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Modes of a twisted optical cavity

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An astigmatic optical resonator consists of two astigmatic mirrors facing each other. The resonator is twisted when the symmetry axes of the mirrors are nonparallel. We present an algebraic method to obtain the complete set of the paraxial eigenmodes of such a resonator. Basic ingredients are the complex eigenvectors of the four-dimensional transfer matrix that describes the transformation of a ray of light over a roundtrip of the resonator. The relation between the fundamental mode and the higher-order modes is expressed in terms of raising operators in the spirit of the ladder operators of the quantum harmonic oscillator.

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I. INTRODUCTION

Beams of light with special structures arouse special interest. Examples are light beams carrying orbital angular momentum [1] or containing optical vortices [2]. For a quantized field, the multidimensional nature of the spatial degrees of freedom allows entangled states with higher degrees of freedom, which can be useful for quantum information [3,4]. Structured light beams can also be used to manipulate small particles.

In this paper we wish to investigate the structure of modes in an optical resonator with general astigmatism. The simplest form of an optical resonator consists of two spherical mirrors that face each other. A mode of such a resonator is a field distribution that reproduces itself after a round trip, bouncing back and forth between the mirrors [5]. The usual approach to the problem of finding the eigenmodes of an optical resonator is by considering the wave equation and imposing boundary conditions. In the paraxial limit we can use the paraxial wave equation, which has the Huygens-Fresnel integral equation as its integral form. The boundary condition is that the electric field vanishes at the surface of the mirrors, which implies that the mirror surfaces match a nodal plane of the standing wave that is formed by a bouncing traveling wave. Conversely, a Gaussian paraxial beam, which has spherical wave fronts, can be trapped between two spherical mirrors that coincide with a wave front. This imposes a condition on the curvatures and the spacing L of the mirrors. When the radii of curvature are R_1 and R_2, the condition is simply [5]

$$0 \leq g_1 g_2 \leq 1, \tag{1}$$

where the parameters g_1 and g_2 are defined by

$$g_i = 1 - \frac{L}{R_i} \tag{2}$$

for $i=1$, 2. This is precisely the stability condition of the resonator. A stable resonator is a periodic focusing system, so that it supports stable ray patterns. Such a resonator has a complete set of Hermite-Gaussian modes, with a simple Gaussian fundamental mode. For a two-mirror resonator with radii of curvature R_i and a spacing L obeying the stability condition (1), the modes are characterized by the Rayleigh range b and the round-trip Gouy phase χ that are given by [5]

$$\frac{b^2}{L^2} = \frac{g_1 g_2 (1 - g_1 g_2)}{(g_1 + g_2 - 2 g_1 g_2)^2}, \quad \cos\frac{\chi}{2} = \pm\sqrt{g_1 g_2}. \tag{3}$$

The plus sign is taken if both g_1 and g_2 are positive whereas the minus sign is taken when both are negative. The wave numbers of the Hermite-Gaussian modes HG_{nm} with transverse mode numbers n and m are determined by the requirement that the phase of the field changes over a round trip by a multiple of 2π. This gives the resonance condition

$$2kL - (n + m + 1)\chi = 2\pi q \tag{4}$$

for the wave number k, with an integer longitudinal mode index q.

It is a simple matter to generalize this method in the case of astigmatic mirrors, provided that the mirror axes are parallel. Each mirror i can be described by two radii of curvature $R_{i\xi}$ and $R_{i\eta}$, corresponding to the curvatures along the two axes. In this case of simple astigmatism the paraxial field distribution separates into a product of two contributions, corresponding to the two transverse dimensions. Stability requires that each of the two dimensions obey the stability condition (1) for the parameters $g_{i\xi}$ and $g_{i\eta}$, and each dimension has its own Rayleigh range and Gouy phase. The resonance condition for a resonator with simple astigmatism takes the modified form

$$2kL - \left(n + \frac{1}{2}\right)\chi_\xi - \left(m + \frac{1}{2}\right)\chi_\eta = 2\pi q. \tag{5}$$

The situation is considerably more complex when the axes of the two astigmatic mirrors are not parallel. In this case of a twisted resonator, the light traveling back and forth between the mirrors displays general astigmatism, which is characterized by the absence of symmetry planes through the optical axis. Also in this case the stability condition and the structure of the modes is in principle determined by the condition that the mirror surfaces match a wave front of a traveling astigmatic beam, but it is not simple to derive the mode structure and the resonance frequencies from this condition. In the case of twisted resonators or lens guides the structure of the fundamental Gaussian mode and of the propagation of rays has been studied by several workers using analytical techniques [6–9]. The light traveling back and forth between the mirrors is described by a Gaussian field pattern. It has rotating elliptic intensity distributions, and wave fronts that are elliptic or hyperbolic [10].

Recently a general discussion has been given of freely propagating general astigmatic modes of all orders [11]. The method has a simple algebraic structure, and it is based on the use of astigmatic ladder operators that obey bosonic commutation rules. This operator technique greatly facilitates the evaluation of the propagating fundamental modes, and it also allows to evaluate higher-order modes. Moreover, it clarifies the amount of freedom one has in selecting a complete basis of modes for a given astigmatic fundamental mode.

In the present paper we apply this operator technique to study the modes to all orders of twisted resonators. In this case, the basis set of modes is fixed by the geometric properties of the resonator, consisting of the elliptical shape of the two astigmatic mirrors, the mirror separation and the relative orientation of the mirrors. Rather than using the condition that the wave fronts match the mirror surfaces, our method is entirely based on the eigenvalues and eigenvectors of the four-dimensional transfer matrix that specifies the transformation of a ray after a round trip through the cavity. This matrix generalizes the *ABCD* matrix, specifying the propagation of a ray in one transverse dimension. We discuss the relevant properties of this transfer matrix in Sec. II. After a brief discussion of paraxial wave optics in an astigmatic resonator in Sec. III, we give in Sec. IV an operator description of fundamental Gaussian modes and higher-order modes. Here we demonstrate that the resonator modes can be directly expressed in terms of the properties of the transfer matrix. Finally, we analyze in Sec. V the physical properties of the modes.

II. PARAXIAL RAY OPTICS

A. One transverse dimension

In paraxial geometric optics a light beam in vacuum is assumed to consist of a pencil of rays. In each transverse plane a ray is specified by a transverse position x and an angle $\theta = \partial x / \partial z$ where z is the longitudinal coordinate. This angle gives the propagation direction of the ray with respect to the optical axis of the system. Both the position x and the angle θ transform under propagation through free space and when passing optical elements. In the paraxial approximation ($\theta \ll 1$) this transformation is linear and can be represented by a 2×2 matrix acting on a column vector that contains the transverse position x and the angle θ

$$\begin{pmatrix} x' \\ \theta' \end{pmatrix} = M \begin{pmatrix} x \\ \theta \end{pmatrix}. \tag{6}$$

Here M is a matrix that transforms the input beam of the optical system into the output beam. The matrices that represent various optical elements can be found in any textbook on resonator optics, for instance [5]. The transfer matrix for propagation through free space over a distance z is described by

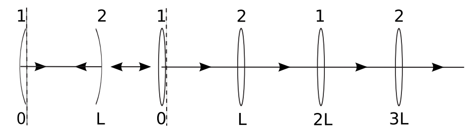

FIG. 1. Unfolding an optical resonator into an equivalent periodic lens guide; the mirrors are replaced by lenses with the same focal lengths and the reference plane is indicated by the dashed line.

$$M_f(z) = \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}. \tag{7}$$

The trajectory that corresponds to this transformation is a straight line with the angle θ, where the transverse position $x' = x + \theta z$ changes linearly with the distance z. The transformation of a ray through a Gaussian thin lens is given by

$$M_l(f) = \begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix}, \tag{8}$$

where f is the focal length of the lens which is taken positive for a converging lens. The transverse position is invariant under this transformation. The angle θ which gives the propagation direction changes abruptly at the location of the lens. It can be easily shown that this transformation reproduces the thin-lens equation.

