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We generalize the concept of an optical cavity mode to the case of an astigmatic cavity that rotates about its optical axis. We show that the modes of such a cavity are both spatially and spectrally confined and use an algebraic method to study their spatial and spectral structure. Our method involves ladder operators in the spirit of the quantum-mechanical harmonic oscillator. It hinges upon their algebraic properties as well as on the group-theoretical properties of the ray ($ABCD$) matrix that describes the time-dependent ray dynamics of the rotating cavity.

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I. INTRODUCTION

Light with a complex spatial structure has applications in different branches of physics. From a quantum information point of view the spatial degrees of a photon can be used to encode and manipulate quantum states in a Hilbert space of many dimensions. An example is provided by the eigenmodes of orbital angular momentum of light [1,2]. This angular momentum, which arises from the phase structure of monochromatic light beams, can also be used to manipulate small particles [3,4].

The (possibly very complex) spatial structure of the modes of a two-mirror cavity is determined by the boundary condition that the electric field must vanish on the mirror surfaces. For monochromatic beams this implies that the wave fronts (surfaces of equal phase) of the standing wave inside the cavity must fit onto the mirror surfaces. The common approach to finding the modes of a paraxial optical cavity is by considering the free propagation of a Gaussian beam and requiring the wave fronts of the beam to fit onto the mirror surfaces. This is straightforward in the case of spherical mirrors [5]. The resulting equation can be solved to obtain the beam parameters. The possible values of the wave number k follow from the resonance condition that the total phase that is picked up after each round trip is an integer multiple of 2π [5]. This approach allows for generalization to the case of astigmatic or cylindrical mirrors, which are curved differently in different directions, provided that the mirrors are aligned. Fundamental difficulties arise in the case of cavities with astigmatic mirrors in nonparallel alignment [6]. As a result of the twist of such a cavity, the modes are twisted as well [7].

An additional source of complexity arises when the boundaries of an optical cavity depend on time. Time-dependent optical cavities are of fundamental (and historical) importance since they provide a very accurate way to observe (violations of) local Lorentz invariance [8,9]. Optical cavities with vibrating mirrors have been studied in detail, especially in the context of the dynamical modification of the Casimir effect [10,11]. The resonant coupling of the modes of such a cavity to the vibration of the mirrors has also been studied [12]. The interplay between physical rotation of an optical cavity and wave chaos has been discussed in a recent paper [13].

In the present paper we analyze the time-dependent modes of an astigmatic two-mirror cavity that is rotating at a uniform velocity about its optical axis. It remains true that the mode structure is determined by the boundary condition that the electric field must vanish on the mirror surfaces. Since modes are usually defined as stationary solutions of a wave equation, the concept of a mode requires special attention in this case. As opposed to vibrations, uniform rotations of the mirrors give rise to a homogeneous time dependence of the cavity. As a result all times (and therefore all round trips) are equivalent. In this special case it is natural to assume that the modes adopt the time dependence of the cavity so that they rotate along with the mirrors. We show that this property can be used as a defining property of the modes and derive explicit expressions that we apply to study their spatial and spectral structure.

II. TIME-DEPENDENT PARAXIAL PROPAGATION

The description of the propagation of optical modes is greatly simplified by making paraxial approximations, which are almost always justified in experimental situations. Here we summarize the perturbative derivation of the paraxial approximation that is due to Lax *et al.* [14] and its generalization to the time-dependent case by Deutsch and Garrison [15]. This helps us to ensure the consistency of our approach, in that we retain all terms up to the same order.

