

# Monotonicity and Boundedness in general Runge-Kutta methods

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Luca Ferracina

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promotors: Prof.dr. M.N. Spijker  
Prof.dr. J.G. Verwer (UvA/CWI)

referent: Dr. W. Hundsdorfer (CWI)

overige leden: Prof.dr. G. van Dijk  
Prof.dr.ir. L.A. Peletier  
Prof.dr. S.M. Verduyn Lunel

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papà, mamma, Marco, Anna.*



# **Monotonicity and Boundedness in general Runge-Kutta methods**

THOMAS STIELTJES INSTITUTE  
FOR MATHEMATICS



# Preface

This thesis consists of an introduction and four papers which appeared (or were submitted for publication) in scientific journals. The introduction has been written with the intention to be understandable also for the reader who is not specialized in the field. The papers, which are listed below, are essentially self-contained, and each of them may be read independently of the others.

FERRACINA L., SPIJKER M.N. (2004): Stepsize restrictions for the total-variation-diminishing property in general Runge-Kutta methods, *SIAM J. Numer. Anal.* **42**, 1073–1093.

FERRACINA L., SPIJKER M.N. (2005): An extension and analysis of the Shu-Osher representation of Runge-Kutta methods, *Math. Comp.* **249**, 201–219.

FERRACINA L., SPIJKER M.N. (2005): Computing optimal monotonicity-preserving Runge-Kutta methods, submitted for publication, report Mathematical Institute, Leiden University, MI 2005-07.

FERRACINA L., SPIJKER M.N. (2005): Stepsize restrictions for total-variation-boundedness in general Runge-Kutta procedures, *Appl. Numer. Math.* **53**, 265–279.





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