The transformation matrix of a sequence of first-order optical elements can be constructed by multiplying the matrices that correspond to the several elements in the correct order. Closed optical systems such as a resonator can be unfolded into an equivalent periodic lensguide, as indicated in Fig. 1. The mirrors are replaced by thin lenses with the same focal lengths. One period of the lens guide is equivalent to a single roundtrip through the resonator. When we choose the transverse reference plane just right of mirror 1 (or lens 1), we can construct the matrix that describes the transformation of a single roundtrip in the form

$$M = M_l(f_1) M_f(L) M_l(f_2) M_f(L). \tag{9}$$

Here L is the distance between the two endmirrors of the resonator, and $f_{1,2}$ are the focal lengths of the mirrors that are related to the radii of curvature by $f_{1,2} = R_{1,2}/2$.

All matrices corresponding to a lossless optical element are real and have a unit determinant. It follows that this is also true for the transfer matrix of any lossless composite system. In case of one transverse dimension these are the only defining properties of the transfer matrix so that the reverse of the above statement is also true: any real matrix that has a unit determinant corresponds to the transformation of a lossless optical system that can be constructed from first-order optical elements.

An important characteristic of optical resonators is whether they are stable or not. In many cases a resonator will support only ray paths that are strongly diverging or converging after a few roundtrips. Only in specific cases does a resonator support a stable ray pattern. Usually the stability criterion of an optical resonator is formulated in terms of the parameters that fix the geometry, i.e., the radii $R_{1,2}$ of curvature of the mirrors and the distance L between them. It is more convenient to relate the stability of a resonator to the eigenvalues m_1 and m_2 of the transfer matrix M for a single roundtrip. Since $\det M=1$, also $m_1 m_2=1$. If we assume that these eigenvalues are different, the corresponding eigenvectors μ_1 and μ_2 are linearly independent, so that an arbitrary input ray can be written as a linear combination

$$r_0 = a_1\mu_1 + a_2\mu_2. \tag{10}$$

After n roundtrips through the resonator this ray transforms to

$$r_n = M^n r_0 = a_1 m_1^n \mu_1 + a_2 m_2^n \mu_2. \tag{11}$$

From this transformation of a ray through the resonator it is clear that the absolute value of the eigenvalues determines the magnification of the ray. It follows that a resonator is stable only if the absolute value of both eigenvalues is equal to 1.

In the case of a nondegenerate transfer matrix M these conditions of the eigenvalues require that the eigenvalues, and therefore also the eigenvectors are complex. Since M is a real matrix, the two eigenvectors as well as the two eigenvalues must be each other's complex conjugates, so that

$$\mu_1 = \mu_2^* = \mu, \quad m_1 = m_2^* = \exp(i\chi) = m. \tag{12}$$

The phase angle χ is the roundtrip Gouy phase of the resonator, which determines the spectrum of the resonator in paraxial wave optics.

The expression (10) for a real ray r takes the form

$$r_0 = 2\,\mathrm{Re}(a\mu). \tag{13}$$

With Eq. (11) this leads to the expression

$$r_n = 2\,\mathrm{Re}(a\mu e^{in\chi}), \tag{14}$$

for the transformed ray after n roundtrips. This shows that both the position and the direction of the ray at successive passages of the reference plane display a discrete oscillatory behavior. An interesting case arises when the Gouy phase χ is a rational fraction of 2π, i.e., if

$$\chi = \frac{2\pi K}{N}, \tag{15}$$

where K and N are integers. Then the two eigenvalues M^N are both equal to 1, so that $M^N=1$. Inside a resonator this means that the trajectory of a ray will form a closed path after N roundtrips.

For a different choice of the reference plane, the roundtrip transfer matrix M takes a different form. The two forms are related by a transformation determined by the transfer matrix from one reference plane to the other. The same transformation also couples the eigenvectors. The eigenvalues are independent of the choice of the reference plane.

B. Two transverse dimensions

The description of the previous subsection can be generalized to account for the presence of two transverse dimensions x and y. Then both the location and the direction of a ray passing a transverse plane become two-dimensional vectors. The transverse coordinate is indicated as $R=(x,y)$ and $\Theta=(\theta_x,\theta_y)$ are the angles that determine the propagation direction in the xz and yz planes. Likewise, the transformation from an input plane to an output plane of an optical system is represented by a 4×4 transfer matrix, in the form

$$\begin{pmatrix} R' \\ \Theta' \end{pmatrix} = M \begin{pmatrix} R \\ \Theta \end{pmatrix}. \tag{16}$$

For a nonastigmatic optical element the 4×4 matrix is obtained by multiplying the four elements of the 2×2 transformation matrix with a 2×2 unit matrix U. For instance, the transformation for free propagation through free space over a distance z can be expressed as

$$M_f(z) = \begin{pmatrix} \mathsf{U} & z\mathsf{U} \\ 0 & \mathsf{U} \end{pmatrix}. \tag{17}$$

In case of an astigmatic optical element, at least some part of the transfer matrix is not proportional to the identity matrix. The most relevant example is an astigmatic thin lens. Its transformation matrix can be written as

$$M_l(\mathsf{F}) = \begin{pmatrix} \mathsf{U} & 0 \\ -\mathsf{F}^{-1} & \mathsf{U} \end{pmatrix}, \tag{18}$$

where F is a real and symmetric 2×2 matrix. The eigenvalues of F are the focal lengths of the lens and the corresponding mutually orthogonal real eigenvectors fix the orientation of the lens in the transverse plane.

Again, the transfer matrix that describes a composite optical system can be constructed by multiplying the matrices that represent the optical elements in the right order. The transfer matrix that describes the transformation of a roundtrip through an astigmatic resonator can be obtained by unfolding the resonator into the corresponding lens guide and multiplying the matrices that represent the transformations of the different elements in the correct order

$$M = M_l(\mathsf{F}_1) M_f(L) M_l(\mathsf{F}_2) M_f(L). \tag{19}$$

Here L is again the distance between the two mirrors and $\mathsf{F}_{1,2}$ are the matrices that describe the surfaces of the mirrors.

If for each mirror the two focal lengths are the same, the resonator is cylindrically symmetric. If one of the two mirrors has two different focal lengths or if both mirrors have two different focal lengths but with the same orientation, the resonator has two symmetry planes and is said to have *simple astigmatism*. If this is not the case, i.e., if each of the two mirrors are astigmatic, and when their orientations are not parallel, there are no symmetry planes and the resonator has *general astigmatism*.