The spatial structure of a paraxial beam is characterized by a vector field $\mathbf{u}(\mathbf{r},t)$ that describes the slowly varying components of the electric field. The field $\mathbf{u}$ defines the electric-field component of the light field by

$$\mathbf{E}(\mathbf{r},t) = \mathrm{Re}\{E_0\mathbf{u}(\mathbf{r},t)\exp(ikz - i\omega t)\}, \tag{1}$$

where E_0 is an amplitude factor and $\omega = ck$ is the frequency of the carrier wave, with k the wave number. In vacuum the electric field obeys the wave equation

$$\nabla^2\mathbf{E} = \frac{1}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2} \tag{2}$$

with the additional requirement that it has a vanishing divergence

$$\nabla \cdot \mathbf{E} = 0. \tag{3}$$

Essential for the paraxial approximation is that the beam has a small opening angle, which we indicate by the smallness parameter δ. Then the beam waist is of the order of the parameter γ, and the diffraction length (or Rayleigh range) is of the order of the parameter b, where

$$\frac{1}{k} = \delta\gamma = \delta^2 b. \tag{4}$$

So the diffraction length is much larger than the beam waist, which is much larger than the wavelength. The smallness of δ ensures that the variations of the profile $\mathbf{u}(\mathbf{r},t)$ with the longitudinal coordinate z are slow compared to the variations with the transverse coordinates $R=(x,y)$, which are, in turn, slow compared to the variations of the carrier wave $\exp(ikz)$ with z.

Time dependence of the spatial profile gives rise to frequency components, so that time-dependent paraxial modes have spectral structure in addition to their spatial structure. The concept of a mode loses its meaning if the difference in diffraction of the frequency components becomes significant, i.e., if the diffraction due to the time dependence of the profile becomes important. Conversely, we shall show that a mode remains a useful concept when the time scale for variation of the cavity boundaries is slower than the transit time through the focal range of the beam. This transit time is of the order of

$$a = \frac{b}{c} = \frac{\delta^2}{\omega}. \tag{5}$$

In order to obtain the time-dependent paraxial wave equation the profile is expanded in powers of the opening angle δ

$$\mathbf{u} = \sum_{n=0}^{\infty} \delta^n \mathbf{u}^{(n)}. \tag{6}$$

In order to account for the relative order of magnitudes of the derivatives, it is convenient to introduce the scaled variables $\xi=x/\gamma$, $\eta=y/\gamma$, $\zeta=z/b$, and $\tau=t/a$. In these variables, the derivatives of $\mathbf{u}$ can be treated as being of the same order in δ. Substituting the expression (1) for the electric field in the wave equation (2) then gives

$$\left(\frac{\partial^2}{\partial\xi^2} + \frac{\partial^2}{\partial\eta^2} + 2i\frac{\partial}{\partial\zeta} + 2i\frac{\partial}{\partial\tau}\right)\mathbf{u} = \delta^2\left(\frac{\partial^2}{\partial\tau^2} - \frac{\partial^2}{\partial\zeta^2}\right)\mathbf{u}, \tag{7}$$

while the transversality condition (3) gives

$$\delta\left(\frac{\partial u_x}{\partial\xi} + \frac{\partial u_y}{\partial\eta}\right) = -\delta^2\frac{\partial u_z}{\partial\zeta} - iu_z. \tag{8}$$

It is natural to assume that to zeroth-order the z component of $\mathbf{u}$ vanishes, and Eq. (8) shows that such a solution can be found. Then to zeroth-order of the paraxial approximation the electric field lies in the transverse plane. In the special case of uniform polarization it can be written as

$$\mathbf{u}^{(0)}(\mathbf{r},t) = \boldsymbol{\epsilon} u(\mathbf{r},t), \tag{9}$$

where the polarization vector $\boldsymbol{\epsilon}$ has a vanishing z component. The scalar function $u(\mathbf{r},t)$ obeys the time-dependent paraxial wave equation

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + 2ik\frac{\partial}{\partial z} + \frac{2ik}{c}\frac{\partial}{\partial t}\right)u(\mathbf{r},t) = 0. \tag{10}$$

The expansion (6) then shows that all even orders of the transverse components are coupled by Eq. (7), and the odd orders can be assumed to vanish. Equation (8) connects odd orders of the z component to the even orders of the transverse components, which implies that all even orders (including the zeroth) of the longitudinal component vanish. The first-order contribution to the profile is longitudinal and by using Eq. (8) this can be expressed in the zeroth term

$$\delta\mathbf{u}^{(1)}(\mathbf{r},t) = \frac{i}{k}\left(\epsilon_x\frac{\partial}{\partial x} + \epsilon_y\frac{\partial}{\partial y}\right)u(\mathbf{r},t)\mathbf{e}_z, \tag{11}$$

where $\mathbf{e}_z$ is the unit vector in the z direction.

