\partial}{\partial t} n(t,x)$ with respect to $2x$.

used to show that $A_0 + B$ generates a C_0-semigroup as well. Finally the spectral theory of positive semigroups (a presentation is given in Section 6.4; see also Arendt *et al.* [1]) yields strong results concerning the asymptotic behaviour for $t \to \infty$.

We formulate structured population models in terms of abstract renewal equations, since we believe that it is very well suited for a subsequent study of the asymptotic behaviour, since we obtain an explicit expression for the resolvent of the generator in terms of Laplace transforms of the building blocks at the population level. By inverse Laplace transformation we then find a representation of the semigroup itself from which it is possible, or even easy, to deduce the asymptotic behaviour for large time.

8.1 Assumptions and derivation of the model

We assume:

- cells never exceed a maximal size, which we normalize to be 1;
- a dividing cell produces two daughter cells, each of which have exactly half of the size of the mother;
- cells cannot divide before reaching a threshold size $\alpha > 0$ and consequently $\alpha/2$ is the minimal cell size;
- "size" is the only parameter for a adequate description of the state of a individual cell.

The following ingredients suffice to describe the behaviour of individual cells:

1. $\mathcal{G}(x) \stackrel{\text{def}}{=}$ the time a cell needs to grow from size $\alpha/2$ to size x;
2. $\mathcal{F}(x) \stackrel{\text{def}}{=}$ the probability that a cell of size $\alpha/2$ reaches size x (that is, without divide or die);
3. $L(x) \stackrel{\text{def}}{=}$ the expected number of daughter cells that a cell of size $\alpha/2$ will produce before reaching size x.

Concerning these ingredients we make the following technical assumptions:

A1. $\mathcal{G}$ is continuous and strictly increasing. $\mathcal{G}(\frac{\alpha}{2}) = 0$.

A2. $\mathcal{F}$ is strictly positive and for $x < 1$, continuous and non-increasing. $\mathcal{F}(\frac{\alpha}{2}) = 1$.

A3. L is continuous and non-decreasing. $L(x) = 0$ for $\frac{\alpha}{2} \leqslant x \leqslant \alpha$ and $L(1) \leqslant 2$.

A4. $\frac{L(1)-L(x)}{\mathcal{F}(x)} \to 0$ as $x \uparrow 1$.

Except from A4, the logic of these assumptions should be clear from the biological interpretation of $\mathcal{G}$, $\mathcal{F}$ and L, but later in this section we shall discuss A4. From the ingredients we construct the following composite ingredients:

4. $X(t,x) \overset{\text{def}}{=}$ the size of a cell at time t, given that the size is x at time zero.

   Then
$$X(t,x) = \mathcal{G}^{-1}(t+\mathcal{G}(x)), \qquad x \geqslant \frac{\alpha}{2}$$
   (with the convention that $X(t,x) = 1$ for $t > \mathcal{G}(1) - \mathcal{G}(x)$; note that this possibility does not occur when $\mathcal{G}(1) = \infty$, i.e., when it takes infinite time to reach the maximum size).

5. $\frac{\mathcal{F}(y)}{\mathcal{F}(x)}$ = the probability that a cell of size x will neither die nor divide before reaching size $y \geqslant x$.

6. $\frac{L(y)-L(x)}{\mathcal{F}(x)}$ = the expected number of daughter cells that a cell of size x will produce before reaching size y. (So we can now understand what A4 means in biological terms.)

7. $\Lambda(x,x_0)(\omega) \overset{\text{def}}{=}$ the expected number of daughter cells with birth size in ω, a measurable subset of $\left[\frac{\alpha}{2}, \frac{1}{2}\right]$, that a cell of size x_0 produces before reaching size x.

   Then
$$\Lambda(x,x_0)(\omega) = \int\limits_{2\omega\cap[x_0,x)} \frac{dL(\xi)}{\mathcal{F}(x_0)} = \frac{1}{\mathcal{F}(x_0)} \int_{2\omega} \chi_{[x_0,x)}(\xi) dL(\xi),$$
   where χ_I denotes the characteristic function of the set I.

8. The expected number of daughter cells, and their distribution with respect to birth size, that a cell of size x_0 produces in a time interval of length t, is described by the measure $\Lambda(X(t,x_0),x_0)$ with support in $\left[\frac{\alpha}{2}, \frac{1}{2}\right]$.

We now present, as an interlude, the relationship between the present ingredients $\mathcal{G}$, $\mathcal{F}$ and L on the one hand, and the rates g, μ and b occurring in (8.1) on the other hand. Formally we have
$$\frac{d}{dx}\mathcal{G}(x) = (g(x))^{-1}$$
$$\frac{1}{\mathcal{F}(x)}\frac{d}{dx}\mathcal{F}(x) = -\frac{b(x)+\mu(x)}{g(x)}$$
$$\frac{1}{\mathcal{F}(x)}\frac{d}{dx}L(x) = -2\frac{b(x)}{g(x)}$$

but of course there relations can only be made precise under additional regularity assumptions on $\mathcal{G}$, $\mathcal{F}$ and L. The inverse relations are

$$\mathcal{G}(x) = \int_{\alpha/2}^{x} \frac{d\xi}{g(\xi)} \tag{8.3}$$

$$\mathcal{F}(x) = \exp\left(- \int_{\alpha/2}^{x} \frac{b(\xi)+\mu(\xi)}{g(\xi)} d\xi \right) \tag{8.4}$$

$$L(x) = 2 \int_{\alpha/2}^{x} \frac{b(\xi)}{g(\xi)} \mathcal{F}(\xi) d\xi \tag{8.5}$$

Note that, for given nonnegative and integrable rates b, μ and strictly positive rate g, the functions $\mathcal{G}$, $\mathcal{F}$ and L, defined respectively by (8.3), (8.4) and (8.5), satisfy assumptions A1–A3. In conclusion of this section we try to relate the biological interpretation of A4 to its mathematical function.

Condition A4 means that a very large cell (i.e. a cell with size close to the maximum observed size, which is normalized to be one) will with high probability not produce any daughter cells. Actually that condition is not satisfied in the setting of Diekmann, Heijmans & Thieme [14]. These authors assumed that μ is bounded while $b(x)$ tends to infinity as x tends to one in such a way that the integral diverges (i.e. $\mathcal{F}(x) \downarrow 0$ as $x \uparrow 1$) and in that case $\frac{L(1)-L(x)}{\mathcal{F}(x)} \to 2$ for $x \uparrow 1$, which means that a large cell will, with high probability, divide before dying. We think that A4 is probably better warranted since in real cell populations an extremely large cell has usually, for some reason, failed to divide in the "normal" way so it seems reasonable to assume that such a cell will not be able to divide at any later instant. Mathematically, the behaviour of individual cells near the singular point $x = 1$ reflects itself at the population level in the continuity properties of the orbits, in particular at $t = 0$. Here we make assumption A4 in order to stay in the realm of (adjoints of) strongly continuous semigroups. See Remark 8.3.

8.2 Building blocks at the population level

We are going to adopt a perturbation approach, that is, we begin by neglecting births, which are given by the non-local-therms in (8.1).

Let, at some time t_0, the population size and composition be described by a measure m on the interval $\left[\frac{\alpha}{2}, 1\right)$. Then m qualifies as the population state at time t_0. A time interval of length t later, the population state is given by the measure $T_0^*(t)m$ defined as follows (cf. points 4 and 5 of section 8.1):

$$(T_0^*(t)m)(\omega) = \int_{[\frac{\alpha}{2},1)} \chi_\omega(X(t,z)) \frac{\mathcal{F}(X(t,z))}{\mathcal{F}(z)} dm(z). \tag{8.6}$$

The use of the $*$ is justified by the observation that indeed the linear operator so defined is the adjoint of the operator $T_0(t)$ acting on $\mathcal{C}_0\big(\big[\frac{\alpha}{2},1\big),\mathbb{C}\big)$ according to

$$(T_0(t)\varphi)(x) = \varphi(X(t,x))\frac{\mathcal{F}(X(t,x))}{\mathcal{F}(x)}. \tag{8.7}$$

where $\mathcal{C}_0\big(\big[\frac{\alpha}{2},1\big),\mathbb{C}\big)$ denotes the space of complex-valued functions on the internal $\big[\frac{\alpha}{2},1\big)$ which are continuous and tend to zero when the argument tends to 1, equipped with the supremum norm. Alternatively we can think of elements of this space as continuous functions defined on $\big[\frac{\alpha}{2},1\big]$ which are zero at 1. As is well known, the dual space can be represented by the space of complex regular Borel measures m on $\big[\frac{\alpha}{2},1\big)$, provided with the total variation norm. The pairing is given by

$$\langle m,\varphi\rangle = \int_{\alpha/2}^{1}\varphi(x)dm(x).$$

Lemma 8.1. $T_0(t)$, *defined by (8.7), constitutes a* C_0*-semigroup of bounded linear operators on* $\mathcal{C}_0\big(\big[\frac{\alpha}{2},1\big),\mathbb{C}\big)$.

Proof. Note that $X(t,x) \geqslant x$ for $t \geqslant 0$. So the expression (8.7) implies at once that

(i) $(T_0(t)\varphi)(x)$ is a continuous function on $\big[\frac{\alpha}{2},1\big)$ as $x \uparrow 1$;

(ii) since $\frac{\mathcal{F}(X(t,x))}{\mathcal{F}(x)} \leqslant 1$, $T_0(t)$ is a bounded operator of norm less than or equal one;

(iii) $T_0(0) = I$;

(iv) $T_0(t)T_0(s) = T_0(t+s)$, since $X(t+s,x) = X(t,X(s,x))$;

(v) for every $\varphi \in \mathcal{C}_0\big(\big[\frac{\alpha}{2},1\big),\mathbb{C}\big)$ we have that

$$\|T_0(t)\varphi - \varphi\| \to 0 \quad \text{as } t \downarrow 0.$$

□

In order to describe births we recall point 8 of section 8.1 and introduce the *reproduction operators*

$$(U_0^*(t)m)(\omega) = \int_{\alpha/2}^{1}\Lambda(X(t,z),z)(\omega)dm(z) \tag{8.8}$$

and their pre-adjoints

$$\begin{aligned}(U_0(t)\varphi)(x) &= \int_{\alpha/2}^{1}\varphi(y)d_y[\Lambda(X(t,x),x)(y)]\\ &= \frac{1}{\mathcal{F}(x)}\int_{\alpha}^{1}\varphi\big(\tfrac{1}{2}\xi\big)\chi_{[x,X(t,x))}(\xi)dL(\xi).\end{aligned} \tag{8.9}$$

Note that $U_0(0) = 0$ and that $U_0^*(t)m$ gives the expected total number of daughter cells, produced in a time interval of length t, by the cells collectively described by m, as well as their distribution with respect to size at birth.

The remainder of this section is devoted to certain technical properties of the operators $U_0(t)$. We define a function $Q = Q(t, x)$ by

$$Q(t,x) = \begin{cases} \frac{L(X(t,x))-L(x)}{\mathcal{F}(x)} & \text{for } \frac{\alpha}{2} \leqslant x, < 1 \\ 0 & \text{for } x = 1. \end{cases} \tag{8.10}$$

Note that A4 guarantees that $Q(t, \cdot)$ is a continuous function. We claim that $t \mapsto Q(t, \cdot)$ is continuous from $\mathbb{R}_+$ into $\mathcal{C}_0\big(\big[\frac{\alpha}{2}, 1\big), \mathbb{C}\big)$. To prove the claim we choose $\epsilon > 0$ and then $\bar{x} < 1$ such that

$$\frac{L(1) - L(x)}{\mathcal{F}(x)} < \frac{\epsilon}{2} \quad \text{for all } x \in [\bar{x}, 1).$$

We can subsequently choose $\delta > 0$ such that

$$|L(X(t_1, x)) - L(X(t_2, x))| < \epsilon \mathcal{F}(\bar{x})$$

for all $x \in \big[\frac{\alpha}{2}, \bar{x}\big]$ and $|t_l - t_1| < \delta$. Then, provided $|t_1 - t_2| < \delta$, we have, for $x \in \big[\frac{\alpha}{2}, \bar{x}\big]$, the estimate

$$|Q(t_1, x) - Q(t_2, x)| \leqslant \frac{|L(X(t_1, x)) - L(X(t_2, x))|}{\mathcal{F}(\bar{x})} < \epsilon,$$

while for all t_l, t_2 and $\bar{x} \leqslant x < 1$

$$|Q(t_1, x) - Q(t_2, x)| \leqslant Q(t_1, x) + Q(t_2, x) \leqslant 2\frac{L(1) - L(x)}{\mathcal{F}(x)} < \epsilon.$$

Lemma 8.2. *Formula (8.9) defines bounded linear operators $U_0(t)$ on $\mathcal{C}_0\big(\big[\frac{\alpha}{2}, 1\big), \mathbb{C}\big)$ and the mapping $t \mapsto U_0(t)$ is continuous from $\mathbb{R}_+$ into $\mathcal{L}\big(\mathcal{C}_0\big(\big[\frac{\alpha}{2}, 1\big), \mathbb{C}\big)\big)$.*

Proof. From the estimate

$$|(U_0(t)\varphi)(x)| \leqslant \|\varphi\| \; Q(t, \cdot),$$

we infer that $U_0(t)$ is a bounded linear operator on $\mathcal{C}_0\big(\big[\frac{\alpha}{2}, 1\big), \mathbb{C}\big)$, with norm $L(1) \leqslant 2$. Likewise the estimate

$$|U_0(t_1)\varphi - U_0(t_2)\varphi| \leqslant \|\varphi\| \, \|Q(t_1, \cdot) - Q(t_2, \cdot)\|$$

and the observations above show that $t \mapsto U_0(t)$ is continuous. ∎

Remark 8.3. It should now be clear why we made assumption A4: it guarantees that $(U_0(t)\varphi)(x)$ tends to zero for $x \uparrow 1$, i.e. that $U_0(t)$ maps $\mathcal{C}_0\big(\big[\frac{\alpha}{2},1\big),\mathbb{C}\big)$ into itself.

Lemma 8.4. *U_0 is locally of bounded semi-variation, i.e. for given $t > 0$,*

$$V_t(U_0) = \sup\Big\|\sum_{i=1}^{n}\big(U_0(t_i) - U_0(t_{i-1})\big)\varphi_i\Big\|$$

where the supremum is taken over all partitions $0 = t_0 < t_1 < \cdots < t_n = t$ and all $\varphi_i \in \mathcal{C}_0\big(\big[\frac{\alpha}{2},1\big),\mathbb{C}\big)$ with $\|\varphi_i\| \leqslant 1$.

Proof. We compute

$$\begin{aligned}\Big|\sum_{i=1}^{n}\big(U_0(t_i) - U_0(t_{i-1})\big)\varphi_i(x)\Big| &\leqslant \frac{1}{\mathcal{F}(x)}\sum_{i=1}^{n}\big(L(X(t_i,x)) - L(X(t_{i-1},x))\big)\\ &= \frac{L(X(t_i,x)) - L(X(t_{i-1},x))}{\mathcal{F}(x)}\\ &= Q(t_n,x) = Q(t,x)\end{aligned}$$

since $\|\varphi\| \leqslant 1$ and L non-decreasing. ☐

Lemma 8.5. $V_t(U_0) \downarrow 0$ *as* $t \downarrow 0$.