A typical transfer matrix M is real, but not symmetric, so that its eigenvectors cannot be expected to be orthogonal. However, it is easy to check that the transfer matrices (17) and (18) obey the identity

$$M^T G M = G, \tag{20}$$

where G is the antisymmetric 4×4 matrix

$$G = \begin{pmatrix} 0 & \mathsf{U} \\ -\mathsf{U} & 0 \end{pmatrix}. \tag{21}$$

Then the same identity must also hold for a composite optical system, in particular for the total roundtrip transfer matrix (19). This is the defining property of a physical transfer matrix that describes a lossless first-order optical system. In mathematical terms the above criterion defines a *symplectic group* [12]. The matrix M must be in the real symplectic group of 4×4 matrices, denoted as $\mathrm{Sp}(4,\mathbb{R})$. Also in four dimensions the determinant of M is equal to 1.

From the general property (20) of the transfer matrix we can derive some important properties of its eigenvalues and eigenvectors. The eigenvalue relation is generally written as

$$M\mu_i = m_i\mu_i, \tag{22}$$

where μ_i are the four eigenvectors and m_i are the corresponding eigenvalues. By taking matrix elements of the matrix identity (20) between two eigenvectors, we find

$$\mu_i G\mu_j = m_i m_j \mu_i G\mu_j. \tag{23}$$

The matrix element $\mu_i G\mu_i$ vanishes, so this relation gives no information on the eigenvalue for $i=j$. For different eigenvectors $\mu_i \neq \mu_j$, we conclude that either $m_i m_j = 1$, or $\mu_i G\mu_j = 0$. Since M is real, when an eigenvalue m_i is complex, the same is true for the eigenvector μ_i. Moreover, μ_i^* is an eigenvector of M with eigenvalue m_i^*. Provided that the matrix element $\mu_i^* G\mu_i \neq 0$, the eigenvalue must then obey the relation $m_i^* m_i = 1$, so that the complex eigenvalue m_i has absolute value 1.

Just as in the case of one transverse dimension, stability requires that all eigenvalues have absolute value 1. Apart from accidental degeneracies, we conclude that a stable astigmatic resonator has two complex conjugate pairs of eigenvectors μ_1, μ_1^*, and μ_2, μ_2^* with eigenvalues m_1, m_1^*, and m_2, m_2^*, that are determined by

$$m_1 = e^{i\chi_1}, \quad m_2 = e^{i\chi_2}. \tag{24}$$

Hence the eigenvalues now specify two different roundtrip Gouy phase angles, and the complex eigenvectors obey the identities $\mu_1 G\mu_2 = 0$ and $\mu_1^* G\mu_2 = 0$. On the other hand, the matrix elements $\mu_1^* G\mu_1$ and $\mu_2^* G\mu_2$ are usually nonzero. These matrix elements are imaginary, and without loss of generality we may assume that they are equal to the imaginary unit i times a positive real number. This can always be realized, when needed by interchanging μ_1 and μ_1^* (or μ_2 and μ_2^*), which is equivalent to a sign change of the matrix element.

It is practical to normalize the eigenvectors, so that

$$\mu_1^* G\mu_1 = \mu_2^* G\mu_2 = 2i. \tag{25}$$

An arbitrary ray in the reference plane characterized by the real four-dimensional vector

$$r_0 = \begin{pmatrix} R \\ \Theta \end{pmatrix} \tag{26}$$

can be expanded in the four complex basis vectors as

$$r_0 = 2\,\mathrm{Re}(a_1\mu_1 + a_2\mu_2), \tag{27}$$

in terms of two complex coefficients a_1 and a_2. These coefficients can be obtained from a given ray vector r_0 by the identities

$$a_1 = \frac{\mu_1^* G r_0}{2i}, \quad a_2 = \frac{\mu_2^* G r_0}{2i}. \tag{28}$$

This is obvious when one substitutes the expansion (27) in the right-hand sides of Eq. (28).

After n roundtrips, the input ray (27) is transformed into the ray

$$r_n = M^n r_0 = 2\,\mathrm{Re}(a_1\mu_1 e^{in\chi_1} + a_2\mu_2 e^{in\chi_2}). \tag{29}$$

This is a linear superposition of two oscillating terms that pick up an additional phase angle χ_1 and χ_2 after each passage of the reference plane. When the two Gouy phases are rational fractions of 2π with a common denominator N, the ray path will be closed after N roundtrips. Then the resonator can be called degenerate. In this case the hit points of the ray on the mirrors (or in any transverse plane) lie on a well-defined curve. For a resonator that has no astigmatism this curve is an ellipse [5]. The transverse position and the propagation direction of the incoming ray determine the shape of the ellipse and in special cases it can reduce into a straight line or a circle. In case of a degenerate resonator with simple astigmatism the hit points lie on Lissajous curves [13,14]. The ratio of the Gouy phases is equal to the ratio of the numbers of extrema of the curve in the two directions, while the incoming ray and the actual values of the Gouy phases determine its specific shape. The presence of general astigmatism gives rise to skew Lissajous curves, which are Lissajous curves in nonorthogonal coordinates. These properties are illustrated in Fig. 2.

For a resonator with two astigmatic mirrors the two Gouy phases vary when the mirrors are rotated with respect to each other over an angle ϕ. When the two mirrors are identical, and their axes are aligned, the resonator has simple astigmatism, and for both components the focus is halfway between the mirrors. Simple astigmatism also occurs in the antialigned configuration $\phi = \pi/2$, when the axis with the larger curvature of one mirror and the axis with the smaller curvature of the other one lie in a single plane through the optical axis. Then both components necessarily have the same Gouy phase, and their foci lie on opposite sides of the central transverse plane. The two Gouy phases attain extremal values for the aligned and the antialigned configuration. For intermediate orientations the resonator has general astigmatism, with Gouy phases varying between these extreme values, with a crossing occurring in the antialigned geometry. The crossing is avoided when the mirrors are slightly different. The behavior of the Gouy phases as a function of the rotation angle is sketched in Fig. 3.

III. PARAXIAL WAVE OPTICS

We describe the spatial structure of the modes in an astigmatic cavity in the same lens-guide picture that we used for the rays. The longitudinal coordinate in the lens guide is

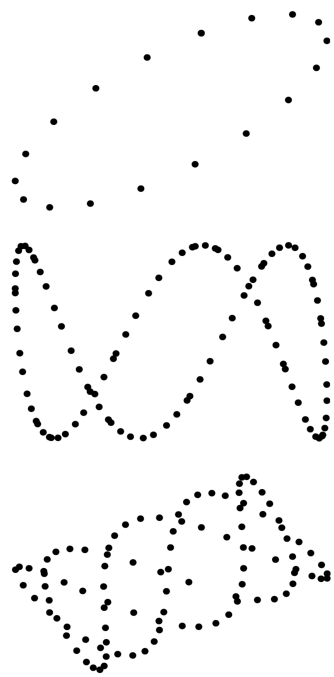

FIG. 2. Hit points at a mirror of a ray in a resonator with degeneracy. The resonator has no astigmatism (above), simple astigmatism (middle), or general astigmatism (below). The resonator without astigmatism consists of two spherical mirrors with focal lengths $\simeq 1.08L$ and $\simeq 2.16L$. The resonator with simple astigmatism consists of two identical aligned astigmatic mirrors with focal lengths $\simeq 1.47L$ and $\simeq 2.94L$. The resonator with general astigmatism consists of two identical mirrors with focal lengths $\simeq 1.075L$ and $\simeq 2.15L$ which are rotated over an angle $\phi = \pi/3$ with respect to each other. In all cases the incoming ray is given by $r_0 = (1, 1.8, 3, 0.02)$.