Up to first order of the paraxial approximation finding the modes of a cavity that has physically rotating mirrors requires solving the time-dependent paraxial wave equation (10) with the boundary condition that the electric field vanishes on the mirror surfaces at all times. The range of validity of this time-dependent wave equation provides a natural upper limit to the rotation frequency of the mirrors. In a typical experimental setup the diffraction length of the modes of an optical cavity is of the order of magnitude of the mirror separation, so that the period of the rotation of the mirror(s) can be at most comparable to the cavity round-trip time. This provides an upper limit for the rotation frequency Ω

$$\Omega \lesssim \frac{c\pi}{L}, \tag{12}$$

where L is the mirror separation and c is the speed of light. The cavity ring-down time (which we leave out of our consideration here) provides a natural lower limit for the rotation frequency of the mirrors.

III. OPERATOR DESCRIPTION OF TIME-DEPENDENT PARAXIAL WAVE OPTICS

A. Operators and transformations

The standard time-independent paraxial wave equation follows if we omit the time derivative in Eq. (10). This has the same structure as the Schrödinger equation for a free particle in two dimensions, with k taking the place of $m/\hbar$ and the longitudinal coordinate z playing the role of time. This analogy can be exploited by adopting the Dirac notation of quantum mechanics to describe classical light beams [16], which naturally leads to an operator description of paraxial wave optics. We show that this description can be generalized to include the time dependence of the profile, even though the time dependence of the beam $u(R,z,t)$ does not have an analog in quantum mechanics.

We associate to the beam profile $u(R,z,t)$ a vector $|u(z,t)\rangle$ in the Hilbert space of transverse modes

$$u(R,z,t) = \langle R|u(z,t)\rangle, \tag{13}$$

where $|R\rangle$ is an eigenstate of the two-dimensional position operator $\hat{R}=(\hat{x},\hat{y})$. The corresponding momentum operator can be represented by $\hat{P}=(\hat{p}_x,\hat{p}_y)=-i(\partial/\partial x,\partial/\partial y)$. The average (or expectation) value of this operator can be shown to correspond to the transverse momentum per unit length per photon (in units of $\hbar$) [17].

The transformations of paraxial propagation and lossless optical elements such as thin lenses can be expressed as unitary transformations in the transverse mode space. Expressing the time-dependent paraxial wave equation (10) in terms of the momentum operators gives

$$\left(\frac{\partial}{\partial z} + \frac{1}{c}\frac{\partial}{\partial t}\right)|u(z,t)\rangle = -\left(\frac{i}{2k}\hat{P}^2\right)|u(z,t)\rangle. \tag{14}$$

The solution of this equation can be expressed as

$$|u(z,t)\rangle = \exp\left(-\frac{iz}{2k}\hat{P}^2\right)|u(0,t-z/c)\rangle = \hat{U}_f(z)|u(0,t-z/c)\rangle, \tag{15}$$

where $\hat{U}_f(z)$ is the unitary operator that describes free propagation of a paraxial beam. This result shows that the time-dependent paraxial wave equation describes the beam propagation while incorporating retardation effects.

A thin spherical lens imposes a Gaussian phase profile and hence the transformation caused by such a lens can be expressed as

$$|u_{\text{out}}\rangle = \exp\left(\frac{ik\hat{R}^2}{2f}\right)|u_{\text{in}}\rangle, \tag{16}$$

where f is the focal length of the lens. The generalization of this transformation to the case of a lens that has astigmatism is given by

$$|u_{\text{out}}\rangle = \exp\left(\frac{ik}{2}\hat{R}\mathbf{F}^{-1}\hat{R}\right)|u_{\text{in}}\rangle = \hat{U}_l(\mathbf{F})|u_{\text{in}}\rangle, \tag{17}$$

where $\mathbf{F}$ is a real and symmetric 2×2 matrix. The eigenvalues of $\mathbf{F}$ are the focal lengths of the lens and the mutually orthogonal real eigenvectors fix the orientation of the lens in the transverse plane.