Proof. Combine the estimate in the proof of Lemma 8.4 with the continuity of $t \mapsto Q(t,\cdot)$ and the fact that $Q(0,\cdot)$ is identically zero. ☐

8.3 The abstract renewal equation

The reproduction operators tell us how many (and what kind of) daughter cells are produced by some given group of cells, which we might call the zeroth generation. In other words, applying the reproduction operators to the zeroth generation we find the first generation. In order to find all births we should iterate this procedure indefinitely and sum the results. So let us define for $n \geqslant 0$

$$U^*_{n+1}(t) = \int_0^t U^*_0(t-\tau)\,d_\tau[U^*_n(\tau)] \tag{8.11}$$

and

$$U^*(t) = \sum_{n=0}^{\infty} U^*_n(t). \tag{8.12}$$

Then $U^*(t)$ deserve to be called the (accumulated) *birth operators*. Note that in our situation the sum in (8.12) is actually finite (though the number of terms depends on t and tends to infinity for $t \to \infty$) since, after one or more divisions, a newborn cell has a size less than α and so needs time before being able to divide. Alternatively and equivalently we can write (8.11)–(8.12) as the renewal equation

$$U^*(t) = U_0^*(t) + \int_0^t U_0^*(t-\tau)dU^*(\tau) \tag{8.13}$$

which states that the births fall into two categories: those produced by cells present at time zero and those produced by cells which themselves were born after time zero. It is easier to study (8.13) in its pre-adjoint form

$$U(t) = U_0(t) + \int_0^t dU(\tau)U_0(t-\tau). \tag{8.14}$$

In Diekmann, Gyllenberg & Thieme [12] it is shown that the Stieltjes convolution product is well-defined on the set $\mathcal{A}$ of all uniformly continuous families U of bounded linear operators which are locally of bounded semi-variation and satisfy $U(0) = 0$, and that the equation (8.14) admits a unique solution $U \in \mathcal{A}$ for given $U_0 \in \mathcal{A}$ with $\lim_{t\downarrow 0} V_t(U_0) = 0$. In this context, U is called the resolvent kernel for U_0 and U is given by the appropriate series of iterated convolutions of U_0 with itself (with, in general, convergence on $\mathbb{R}_+$ with respect to some exponentially weighted norm). Recalling Lemmas 8.2, 8.4 and 8.5 and taking adjoints we arrive at the following conclusion:

Theorem 8.6. *The renewal equation (8.13) admits a unique solution, which is given by (8.12) with $U_n^*(t)$ defined by (8.11).*

8.4 Population development

The operators $T_0^*(t)$ tell us how the zeroth generation changes in number (due to death and division) and in cell size (due to cell growth). In order to obtain this information for the whole population, we have to take newborns and their subsequent fate into account. So we introduce operators

$$T^*(t) = T_0^*(t) + \int_0^t T_0^*(t-\tau)dU^*(\tau) \tag{8.15}$$

and their pre-adjoints

$$T(t) = T_0(t) + \int_0^t dU(\tau)T_0(t-\tau). \tag{8.16}$$

The interpretation guarantees that these operator families have the semi-group property. Distrustful as mathematicians are, we shall now verify this formally and, as a bonus, we pinpoint the crucial relationship between T_0 and U_0.

Lemma 8.7. *U_0 is a* step response *for T_0, i.e*

$$U_0(t+s) = U_0(s) + T_0(s)U_0(t).$$

By duality, U_0^* is a cumulative output family *for T_0^*, i.e.*

$$U_0^*(t+s) = U_0^*(s) + U_0^*(t)T_0^*(s).$$

Proof.

$$\begin{aligned}
(T_0(s)U_0(t)\varphi)(x) &= \frac{\mathcal{F}(X(s,x))}{\mathcal{F}(x)} \frac{1}{\mathcal{F}(X(s,x))} \int_\alpha^1 \varphi\big(\tfrac{\xi}{2}\big)\chi_{[X(s,x),X(t,X(s,x)))}(\xi)dL(\xi) \\
&= \frac{1}{\mathcal{F}(x)} \int_\alpha^1 \varphi\big(\tfrac{\xi}{2}\big)\chi_{[X(s,x),X(t,X(s,x)))}(\xi)dL(\xi) \\
&= \frac{1}{\mathcal{F}(x)} \int_\alpha^1 \varphi\big(\tfrac{\xi}{2}\big)\big[\chi_{[x,X(t+s,x))}(\xi) - \chi_{[x,X(s,x))}(\xi)\big]dL(\xi) \\
&= (U_0(t+s)\varphi)(x) - (U_0(s)\varphi)(x).
\end{aligned}$$

□

Theorem 8.8. *U is a step response for T and $T(t)$ is a C_0-semigroup.*

Proof. Using (8.14) and the fact that U_0 is a step response for T_0 we can write

$$\begin{aligned}
U(t+s) &= U_0(t+s) + \int_0^{t+s} dU(\tau)U_0(t+s-\tau) \\
&= U_0(t) + T_0(t)U_0(s) + \int_0^t dU(\tau)\big[U_0(t-\tau) + T_0(t-\tau)U(s)\big] \\
&\qquad + \int_t^{t+s} dU(\tau)U_0(t+s-\tau) \\
&= U_0(t) + T_0(t)U(s) + \int_0^t dU(\tau)\big[U_0(t-\tau) + T_0(t-\tau)U_0(s)\big] \\
&\qquad + \int_0^s d_\tau\big[U(\tau+t) - U(t)\big]U_0(s-\tau) \\
&= U_0(t) + \underbrace{\int_0^t dU(\tau)U_0(t-\tau)}_{U(t)-U_0(t)} \\
&\qquad + \underbrace{\left\{T_0(t) + \int_0^t dU(\tau)T_0(t-\tau)\right\}}_{T(t)} U_0(s)
\end{aligned}$$

$$
\begin{aligned}
&+ \int_0^s d_\tau \big[U(\tau + t) - U(t)\big] U_0(s-\tau) \\
&= U(t) + T(t)U_0(s) + \int_0^s d_\tau \big[U(\tau + t) - U(t)\big] U_0(s-\tau).
\end{aligned}
$$

So, if we fix t and set

$$
\begin{aligned}
W(s) &= U(t+s) - U(t) \\
W_0(s) &= T(t)U_0(s)
\end{aligned}
$$

we have

$$
W = W_0 + dW * U_0
$$

Since U is the resolvent kernel corresponding to U_0 we can compute (cf. Gripenberg *et al.* [21], section 9.3)

$$
\begin{aligned}
W &= W_0 + dW * U_0 \\
&= W_0 + dW * [U - dU * U_0] \\
&= W_0 + dW * U - dW * [U * dU_0] \\
&= W_0 + dW * U - dW * [U_0 * dU] \\
&= W_0 + dW * U - [dW * U_0] * dU \\
&= W_0 + dW * U - [W - W_0] * dU \\
&= W_0 + dW_0 * U
\end{aligned}
$$

which, in terms of the original variables, reads

$$
U(t-s) - U(t) = T(t)U_0(s) + \int_0^s T(t) dU_0(\sigma) U(s-\sigma) = T(t)U(s).
$$

Now $T(0) = T_0(0) = I$ and the strong continuity of T_0 follows from the strong continuity of $t \mapsto U(t)$. It remains to verify that $T(t+s) = T(t)T(s)$, which we do by formula manipulation:

$$
\begin{aligned}
T(t+s) &= T_0(t+s) + \int_0^{t+s} dU(\tau) T_0(t+s-\tau) \\
&= T_0(t)T_0(s) + \int_0^t dU(\tau) T_0(t-\tau) T_0(s) \\
&\qquad + \int_0^s d_\tau \underbrace{[U(t+\tau) - U(t)]}_{T(t)U(\tau)} T_0(s-\tau) \\
&= \underbrace{\left\{ T_0(t) + \int_0^t dU(\tau) T_0(t-\tau) \right\}}_{T(t)} T_0(s)
\end{aligned}
$$

$$
\begin{aligned}
&\quad + \int_0^s T(t) dU(\tau) T_0(s-\tau) \\
&= T(t) \left\{ T_0(s) + \int_0^s dU(\tau) T_0(s-\tau) \right\} \\
&= T(t) T(s)
\end{aligned}
$$

□

8.5 Asymptotic behaviour

Throughout this section we concentrate on the case $\mathcal{G}(1) < \infty$. In most of this section we assume that $\alpha \geqslant \frac{1}{2}$.

As we have done in Chapters 1 and 2, the standard approach in the analysis of the large time behaviour of a semigroup of operators is to analyze the spectrum of its infinitesimal generator. It is possible to determine the resolvent of the generator more or less directly by applying the Laplace transform to the defining equations of the semigroup. Subsequently it only remains to analyze the singularities of the resolvent.

Recalling the identity, for $\operatorname{Re} \lambda$ sufficiently large,

$$
(\lambda I - A)^{-1} = \int_0^\infty e^{-\lambda \tau} T(\tau) d\tau = \widehat{T}(\lambda), \tag{8.17}
$$

we find, upon applying the Laplace transform to (8.16) and (8.14), the equations

$$
(\lambda I - A)^{-1} = (\lambda I - A_0)^{-1} + \widehat{dU}(\lambda)(\lambda I - A_0)^{-1} \tag{8.18}
$$

$$
\widehat{dU}(\lambda) = \widehat{dU}_0(\lambda) + \widehat{dU}(\lambda)\widehat{dU}_0(\lambda) \tag{8.19}
$$

where $\widehat{dU}(\lambda)$ is, by definition, the Laplace-Stieltjes transform of the measure dU, *i.e.*

$$
\widehat{dU}(\lambda) = \int_0^\infty e^{-\lambda \tau} dU(\tau) \tag{8.20}
$$

and with identical definitions for the objects with index zero. (Theorem II.2.1 in Widder [60] gives that $\widehat{dU}(\lambda)$ is defined in a half plane provided that U is exponentially bounded; see Diekmann, Gyllenberg & Thieme [12], Proposition 3.2.d. for the proof that U and U_0 satisfy this hypothesis.) From (8.19) we find that

$$
I + \widehat{dU}(\lambda) = (I - \widehat{dU}_0)^{-1} \tag{8.21}
$$

Combining (8.18) and (8.21) we deduce at once that

$$
(\lambda I - A)^{-1} = (I - \widehat{dU}_0(\lambda))^{-1}(\lambda I - A_0)^{-1} \tag{8.22}
$$

Our next step is to derive explicit representations for the operators at the right hand side of (8.22). Combining (8.17), with index zero, and (8.7) we find

$$\begin{aligned}
[(\lambda I - A_0)^{-1}\psi](x) &= \int_0^\infty e^{-\lambda\tau}\big(T_0(\tau)\psi\big)(x)d\tau \\
&= \int_0^\infty e^{-\lambda\tau}\mathcal{F}(X(\tau,x))\psi(X(\tau,x))d\tau \\
&= \frac{1}{\mathcal{F}(x)}\int_0^\infty e^{-\lambda\tau}\mathcal{F}(X(\tau,x))\psi(X(\tau,x))d\tau \\
&= \frac{e^{\lambda\mathcal{G}(x)}}{\mathcal{F}(x)}\int_x^1 e^{-\lambda\mathcal{G}(\xi)}\mathcal{F}(\xi)\psi(\xi)d\mathcal{G}(\xi) \qquad (8.23)
\end{aligned}$$

So if $\mathcal{G}(1) < \infty$ the function $\lambda \mapsto (\lambda I - A_0)^{-1}$ is entire and, as far as singularities are concerned, we can concentrate on $(I - \widehat{dU}_0(\lambda))^{-1}$. Combining (8.20), with index zero, and (8.9) we find after a simple computation that

$$\begin{aligned}
(\widehat{dU}_0(\lambda)\psi)(x) &= \int_0^\infty e^{-\lambda\tau}d_\tau(U_0(\tau)\psi(x)) \\
&= \cancelto{0}{e^{-\lambda\tau}U_0(\tau)\psi(x)\Big|_{\tau=0}^{\infty}} \quad \text{($U_0(0) = 0$ and $\lim_{t\to\infty} e^{-\lambda t} = 0$)} \\
&\quad - \int_0^\infty (-\lambda e^{-\lambda\tau})(U_0(\tau)\psi)(x)d\tau \\
&= \int_0^\infty \frac{1}{\mathcal{F}(x)}\Big(\int_\alpha^1 \psi(\tfrac{1}{2}\xi)\chi_{[x,X(\tau,x))}(\xi)dL(\xi)\Big)\lambda e^{-\lambda\tau}d\tau
\end{aligned}$$

using that $X(\tau,x) = 1$ for $\tau > \mathcal{G}(1) - \mathcal{G}(x)$

$$\begin{aligned}
&= \int_0^{\mathcal{G}(1)-\mathcal{G}(x)} \frac{1}{\mathcal{F}(x)}\Big(\int_\alpha^1 \psi(\tfrac{1}{2}\xi)\chi_{[x,X(\tau,x))}(\xi)dL(\xi)\Big)\lambda e^{-\lambda\tau}d\tau \\
&\quad + \int_{\mathcal{G}(1)-\mathcal{G}(x)}^\infty \frac{1}{\mathcal{F}(x)}\Big(\int_\alpha^1 \psi(\tfrac{1}{2}\xi)\chi_{[x,1)}(\xi)dL(\xi)\Big)\lambda e^{-\lambda\tau}d\tau. \qquad (8.24)
\end{aligned}$$

By changing variables $\gamma = X(\tau,x) \iff \tau = \mathcal{G}(\gamma) - \mathcal{G}(x)$ with $d\tau = d\mathcal{G}(\gamma)$, that $\chi_{[x,\gamma)}(\xi) = H(\xi - x)(1 - H(\xi - \gamma))$, where H is the Heaviside function, and applying Fubini's Theorem, we compute the first integral on the right hand side of (8.24)

$$
\begin{aligned}
\int_0^{\mathcal{G}(1)-\mathcal{G}(x)} & \frac{1}{\mathcal{F}(x)}\Big(\int_\alpha^1 \psi(\tfrac{1}{2}\xi)\chi_{[x,X(\tau,x))}(\xi)dL(\xi)\Big)\lambda e^{-\lambda\tau}d\tau \\
&= \frac{1}{\mathcal{F}(x)}\int_x^1\Big(\int_\alpha^1 \psi(\tfrac{1}{2}\xi)\chi_{[x,\gamma)}(\xi)dL(\xi)\Big)\lambda e^{-\lambda(\mathcal{G}(\gamma)-\mathcal{G}(x))}d\mathcal{G}(\gamma) \\
&= \frac{e^{\lambda\mathcal{G}(x)}}{\mathcal{F}(x)}\int_x^1\Big(\int_\alpha^1 \psi(\tfrac{1}{2}\xi)H(\xi-x)(1-H(\xi-\gamma))dL(\xi)\Big)\lambda e^{-\lambda\mathcal{G}(\gamma)}d\mathcal{G}(\gamma) \\
&= \frac{e^{\lambda\mathcal{G}(x)}}{\mathcal{F}(x)}\int_\alpha^1\Big(\int_x^1 (1-H(\xi-\gamma))\lambda e^{-\lambda\mathcal{G}(\gamma)}d\mathcal{G}(\gamma)\Big)H(\xi-x)\psi(\tfrac{1}{2}\xi)dL(\xi) \\
&= \frac{e^{\lambda\mathcal{G}(x)}}{\mathcal{F}(x)}\int_\alpha^1\Big(\overbrace{\int_x^1 \lambda e^{-\lambda\mathcal{G}(\gamma)}d\mathcal{G}(\gamma)}^{=-e^{\lambda\mathcal{G}(\gamma)}\big|_{\gamma=x}^{1}=e^{-\lambda\mathcal{G}(x)}-e^{-\lambda\mathcal{G}(1)}} \\
&\qquad -\overbrace{\int_x^1 H(\xi-\gamma)\lambda e^{-\lambda\mathcal{G}(\gamma)}d\mathcal{G}(\gamma)}^{=-H(\xi-x)\int_x^\xi \lambda e^{-\lambda\mathcal{G}(\gamma)}d\mathcal{G}(\gamma)=H(\xi-x)(e^{-\lambda\mathcal{G}(\xi)}-e^{-\lambda\mathcal{G}(x)})}\Big)H(\xi-x)\psi(\tfrac{1}{2}\xi)dL(\xi) \\
&= \frac{e^{\lambda\mathcal{G}(x)}}{\mathcal{F}(x)}\int_\alpha^1\big(-e^{-\lambda\mathcal{G}(1)}+H(\xi-x)e^{-\lambda\mathcal{G}(\xi)}\big)H(\xi-x)\psi(\tfrac{1}{2}\xi)dL(\xi) \\
&\quad + \frac{e^{\lambda\mathcal{G}(x)}}{\mathcal{F}(x)}\int_\alpha^1 e^{-\lambda\mathcal{G}(x)}\cancelto{0}{(1-H(\xi-x))H(\xi-x)}\psi(\tfrac{1}{2}\xi)dL(\xi) \\
&= \frac{e^{\lambda\mathcal{G}(x)}}{\mathcal{F}(x)}\int_\alpha^1\big(-e^{-\lambda\mathcal{G}(1)}+e^{-\lambda\mathcal{G}(\xi)}\big)H(\xi-x)\psi(\tfrac{1}{2}\xi)dL(\xi) \\
&= -\frac{e^{\lambda\mathcal{G}(x)-\lambda\mathcal{G}(1)}}{\mathcal{F}(x)}\int_\alpha^1 H(\xi-x)\psi(\tfrac{1}{2}\xi)dL(\xi) \\
&\quad + \frac{e^{\lambda\mathcal{G}(x)}}{\mathcal{F}(x)}\int_\alpha^1 e^{-\lambda\mathcal{G}(\xi)}H(\xi-x)\psi(\tfrac{1}{2}\xi)dL(\xi). \qquad (8.25)
\end{aligned}
$$