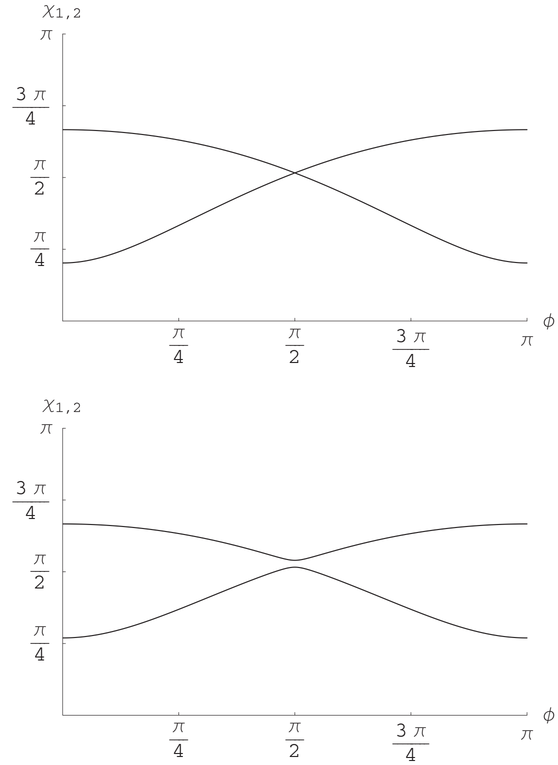

FIG. 3. The dependence of the two Gouy phases on the relative orientation of two identical (top) and two slightly different (bottom) astigmatic mirrors. In the top figure the mirrors are identical with focal lengths $f_\xi = L$ and $f_\eta = 10L$, with ξ and η indicating the principal axes of the mirrors. In the bottom figure the second mirror has focal lengths $f_\xi = L$ and $f_\eta = 4L$. Rotation angle $\phi = 0$ corresponds to the orientation for which the mirrors are aligned.

indicated by z, and R denotes the two-dimensional transverse position. A monochromatic beam of light with uniform polarization in the paraxial approximation is characterized by the expression

$$\mathbf{E}(\mathbf{r},t) = \mathrm{Re}[E_0 \epsilon u(R,z) \exp(ikz - i\omega t)] \tag{30}$$

for the electric field. It contains a carrier wave with wave number k and frequency $\omega = ck$, a normalized complex polarization vector ϵ and an amplitude E_0. The magnetic field is given by the analogous expression [15]

$$\mathbf{B}(\mathbf{r},t) = \frac{1}{c}\mathrm{Re}[E_0(\hat{z} \times \epsilon) u(R,z) \exp(ikz - i\omega t)]. \tag{31}$$

The transverse spatial structure of the beam for each reference plane is determined by the normalized profile $u(R,z)$. During propagation in free space, the z dependence of the profile is governed by the paraxial wave equation

$$\left(\nabla^2 + 2ik\frac{\partial}{\partial z}\right) u(R,z) = 0. \tag{32}$$

In a region of free propagation, the transverse profile $u(R,z)$ varies negligibly with z over a wavelength. On the other hand, u changes abruptly at the position of a thin lens. The effect of an astigmatic lens is given by the input-output relation for the beam profile

$$u_{\mathrm{out}}(R) = \exp\left(\frac{ik}{2} R \mathsf{F}^{-1} R\right) u_{\mathrm{in}}(R), \tag{33}$$

where the real symmetric matrix F specifies the orientation and the focal lengths of the lens. Again, an astigmatic lens in the lens-guide models an equivalent astigmatic mirror in the resonator.

The wave equation (32) has the same structure as the Schrödinger equation for a free particle in two spatial dimensions, where now the longitudinal coordinate z plays the role of the time. This analogy suggests to use the mathematical structure of quantum mechanics with Dirac's notation for state vectors also for the propagation of paraxial modes [16]. Since the beam profile $u(R,z)$ is analogous to the particle wave function, we associate to the profile a state vector $|u(z)\rangle$, so that

$$u(R,z) = \langle R | u(z) \rangle, \tag{34}$$

where $|R\rangle$ is an eigenstate of the two-dimensional position operator $\hat{R} = (\hat{x}, \hat{y})$. The corresponding momentum operator is $\hat{P} = (\hat{p}_x, \hat{p}_y) = -i(\partial/\partial x, \partial/\partial y)$. The average (or expectation) value of this operator can be shown to correspond to the transverse momentum of the light beam [17]. Notice that the ratio of the transverse and longitudinal momentum $\hat{P}/k$ is the wave optical analogue of the angles that determine the propagation direction of a ray. Therefore, the operator corresponding to the vector Θ, which specifies the direction of the beam, is simply $\hat{\Theta}=\hat{P}/k$. The components of $\hat{R}$ and $\hat{P}$ satisfy the canonical commutation relations

$$[\hat{x},\hat{p}_x]=[\hat{y},\hat{p}_y]=i. \tag{35}$$

The effect of free propagation and of astigmatic lenses can be represented by unitary operators acting on the state vectors $|u\rangle$ in Hilbert space. The paraxial wave equation (32) can be represented in operator notation as

$$\frac{d}{dz}|u(z)\rangle=-\frac{i}{2k}\hat{P}^2|u(z)\rangle. \tag{36}$$

Hence free propagation over a distance z has the effect

$$|u(z_0+z)\rangle=\hat{U}_f(z)|u(z_0)\rangle \tag{37}$$

with

$$\hat{U}_f(z)=\exp\left(-\frac{iz}{2k}\hat{P}^2\right), \tag{38}$$

while the effect of an astigmatic lens can be expressed as

$$|u_{\text{out}}\rangle=\hat{U}_l(\mathsf{F})|u_{\text{in}}\rangle \tag{39}$$

with

$$\hat{U}_l(\mathsf{F})=\exp\left(\frac{ik\hat{R}\mathsf{F}^{-1}\hat{R}}{2}\right). \tag{40}$$

The unitary transformation of an optical system can be constructed by multiplying the operators representing the elements in the proper order. Therefore, the unitary transformation describing a single roundtrip through the astigmatic resonator can be written as

$$\hat{U}=\hat{U}_l(\mathsf{F}_1)\hat{U}_f(L)\hat{U}_l(\mathsf{F}_2)\hat{U}_f(L), \tag{41}$$

where the reference plane is the same as sketched in Fig. 1. It is clear that other unitary roundtrip operators can be constructed for different reference planes. For different choices the operators are related by unitary transformations.

The variation of the position $R(z)$ and the direction $\Theta(z)$ of a ray during propagation must be reproduced by the variation of the average value of $\hat{R}$ and $\hat{\Theta}=\hat{P}/k$ over the beam profile. Since the variation of this profile during propagation is governed by the evolution operator $\hat{U}(z)$, the propagation of a ray in geometric optics should be reproduced by the expectation value of $\hat{U}^\dagger\hat{R}\hat{U}$ and $\hat{U}^\dagger\hat{\Theta}\hat{U}$, in analogy to the Heisenberg picture of quantum mechanics. Therefore, the wave-optical propagation operator $\hat{U}$ and the ray-optical transfer matrix M must be related by

$$\hat{U}^\dagger\begin{pmatrix}\hat{R}\\ \hat{\Theta}\end{pmatrix}U=M\begin{pmatrix}\hat{R}\\ \hat{\Theta}\end{pmatrix}. \tag{42}$$

It is easy to check that this relation holds in the case of free propagation (described by $\hat{U}_f$ and M_f) and for astigmatic lenses (which is represented by $\hat{U}_l$ and M_l). From this one verifies that the relation (42) must hold generally for any optical system that is composed of regions of free propagation, interrupted by astigmatic lenses (or mirrors).