B. Frequency combs

The operator that rotates a scalar function around the z axis in the positive (counterclockwise) ϕ direction can be expressed in the transverse mode space as

$$\hat{U}_r(\alpha) = \exp(-i\alpha\hat{J}_z), \tag{18}$$

where α is the rotation angle and $\hat{J}_z=\hat{R}\times\hat{P}=\hat{x}\hat{p}_y-\hat{y}\hat{p}_x=-i\partial/\partial\phi$ is the z component of the angular momentum operator. The inverse of this rotation is a rotation in the opposite direction, i.e., $\hat{U}^\dagger(\alpha)=\hat{U}(-\alpha)$. The transformation of a rotated lens can be expressed as

$$\hat{U}_r(\alpha)U_l(\mathbf{F})\hat{U}_r^\dagger(\alpha). \tag{19}$$

This (anti-Heisenberg) transformation property makes sense if one realizes that rotating a lens is equivalent to rotating the profile in the opposite direction, applying the lens and rotating the profile backward. The beam transformation caused by an astigmatic lens (17) is a function of the position operator $\hat{R}=(\hat{x},\hat{y})$ only. The anti-Heisenberg transformation of the position operator under the rotation about the z axis (18) can be expressed as

$$\hat{U}_r(\alpha)\hat{R}\hat{U}_r^\dagger(\alpha) = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}\hat{R} = \mathbf{R}^T(\alpha)\hat{R}, \tag{20}$$

with $\mathbf{R}(\alpha)$ the two-dimensional rotation matrix. By using this transformation property of the position operators, the transformation of a rotated lens (19) can be expressed as

$$\hat{U}_l(\mathbf{R}(\alpha)\mathbf{F}\mathbf{R}^T(\alpha)). \tag{21}$$

For a lens rotating at angular velocity Ω, the rotation angle is $\alpha=\Omega t$, so that the time-dependent beam transformation caused by the rotating lens is given by $\hat{U}_l(\mathbf{F}(t))$, where $\mathbf{F}(t)=\mathbf{R}(\Omega t)\mathbf{F}(0)\mathbf{R}^T(\Omega t)$. Without loss of generality we can choose the real and symmetric matrix $\mathbf{F}(t)$ diagonal at $t=0$

$$\mathbf{F}(0) = \begin{pmatrix} f_\xi & 0 \\ 0 & f_\eta \end{pmatrix}. \tag{22}$$

By using Eqs. (19)–(22) and introducing cylindrical coordinates with $x=\rho\cos\phi$, $y=\rho\sin\phi$, the time-dependent transformation of a rotating lens can be expressed as

$$\hat{U}_l(\mathbf{F}(\mathbf{t})) = \exp\left[-\frac{ik\rho^2}{4}(f_\xi^{-1}+f_\eta^{-1}) - \frac{ik\rho^2}{4}(f_\xi^{-1}-f_\eta^{-1})\cos(2\Omega t-2\phi)\right]. \tag{23}$$

Since $\hat{U}_l$ is periodic with period π/Ω it follows that a rotating lens introduces frequency sidebands at frequencies $\omega\pm 2p\Omega$, with integer p in a monochromatic optical field [18].

IV. MODES IN ROTATING CAVITIES

The transformation of a sequence of lossless optical elements can be constructed by multiplying the transformations of the elements and free propagation over the distances between them in the correct order. If any one of the transformations is time dependent, retardation effects must be incorporated.

A. Lens guide picture

In order to describe the evolution of a profile vector $|u(z,t)\rangle$ inside the cavity it is convenient to unfold the cavity into an equivalent lens guide. As illustrated in Fig. 1, the mirrors are replaced by lenses with the same focal lengths. Rather than describing the bouncing back and forth inside the cavity we describe the propagation along the axis of the

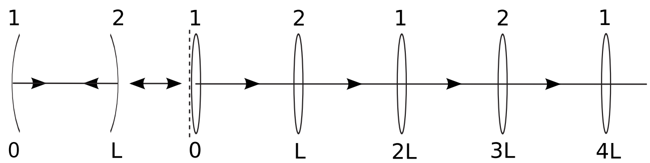

FIG. 1. Unfolding an optical cavity into an equivalent periodic lens guide. The mirrors are replaced by lenses with the same focal lengths and the $z=0$ plane is indicated by the dashed line.