Now we compute the other integral on (8.24)

$$
\begin{aligned}
\int_{\mathcal{G}(1)-\mathcal{G}(x)}^\infty & \frac{1}{\mathcal{F}(x)}\Big(\int_\alpha^1 \psi(\tfrac{1}{2}\xi)\chi_{[x,1)}(\xi)dL(\xi)\Big)\lambda e^{-\lambda\tau}d\tau \\
&= \frac{1}{\mathcal{F}(x)}\Big(\int_\alpha^1 \psi(\tfrac{1}{2}\xi)H(\xi-x)dL(\xi)\Big)\int_{\mathcal{G}(1)-\mathcal{G}(x)}^\infty \lambda e^{-\lambda\tau}d\tau \\
&= \frac{e^{\lambda\mathcal{G}(x)-\lambda\mathcal{G}(1)}}{\mathcal{F}(x)}\int_\alpha^1 \psi(\tfrac{1}{2}\xi)H(\xi-x)dL(\xi). \qquad (8.26)
\end{aligned}
$$

Combining (8.24), (8.25) and (8.26), we obtain

$$(\widehat{dU}_0(\lambda)\psi)(x) = \frac{e^{\lambda\mathcal{G}(x)}}{\mathcal{F}(x)} \int_\alpha^1 H(\xi - x)\psi\big(\tfrac{1}{2}\xi\big)e^{-\lambda\mathcal{G}(\xi)}dL(\xi). \tag{8.27}$$

Note that the image of ψ under $\widehat{dU}_0(\lambda)$ depends only on the restriction of ψ to the interval $\big[\frac{\alpha}{2}, \frac{1}{2}\big]$. This can be exploited when studying the invertability of $I - \widehat{dU}_0(\lambda)$.

In the remainder of this chapter we restrict our attention to the relatively simple case $\alpha \geqslant \frac{1}{2}$ while noting that, provided one adds some notation and a bit of linear algebra, essentially the same ideas allow us to handle any case $\alpha \geqslant 2^{-k}$, $k \in \mathbb{N}$ (see in particular Heijmans [29]).

When $\alpha \geqslant \frac{1}{2}$, $\xi \geqslant \alpha$ and $x \leqslant \frac{1}{2}$, necessarily $\xi \geqslant x$ and the Heaviside function in (8.27) always takes the value one. As a consequence, $\widehat{dU}_0(\lambda)$, considered as an operator on $\mathcal{C}\big(\big[\frac{\alpha}{2}, \frac{1}{2}\big), \mathbb{C}\big)$, has one-dimensional range. Exploiting this fact we derived the following explicit representation for the inverse of $I - \widehat{dU}_0(\lambda)$.

Lemma 8.9. *For $\lambda \in \mathbb{C}$, let $\widehat{dV}(\lambda) : X \to \mathbb{C}$, $\widehat{dW}(\lambda) \in X$ and $\widehat{dk}(\lambda) \in \mathbb{C}$ be defined by*

$$\widehat{dV}(\lambda)\psi = e^{\lambda\mathcal{G}(\alpha)} \int_\alpha^1 \psi\big(\tfrac{1}{2}\xi\big)e^{-\lambda\mathcal{G}(\xi)}dL(\xi) \tag{8.28}$$

$$\widehat{dW}(\lambda)(x) = \frac{e^{\lambda(\mathcal{G}(x)-\mathcal{G}(\alpha))}}{\mathcal{F}(x)} \int_\alpha^1 \frac{e^{\lambda(\mathcal{G}(\frac{1}{2}\xi)-\mathcal{G}(\xi))}}{\mathcal{F}(\frac{1}{2}x)} H(\xi - x)dL(\xi) \tag{8.29}$$

$$\widehat{dk}(\lambda) = \int_\alpha^1 \frac{e^{\lambda(\mathcal{G}(\frac{1}{2}\xi)-\mathcal{G}(\xi))}}{\mathcal{F}(\frac{1}{2}x)} dL(\xi) \tag{8.30}$$

then for $\operatorname{Re}\lambda$ *sufficiently large*

$$(I - \widehat{dU}_0(\lambda))^{-1} = I + \widehat{dU}_0(\lambda) + \frac{1}{1 - \widehat{dk}(\lambda)}\widehat{dW}(\lambda)\widehat{dV}(\lambda). \tag{8.31}$$

Proof. We observed that $\widehat{dU}_0(\lambda)\psi$ depends only on the restriction of ψ to $\big[\frac{\alpha}{2}, \frac{1}{2}\big]$, and when $\alpha > 1/2$, for $x \leqslant 1/2$,

$$(\widehat{dU}_0(\lambda)\psi)(x) = \frac{e^{\lambda\mathcal{G}(x)}}{\mathcal{F}(x)} \int_\alpha^1 \psi\big(\tfrac{1}{2}\xi\big)e^{-\lambda\mathcal{G}(\xi)}dL(\xi).$$

We conclude that

$$[\widehat{dU}_0(\lambda)]^3 = \widehat{dk}(\lambda)[\widehat{dU}_0(\lambda)]^2. \tag{8.32}$$

We can split the parts of $([\widehat{dU}_0(\lambda)]^2\psi)(x)$ that explicitly depends on ψ and x:

$$[\widehat{dU}_0(\lambda)]^2 = \widehat{dW}(\lambda)\widehat{dV}(\lambda). \tag{8.33}$$

(It is easy to see this since $\widehat{dW}(\lambda) = \widehat{dU}_0(\lambda)(e^{\lambda\mathcal{G}(\cdot)}/\mathcal{F}(\cdot))$ and for $x \leqslant 1/2$ we have $\widehat{dV}(\lambda)\psi\,\frac{e^{\lambda\mathcal{G}(x)}}{\mathcal{F}(x)} = [\widehat{dU}_0(\lambda)\psi](x)$.) For $\operatorname{Re}\lambda$ sufficiently large we have that $(1 - \widehat{dk}(\lambda)) \neq 0$ and we can combine (8.32) and (8.33) to obtain

$$\frac{I - \widehat{dU}_0(\lambda)}{1 - \widehat{dk}(\lambda)}\,\widehat{dW}(\lambda)\widehat{dV}(\lambda) = [\widehat{dU}_0(\lambda)]^2.$$

An easy evaluation shows that

$$\begin{aligned}
&(I - \widehat{dU}_0(\lambda))\Big(I + \widehat{dU}_0(\lambda) + \frac{1}{1 - \widehat{dk}(\lambda)}\widehat{dW}(\lambda)\widehat{dV}(\lambda)\Big) = \\
&\Big(I + \widehat{dU}_0(\lambda) + \frac{1}{1 - \widehat{dk}(\lambda)}\widehat{dW}(\lambda)\widehat{dV}(\lambda)\Big)(I - \widehat{dU}_0(\lambda)) = I
\end{aligned}$$

and the lemma is proved. □

Corollary 8.10. $\sigma(A) = P\sigma(A) = \{\lambda \in \mathbb{C} : \widehat{dk}(\lambda) = 1\}$

Proof. The assumption $\mathcal{G}(1) < \infty$ guarantees that $\widehat{dU}_0$ (and consequently $\widehat{dW}_0$), $\widehat{dV}_0$ and $\widehat{dk}$ are entire functions of λ. The same is true for $(\lambda I - A_0)^{-1}$, see (8.23). So combining (8.22) with (8.31), we see that the resolvent of A is regular for all λ with $\widehat{dk}(\lambda) \neq 1$ and that it has (non-removable) singularities at those λ for which $\widehat{dk}(\lambda) = 1$. Since $\widehat{dk}$ is an analytic function of λ, these singularities are necessarily isolated poles of finite order. The associated projection operator has finite rank, hence all points in the spectrum are eigenvalues. □

In order to justify the suggestive notation and for future use we introduce functions of time $V(t) : X \to \mathbb{C}$, $W(t) \in X$ and $k(t) \in \mathbb{C}$:

$$\begin{aligned}
V(t)\psi &= \int_{\alpha}^{X(t,\alpha)} \psi\big(\tfrac{1}{2}\xi\big)\,dL(\xi) \\
&= \int_{\alpha}^{1} H(t + \mathcal{G}(\alpha) - \mathcal{G}(\xi))\psi\big(\tfrac{1}{2}\xi\big)\,dL(\xi) \qquad (8.34)
\end{aligned}$$

$$\begin{aligned}
W(t)(x) &= \frac{1}{\mathcal{F}(x)} \int\limits_{\{\xi\in[\alpha,1):\mathcal{G}(\xi)-\mathcal{G}(\frac{1}{2}\xi)<t+\mathcal{G}(x)-G(\alpha)\}} \frac{H(\xi - x)}{\mathcal{F}(\frac{1}{2}\xi)}\,dL(\xi) \\
&= \frac{1}{\mathcal{F}(x)} \int_{\alpha}^{1} \frac{H(t + \mathcal{G}(x) - G(\alpha) - \mathcal{G}(\xi) + \mathcal{G}(\frac{1}{2}\xi))H(\xi - x)}{\mathcal{F}(\frac{1}{2}\xi)}\,dL(\xi) \qquad (8.35)
\end{aligned}$$

$$k(t) = \int\limits_{\{\xi\in[\alpha,1):\mathcal{G}(\xi)-\mathcal{G}(\frac{1}{2}\xi)<t\}} \frac{1}{\mathcal{F}(\frac{1}{2}\xi)}\,dL(\xi)$$

$$= \int_\alpha^1 \frac{H(t - \mathcal{G}(\xi) + \mathcal{G}(\frac{1}{2}\xi))}{\mathcal{F}(\frac{1}{2}\xi)} dL(\xi) \tag{8.36}$$

One can use either Theorem C on page 163 of Halmos [28] or, alternatively, apply the inverse Laplace transform to the functions of λ, cf. Widder [60], to verify the assertion of the next lemma that $\widehat{dV}$, $\widehat{dW}$ and $\widehat{dk}$ are indeed the Laplace-Stieltjes transform of V, W and k, respectively. Note that we have incorporated a factor $e^{\lambda\mathcal{G}(\alpha)}$ in $V(\lambda)$, compensated by a factor $e^{-\lambda\mathcal{G}(\alpha)}$ in $W(\lambda)$, in order that V has support in $[0,\infty)$ (and without "destroying" that property for W, since $\mathcal{G}(x) - \mathcal{G}(\alpha) + \mathcal{G}\big(\frac{1}{2}\xi\big) - \mathcal{G}(\xi) < 0$ for $\xi \geqslant \max\{\alpha, x\}$.

Lemma 8.11.

(i) $$\widehat{dV}(\lambda) = \int_0^\infty e^{-\lambda t} dV(t)$$

(ii) $$\widehat{dW}(\lambda) = \int_0^\infty e^{-\lambda t} dW(t)$$

(iii) $$\widehat{dk}(\lambda) = \int_0^\infty e^{-\lambda t} dk(t)$$

8.5.1 Analysis of the spectrum

There are now two cases to be distinguished:

- The lattice case;
- The non-lattice case.

In the lattice case the kernel k is constant except for (finitely many) jumps which all occur at integer multiples of some number c. In other words, the measure dk is concentrated on a set of points which are commensurable, with a greatest common factor which we call c (since k has compact support the set of points is actually finite). As a consequence, $\widehat{dk}(\lambda)$ is periodic along lines $\operatorname{Re}\lambda = \text{constant}$, with period $\frac{2\pi}{c}$. Therefore the roots of $\widehat{dk}(\lambda) = 1$ occur in countably infinite groups on vertical lines in the complex plane. Since k is a nontrivial non-decreasing function of t, there exists precisely one real root, which we call λ_d. On the line $\operatorname{Re}\lambda = \lambda_d$ we have roots $\lambda = \lambda_d + \frac{2n\pi i}{c}$, $n \in \mathbb{Z}$, which form a group under addition. All other roots have real part less than λ_d.

As we shall show below by explicit calculations for a special lattice case, asymptotically for $t \to \infty$ we have

$$T(t)\phi \sim e^{\lambda_d} S(t) P_{\ell(\lambda_d)}\phi \tag{8.37}$$

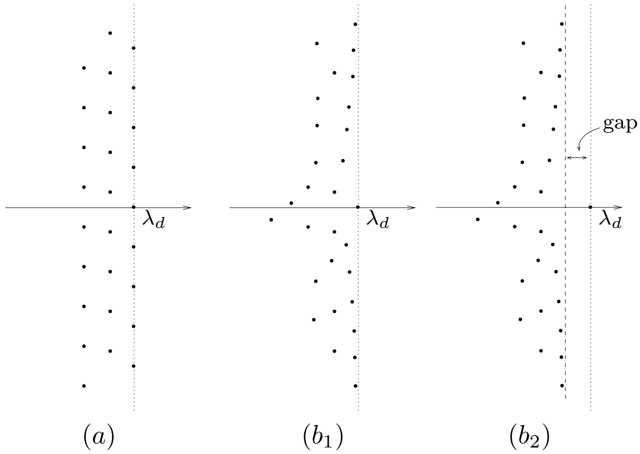

Figure 8.1: Sketch of the location of roots of $1 - \widehat{dk}(\lambda)$: (a) illustrates when k is lattice, where there is a constant c such that $\lambda_d + \frac{2j\pi}{c}$ is a root, and no other roots have real part greater than λ_d; when k is non-lattice, all roots $\lambda \neq \lambda_d$ satisfy $\operatorname{Re}\lambda < \lambda_d$, but there are (simple) examples where there is no "spectral gap", that is, there exists a sequence λ_j of roots such that $\lim_{j\to\infty} \operatorname{Re}\lambda_j = \lambda_d$, as illustrated in (b_1); the case where there is the spectral gap is illustrated in (b_2).