It is noteworthy that the general property (20) of transfer matrices M can be reproduced by using this relation (42), combined with the fact that the Heisenberg-transformed operators $\hat{U}^\dagger\hat{R}\hat{U}$ and $\hat{U}^\dagger\hat{P}\hat{U}$ obey the canonical commutation rules (35).

IV. OPERATOR DESCRIPTION OF GAUSSIAN MODES

A characteristic of the paraxial wave equation is that a transverse beam profile with a Gaussian shape retains its Gaussian structure, both during propagation in free space and when passing lenses (or mirrors) described by the transformation of Eq. (40). This is the general structure of fundamental modes. Higher-order modes have the form of the same Gaussian function multiplied by Hermite polynomials in each transverse coordinate [5]. This provides the basis of nonastigmatic Hermite-Gaussian modes, which can be rearranged to yield the basis of Laguerre-Gaussian modes. There is a clear similarity between these bases of Gaussian modes and the stationary states of an isotropic quantum-mechanical harmonic oscillator in two dimensions. The modes of different order are connected by ladder operators, in analogy to the algebraic description of the quantum harmonic oscillator [18]. These ladder operators are linear combinations of the components of the position operator $\hat{R}$ and momentum operator $\hat{P}$ (or $\hat{\Theta}=\hat{P}/k$). This description has also been generalized for astigmatic modes [11].

In order to obtain expressions for cavity modes we shall demonstrate that the ladder operators connecting the modes can be obtained from the eigenvectors of the transfer matrix M.

A. Gaussian modes in one transverse dimension

For simplicity, we first consider a single period of the lensguide that represents the resonator described in Sec. II A, with one transverse dimension. It has been shown in Ref. [18] that for an arbitrary paraxial optical system the higher-order Gaussian modes are obtained by repeated application of a raising operator $\hat{a}^\dagger$, acting on the fundamental mode. The raising operator is the Hermitian conjugate of the lowering operator

$$\hat{a}(z)=\sqrt{\frac{k}{2}}\left(K\hat{x}+iB\frac{\hat{p}}{k}\right)=\sqrt{\frac{k}{2}}(K\hat{x}+iB\hat{\theta}), \tag{43}$$

where k is the wave number, and the z dependence of the ladder operators is determined only by the variation of the complex parameters B and K as a function of the longitudinal coordinate z. These parameters also determine the z-dependent profile of the fundamental mode

$$u_0(x,z) = \sqrt{\frac{k}{B\sqrt{\pi}}}\exp\left(-\frac{kKx^2}{2B}\right). \tag{44}$$

Notice that the notation is slightly different as compared with Ref. [18], where the symbols $\beta = B/\sqrt{k}$, $\kappa = K\sqrt{k}$ have been used. The reason is that the parameters B and K have a purely geometric significance, in that they are fully determined by the geometric properties of the resonator, the length L, and the focal lengths $f_{1,2}$, independent of k. These parameters B and K determine the diffraction length (or Rayleigh range) and the radii of curvature of the wave fronts.

For each value of z, the ladder operators $\hat{a}$ and $\hat{a}^\dagger$ must obey the bosonic commutation rule

$$[\hat{a}(z), \hat{a}^\dagger(z)] = 1, \tag{45}$$

which requires that K and B obey the normalization identity

$$KB^* + BK^* = 2. \tag{46}$$

With this condition, the fundamental mode profile (44) is normalized, in the sense that $\int dx |u(x,z)|^2 = 1$ for all values of z. Moreover, the lowering operator (43) gives zero when acting on the fundamental mode, so that $\hat{a}(z)|u_0(z)\rangle = 0$.

The z-dependent propagation operator $\hat{U}(z)$ is defined to transform the beam profile in the reference plane at $z=0$ into the profile in another transverse plane at position z. Then $|u(z)\rangle = \hat{U}(z)|u(0)\rangle$ describes a light beam propagating through the optical system. This means that in the regions of free propagation between the lenses, $|u(z)\rangle$ solves the paraxial wave equation, while it picks up the appropriate phase factor when passing through a lens. The z dependence of the parameters B and K must be chosen in such a way that the ladder operators $\hat{a}(z)$ and $\hat{a}^\dagger(z)$ acting on a z-dependent mode $|u(z)\rangle$ create another mode that solves the wave equation. This condition is summarized as

$$\hat{a}(z)|u(z) = \hat{U}(z)\hat{a}(0)|u(0), \tag{47}$$

which in view of the unitarity of the propagation operator is equivalent to the operator identity

$$\hat{a}(z) = \hat{U}(z)\hat{a}(0)\hat{U}^\dagger(z). \tag{48}$$

When this is the case, a complete orthogonal set of higher-order modes is obtained in terms of the raising operator and the fundamental mode, in the well-known form

$$|u_n(z)\rangle = \frac{1}{\sqrt{n!}}[\hat{a}^\dagger(z)]^n |u_0(z)\rangle. \tag{49}$$

In Ref. [18] it has been shown that the condition (46) implies that the parameter K is constant in a region of free propagation, while B has the derivative $dB/dz = iK$. Upon passage through a lens with focal length f, B does not change, whereas K modifies according to the relation $K_{\text{out}} = K_{\text{in}} + iB/f$. It is noteworthy that this z dependence of the parameters can be summarized by the statement that the transformation of the two-dimensional vector (B, iK) during propagation is identical to the transformation of a ray (x, θ). This transformation is described by the transfer matrix $M(z)$ that corresponds to $\hat{U}(z)$ in accordance with Eq. (42), so that

$$\begin{pmatrix} B(z) \\ iK(z) \end{pmatrix} = M(z)\begin{pmatrix} B(0) \\ iK(0) \end{pmatrix}. \tag{50}$$

Now it is straightforward to obtain the modes and the eigenfrequencies of the resonator. The condition for a mode is that the mode profile $u(x,z)$ reproduces after a roundtrip, up to a phase factor. This is accomplished when the two-dimensional vector $(B(0), iK(0)) = \mu$ is an eigenvector of the roundtrip transfer matrix $M = M(2L)$, after proper normalization of μ to ensure that B and K obey the identity (46). (In the case that $BK^* + KB^*$ turns out to be negative, we just take the other eigenvector μ^* instead of μ.) With this choice, the fundamental mode obeys the relation $|u_0(2L)\rangle = |u_0(0)\rangle\exp(-i\chi/2)$, and the lowering operator transforms after a roundtrip as $\hat{a}(2L) = \hat{a}(0)\exp(i\chi)$, with χ the roundtrip Gouy phase. The nth-order mode (49) then obeys the well-known relation

$$|u_n(2L)\rangle = e^{-i(n+1/2)\chi}|u_n(0)\rangle. \tag{51}$$

As indicated in Eq. (30), the complex electric field, which should reproduce exactly after a roundtrip, is proportional to $u_n(x,z)\exp(ikz)$, so that the resonance condition reads

$$2Lk - (n + 1/2)\chi = 2\pi q, \tag{52}$$

where the integer number q plays the role of the longitudinal mode number. This relation defines the frequencies of the eigenmodes $\omega = ck$.