lens guide, with coordinate z. The dashed line on the left of the first lens in Fig. 1 indicates the input plane of the lens guide, which is positioned at $z=0$. The profile in any transverse plane of the lens guide is connected to the profile in the input plane by a unitary transformation. Just as in Eq. (15), this time-dependent connection involves a retardation time, as described by

$$|u(z,t)\rangle = \hat{U}(z,t)|u(0,t-z/c)\rangle. \tag{24}$$

The unitary operator $\hat{U}(z,t)$ is constructed by successive application of the transformations of the optical elements and free propagation that are in between the reference plane and the z plane in the correct order. We need only two different transformation operators for the lenses in the lens guide, which we denote for simplicity as $\hat{U}_1(t)$ and $\hat{U}_2(t)$. For the lens guide that corresponds to a cavity rotating at the uniform angular velocity Ω, these time-dependent operators are given by Eq. (19) with rotation angle $\alpha=\Omega t$, so that

$$\hat{U}_i(t) = \hat{U}_r(\Omega t)\hat{U}_l(\mathbf{F}_i(0))\hat{U}_r^\dagger(\Omega t), \tag{25}$$

with $i=1,2$ labeling the two lens types.

Since the orientations of the rotating lenses depend on time, retardation effects must be included. As an example, we give the operator that connects the profile vectors in transverse planes that are separated by one period of the lens guide

$$\hat{U}(2L,t) = \hat{U}_f(L)\hat{U}_2(t-L/c)\hat{U}_f(L)\hat{U}_1(t-2L/c). \tag{26}$$

Obviously, all lenses that correspond to the same mirror of the cavity have the same orientation at any instant of time. Nevertheless, as a result of retardation the orientation of two lenses that correspond to the same mirror of the cavity is perceived differently by a light pulse that propagates through the lens guide.

B. Rotating modes

Cavity modes are resonant field distributions inside an optical cavity. In a stationary cavity, a field pattern is resonant if it repeats itself after a round trip through the cavity. In the case of a rotating cavity, the requirement is that the field pattern in the lens guide is the same in every period, at a single given instant of time. This implies that the mode vector $|u(0,t)\rangle$ in the reference plane for a given value of t repeats itself after one period $2L$, apart from a phase factor

$$|u(2L,t)\rangle = \exp(i\chi)|u(0,t)\rangle. \tag{27}$$

The phase χ generalizes the Gouy phase for the round trip in a stationary cavity [5] to the case of time-dependent paraxial propagation through a periodic lens guide with rotating lenses. When the lens guide is rotating at a uniform velocity Ω, this mode criterion (27) can be obeyed only if the mode pattern rotates along with the lenses, so that the time dependence of the mode vector is determined by

$$|u(z,t)\rangle = \hat{U}_r(\Omega t)|v(z)\rangle. \tag{28}$$

Then we can eliminate time by introducing the z-dependent profile

$$|v(z)\rangle = |u(z,0)\rangle, \tag{29}$$

which has the significance of the beam profile in the rotating frame. By combining the relation (28) with Eqs. (24)–(26), we find that the propagation of a beam profile in this frame is governed by the general relation

$$|v(z)\rangle = \hat{U}(z,0)\hat{U}_r(-\Omega z/c)|v(0)\rangle. \tag{30}$$

In the special case of the propagation over one period this gives

$$|v(2L)\rangle = \hat{U}_{\text{round}}\hat{U}_r^\dagger(\Omega t)|u(0,t)\rangle = \hat{U}_{\text{round}}|v(0)\rangle. \tag{31}$$

The operator $\hat{U}_{\text{round}}$ is given by the expression

$$\hat{U}_{\text{round}} = \hat{U}_f(L)\hat{U}_r(-\Omega L/c)\hat{U}_2(0)\hat{U}_f(L)\hat{U}_r(-\Omega L/c)\hat{U}_1(0). \tag{32}$$

It has the significance of the transformation operator over a round trip in the rotating frame. The analogous transformation operator at an arbitrary time figures in Eq. (31). Notice that the operators for free evolution $\hat{U}_f$ are denoted as a function of length, the lens operators $\hat{U}_i$ as a function of time, and the rotation operators $\hat{U}_r$ as a function of angle. The product $\hat{U}_f(L)\hat{U}_r(-\Omega L/c)$ can be viewed as the operator for free propagation over a distance L in the corotating frame.