⋆ ⋆ ⋆

where $P_{\ell(\lambda_d)}$ is a projection onto an infinite dimensional subspace which is isomorphic to the space of c-periodic continuous functions and $S(t)$ is a group of operators conjugate to translation as a group acting on the c-periodic functions.

The argument which shows that there exists precisely one real root λ_d of the characteristic equation $\widehat{dk}(\lambda) = 1$ does not require k to be lattice. In the non-lattice case there are no other roots on the line $\operatorname{Re}\lambda = \lambda_d$ and all roots different from λ_d satisfy $\operatorname{Re}\lambda < \lambda_d$. We shall then find that

$$T(t)\phi \sim e^{\lambda_d} S(t) P_{\lambda_d}\phi \tag{8.38}$$

where P_{λ_d} is a projection operator with one-dimensional range.

This dichotomy of two types of possible behaviour, (8.38) the rule and (8.37) the exception, is a general phenomenon for positive semigroups, see Arendt *et al.* [1], section C-IV.2.

In order to gain some intuitive understanding of (8.37) and (8.38) we shall first analyze the problem from the point of view of inverse Laplace transformation applied to (8.22), using (8.31). Since k is a non-decreasing function of λ, then $\widehat{dk}'(\lambda_d) \neq 0$ or, in other words, λ is a simple root of the equation $\widehat{dk}(\lambda) = 1$. Hence

$(\lambda I - A)^{-1}$ has a first order pole in $\lambda = \lambda_d$ with residue

$$P_{\lambda_d} = \frac{\widehat{dW}(\lambda_d)\widehat{dV}(\lambda_d)(\lambda_d I - A_0)^{-1}}{-\widehat{dk}'(\lambda_d)}. \tag{8.39}$$

The range of this operator P_{λ_d} is spanned by $\widehat{dW}(\lambda_d)$. More precisely we have

$$P_{\lambda_d}\psi = C(\psi)\widehat{dW}(\lambda_d) \tag{8.40}$$

where

$$C(\psi) = \frac{e^{\lambda\mathcal{G}(\alpha)}\int_\alpha^1 \frac{e^{\lambda(\mathcal{G}(\frac{1}{2}\xi)-\mathcal{G}(\xi))}}{\mathcal{F}(\frac{1}{2}\xi)}\int_{\frac{1}{2}\xi}^1 e^{-\lambda\mathcal{G}(\eta)}\mathcal{F}(\eta)\psi(\eta)d\mathcal{G}(\eta)dL(\xi)}{\int_\alpha^1\left(\mathcal{G}(\xi)-\mathcal{G}(\frac{1}{2}\xi)\right)\frac{e^{\lambda(\mathcal{G}(\frac{1}{2}\xi)-\mathcal{G}(\xi))}}{\mathcal{F}(\frac{1}{2}\xi)}dL(\xi)}. \tag{8.41}$$

So either invoking general results about Dunford integrals (see Yosida [62], section VIII.7,8) or checking that $\widehat{dk}(\lambda_d) = 1$ implies that $C(\widehat{dW}(\lambda_d)) = 1$, we conclude that P_{λ_d} is a projection operator with one-dimensional range. When the real parts of all other roots of $k(\lambda) = 1$ are uniformly bounded away from λ_d one can prove that

$$\|e^{-\lambda_d t}T(t) - P_{\lambda_d}\| \to 0 \quad \text{as } t \to \infty, \tag{8.42}$$

that is, convergence in the operator norm. In the present generality, however, this need not be the case. The behaviour of $\widehat{dk}(\lambda)$ for $\operatorname{Im}\lambda \to \pm\infty$ may be quite complicated if k has a non-discrete singular part (see Lyons [40], Rudin [50]; also see section 4.4 in Gripenberg, Londen & Staffans [21]). Moreover, if, for instance, k is constant except for two jumps which occur in points which are incommensurable (i.e. rationally independent) then there must exist a sequence of roots λ_n such that $\operatorname{Im}\lambda_n \to \pm\infty$, $\operatorname{Re}\lambda_n \uparrow \lambda_d$, essentially since $\widehat{dk}(\lambda)$ is almost periodic as a function of $\operatorname{Im}\lambda$ (see the proof of Proposition 4.1 in Gyllenberg [22]) As we shall show below using a result of Feller [17] on the renewal equation, we then still can have that (8.38) holds (i.e. strong convergence) but (8.42) is no longer true (indeed Webb [59] proves that uniform convergence implies exponential estimates for the remainder and these cannot hold in the situation described above; see Gyllenberg & Webb[25] for another example of strong but non-uniform convergence).

In order to have a lattice kernel k we should have that $\mathcal{G}\left(\frac{1}{2}\xi\right) - \mathcal{G}(\xi)$ is constant on intervals on which L is strictly increasing. In order to simplify both the discussion and the calculations we shall restrict our attention to the case that $\mathcal{G}\left(\frac{1}{2}\xi\right) - \mathcal{G}(\xi) = c$ for $\alpha \leqslant \xi < 1$. It is then immediate that the residues in $\lambda = \lambda_d + \frac{2n\pi i}{c}$ are of the form

$$c_n e^{\frac{2n\pi i}{c}\mathcal{G}(x)}\frac{e^{\lambda_d}\mathcal{G}(x)}{\mathcal{F}(x)}\int_\alpha^1 \frac{H(\xi - x)}{\mathcal{F}(\frac{1}{2}\xi)}dL(\xi) \tag{8.43}$$

So, provided the c_n are indeed the Fourier coefficients of a c-periodic function z, which depends on the initial condition ψ, we find that one can associate with the whole group of poles the operator $P_{\ell(\lambda_d)}$ "defined" by

$$(P_{\ell(\lambda_d)}\psi)(x) = \frac{e^{\lambda_d}\mathcal{G}(x)}{\mathcal{F}(x)} \int_\alpha^1 \frac{H(\xi - x)}{\mathcal{F}(\frac{1}{2}\xi)} dL(\xi) z_\psi(\mathcal{G}(x)). \tag{8.44}$$

Let us try to understand in biological terms what the condition $\mathcal{G}(\xi) - \mathcal{G}\left(\frac{1}{2}\xi\right) = c$ means. When $\alpha \geqslant 1/2$, every newborn cell necessarily has to pass the size $1/2$ before it can possibly divide. So we can base our book-keeping as well on the "traffic" of cells at $x = 1/2$ (actually this is the biological "reason" for the "essentially one-dimensional range" property of $\widehat{dU}_0(\lambda)$ and the fact that the problem reduces to a one-dimensional renewal equation). Let us start a clock when a cell passes $x = 1/2$ and stop that clock when any of its daughters passes $1/2$. Suppose the cell divides at $\xi \geqslant \alpha$. Then the clock shows time $\mathcal{G}(\xi) - \mathcal{G}(1/2)$ when it divides. At that time the two daughters start to grow with size $\xi/2$ and so they will reach size $1/2$ after a time interval of length $\mathcal{G}(1/2) - \mathcal{G}(\xi/2)$. Hence the clock will stop at time $\mathcal{G}(\xi) - \mathcal{G}(\xi/2)$. Clearly when this time is independent of ξ, all cells that pass $x = 1/2$ simultaneously will have offspring that pass $x = 1/2$ simultaneously and likewise time differences are preserved from generation to generation (in fact the renewal equation reduces to a difference equation in this case; see also Metz & Diekmann (eds.) [41], section V.11). So there is no "smoothing", no (eventual) compactness. Clearly any modification of our assumptions, however slight, may introduce some smoothing and thereby change the "periodic" asymptotic behaviour into the "normal" (for positive semigroups) one of convergence to a stable distribution. Such perturbations were studied by Heijmans [29], who considered asymmetric division, and Gyllenberg & Webb [23, 24], Rossa [47], who considered the effect of quiescence.

Below we shall derive, by explicit calculations, an expression for z_ψ from which it then directly follows that $P_{\ell(\lambda_d)}$ is a projection operator. In addition these calculations will show that (8.37) holds. Under additional regularity conditions on k an alternative proof follows from the results of Kaashoek & Verduyn Lunel [36].

After this long exposition about the singularities of the resolvent and their relation to the two types of behaviour (either a stable distribution or periodic continuation of an infinite dimensional amount of information about the initial condition), we shall now finally formulate and prove two theorems. The method of proof is slightly different: we use the resolvent kernel r of the kernel k to translate (8.22) and (8.31) into an explicit representation for $T(t)$ and subsequently use results of Feller [17] about the one-dimensional renewal equation to deduce the asymptotic behaviour of $T(t)$ for $t \to \infty$.

Theorem 8.12. *Assume that k is non-lattice. Let λ_d be the unique real root of the* *equation* $\widehat{dk}(\lambda) = 1$. *Then for any* $\phi \in X$

$$\lim_{t\to\infty} \|e^{-\lambda_d t}T(t)\phi - P_{\lambda_d}\phi\| = 0,$$

where P_{λ_d} *is defined by (8.40)–(8.41).*

Proof. Let r denote the resolvent kernel of k, i.e.

$$r = \sum_{n=0}^{\infty} k^{n*} \tag{8.45}$$

where by definition $r^{0*} = H$, the Heaviside function, and $r^{(n+l)*} = dr * r^{n*}$, $n \geqslant 0$. So equivalently r is the unique solution of the renewal equation

$$r = H + dk * r \tag{8.46}$$

and from this we infer that

$$\widehat{r}(\lambda) = (1 - \widehat{dk}(\lambda))^{-1} \tag{8.47}$$

or, in words, $(1 - \widehat{dk}(\lambda))^{-1}$ is the Laplace-Stieltjes transform of r. Combining (8.22), (8.31) and (8.47), we find the representation

$$\begin{aligned} T(t) = T_0(t) &+ \int_0^t dU_0(\tau)T_0(t-r) \\ &+ \int_0^t dr(\tau)\int_0^{t-\tau} dW(\sigma)\int_0^{t-r-\sigma} dV(\eta)T_0(t-\tau-\sigma-\eta). \end{aligned} \tag{8.48}$$

Since $\mathcal{G}(1) < \infty$, the first two terms become zero in finite time whereas the operators $V(t)$ and $W(t)$ are independent of t, once t is large enough. It follows that the asymptotic behaviour of $T(t)$ is completely determined by the asymptotic behaviour of $r(t)$. For this we consult Feller [17], Chapter XI. In order to adjust to the setting of that chapter we define

$$\tilde{r}(t) = \int_0^t e^{-\lambda_d \tau} dr(\tau), \qquad \tilde{k}(t) = \int_0^t e^{-\lambda_d \tau} dk(\tau). \tag{8.49}$$

Then $k(\infty) = 1$ and (8.46) can be reformulated as

$$\tilde{r} = H + d\tilde{k} * \tilde{r}. \tag{8.50}$$

Since, by assumption, k is non-lattice (or non-arithmetic in the terminology of Feller) we deduce from Feller's first form of the Renewal Theorem that for every $h > 0$

$$\tilde{r}(t) - \tilde{r}(t-h) \to \frac{h}{\mu} \quad \text{as } t \to \infty \tag{8.51}$$

where

$$\mu \stackrel{\text{def}}{=} \int_0^\infty \tau e^{-\lambda_d \tau} dk(\tau). \tag{8.52}$$

It follows that for any f with compact support

$$\lim_{t\to\infty} e^{-\lambda_d t} \int_0^t dr(\tau) f(t-\tau) = \frac{1}{\mu} \int_0^\infty f(\tau) e^{\lambda_d \tau} d\tau \tag{8.53}$$

which is (a weak version of) Feller's alternative form of the Renewal Theorem. Applying this to (8.48) we find that

$$\lim_{t\to\infty} e^{-\lambda_d t} T(t)\phi = \frac{1}{\mu} \widehat{dW}(\lambda_d) \widehat{dV}(\lambda_d)(\lambda_d I - A_0)^{-1}\phi. \tag{8.54}$$

Using the explicit formulas for all factors we finally conclude that the right-hand side of (8.54) equals $P_{\lambda_d}\phi$, where P_{λ_d} is defined by (8.40)–(8.41). □

Theorem 8.13. *Assume that $\mathcal{G}(\xi) - \mathcal{G}\big(\tfrac{1}{2}\xi\big) = c$ for $\alpha \leqslant \xi < 1$. For given $\phi \in X$ define a c-periodic function $z_\phi : \mathbb{R} \to \mathbb{R}$ by the formulas*

$$\begin{aligned} z_\phi(\sigma) = \Theta^{-2} e^{-\lambda_d \sigma} \Big\{ &\phi(2\mathcal{G}^{-1}(\sigma))\mathcal{F}(2\mathcal{G}^{-1}(\sigma)) \\ &+ \phi(\mathcal{G}^{-1}(\sigma))\mathcal{F}(\mathcal{G}^{-1}(\sigma)) \int_\alpha^{2\mathcal{G}^{-1}(\sigma)} \frac{dL(\xi)}{\mathcal{F}(\frac{1}{2}\xi)}, \quad 0 \leqslant \sigma \leqslant \mathcal{G}^{-1}\big(\tfrac{1}{2}\big) \end{aligned} \tag{8.55}$$

$$z_\phi(\sigma) = \Theta^{-1} e^{-\lambda_d \sigma} \phi(\mathcal{G}^{-1}(\sigma))\mathcal{F}(\mathcal{G}^{-1}(\sigma)), \qquad \mathcal{G}^{-1}\big(\tfrac{1}{2}\big) \leqslant \sigma < c \tag{8.56}$$

and periodic continuation. Here

$$\Theta \stackrel{\text{def}}{=} \int_\alpha^1 \frac{dL(\xi)}{\mathcal{F}(\frac{1}{2}\xi)}. \tag{8.57}$$

Then for sufficiently large t we have explicitly

$$(T(t)\phi)(x) \approx e^{\lambda_d t} z_\phi(t + \mathcal{G}(x)) \frac{e^{\lambda_d \mathcal{G}(x)}}{\mathcal{F}(x)} \int_\alpha^1 \frac{H(\xi - x)}{\mathcal{F}(\frac{1}{2}\xi)} dL(\xi). \tag{8.58}$$

Proof. First recall that $\mathcal{G}\big(\frac{\alpha}{2}\big) = 0$ and therefore $c = \mathcal{G}(a) > \mathcal{G}\big(\frac{1}{2}\big)$. Next recall that $\phi(1) = 0$, which guarantees that z is continuous in $\sigma = \mathcal{G}\big(\frac{1}{2}\big)$ and $z_\phi(0) = z_\phi(c)$. Under our assumption that $\mathcal{G}(\xi) - \mathcal{G}\big(\frac{1}{2}\xi\big) = c$ we find that

$$\left(\int_0^t dV(\eta) T_0(t-\eta)\phi \right) = \phi(\mathcal{G}^{-1}(t))\mathcal{F}(\mathcal{G}^{-1}(t)) \int_\alpha^{\mathcal{G}^{-1}(t+c)} \frac{dL(\xi)}{\mathcal{F}(\frac{1}{2}\xi)} \tag{8.59}$$

and subsequently that

$$\begin{aligned}
&\Big(\int_0^t dW(\tau)\int_0^{t-\tau} dV(\eta)T_0(t-\tau-\eta)\phi\Big)(x)\\
&\qquad = \phi(\mathcal{G}^[-1](t+\mathcal{G}(x)-2c))\mathcal{F}(\mathcal{G}^{-1}(t+\mathcal{G}(x)-2c)) \qquad (8.60)\\
&\qquad\quad \cdot \frac{1}{\mathcal{F}(x)}\int_\alpha^1 \frac{H(\xi-x)}{\mathcal{F}(\frac{1}{2}\xi)}dL(\xi)\int_\alpha^{\mathcal{G}^{-1}(t+\mathcal{G}(x)-c)}\frac{dL(\xi)}{\mathcal{F}(\frac{1}{2}\xi)}
\end{aligned}$$