In conclusion, we have shown that the eigenmodes are determined by the values of the parameters B and K, such that in the reference plane the vector $(B(0), iK(0))$ is identical to the normalized eigenvector μ of the roundtrip transfer matrix M. The z dependence of the parameters $B(z)$ and $K(z)$ is governed by the transfer matrix between the reference plane and the transverse plane z. This is equivalent to the statement that the vector $(B(z), iK(z))$ coincides with the eigenvector of the transfer matrix for a roundtrip starting in the transverse plane z. Different modes of the resonator have different wave numbers k, but they are all characterized by the same parameters B and K.

Before turning to the case of two transverse dimensions, it is illuminating to relate the z dependence of the ladder operators to their structure in terms of the asymmetric matrix G. In the present case of one transverse dimension, this matrix as defined in Eq. (21) is two-dimensional, just as the vectors (B, iK) and (x, θ). Then the property (20) of M is just equivalent to the statement that $\det M = 1$. Also for a single transverse dimension the transfer matrix M is linked to the propagation operator $\hat{U}$ by the identity (42). We can rewrite the expression (43) for the lowering operator as

$$\hat{a}(z) = i\sqrt{\frac{k}{2}}(B(z),iK(z))G\begin{pmatrix}\hat{x}\\ \hat{\theta}\end{pmatrix}. \tag{53}$$

When we substitute this expression in the transformation rule (47) for $\hat{a}$, while using the two-dimensional version of the relation (42), we obtain

$$\hat{a}(z) = i\sqrt{\frac{k}{2}}(B(0),iK(0))M^T(z)G\begin{pmatrix}\hat{x}\\ \hat{\theta}\end{pmatrix}, \tag{54}$$

where we used the identity (20) in the form $GM^{-1}=M^TG$. The equivalence of Eqs. (53) and (54) is in obvious accordance with the identity (50).

B. Astigmatic Gaussian modes

The formulation that we have given for the modes in one transverse dimension allows a direct generalization to two transverse dimensions. In that case we must have two lowering operators rather than one. Since these operators must return to their initial form after a full roundtrip, they must be determined by the eigenvectors of the roundtrip transfer matrix M. In analogy to the expression (53), we introduce the two z-dependent lowering operators

$$\hat{a}_i(z) = i\sqrt{\frac{k}{2}}\mu_i M^T(z)G\begin{pmatrix}\hat{R}\\ \hat{\Theta}\end{pmatrix}, \tag{55}$$

in terms of the two eigenvectors μ_i of M $(i=1,2)$. By the same argument as given for Eq. (54), these operators obey the transformation rule (48), and in the reference plane at $z=0$ they are given by

$$\hat{a}_i(0) = i\sqrt{\frac{k}{2}}\mu_i G\begin{pmatrix}\hat{R}\\ \hat{\Theta}\end{pmatrix}. \tag{56}$$

Over a full roundtrip, they transform as

$$\hat{a}_i(2L) = \hat{a}_i(0)e^{i\chi_i}, \tag{57}$$

in terms of the eigenvalues of μ_i. By using the identities (23), one verifies that the ladder operators obey the commutation rules

$$[\hat{a}_i(z),\hat{a}_i^\dagger(z)] = 1. \tag{58}$$

By using the identities $\mu_1^* G\mu_2=\mu_1 G\mu_2=0$, we find that other commutators vanish, so that $[\hat{a}_2,\hat{a}_1^\dagger]=[\hat{a}_2,\hat{a}_1]=0$.

Just as in Ref. [11], for notational convenience we combine the two lowering operators into a vector of operators

$$\hat{A} = \begin{pmatrix}\hat{a}_1\\ \hat{a}_2\end{pmatrix}, \tag{59}$$

for all values of z. In analogy to Eq. (43), this can be written as

$$\hat{A} = \sqrt{\frac{k}{2}}\left(\mathsf{K}\hat{R} + i\mathsf{B}\frac{\hat{P}}{k}\right) = \sqrt{\frac{k}{2}}(\mathsf{K}\hat{R} + i\mathsf{B}\hat{\Theta}), \tag{60}$$

where now B and K are 2×2 z-dependent matrices in the transverse plane. Comparison with Eq. (55) shows that in the reference plane $z=0$ the two matrices $\mathsf{B}(0)$ and $\mathsf{K}(0)$ can be combined into a single 2×4 matrix, where the two rows coincide with the transposed eigenvectors μ_i^T. This gives the formal identification

$$(\mathsf{B}(0),i\mathsf{K}(0)) = \begin{pmatrix}\mu_1^T\\ \mu_2^T\end{pmatrix}. \tag{61}$$

Equation (60) then shows that the z dependence of B and K is formally expressed as

$$(\mathsf{B}(z),i\mathsf{K}(z)) = (\mathsf{B}(0),i\mathsf{K}(0))M^T(z), \tag{62}$$

where in the right-hand side a 2×4 matrix multiplies a 4×4 matrix, producing a 2×4 matrix. The behavior of M as a function of z is fully determined by the expressions (17) and (18) for free propagation and at passage of a lens. It follows that during free propagation, K is constant, while B obeys the differential equation $d\mathsf{B}/dz=i\mathsf{K}$. At passage through a lens with focal matrix F, B does not change, whereas the change in K is given by

$$\mathsf{K}_{\text{out}} = \mathsf{K}_{\text{in}} + i\mathsf{B}\mathsf{F}^{-1}. \tag{63}$$

The fundamental mode $|u_{00}(z)\rangle$ in the astigmatic resonator is defined by the requirement that it obeys the paraxial wave equation, and that it gives zero when operated on by the lowering operators $\hat{a}_i$. It is easy to check that these conditions are verified by the normalized mode function

$$u_{00}(R,z) = \sqrt{\frac{k}{\pi \det\mathsf{B}}}\exp\left(-\frac{k}{2}R\mathsf{B}^{-1}\mathsf{K}R\right) \tag{64}$$

in terms of the z-dependent matrices B and K. The matrix $\mathsf{B}^{-1}\mathsf{K}$ is symmetric, as can be checked by using the properties of the eigenvectors μ_i derived in Sec. II. This expression generalizes the description of freely propagating astigmatic beams given in Ref. [11]. Because of the definitions of B and K in terms of the eigenvectors of the roundtrip matrix M, the fundamental mode returns to itself after a roundtrip, as expressed by

$$|u_{00}(2L)\rangle = |u_{00}(0)\rangle\exp[-i(\chi_1+\chi_2)/2]. \tag{65}$$

Higher-order modes are defined by repeated application of the raising operators, which gives

$$|u_{nm}(z)\rangle = \frac{1}{\sqrt{n!\,m!}}[\hat{a}_1^\dagger(z)]^n[\hat{a}_2^\dagger(z)]^m|u_{00}(z). \tag{66}$$

The set of modes functions $|u_{nm}(z)\rangle$ is complete and orthonormal in each transverse plane. Because of the roundtrip properties (57) of the ladder operators, the modes transform over a roundtrip as

$$|u_{nm}(2L)\rangle = e^{-i(n+1/2)\chi_1 - i(m+1/2)\chi_2}|u_{nm}(0). \tag{67}$$