(where the expression should be interpreted as zero for $t+\mathcal{G}(x) < 2c$ and $t+\mathcal{G}(x) > 2c+\mathcal{G}(1)$). Since $k(t) = \Theta H(t-c)$ we find for the resolvent r defined by (8.45) that

$$r(t) = \sum_{j-0}^{\infty} \Theta^j H(t-jd). \qquad (8.61)$$

Combining (8.48), (8.60) and (8.61) we see that for $t > 2\mathcal{G}(1)$, when the first two terms at the right hand side of (8.48) are zero, we have

$$\begin{aligned}
&(T(t)\phi)(x)\\
&\quad = \frac{1}{\mathcal{F}(x)}\int_\alpha^1 \frac{H(\xi-x)}{\mathcal{F}(\frac{1}{2}\xi)}dL(\xi)\\
&\qquad \cdot \sum_{j=0}^{\infty}\Big(\Theta^j\phi(\mathcal{G}^{-1}(t+\mathcal{G}(x)-(2+j)c)) \qquad (8.62)\\
&\qquad\qquad \cdot \mathcal{F}(\mathcal{G}^{-1}(t+\mathcal{G}(x)-(2+j)c))\int_0^{\mathcal{G}^{-1}(t+\mathcal{G}(x)-(1+j)c)}\frac{dL(\xi)}{\mathcal{F}(\frac{1}{2}\xi)}\Big)
\end{aligned}$$

where the j-th term of the sum is different from zero if and only if

$$(2+j)c < t+\mathcal{G}(x) < (2+j)c+\mathcal{G}(1). \qquad (8.63)$$

Now note that at most two terms can contribute at the same time since $\mathcal{G}(1) = c+\mathcal{G}\big(\frac{1}{2}\big) < c+\mathcal{G}(\alpha) = 2c$. In fact only the j-th term contributes for $(j+1)c+\mathcal{G}(1) < t+\mathcal{G}(x) < (j+3)c$, whereas both the j-th and $(j+1)$-th term contribute for $(j+3)c < t+\mathcal{G}(x) < (j+2)c+\mathcal{G}(1)$. It follows that we have periodicity of period c modulo multiplication by Θ. It only remains to note that $\Theta e^{-\lambda_d c} = \widehat{dk}(\lambda_d) = 1$ and to exploit the periodicity to rewrite (8.62) as (8.58) with z_ϕ given by (8.55) and (8.56). $\square$

Corollary 8.14. *When $\mathcal{G}(\xi)-\mathcal{G}\big(\frac{1}{2}\xi\big) = c$ for $\alpha \leqslant \xi < 1$, the projection operator $P_{\ell(\lambda_d)}$ takes the form*

$$\big(P_{\ell(\lambda_d)}\psi\big)(x) = \begin{cases}
\Theta^{-1}\Big(\frac{\mathcal{F}(2x)}{\mathcal{F}(x)}\psi(2x)+\int_\alpha^{2x}\frac{dL(\xi)}{\mathcal{F}(\frac{1}{2}\xi)}\psi(x)\Big), & \frac{\alpha}{2}\leqslant x<\frac{1}{2}\\
\psi(x), & \frac{1}{2}\leqslant x<\alpha\\
\Theta^{-1}\Big(\frac{\mathcal{F}(\frac{x}{2})}{\mathcal{F}(x)}\int_\alpha^x\frac{dL(\xi)}{\mathcal{F}(\frac{1}{2}\xi)}\int_x^1\frac{dL(\xi)}{\mathcal{F}(\frac{1}{2}\xi)}\psi\big(\frac{x}{2}\big)+\int_\alpha^1\frac{dL(\xi)}{\mathcal{F}(\frac{1}{2}\xi)}\psi(x)\Big), & \alpha\leqslant x<1
\end{cases}$$

8.6 Comments

Different authors have made different assumptions concerning g, μ and b as well as different choices for the (population) state space, ranging from L_1 (with or without a weight function related to g and b) to the space of finite Borel measures considered as the dual space of the space of continuous functions. Jagers [34] shows how the model fits into the framework of multi-type branching processes and he obtains strong conclusions from the general theory of such processes.

Similar cell cycle models have been formulated by Arino & Kimmel [2] as integral equations; in Arino & Kimmel [3] these authors have elaborated in detail the relationship between various formulations. See Kimmel [37, 38] for earlier work in the spirit of the approach that we take here. See Tyson & Diekmann [54] for an application to actual data.

Diekmann, Gyllenberg & Thieme [12, 13] argued that abstract Stieltjes renewal equations are an attractive alternative to functional partial equations, like (8.1), to formulate structured population models. Although the full strength of the alternative manifests itself only in time-dependent linear and in nonlinear problems, some of the advantages are evident in the context of autonomous linear problems.

Appendix A

The Hale bilinear form and the duality between $\mathcal{C}$ and $\mathcal{C}'$

In this chapter we justify the duality between the spaces $\mathcal{C}$ and $\mathcal{C}'$ though a bilinear form first introduced by Hale. See Chapter 9 of Hale & Verduyn Lunel [27]. See also Hale [26].

A.1 Representation of a FDE as a renewal equation

In Section 1.4, from Lemma 1.12 and the discussion after following it, we have represented the FDE

$$\begin{cases} \frac{d}{dt} M x_t = L x_t, & t \geqslant 0, \\ x_0 = \varphi, & \varphi \in \mathcal{C}, \end{cases} \tag{A.1}$$

where $L, M : \mathcal{C} \to \mathbb{C}^n$ are linear continuous, given respectively by

$$L\varphi = \int_0^r d\eta(\theta)\varphi(-\theta), \qquad M\varphi = \varphi(0) - \int_0^r d\mu(\theta)\varphi(-\theta), \tag{A.2}$$

with $\eta, \mu \in NBV\big([0,r], \mathbb{C}^{n\times n}\big)$ and μ continuous at zero, as the renewal equation

$$x = dk * x + F\varphi, \tag{A.3}$$

where $k \in NBV\big([0,r], \mathbb{C}^{n\times n}\big)$ is given by

$$k(\theta) = \mu(\theta) + \int_0^t \eta(\theta) d\theta. \tag{A.4}$$

and $F : \mathcal{C} \to L^\infty$ is given by

$$F\varphi(t) = M\varphi + \int_t^r d\mu(\theta)\varphi(t-\theta) + \int_0^t \Big[\int_s^r d\eta(\theta)\varphi(s-\theta)\Big] ds, \quad t \geqslant 0. \tag{A.5}$$

We have that F maps the initial condition $\varphi \in \mathcal{C}$ into the corresponding forcing function of the renewal equation (A.3). For $\varphi \in \mathcal{C}$ one has that $F\varphi(\cdot)$ is constant on $[r, \infty)$, $F\varphi(0) = \varphi(0)$ and $F\varphi + \mu \cdot \varphi(0)$ is continuous. We have seen also that FDE (A.1) is also equivalent to the renewal in the space of

$$y = dk * y + F_0\varphi, \tag{A.6}$$

where $F_0 : \mathcal{C} \to \{f \in \mathcal{C}([0,\infty), \mathbb{C}^n) : f(0) = 0\}$ is given by

$$F_0(\varphi) \stackrel{\text{def}}{=} F\varphi - \varphi(0) + k \cdot \varphi(0). \tag{A.7}$$

A computation shows that

$$F_0\varphi(t) = F\big(\varphi - \varphi(0)\big)(t) + \eta(r)\varphi(0)t.$$

From Theorem 1.3, we obtain that the solution y satisfies $y(0) = 0$ and y is continuous in $[0, \infty)$. The correspondence between the solutions of (A.3)–(A.6) is given by $x = y + \varphi(0)$. From Lemma 1.7, there exists a unique resolvent kernel $R \in NBV\big([0, r], \mathbb{C}^{n\times n}\big)$, *i.e.*,

$$dR - dk = dR * dk.$$

There is a one-to-one correspondence between *restrictions of solutions of renewal equations to* $[0, \infty)$ and *forcing functions*, namely

$$x \mapsto f = x - dk * x \qquad \text{and} \qquad f \mapsto x = f - dR * f$$

In equations (A.3) and (A.6), by x and y we mean such restrictions. We observe that there is *not* a one-to-one correspondence between initial conditions and forcing functions. By construction, the forcing function can also be obtained by applying F to the initial condition φ. See Figure A.1.

In Section 1.5, we discussed the space $\mathcal{F}$ of forcing functions for the representation of FDE (A.1) as the renewal equation (A.3). We defined $\mathcal{F}$ as the set of $\mathbb{C}^n$-valued functions f on $[0, \infty)$ such that there exists $g \in \mathcal{C}([0,\infty), \mathbb{C}^n)$ such that $f = g + \mu \cdot g(0)$ and $f(t) = f(r)$ for $t \geqslant r$.

A.2 The Transposed Equation

In an initial value problem like FDE A.1, the "evolution" is done by translation of the solutions. One defines the solution semigroup $T(t)\varphi$, for $\varphi \in \mathcal{C}$, by $T(t)\varphi = x_t(\cdot, \varphi)$. This semigroup is strongly continuous, that is,

$$\lim_{t\to 0} T(t)\varphi = \varphi,$$

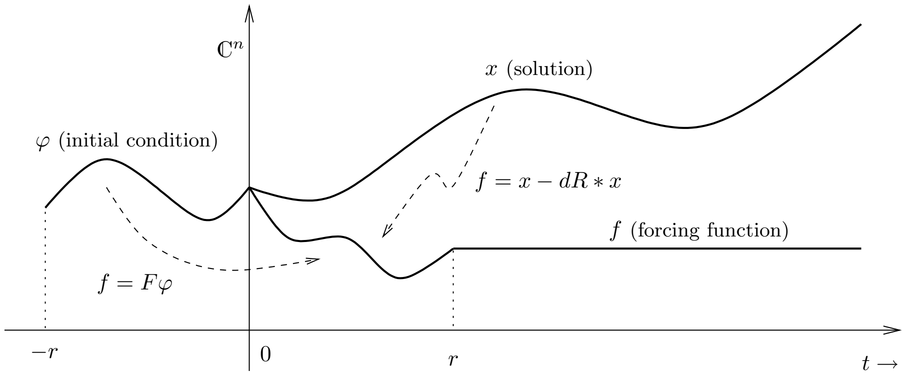

Figure A.1: Functional equation (A.1) is equivalent to renewal equation (A.3), where forcing function f can be obtained either from the initial condition by $f = F\varphi$ or by the restriction of the solution to $[0, \infty)$ by $f = x - dk * x$.

⋆ ⋆ ⋆

for all $\varphi \in \mathcal{C}$. It is possible to isolate the part that explicitly depends on the initial condition, and from there we obtain the forcing function of the representation of FDE as a renewal equation, as we have done in Lemma 1.12. By construction, evolution of initial condition (by translation) is equivalent to evolution of the forcing function (by computing the forcing function corresponding to the piece of solution.) It was realized that the dual semigroup $T^*(t)$ acts by evolving the forcing function of an equation related to (A.1), the so-called *transposed equation*, see Hale & Verduyn Lunel [27]. It turns out to be simpler to study the duality between the transposed and the original equation than develop the adjoint theory for functional equations.

The transposed equation can be defined in several equivalent ways. One is to consider the transpose μ^{T} and η^{T} of the matrices μ and η used in (1.3) and (1.4). In this case the transposed equation takes the form

$$\begin{cases} \dfrac{d}{dt}\Big[z(t) - \displaystyle\int_0^r d\mu^{\mathrm{T}}(\theta) z(t-\theta)\Big] = \displaystyle\int_0^r d\eta^{\mathrm{T}}(\theta) z(t-\theta), & t \geqslant 0, \\ \qquad\qquad z_0 = \varphi, & \varphi \in \mathcal{C}. \end{cases}$$

This form has $\mathcal{C}$ as it state space too. A second approach, that will be considered here, has $\mathcal{C}' \stackrel{\text{def}}{=} \mathcal{C}\big([0,r], \mathbb{C}^{n*}\big)$ as its state space. We define the operators M' and L' by

$$M'\psi = \psi(0) - \int_0^r \psi(\xi) d\mu(\xi), \qquad L'\psi = \int_0^r \psi(\xi) d\eta(\xi), \tag{A.8}$$

where μ and η are the same measures as in the definition of M and L in (1.4).

The transpose of (1.3) is defined to be

$$\begin{cases} \frac{d}{dt}\big[M'y(t+\cdot)\big] = -L'y(t+\cdot), & t \leqslant 0, \\ \qquad\qquad y^0 = \psi, & \psi \in \mathcal{C}'. \end{cases} \tag{A.9}$$

Remark A.1. For the sake of readability, we motivate the choice of taking solutions on backward time. The phenomenon described bellow can not be observed in autonomous equations, like the ones that we study. For a *non-autonomous* equation, that is, the equation depends on time, the solution is no longer a one-parameter semigroup, but two-parameter. For $s \leqslant t$ and $\varphi \in \mathcal{C}$, denote by $T(s,t)\varphi = x_t(\cdot; s, \varphi)$ the solution of an non-autonomous FDE through (s,φ) at time t, and let $T^*(t,s)$ denote the dual of $T(s,t)$ — pay attention that s and t are swapped in T^*. For $f \in \mathcal{C}^*$ and $r \leqslant s \leqslant t$,

$$\big\langle f, T(r,s)T(s,t)\varphi\big\rangle = \big\langle T^*(s,r)f, T(s,t)\varphi\big\rangle = \big\langle T^*(t,s)T^*(s,r)f, \varphi\big\rangle.$$

This shows that T^* is a backward semigroup. Since the transposed equation is related to T^*— it is the dual of T with respect to some bilinear form — it shares the backward nature. We keep this behaviour for autonomous equations. The minus sign in front of L' in (A.9) comes from the derivative of t in inverted time direction.

For each $f \in \mathcal{F}$, (1.33) has a unique solution $x(\cdot; f)$ defined on $\mathbb{R}_+$ (see Chapter 9 of [27]). We can define a semigroup $\{S(t)\}_{t\geqslant 0}$ on $\mathcal{F}$ by

$$S(t)f = Fx_t(\cdot; f), \tag{A.10}$$

that is, $S(t)f$ is the forcing function corresponding to the solution $x_t(\cdot; f)$ of (1.33). By construction, for $\varphi \in \mathcal{C}$, the solution $x(\cdot; F\varphi)$ of (1.28) also satisfies (1.3). So the following diagram commutes.

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\ F\ } & \mathcal{F} \\ {\scriptstyle T(t)}\big\downarrow & & \big\downarrow{\scriptstyle S(t)} \\ \mathcal{C} & \xrightarrow[F]{\quad} & \mathcal{F} \end{array}$$

If we define $T^*(t) = T(t)^*$ for $t \geqslant 0$, then $T^*(t)$ defines a semigroup (however, generally not strongly continuous; see[15]) in $\mathcal{C}^* = NBV\big([0,r], \mathbb{C}^{n*}\big)$. One could ask for the differential equation of which $T^*(t)$ is the solution semigroup. One sees that $\mathcal{C}^*$ and $\mathcal{F}$ are spaces with the same nature (both are spaces of normalized bounded variation into a n-dimensional complex space). It turns out that there is indeed a close connection between $T^*(t)$ and $S^{\mathrm{T}}(t)$, the semigroup defined in a similar manner as $S(t)$, but starting with the transposed equation (A.9).

Proceeding analogously as in Section 1.5, we present the space of forcing functions for the transposed equation, which we denote by $\mathcal{F}'$.

Definition A.2. We define $\mathcal{F}'$ as the set of functions $g : (-\infty, 0] \to \mathbb{C}^{n*}$ such that there is a continuous function $h : (-\infty, 0] \to \mathbb{C}^{n*}$ such that

$$g(t) = h(t) - h(0)\mu(-t). \tag{A.11}$$

and $g(t) = g(-r)$ for $t \leqslant -r$.

It is clear that $g(0) = h(0)$, and therefore we can rewrite (A.11) as

$$h(t) = g(t) + g(0)\mu(-t). \tag{A.12}$$

Proceeding as in Lemma 1.12, we rewrite the transposed equation as a renewal equation.

Lemma A.3. *The transposed equation (A.9) is equivalent to the following renewal equation*

$$y(t) = \int_0^t y(\theta)\big[d\mu(s-\theta) + \eta(s-\theta)d\theta\big] + F^{\mathrm{T}}\psi(t), \quad t \leqslant 0, \tag{A.13}$$

where

$$F^{\mathrm{T}}\psi(t) = M'\psi + \int_{-t}^r \psi(t+\xi)d\mu(\xi) - \int_0^t \left[\int_{-s}^r \psi(s+\xi)d\eta(\xi)\right]ds, \quad t \leqslant 0, \tag{A.14}$$

maps $\psi \in \mathcal{C}'$ onto the space $\mathcal{F}'$.

Proof. As was done in the proof of Lemma 1.12, if $y(\cdot;\psi)$ is the solution of (A.9) through $\psi \in \mathcal{C}'$, for $t > 0$ and $\gamma \in NBV\big([0,r], \mathcal{L}(\mathbb{C}^n)\big)$, we can write

$$\int_0^r y(-t+\theta)d\gamma(\theta) = \int_0^t y(-t+\theta)d\gamma(\theta) + \int_t^r \psi(-t+\theta)d\gamma(\theta). \tag{A.15}$$

From equation (A.9), we have

$$M'y(t+\cdot) - M'\psi = -\int_0^t L'y(s+\cdot)ds = \int_0^{-t} L'y(-s+\cdot)ds. \tag{A.16}$$

We can isolate $y(t)$ on (A.16) from the definition of M', use (A.15) and apply

Fubbini

$$\begin{aligned}
y(t) &= \int_0^r y(t+\xi)d\mu(\xi) + M'\psi + \int_0^{-t}\left[\int_0^r y(-s+\xi)d\eta(\xi)\right]ds \\
&= \int_0^{-t} y(t+\xi)d\mu(\xi) + \int_{-t}^r \psi(t+\xi)d\mu(\xi) + M'\psi \\
&\qquad + \int_0^{-t}\left[\int_0^s y(-s+\xi)d\eta(\xi) + \int_s^r \psi(-s+\xi)d\eta(\xi)\right]ds \\
&= \int_0^{-t} y(t+\xi)d\mu(\xi) + \int_0^{-t}\left[\int_0^s y(-s+\xi)d\eta(\xi)\right]ds \\
&\qquad + M'\psi + \int_{-t}^r \psi(t+\xi)d\mu(\xi) + \int_0^{-t}\left[\int_s^r \psi(-s+\xi)d\eta(\xi)\right]ds \\
&= \int_0^{-t} y(t+\xi)d\mu(\xi) + \int_0^{-t}\left[\int_\xi^{-t} y(-s+\xi)ds\right]d\eta(\xi) + F^{\mathrm{T}}\psi(-t) \\
&= \int_0^{-t} y(t+\xi)d\mu(\xi) + \int_0^{-t}\left[\int_0^{-t-\xi} y(\tau)d\tau\right]d\eta(\xi) + F^{\mathrm{T}}\psi(-t) \\
&= \int_0^{-t} y(t+\xi)d\mu(\xi) + \cancelto{0}{\left[\int_0^{-t-\xi} y(\tau)d\tau\right]\eta(\xi)\,\Big|_{\xi=0}^{-t}} \\
&\qquad - \int_0^{-t} y(-t+\xi)\eta(\xi)d\xi + F^{\mathrm{T}}\psi(-t) \\
&= \int_0^{-t} y(t+\xi)d\mu(\xi) - \int_0^{-t} y(-t+\xi)\eta(\xi)d\sigma + F^{\mathrm{T}}\psi(-t).
\end{aligned}$$

To show that F^{T} maps $\mathcal{C}\big([0,r],\mathbb{C}^{n*}\big)$ into $\mathcal{F}'$, it suffices to show, for $t \leqslant 0$, that $F^{\mathrm{T}}\psi(t) + \psi(0)\mu(t)$, as we see in (A.12). We observe that the only term in the right hand side of (A.14) that is not continuous is the first integral. As we did in the proof of Lemma 1.12, we extend ψ on $(-\infty, r]$ by setting $\psi(t) = \psi(0)$ for $t \leqslant 0$. Since "translation" is a continuous operation, the map

$$t \mapsto \int_0^r \psi^{-t}(\xi)d\mu(\xi) = \psi(o)\cdot\mu(-t) + \int_{-t}^r \psi(t+\xi)d\mu(\xi), \quad t \leqslant 0$$

is continuous. Hence, we obtain that $F^{\mathrm{T}}\psi(t) + \psi(0)\cdot\mu(-t)$, for $t \leqslant 0$. Since $F^{\mathrm{T}}\psi(0) = \psi(0)$, it follows that $F^{\mathrm{T}}\psi \in \mathcal{F}'$. $\square$

A.3 Duality between $\mathcal{C}$ and $\mathcal{C}'$

As before, for each $g \in \mathcal{F}'$, the equation

$$y - y * dk = g \tag{A.17}$$

has a unique solution $y(\cdot, g)$ defined in $\mathbb{R}_-$, and we can define a semigroup $S^{\mathrm{T}}(t)$ on $\mathcal{F}'$ by

$$S^{\mathrm{T}}(t)g = F^{\mathrm{T}} y^t(\cdot, g),$$

that is, $S^{\mathrm{T}}(t)g$ is the forcing function corresponding to the solution $y^t(\cdot, g)$ of (A.17). By construction, for $\psi \in \mathcal{C}'$, the solution $y(\cdot, F^{\mathrm{T}}\psi)$ also satisfies (A.9) and therefore the following diagram commutes

$$\begin{array}{ccc} \mathcal{C}' & \xrightarrow{F^{\mathrm{T}}} & \mathcal{F}' \\ {\scriptstyle T^{\mathrm{T}}(t)}\big\downarrow & & \big\downarrow{\scriptstyle S^{\mathrm{T}}(t)} \\ \mathcal{C}' & \xrightarrow[F^{\mathrm{T}}]{} & \mathcal{F}' \end{array}$$

One can show that $S^{\mathrm{T}}(t) = T^*(t)$. More precisely, the actions of the operators $S^{\mathrm{T}}(t)$ and $T^*(t)$ coincide when applied to functions in the intersection of their domains. These facts are shown in Chapter 9 of Hale & Verduyn Lunel [27]. Therefore

$$F^{\mathrm{T}} T^{\mathrm{T}}(s)\psi = T^*(s) F^{\mathrm{T}}\psi, \qquad \psi \in \mathcal{C}'. \tag{A.18}$$

Relation (A.18) suggests the introduction of a special bilinear form between $\mathcal{C}$ and $\mathcal{C}'$. The introduction of the bilinear form (A.19) on next lemma allows us to use the duality between A and A^{T}, instead of, between A and the much more complicated operator A^*.

Lemma A.4 (Hale bilinear form). *Consider the bilinear form* $(\cdot,\cdot) : \mathcal{C}' \times \mathcal{C} \to \mathbb{C}$ *defined by*

$$\begin{aligned}(\psi, \varphi) = \psi(0)\varphi(0) - \int_{-r}^{0} d_\theta \Big[\int_{-r}^{\theta} \psi(\theta - \xi) d\mu(\xi)\Big]\varphi(\theta) \\ + \int_{-r}^{0}\int_{-r}^{\theta} \psi(\theta - \xi) d\eta(\xi)\varphi(\theta) d\theta.\end{aligned} \tag{A.19}$$

Then the transposed operator A^{T} *satisfies*

$$(\psi, A\varphi) = (A^{\mathrm{T}}\psi, \varphi), \qquad \psi \in \mathcal{D}(A^{\mathrm{T}}), \qquad \varphi \in \mathcal{D}(A).$$

Proof. For $\psi \in \mathcal{C}'$ and $\varphi \in \mathcal{C}$, define

$$\begin{aligned}
(\psi, \varphi) &\stackrel{\text{def}}{=} -\big\langle F^{\mathrm{T}}\psi, \varphi \big\rangle \\
&= -\int_{-r}^{0} d[F^{\mathrm{T}}\psi(\theta)]\varphi(\theta) \\
&= -\int_{-r}^{0} d_\theta \Big[M'\psi + \int_{-r}^{\theta} \psi(\theta - \xi) d\mu(\xi) + \int_{\theta}^{0} \int_{-r}^{\sigma} \psi(\sigma - \xi) d\eta(\xi)\, d\sigma \Big] \varphi(\theta) \\
&= F^{\mathrm{T}}\psi(0)\varphi(0) - \int_{-r}^{0} d_\theta \Big[\int_{-r}^{\theta} \psi(\theta - \xi) d\mu(\xi) \Big] \varphi(\theta) \\
&\qquad\qquad - \int_{-r}^{0} d_\theta \Big[\int_{\theta}^{0} \int_{-r}^{\sigma} \psi(\sigma - \xi) d\eta(\xi)\, d\sigma \Big] \varphi(\theta) \\
&= \psi(0)\varphi(0) - \int_{-r}^{0} d_\theta \Big[\int_{-r}^{\theta} \psi(\theta - \xi) d\mu(\xi) \Big] \varphi(\theta) \\
&\qquad\qquad + \int_{-r}^{0} \int_{-r}^{\theta} \psi(\theta - \xi) d\eta(\xi) \varphi(\theta) d\theta
\end{aligned}$$

from where Hale bilinear form (A.19) arises. Using (A.18), for $\psi \in \mathcal{D}\big(A^{\mathrm{T}}\big)$ and $\varphi \in \mathcal{D}\big(A\big)$, we have

$$(\psi, A\varphi) = -\big\langle F^{\mathrm{T}}\psi, A\varphi \big\rangle = -\big\langle A^{*}F^{\mathrm{T}}\psi, \varphi \big\rangle = -\big\langle F^{\mathrm{T}}A^{\mathrm{T}}\psi, \varphi \big\rangle = (A^{\mathrm{T}}\psi, \varphi).$$