The resonance condition for the electric field that is proportional to $\exp(ikz)u_{nm}(R,z)$ gives the condition for the wave number

$$2kL - (n + 1/2)\chi_1 - (m + 1/2)\chi_2 = 2\pi q, \tag{68}$$

so that the frequency of the mode specified by the transverse mode numbers n and m, and the longitudinal mode number q is

$$\omega = \frac{c}{2L}\left[2\pi q + \left(n + \frac{1}{2}\right)\chi_1 + \left(m + \frac{1}{2}\right)\chi_2\right]. \tag{69}$$

Obviously, the general astigmatism does not show up in the frequency spectrum of the resonator. All that can be seen is the presence of two different roundtrip Gouy phases. There are two different ways in which the corresponding frequency spectrum can be degenerate. For a resonator that has cylinder symmetry the two eigenvalue spectrum of the transfer matrix is degenerate (i.e., if $\chi_1 = \chi_2$) and its modes are frequency degenerate in the total mode number $n+m$. As a result any linear combination of eigenmodes with the same total mode number is an eigenmode too. The second kind of degeneracy arises when one of the Gouy phases is a rational fraction of 2π. Then the combs of modes at different values of q overlap so that many different modes appear at the same frequency.

V. PHYSICAL PROPERTIES OF THE EIGENMODES

A. Symmetry properties

So far we have described the modes as a periodic solution of the paraxial equation in the lens guide that is equivalent to the resonator. The electric and magnetic field in the resonator are obtained by refolding the periodic lensguide fields (30) and (31). The fields in two successive intervals with length L in the lens guide then give the fields propagating back and forth within the resonator. The electric field (30) in the lens guide then gives the expression for the field in the resonator

$$\mathbf{E}_{\rm res}(\mathbf{r},t) = \mathrm{Re}[E_0 \epsilon f(R,z) i \exp(-i\omega t)] \tag{70}$$

for $0 < z < L$, with

$$f(R,z) = \frac{1}{i}[u(R,z)\exp(ikz) - u(R,-z)\exp(-ikz)]. \tag{71}$$

The minus sign in Eq. (71) ensures that the mirror surfaces coincide with a nodal plane. This follows from the relation (63) between the input and the output of a lens. Applied to the lens at $z=0$, this shows that in the lens guide the transverse profile $u(R,0^{\pm})$ just left and right of lens 1 can be written as

$$u(R,0^{\pm}) = u_1(R)\exp(\mp ikR\mathsf{F}_1^{-1}R/4), \tag{72}$$

where u_1 may be viewed as the transverse profile halfway lens 1. Substitution in Eq. (71) shows that the resonator field f near mirror 1 is given by $2u_1(R)\sin(kz - kR\mathsf{F}_1^{-1}R/4)$. Since the value of k obeys the resonance condition (68), which makes $u(R,z)\exp(ikz)$ periodic, a similar argument holds for mirror 2. When $u_2(R)$ is defined as the periodic lens guide field $u(R,z)\exp(ikz)$ at the plane halfway lens 2, the resonator field f near mirror 2 (where $z \approx L$) is $2u_2(R)\sin[k(z-L+kR\mathsf{F}_2^{-1}R/4)]$.

The corresponding expression for the magnetic field in the resonator is

$$\mathbf{B}_{\rm res}(\mathbf{r},t) = \mathrm{Re}[E_0(\hat{z}\times\epsilon) b(R,z)\exp(-i\omega t)], \tag{73}$$

with

$$b(R,z) = \frac{1}{c}[u(R,z)\exp(ikz) + u(R,-z)\exp(-ikz)], \tag{74}$$

for $0 < z < L$. The expression (74) for the magnetic field has a plus sign, arising from the fact that the propagation direction $\hat{z}$ in Eq. (31) is replaced by $-\hat{z}$ for the field component propagating in the negative direction. Near mirror 1, the magnetic field function b is given by $2u_1(R)\cos(kz - kR\mathsf{F}_1^{-1}R/4)/c$, while near mirror 2 we find $b(R,z) = 2u_2(R)\cos[k(z-L+kR\mathsf{F}_2^{-1}R/4)]/c$.

The paraxial field in the resonator as described here arises from refolding a periodic field in the lens guide that propagates in the positive direction. We could just as well start from a lens-guide field propagating in the negative direction. Such a field is obtained by replacing $u(R,z)\exp(ikz)$ by its complex conjugate in Eq. (30). This leads to an alternative expression for the resonator field in the form (70) with f given by $[u^*(R,z)\exp(-ikz) - u^*(R,-z)\exp(ikz)]/i$. For a nondegenerate mode, this alternative expression for f must be proportional to the expression (71). This leads to the symmetry relation

$$u(R,-z) = u^*(R,z), \tag{75}$$

apart from an overall phase factor. This shows that the mode functions $f(R,z)$ and $b(R,z)$ are real, so that they can be expressed as

$$f(R,z) = 2\,\mathrm{Im}[u(R,z)\exp(ikz)],$$

$$b(R,z) = \frac{2}{c}\,\mathrm{Re}[u(R,z)\exp(ikz)]. \tag{76}$$

From Eqs. (70) and (73) we find that in a nondegenerate paraxial mode of a two-mirror resonator the electric and the magnetic field can be written as

$$\mathbf{E}_{\rm res}(\mathbf{r},t) = -f(R,z)\mathrm{Im}[E_0\epsilon\exp(-i\omega t)],$$

$$\mathbf{B}_{\rm res}(\mathbf{r},t) = b(R,z)\mathrm{Re}[E_0(\hat{z}\times\epsilon)\exp(-i\omega t)], \tag{77}$$

which is the product of a real function of position and a real function of time. Both fields take the form of a standing wave, with phase difference $\pi/2$. The curved transverse nodal planes of the electric field are determined by the requirement that $u(R,z)\exp(ikz)$ is real. These nodal planes coincide with the antinodal planes of the magnetic field.

B. Shape of the modes

It is interesting to notice that the real and the imaginary part of the complex propagating field in the lens guide correspond to the electric and magnetic field in the resonator, as given by the expression

$$u(R,z)\exp(ikz) = cb(R,z) + if(R,z). \tag{78}$$

This shows that the nodal planes of the electric or the magnetic field in the resonator are wave fronts of the traveling wave in the lens guide.