$\square$

Appendix B

Source code for the Maple library FDESpectralProj

This code was developed with Maple release 9, but should work in older versions like 5 without any problem.

```
# begin of ‘FDESpectralProj.mpl’

###################################################################
# Maple(R) Library ‘FDESpectralProj’
#
# Compute the Spectral Projection for linear autonomous
# functional differential equations of neutral type with
# discrete time delays.
#
# Copyright 2004 by Miguel V. S. Frasson
#
###################################################################

# Package initialization

# A maple package is defined by a variable that contains a table,
# where each entry is a object (often, procedures).
FDESpectralProj:= table():

# The ‘init’ function is executed when package is loaded
FDESpectralProj[init]:= proc() with(LinearAlgebra): end proc:

# Default max order for zeros. Avoid infinite loop.
FDESpectralProj[MaxOrderZero]:= 100:

###################################################################
#######  Zeros of functions and derivatives of fractional  ########
#######      functions in points of differentiability      ########
###################################################################
```

```

########################## OrderOfZero #########################
#      (expression, variable, point [, maxorder=<number>])
#
# Description
#
# Determine the order of the zero point of (nonzero) expression as
# function of variable. A function f(x) has a as zero of order n if
# the derivatives of f  up to order n-1 are zero on a  and the next
# derivative is nonzero. The function identically zero have
# infinity order.
#
# Parameters
#
# expression - expression that should depend on variable
# variable - a (not assigned) variable
# point - a expression (should be just a constant) where the
#    derivatives of expression with respect to variable are going
#    to be evaluated.
# maxorder - optional parameter that determines a limit for the
#    computation of derivatives.

FDESpectralProj[OrderOfZero]:= proc( expression::algebraic,
                                     variable::name,
                                     point::algebraic )

local i, MaxOrder;
global FDESpectralProj;

  MaxOrder:= FDESpectralProj['MaxOrderZero']; # default value

## parsing optional parameters ##
  if not typematch([args[4..nargs]],
    'list'({'identical'('maxorder')='MaxOrder'::'nonnegint'}))
  then
    ERROR( sprintf("%s %s '%q'",
      "invalid set of optional parameters; the options",
      "should set `maxorder', like `maxorder=15', but got",
      args[4..nargs]))
  end if;

  if expression = 0 then
    RETURN(infinity)
  else
    if simplify(eval(expression,variable=point))=0 then
      i:=1;
      while simplify(eval(diff(expression,variable$i),
                 variable=point))=0 and i< MaxOrder do
        i:= i+1;
      end do;
      if i >= MaxOrder then
        if simplify(eval(diff(expression,variable$i),
                variable=point))=0 then
```

```
          if nargs > 3 then
            ERROR( sprintf("%s %s %d. %s %s",
              "1st parameter (expression) has a zero of",
              "order greater than given `maximum order'",
              MaxOrder,
              "Tip: provide a larger number for optional",
              "parameter `maxorder=<number>'."))
          else
            ERROR( sprintf("%s %s %d. %s %s",
              "1st parameter (expression) has a zero of",
              "order greater than default `maximum order'",
              MaxOrder,
              "Tip: provide a larger number for optional",
              "parameter `maxorder=<number>'."))
          end if;
        else
          RETURN(i)
        end if
      else
        RETURN(i)
      end if
    else
      RETURN(0)
    end if;
  end if;
end proc:

######################         MatrixOrderOfZero  ######################
#      (MatrixExpr, variable, point [, maxorder=<number>])
#
# Same as OrderOfZero, but for matrices.
#
# Parameters
#
# MatrixExpr - Matrix that should depend on variable
# variable - a (not assigned) variable
# point - a expression (should be just a constant) where the
#    derivatives of expression with respect to variable are going
#    to be evaluated.
# maxorder - optional parameter that determines a limit for the
#    computation of derivatives.

FDESpectralProj[MatrixOrderOfZero]:= proc( MatrixExpr::Matrix,
                                           variable::name,
                                           point::algebraic )
local i, j, MaxOrder, result, dimh, dimv;
global FDESpectralProj;

  MaxOrder:= FDESpectralProj['MaxOrderZero']; # default value

## parsing optional parameters ##
  if not typematch([args[4..nargs]],
    'list'({'identical'('maxorder')='MaxOrder'::'nonnegint'}))
```

```
  then
    ERROR( sprintf("%s %s '%q'",
      "invalid set of optional parameters; the options",
      "should set `maxorder', like `maxorder=15', but got",
      args[4..nargs]))
  end if;

  dimh,dimv:= LinearAlgebra['Dimension'](MatrixExpr);

  result:= infinity;

  for i from 1 to dimh do
    for j from 1 to dimv do
      result:= eval(min(result,
                        FDESpectralProj['OrderOfZero'](
                              MatrixExpr[i,j],
                              variable,
                              point,maxorder=MaxOrder)));
    end do;
  end do;
  RETURN(eval(result));
end proc:

####################  EvalDerivsFracFunction  ####################
#     (numerator, denominator, variable, point [, options])
#
# Description
#
# Return a list with the value of fractional function
# numerator/denominator and its derivatives with respect to
# variable from 0 up to order order (optional parameter
# order=<number>, default 0) when variable is evaluated to
# point.  In order to numerator/denominator been analytic at
# variable=point, the order of point as zero of  numerator
# must be greater or equal than the order of point as zero of
# numerator. This routine uses L'Hospital rule to compute
# these values.
#
# Parameters
#
# numerator - expression that should depend on variable.
# denominator - expression that should depend on variable.
# variable - the variable, a name (symbol)
# point - point of evaluation of
#   d^j/d variable^j (numerator/denominator), j=0,.., order.
#
# Optional parameters
#
# order=number - order up to which derivatives are computed
# maxorderofzero=number - passed as parameter of procedure
#    OrderOfZero as maxorder

FDESpectralProj[EvalDerivsFracFunction]:=
```

```
                              proc( numerator::algebraic,
                                    denominator::algebraic,
                                    variable::name,
                                    point::algebraic)

global FDESpectralProj;
local oozden,ooznum,diffnum,diffden,ord,MaxOrderOfZero,den_res,i,j;

# defaults for order and maxorder
ord:=0;
MaxOrderOfZero:= eval(FDESpectralProj['MaxOrderZero']);

## parsing optional parameters ##
if not typematch([args[5..nargs]], 'list'(
    {'identical'('maxorderofzero')='MaxOrderOfZero'::'nonnegint',
     'identical'('order')='ord'::'nonnegint'})) then
  ERROR( sprintf("%s %s `%q'",
    "invalid set of optional parameters; the options should set",
    "`maxorderofzero=<number>' or `order=<number>', but got",
    args[5..nargs]))
end if;

oozden:= FDESpectralProj[OrderOfZero]
             (denominator,variable,point,maxorder=MaxOrderOfZero);
ooznum:= FDESpectralProj[OrderOfZero]
             (numerator,variable,point,maxorder=MaxOrderOfZero);

if oozden > ooznum then
  ERROR(sprintf("%s %a at %a %s %s",
    "Order of Zero of 1st parameter with respect to",
    variable, point,
    "is greater than order of zero of 2nd parameter on same",
    "context, causing division by zero"))
else
  if oozden=0 then
    diffnum[0]:= simplify(numerator);
    diffden[0]:= simplify(denominator);
  else
    diffnum[oozden]:= simplify(diff(numerator,variable$oozden));
    diffden[oozden]:= simplify(diff(denominator,variable$oozden));
  end if;
  for i from oozden+1 to oozden+1+ord do
     diffnum[i]:= simplify(diff(diffnum[i-1],variable));
     diffden[i]:= simplify(diff(diffden[i-1],variable));
  end do;
  # eval at point and get coeffs of Taylor expansion
  for i from oozden to oozden+1+ord do
     diffnum[i]:=
        simplify(eval(diffnum[i],variable=point)/factorial(i));
     diffden[i]:=
        simplify(eval(diffden[i],variable=point)/factorial(i));
  end do;
```

```
  den_res[0]:= diffnum[oozden];
  for i from 1 to ord do
    den_res[i]:= diffnum[oozden+i]*diffden[oozden]^i;
    for j from 0 to (i-1) do
      den_res[i]:= den_res[i]
                   - den_res[j]*diffden[oozden]^(i-1-j)
                              *diffden[oozden+i-j];
    end do;
    den_res[i]:= simplify(den_res[i]);
  end do;
  # preparing result
  for i from 0 to ord do
    den_res[i]:= simplify(den_res[i]
                        *factorial(i)/diffden[oozden]^(i+1));
  end do;
  RETURN(convert(den_res,list));
end if;
end proc:

#################  MatrixEvalDerivsFracFunction  #################
#   ( Mnumerator, denominator, variable, point [, options])
#
# Description
#
# Same as EvalDerivsFracFunction, but works
# entriwise. Return a list with the value of fractional function
# Mnumerator/denominator, with Mnumerator of type Matrix, and its
# derivatives with respect to variable from 0 up to order order,
# when variable is evaluated to point. In order to
# Mnumerator/denominator been analytic at variable=point, the order
# of point as zero of all entries of Mnumerator must be greater or
# equal than the order of point as zero of denominator. This
# routine uses L'Hospital rule t compute these values.
#
# Parameters
#
# Mnumerator - Matrix such that each entry is a expression, that
#     should depend on variable.
# denominator - expression that should depend on variable.
# point - point of evaluation of
#     d^j/d variable^j (numerator/denominator), j=0,.., order.
#
# Optional parameters
#
# order=number - order up to which derivatives are computed
# maxorderofzero=number - passed as parameter of procedure
#    OrderOfZero as maxorder

FDESpectralProj[MatrixEvalDerivsFracFunction]:=
                                     proc( numerator::Matrix,
                                           denominator::algebraic,
                                           variable::name,
                                           point::algebraic)
```

```
global FDESpectralProj;
local i,j,k,dimh,dimv,Mresult,algnum,algresult,MaxOrderOfZero,ord;

ord:= 0;

MaxOrderOfZero:= FDESpectralProj['MaxOrderZero'];

## parsing optional parameters ##
if not typematch([args[5..nargs]],'list'(
   {'identical'('maxorderofzero')='MaxOrderOfZero'::'nonnegint',
    'identical'('order')='ord'::'nonnegint'})) then
  ERROR( sprintf("%s %s `%q'",
    "invalid set of optional parameters; the options should set",
    "`maxorderofzero=<number>' or `order=<number>', but got",
    args[5..nargs]))
end if;
dimh,dimv:= LinearAlgebra['Dimension'](numerator);
Mresult:= [ seq( Matrix(dimh,dimv), i=0..ord ) ];
for i from 1 to dimh do
  for j from 1 to dimv do
    algnum:= numerator[i,j];
    algresult:= FDESpectralProj['EvalDerivsFracFunction']
                     (algnum,denominator,variable,point,
                      'order'=ord,'maxorderofzero'=MaxOrderOfZero);
    for k from 1 to ord+1 do
      Mresult[k][i,j]:= algresult[k];
    end do;
  end do;
end do;
RETURN(Mresult);
end proc:

####################################################################
##################  Functions related with FDEs  ##################
####################################################################

######################  SystemConsistent  ######################
#                     (Mmat, Mdel, Lmat, Ldel)
#
# Checks for the consistency of a FDE described as above given by
# four lists, where the 1st and 3rd are lists of matrices (see
# function Matrix) or expressions and the 2nd and 4th args are
# lists of positive numbers (the delays), and 1st and 2nd have the
# same number of elements, and the same for the 3rd and 4th lists.
#
# Parameters
#
# Mmat - list of numbers or matrices of same dimension.
# Mdel - list of (positive) time delays. It must have the same
#    number of elements of Mmat to return true.
# Lmat - list of numbers or matrices of same dimension.
# Ldel  - list of (positive) time delays. It must have the same
```

```
#    number of elements of Lmat to return true.

FDESpectralProj[SystemConsistent] :=
                           proc( Mmat::list(Or(algebraic,Matrix)),
                                 Mdel::list(algebraic),
                                 Lmat::list(Or(algebraic,Matrix)),
                                 Ldel::list(algebraic) )
local Ok,dim,j;

Ok:= evalb( nops(Mmat) = nops(Mdel) and nops(Lmat) = nops(Ldel) );
if type(Mmat[1],'Matrix') and Ok then
  dim:= LinearAlgebra['Dimension'](Mmat[1]);
  for j from 2 to nops(Mmat) do
    Ok:= evalb( Ok and type(Mmat[j],'Matrix') );
    if Ok then
      Ok:= evalb( dim = LinearAlgebra['Dimension'](Mmat[j]) );
    end if;
  end do;
  if Ok then
    for j from 1 to nops(Lmat) do
      Ok:= evalb( Ok and type(Lmat[j],'Matrix') );
      if Ok then
        Ok:= evalb( dim = LinearAlgebra['Dimension'](Lmat[j]) );
      end if;
    end do;
  end if;
end if;
RETURN(Ok);
end proc:

#####################  DiscreteTimeOperator  #####################
#             (mat, del, function [, dimension=value])
#
# Evaluate a operator described by lists mat and del in function.
# Typically, ‘mat’ would be ‘Mmat’ or ‘Lmat’, described before, and
# ‘del’ should be ‘Mdel’ or ‘Ldel’.
#
# Paramenters
#
# mat - list of numbers or matrices of same dimension.
# del - list of (positive) time delays; it should have the same
#    number of elements of mat.
# function -  a function that evaluates to an algebraic (scalar)
#    expression or a vector of functions.
#
# Optional parameter
#
# dimension=value - in case of empty lists mat and del, then this
#    option tell the type of the zero to be returned; the options
#    for ‘value’ are ‘scalar’ (0 is returned) or a positive integer
#    (matrix is returned); this option is ignored if mat is non
#    empty.  If omitted, ‘scalar’ is assumed.
```

```
FDESpectralProj[DiscreteTimeOperator]:=
                      proc( mat::list(Or(algebraic,Matrix)),
                            del::list(algebraic),
                            function)
local i, dim, temp;
global FDESpectralProj;

## parsing optional parameters ##
  if not typematch([args[4..nargs]],
    'list'({'identical'('dimension')='dim'::
             Or('nonnegint','identical'('scalar'))}))
  then
    ERROR( sprintf("%s %s %s '%q'",
      "invalid set of optional parameters; the optional parameter",
      "should set `dimension' with `scalar' or a positive integer, "
      "but got",
      args[4..nargs]))
  end if;

  if evalb(nops(mat)>0) then
    if evalb( type(mat[1],'Matrix') ) then
      dim:= LinearAlgebra['RowDimension'](mat[1]);
    else
      dim:= 'scalar';
    end if;
  else
    if evalb(not(assigned(dim))) then
      dim:= 'scalar';
    end if;
  end if;

  if evalb(nops(mat)>0) then
    if evalb(dim='scalar') then
      temp:= 0;
      for i from 1 to nops(mat) do
        temp:= temp + mat[i]*function(-del[i]);
      end do;
    else
      temp:= Vector(dim);
      for i from 1 to nops(mat) do
        temp:= temp + LinearAlgebra['MatrixVectorMultiply'](
                   mat[i],
                   Vector([seq(function[j](-del[i]),j=1..dim)]));
      end do;
    end if;
    RETURN(simplify(temp));
  else
    if evalb(dim='scalar') then
      RETURN(0);
    else
      RETURN(Vector(dim));
    end if;
  end if;
```

```
end proc:

##########################  CharMatrix  ##########################
#                (Mmat, Mdel, Lmat, Ldel, variable)
#
# Compute the characteristic matrix of the autonomous Neutral FDE
# given by parameters.
#
# Parameters
#
# Mmat, Mdel, Lmat, Ldel - description of FDE as before
# variable - variable of the characteristic matrix.

FDESpectralProj[CharMatrix]:= proc(Mmat,Mdel,Lmat,Ldel,variable)
local thesum, dimsystem, i, j, Mlen, Llen;
if FDESpectralProj[SystemConsistent]( Mmat, Mdel, Lmat, Ldel ) then
  Mlen:= nops(Mmat);
  Llen:= nops(Lmat);
  thesum:= 0;
  for i from 1 to Mlen do
    thesum:= thesum + variable*exp(-variable*Mdel[i])*Mmat[i]
  end do;
  for i from 1 to Llen do
    thesum:= thesum - exp(-variable*Ldel[i])*Lmat[i]
  end do;
  if type(Mmat[1],'Matrix') then
    dimsystem:= LinearAlgebra['Dimension'](Mmat[1]);
    for i from 1 to dimsystem[1] do
      for j from 1 to dimsystem[2] do
        thesum[i,j]:= collect(thesum[i,j],['exp',variable],simplify);
      end do;
    end do;
  else
    thesum:= collect(thesum,['exp',variable],simplify);
  end if;
  RETURN(thesum)
else
  ERROR( sprintf("%s %s %s %s",
    "System inconsistent -- in the sense that the lists do not",
    "have compatible number of elements. The number of elements",
    "of 1st and 2nd parameters must be the equal, and the same",
    "for the 3rd and 4th."))
end if;
end proc:

#########################  CharEquation  #########################
#                (Mmat, Mdel, Lmat, Ldel, variable)
#
# Description
#
# Return the characteristic equation of and neutral FDE, as
# described in function FDECharMatrix (see above).
#
```

```
# Parameters
#
# Mmat, Mdel, Lmat, Ldel - description of FDE as before
# variable - variable of the characteristic equation.

FDESpectralProj[CharEquation]:= proc(Mmat,Mdel,Lmat,Ldel,variable)
local chareq_temp;
if type(Mmat[1],Matrix) then
  chareq_temp:=
    simplify(LinearAlgebra['Determinant'](
      FDESpectralProj[CharMatrix](Mmat,Mdel,Lmat,Ldel,variable)))
else
  chareq_temp:=
    simplify(FDESpectralProj['CharMatrix']
      (Mmat,Mdel,Lmat,Ldel,variable))
end if;
RETURN(collect(chareq_temp,['exp',variable],simplify));
end proc:

###########################  Vfunction  ###########################
#                (Mmat, Mdel, Lmat, Ldel, variable)
#
# Compute the function V used to determine if some system has a
# simple dominant eigenvalue.  It assumes that the time delays are
# ordered, and that Mmat[1]=1 and Mdel[1]=0, that is, the V function
# defined in the thesis is possible.
#
# Parameters
#
# Mmat, Mdel, Lmat, Ldel - description of FDE as before
# variable - the variable that plays the role of lambda.

FDESpectralProj[Vfunction]:= proc(Mmat, Mdel, Lmat, Ldel, variable)

local i, value;

value:= 0;

if evalb( not ((Mmat[1]=1) and (Mdel[1]=0)) ) then
  WARNING( sprintf("%s%q%s%q.",
     "Mmat[1] should be 1 and Mdel[1] should be 0, but got Mmat[1]=",
     Mmat[1], " and Mdel[1]=", Mdel[1]));
end if;

for i from 2 to nops(Mdel) do
  value:= value + abs(Mmat[i])*(1+abs(variable)*Mdel[i])
                              *exp(-variable*Mdel[i]);
end do;

for i from 1 to nops(Ldel) do
  if evalb(Ldel[i]<>0) then
     value:= value + abs(Lmat[i])*Ldel[i]
                                 *exp(-variable*Ldel[i]);
```

```
  end if;
end do;

RETURN(value);

end proc:

#############################  diffH  #############################
#  (Mmat, Mdel, Lmat, Ldel,  variable, function, intvar, order)
#
# Description
#
# Compute H(z,phi) or its derivatives with respect to z, where H is
# introduced in this thesis.
#
# Parameters
#
# Mmat, Mdel, Lmat, Ldel - description of FDE as before
# variable - variable of the characteristic matrix
# function - (initial) function
# intvar - variable inside integrals
# order - order of the derivative of H

FDESpectralProj[diffH] := proc(Mmat, Mdel, Lmat, Ldel, variable,
                               function, intvar, order)

local int1, int2, int3, i, j, ismatrix, dim, result;

ismatrix:= evalb( type(Mmat[1],'Matrix') );

if ismatrix then
  dim:= LinearAlgebra['RowDimension'](Mmat[1]);
  if order=0 then
    int1:= Vector(dim);
    for i from 1 to nops(Mmat) do
      int1:= int1 + LinearAlgebra['MatrixVectorMultiply']
         (Mmat[i], Vector([seq(function[j](-Mdel[i]),j=1..dim)]));
    end do;
  else
    int1:= Vector(dim);
    for i from 1 to nops(Mmat) do
      if evalb(Mdel[i]<>0) then
        int1:= int1 + LinearAlgebra['MatrixVectorMultiply']
                    (Mmat[i], Vector([
                        seq( int( intvar^(order-1)
                                  * exp(-variable*intvar)
                                  * function[j](intvar-Mdel[i]),
                                  intvar=0..Mdel[i]),
                             j=1..dim)] ) );
      end if;
    end do;
    # the first minus on next formula is due to the for of equation
    # M phi = I phi(0) - int du phi
```

```
      int1:= -(-1)^(order+1) * order * int1;
    end if;
    int2:= Vector(dim);
    for i from 1 to nops(Mmat) do
      if evalb(Mdel[i]<>0) then
        int2:= int2 + LinearAlgebra['MatrixVectorMultiply']
                  (Mmat[i], Vector([
                      seq( int( intvar^order * exp(-variable*intvar)
                                * function[j](intvar-Mdel[i]),
                                intvar=0..Mdel[i]),
                           j=1..dim)] ) );

      end if;
    end do;
    # the first minus on next formula is due to the for of equation
    # M phi = I phi(0) - int du phi
    int2:= - ((-1)^order * variable)* int2;
    int3:= Vector(dim);
    for i from 1 to nops(Lmat) do
      if evalb(Ldel[i]<>0) then
        int3:= int3 + LinearAlgebra['MatrixVectorMultiply']
                  (Lmat[i], Vector([
                      seq( int( intvar^order * exp(-variable*intvar)
                                * function[j](intvar-Ldel[i]),
                                intvar=0..Ldel[i]),
                           j=1..dim)] ) );
      end if;
    end do;
    int3:= ((-1)^order)*int3;
  else
    if order=0 then
      int1:= 0;
      for i from 1 to nops(Mmat) do
        int1:= int1 + `*`(Mmat[i],function(-Mdel[i]));
      end do;
    else
      int1:= 0;
      for i from 1 to nops(Mmat) do
        if evalb(Mdel[i]<>0) then
          int1:= int1 + `*`(Mmat[i],
                            int( `*`(intvar^(order-1) *
                                     exp(-variable*intvar),
                                     function(intvar-Mdel[i])),
                                 intvar=0..Mdel[i] ) )
        end if;
      end do;
      # the first minus on next formula is due to the for of equation
      # M phi = I phi(0) - int du phi
      int1:= -(-1)^(order+1) * order * int1;
    end if;
    int2:= 0;
    for i from 1 to nops(Mmat) do
      if evalb(Mdel[i]<>0) then
```

```
      int2:= int2 + `*`(Mmat[i],
                        int( `*`(intvar^order
                                 * exp(-variable*intvar),
                                 function(intvar-Mdel[i])),
                             intvar=0..Mdel[i] ) )
    end if;
  end do;
  # the first minus on next formula is due to the for of equation
  # M phi = I phi(0) - int du phi
  int2:= -(-1)^order * variable* int2;
  int3:= 0;
  for i from 1 to nops(Lmat) do
    if evalb(Ldel[i]<>0) then
      int3:= int3 + `*`(Lmat[i],
                        int( `*`(intvar^order
                                 * exp(-variable*intvar),
                                 function(intvar-Ldel[i])),
                             intvar=0..Ldel[i] ) )
    end if;
  end do;
  int3:= (-1)^order*int3;
end if;
#result:=int1+int2+int3
RETURN(int1+int2+int3)
end proc:

######################  SpectralProjection  ######################
#  (Mmat, Mdel, Lmat, Ldel, point, function, projvar [, options])
#
# Returns the Spectral projection of the solution of FDE described
# by `Mmat', `Mdel', `Lmat' and `Ldel', at `point', of `function'
# and the projection is an expression in the variable `projvar'.
#
# Optional parameters
#
# intvar=name - the variable of integration (that never plays a
#    role in computiong the integral)
# maxorderofzero - max order of zero of functions involved in
#    subroutines.

FDESpectralProj[SpectralProjection]:= proc(Mmat, Mdel, Lmat, Ldel,
                                         point, function, projvar)

local proj, n, i, j, k, Knum, Kden, dK, dH, dim, Intvar, t,
      partialsum, ismatrix,charmat, chareq, adjmat, MaxOrderOfZero,
      variable, oozeq, oozadj, forceoozeq, forceoozadj;
MaxOrderOfZero:= FDESpectralProj['MaxOrderZero'];
Intvar:= 't';
forceoozeq:= -1;
forceoozadj:= -1;
## parsing optional parameters ##
if not typematch([args[8..nargs]],'list'(
     {'identical'('maxorderofzero')='MaxOrderOfZero'::'nonnegint',
```

```
      'identical'('intvar')='Intvar'::'name',
      'identical'('force_order_chareq')='forceoozeq'::'nonnegint',
      'identical'('force_order_adjcharmat')=
                               'forceoozadj'::'nonnegint'})) then
  ERROR( sprintf("%s %s %s %s `%q'",
    "invalid set of optional parameters; the options should set",
    "`maxorderofzero=<number>' or `intvar=<name>' or"
    "`force_order_chareq'=<number> or"
    "`force_order_adjcharmat'=<number>, but got",
    args[8..nargs]))
end if;
ismatrix:= evalb( type(Mmat[1],'Matrix') );
charmat:= FDESpectralProj[CharMatrix]
                (Mmat, Mdel, Lmat, Ldel, variable);
# computing char. equations and adjoint, simplifying
# computing the dimension of the projection space
if ismatrix then
  dim:= LinearAlgebra['RowDimension'](Mmat[1]);
  chareq:= collect(LinearAlgebra['Determinant'](charmat),
                   ['exp',variable],simplify);
  adjmat:= LinearAlgebra['Adjoint'](charmat);
  for i from 1 to dim do
    for j from 1 to dim do
      adjmat[i,j]:= collect(adjmat[i,j],['exp',variable],simplify);
    end do;
  end do;
  if evalb(forceoozeq=-1) then
    oozeq:= FDESpectralProj['OrderOfZero']
          ( chareq, variable, point, 'maxorder'=MaxOrderOfZero );
  else
    oozeq:= forceoozeq;
  end if;
  if evalb(forceoozadj=-1) then
    oozadj:= FDESpectralProj['MatrixOrderOfZero']
             (adjmat, variable, point, 'maxorder'=MaxOrderOfZero);
  else
    oozadj:= forceoozadj;
  end if;

  n:= oozeq - oozadj;
else
  chareq:= charmat;
  adjmat:= 1;
  if evalb(forceoozeq=-1) then
    oozeq:= FDESpectralProj['OrderOfZero']
          ( chareq, variable, point, 'maxorder'=MaxOrderOfZero );
  else
    oozeq:= forceoozeq;
  end if;
  n:= oozeq;
end if;

Knum:= (variable-point)^n*adjmat;
```

```
Kden:= chareq;

if ismatrix then
  dK:= FDESpectralProj['MatrixEvalDerivsFracFunction']
      (Knum, Kden, variable, point, 'order'=n,
       'maxorderofzero'=MaxOrderOfZero)
else
  dK:= FDESpectralProj['EvalDerivsFracFunction']
      (Knum, Kden, variable, point, 'order'=n,
       'maxorderofzero'=MaxOrderOfZero)
end if;

dH:= [ seq(
        subs(variable=point,
             FDESpectralProj['diffH'](Mmat,Mdel,Lmat,Ldel,
                                     variable,function,Intvar,i)),
           i=0..n) ];
if ismatrix then
  proj:= Vector(dim);
  for j from 0 to n-1 do
    partialsum:= Vector(dim);
    for k from j to n-1 do
      partialsum:= eval( partialsum
                         + 1/factorial(n-1-k)*1/factorial(k-j)*
                           LinearAlgebra['MatrixVectorMultiply']
                                          (dK[n-k],dH[k-j+1]) );
    end do;
    proj:= eval( proj + projvar^j*exp(point*projvar)/j!
                        * partialsum );
  end do;
else
  proj:= 0;
  for j from 0 to n-1 do
    partialsum:= 0;
    for k from j to n-1 do
      partialsum:= eval(partialsum + 1/factorial(n-1-k)
                        *1/factorial(k-j)*dK[n-k]*dH[k-j+1]);
    end do;
    proj:= eval(proj + projvar^j*exp(point*projvar)/j!*partialsum);
  end do;
  proj;
end if;
end proc:

##################################################################
####################  Writing library to file  ###################
##################################################################

# Saving library to the current directory
save(FDESpectralProj, `FDESpectralProj.m` );

# end of `FDESpectralProj.mpl'
```

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Samenvatting

De doelstelling van dit proefschrift is het bestuderen van het lange tijdgedrag van oplossingen van lineaire autonome en periodieke functionaaldifferentiaalvergelijkingen (FDE) van neutraal type. De nadruk ligt op de expliciete berekening (ook symbolisch) van het lange tijdgedrag door gebruik te maken van spectraalprojecties en Dunford calculus. De resultaten zijn ook van toepassing op niet-lineaire vergelijkingen.

In Hoofdstuk 1, bestuderen wij FDE via een analyse van een klasse renewalvergelijkingen (integraal-differentiaal vergelijkingen van Volterra en Volterra-Stieltjes type) die met FDE kan worden geassocieerd. Wij laten het lange tijdgedrag van oplossingen van renewalvergelijkingen zien, op voorwaarde dat het "neutrale deel" voldoet aan een (redelijke) extra hypothese/veronderstelling. Het hoofdstuk is geschreven voor een algemeen publiek en heeft tot doel een overzicht te geven van de belangrijkste resultaten die in dit proefschrift naar voren worden gebracht. Aangezien het moeilijker is om direct expliciete berekeningen voor renewalvergelijkingen te verkrijgen, worden de expliciete formules uitgesteld tot latere hoofdstukken. In hoofdstuk 8 en in Appendix A komen we terug op deze renewalvergelijkingen.

In Hoofdstuk 2 wordt de basis spectraaltheorie voor autonome en periodieke FDE bestudeerd. Wij geven een decompositie van de toestandsruimte $\mathcal{C}$ in een directe som van eindige dimensionale invariante deelruimten — de algemene eigenruimten $\mathcal{M}_\lambda$ behorende bij de eigenwaarden λ van de onderliggende vergelijking — en een complementaire ruimte. Hieruit leiden wij schattingen af voor oplossingen beperkt tot deze invariante deelruimten. Samen met de expliciete projectie op de deelruimte $\mathcal{M}_\lambda$ volgt dan het lange tijdgedrag van oplossingen van FDE.

In Hoofdstuk 3 bestuderen wij de spectraalprojectie (van $\mathcal{C}$ aan $\mathcal{M}_\lambda$) en laten we verschillende methodes zien om de projecties expliciet te berekenen. Eerst berekenen wij de spectraalprojecties, in het autonome geval, voor eenvoudige dominante eigenwaarden, waarbij we gebruik maken van dualiteit via de Hale bilineaire vorm. Vervolgens berekenen wij spectraalprojecties met behulp van de Dunford calculus. Wij brengen deze methode graag naar voren, omdat het een algoritmische wijze levert voor het berekenen van de spectraalprojecties behorende bij eigenwaarden van willekeurige orde, op een eenvoudiger manier dan via het gebruik van de dualiteitsmethode. Bovendien kunnen wij deze laatste methode uitbreiden om ook spectraalprojecties voor klassen periodieke vergelijkingen met vertragingen te berekenen. De algoritmische aard van deze methode staat ons toe om de berekening van de spectraalprojectiemethode voor autonome vergelijkingen te implementeren via een Maple bibliotheekpakket, `FDESpectralProj` genaamd. Aan de hand van voorbeelden laten wij het gebruik van deze bibliotheek `FDESpectralProj` zien in Hoofdstuk 4. De broncode kan in Appendix B gevonden worden.

In Hoofdstuk 5 stellen wij een aantal testen voor om te besluiten of een wortel van een bepaalde karakteristieke vergelijking enkelvoudig en dominant is.

In Hoofdstuk 6 beschrijven wij onze belangrijkste resultaten, namelijk het lange tijdgedrag voor autonome en periodieke FDE en laten wij meer toepassingen zien van de tot zover ontwikkelde basisresultaten. In het bijzonder geven wij een analyse van het lange tijdgedrag van een klasse neutrale FDE in $\mathbb{C}^2$, en maken wij expliciete berekeningen (ook symbolisch) voor een specifiek geval waar de dominante eigenwaarde van de infinitesimale generator van de oplossings-halfgroep een dominant nulpunt van de zesde orde is van de bijbehorende karakteristieke vergelijking.

In Hoofdstuk 7 laten wij andere toepassingen van spectraalprojecties zien in de context van de berekening van invariante variëteiten. Ter illustratie gaan wij in op een aantal niet-lineaire FDE waar linearizatie leidt tot lineaire vergelijkingen met 0 als dominante eigenwaarde van de karakteristieke vergelijking. Wij tonen in detail de (symbolische) berekening van het lange tijdgedrag en berekenen de gewone differentiaalvergelijkingen, die de stroom op de centrumvariëteit in een buurt van de oorsprong geven.

In Hoofdstuk 8, bestuderen wij een gestuctureerd model van de celcyclus. We gebruiken de renewalvergelijkingen en vergelijkbare technieken zoals in Hoofdstuk 1 om het lange tijdgedrag van de oplossing van het model te verkrijgen, zelfs wanneer er geen dominante eigenwaarde (of eindige reeks dominante eigenwaarden) is, zoals in de vorige hoofdstukken, maar wanneer het lange tijdgedrag beheerst wordt door de oplossing die beperkt is tot een oneindig dimensionale deelruimte.

Wij sluiten het proefschrift af met Appendices A and B. In Appendix A, leveren wij een onderbouwing voor een bilineaire vorm die voor het eerst door Hale werd geïntroduceerd. Met betrekking tot deze bilineaire vorm, verkrijgen wij als duale ruimte van $\mathcal{C} \stackrel{\text{def}}{=} \mathcal{C}([-r,0],\mathbb{C}^n)$ de ruimte $\mathcal{C}' \stackrel{\text{def}}{=} \mathcal{C}([0,r],\mathbb{C}^{n*})$, d.w.z., de ruimte van continue functies op $[0,r]$ met waarden in de ruimte van complexe rijvectoren van lengte n. Tot slot laten we in Appendix B de broncode van de Maple-bibliotheek `FDESpectralProj` zien.

Curriculum vitæ

Miguel Vinícius Santini Frasson was born in November 1$^{\text{st}}$, 1977, in Lucélia, province of São Paulo, Brazil. He completed primary and secondary studies in Adamantina. In 1995, he started his studies in mathematics in the ICMC of the Universidade de São Paulo, in São Carlos, Brazil, where he got a "scientific initiation" grant after the first year. His graduation was in July 1998. In the same institute, he finished his Master study, under supervision of prof. dr. Plácido Zoega Táboas, in July of 2000, and the title of his master thesis is "Impulsive systems under the point of view of measure differential equations". From December 2000 until February 2005, he did his PhD research under supervision of prof. dr. Sjoerd M. Verduyn Lunel at the Mathematical Institute of the University of Leiden, The Netherlands, which resulted this thesis.

Outside his research, he spends some of his free time on developing software useful for mathematicians. He is one of the developers of ABNTEX, a group in the Internet to implement the Brazilian standards for the presentation of scientific documents in LaTeX. Also, recently he became one of the developers of AUCTEX, an Emacs mode for writing LaTeX documents, where he implements the tool bar.

In December 2002, he married Marília Juliana R. Vieira, and they are expecting their first child to be born in May 2005. In The Hague, where he lived during the last three years of his stay in The Netherlands, he took part in a Spanish-speaking Neocatechumenal community in Sint Agnes Kerk, a catholic church.