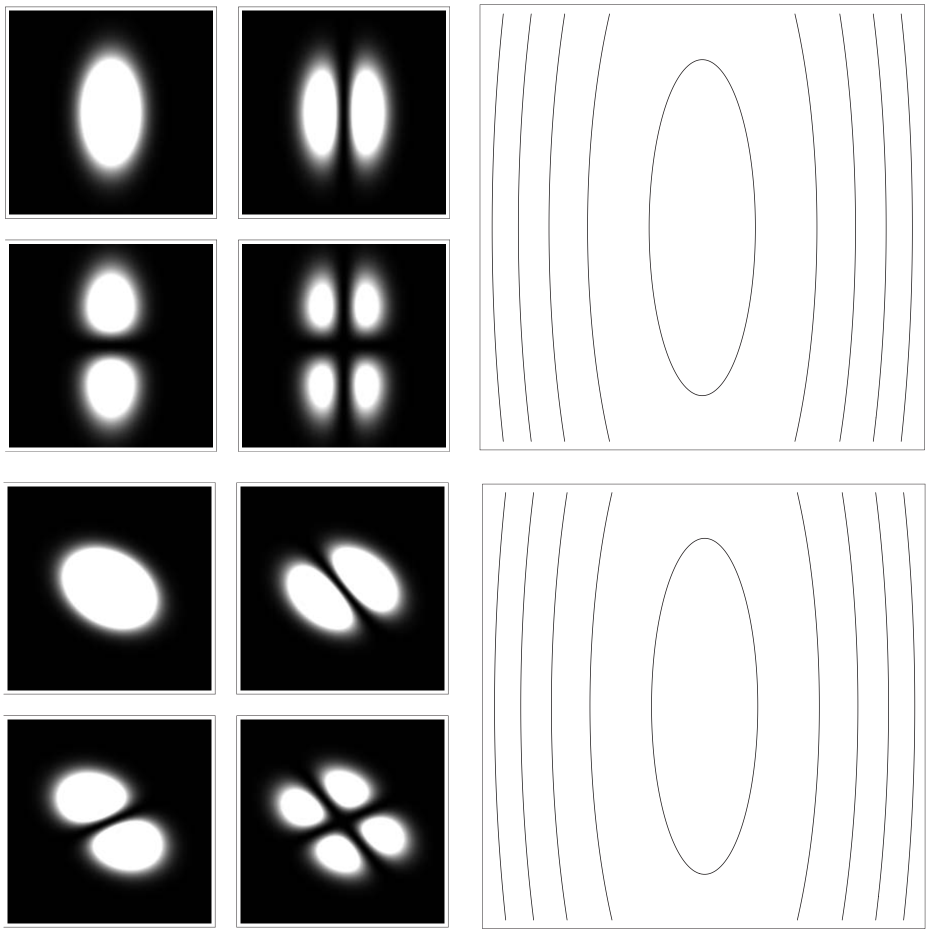

FIG. 4. Intensity (left) and phase pattern (right) of lens-guide modes with simple (top) and general astigmatism (bottom) in the reference plane next to lens 1. The intensity patterns refer to modes with transverse mode numbers $(0,0)$, $(1,0)$, $(0,1)$, and $(1,1)$. The focal lengths of lenses 1 and 2 are $f_{\xi_1}=L$, $f_{\eta_1}=10L$, and $f_{\xi_2}=L$ and $f_{\eta_2}=20L$. In the case of simple astigmatism the lenses are aligned, in the case of general astigmatism the mirrors are rotated over $\pi/3$ with respect to each other. Two neighboring curves of constant phase have a phase difference of $\pi/20$. The phase plots are enlarged by a factor of 3 compared with the intensity plots.

From the symmetry properties it follows that the periodic lens-guide field $u(R,z)\exp(ikz)$ is real in the transverse plane halfway each of the lenses. Since this conclusion holds for eigenmodes of all orders, also the ladder operators $\hat{a}_i$ can be chosen real in these two symmetry planes. As a result, the higher-order modes have the nature of astigmatic Hermite-Gaussian modes, with a pattern of two sets of parallel straight nodal lines. In the language of Ref. [11], the modes in these planes are characterized by a point on the equator of the Hermite-Laguerre sphere. However, nodal lines in these two sets are not orthogonal in the case of a twisted resonator. In the free space between the mirrors of a twisted cavity the modes can attain structure with vortices, arising from an elliptical rather than a linear nature of the distribution of transverse momentum. In the special case of simple astigmatism, the eigenmodes have a Hermite-Gaussian structure in all transverse planes, with rectangular patterns of nodal lines that are aligned to the axes of the two mirrors.

In Fig. 4 we illustrate the structure of the lowest eigenmodes of a resonator with simple and with general astigmatism, in the immediate neighborhood of mirror 1. It is convenient to plot the intensity and the phase pattern in the equivalent lens guide. Intensity plots are shown for modes with transverse mode numbers (n,m) with $n,m=0,1$. Since the wave fronts in the lens-guide field coincide with nodal planes of the cavity fields, the sketched curves of constant phase in the reference plane may also be viewed as the curves of constant height of mirror 1. The figure shows that the orientation of the nodal lines in the reference plane are aligned with the axes of the mirror in the case of simple astigmatism. When the two mirrors do not have parallel axes, so that the resonator is twisted, the orientation of the nodal lines is no longer parallel to that of the mirror, and the pattern of nodal lines is no longer rectangular.

VI. DISCUSSION AND CONCLUSIONS

We have presented an algebraic method to obtain the complete set of orthonormal eigenmodes of an optical resonator with astigmatism. This situation occurs for an optical resonator formed by two astigmatic mirrors. When the axes of the mirrors are parallel, the modes have a factorized form, with each factor corresponding to one transverse dimension. Then the problem is equivalent to the case of a single transverse dimension, where standard techniques can be applied. The situation is significantly different when one of the mirrors is rotated about the optical axis, so that the axes of the two mirrors are no longer parallel. In that case, the mode field propagating back and forth between the mirrors has general astigmatism, for which the description of Ref. [11] applies. An essential ingredient in the characterization of the modes is the four-dimensional transfer matrix M, which describes the transformation of an optical ray over one roundtrip through the resonator. This matrix obeys the identity (20). We argue that the resonator is stable when the eigenvalues of the transfer matrix are complex. Since the transfer matrix is real, the eigenvalues (and the eigenvectors) form two pairs of complex conjugates, and we show that the eigenvalues are unitary. The arguments of the eigenvalues play the role of two Gouy phase angles χ_1 and χ_2, which determine the resonance frequencies according to Eq. (69). The structure of the modes of the resonator are determined by the eigenvectors. They depend on the transverse reference plane that is taken as the start of a roundtrip, while the eigenvalues are invariant. As indicated in Eqs. (61) and (62), two of the four eigenvectors determine two two-dimensional matrices $\mathsf{K}(z)$ and $\mathsf{B}(z)$ that vary along the optical axis in the lens-guide picture. The Gaussian fundamental mode depends on these two matrices according to Eq. (64). Higher order modes arise after repeated application of bosonic raising operators as in Eq. (66), where these operators are specified by Eqs. (59) and (60). These algebraic expressions can be used to calculate directly the amplitude profile of the modes of all orders.

It is noteworthy that the structure of the wave-optical electromagnetic mode field is characterized by the eigenvectors and eigenvalues of the transfer matrix, which is a geometric concept of ray optics. Moreover, although the mode fields belong to the domain of classical electrodynamics, the relation between modes of various orders is determined by ladder operators which generalize the ladder operators that arise in the context of the quantum-mechanical harmonic oscillator. Finally, since the paraxial wave equation for free propagation is identical to the Schrödinger equation of a free particle (in two dimensions), the methods and results of the present paper can also be applied to the evolution of a particle in free space. In the Schrödinger equation, the propagation coordinate z of the paraxial wave equation is replaced by time, while the two transverse coordinates x and y are replaced by the three spatial coordinates. This allows us to obtain complete orthogonal sets of exact solutions of the Schrödinger equation in terms of nonisotropic Gaussians and ladder operators, where the 2×2 matrices in the present paper are replaced by 3×3 matrices. In that case, we obtain three lowering operators for each instant of time.

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