Now the mode criterion (27) in the reference plane $z=0$ in the rotating frame is obeyed by the eigenvectors of the round-trip operator $\hat{U}_{\text{round}}$. Once these mode vectors are determined, we can use the propagation equation (15) and the time dependence (28) to obtain the shape of the modes at other time instants, and at any position within a period of the lens guide. The eigenvalues, which are specified by the phase angles χ, determine the resonance frequencies of the modes. In the next section we shall indicate how the eigenmodes can be obtained explicitly from a ladder-operator method.

V. RAY MATRICES AND LADDER OPERATORS

A. Time-dependent ray matrices

The transverse spatial structure of paraxial modes in cavities with spherical mirrors is known to be similar to the spatial structure of the stationary states of a two-dimensional quantum harmonic oscillator [5]. Complete sets of modes can be generated by using bosonic ladder operators [19]. In a recent paper we used a ladder-operator method to find explicit expressions of all the modes of an optical cavity that has nonorthogonal astigmatism [7]. These ladder operators are conveniently expressed in terms of the eigenvectors of the ray matrix for one period in the lens guide, or, equivalently, for one round trip in the cavity. Here we generalize this approach to account for the time dependence that arises from the rotation of the cavity. In this case, the ray matrices also depend on time.

In geometric paraxial optics, a light ray is specified by its position $R=(x,y)$ and its direction $\Theta=(\partial x/\partial z,\partial y/\partial z)$ in the transverse plane z [5]. They are combined into a four-dimensional column vector

$$r(z)=\begin{pmatrix} R(z) \\ \Theta(z) \end{pmatrix}. \tag{33}$$

The ray matrix $M(z)$ is the 4×4 ray matrix that describes the transformation of a ray through the lens guide from the reference plane at $z=0$ to the transverse plane z. It is the generalization of the $ABCD$ matrix [5] to two transverse dimensions. In wave optics, the position of a light beam is the expectation value of the operator $\hat{R}$, while its direction is the expectation value of $\hat{P}/k$, which is the ratio of the transverse and the longitudinal momentum. Therefore, the operator for the direction of the beam is $\hat{\Theta}=\hat{P}/k$. This is confirmed by the fact that the Heisenberg propagation of the operator vector $(\hat{R},\hat{\Theta})$ reproduces the ray matrix, as exemplified by the identity

$$\hat{U}^{\dagger}(z)\begin{pmatrix} \hat{R} \\ \hat{\Theta} \end{pmatrix}\hat{U}(z)=M(z)\begin{pmatrix} \hat{R} \\ \hat{\Theta} \end{pmatrix}. \tag{34}$$

Here the propagation operator $\hat{U}$ acts on the operator nature of $\hat{R}$ and Θ, while the matrix M acts on the four-dimensional ray vector. This relation may be viewed as the optical analog of the Ehrenfest theorem in quantum mechanics. Note that the commutation relations for the components of the position and direction operators take the form

$$[\hat{R}_x,\hat{\Theta}_x]=[\hat{R}_y,\hat{\Theta}_y]=i/k. \tag{35}$$

The ray matrix $M(z)$ for propagation from the plane $z=0$ to the plane z is the product of the ray matrices for the regions of free evolution and for the lenses in between these planes in the right order. These ray matrices can be found in any textbook on optics. The ray matrix for free propagation is described by

$$\hat{U}_f^{\dagger}(z)\begin{pmatrix} \hat{R} \\ \hat{\Theta} \end{pmatrix}\hat{U}_f(z)=\begin{pmatrix} \mathbf{1} & z\mathbf{1} \\ \mathbf{0} & \mathbf{1} \end{pmatrix}\begin{pmatrix} \hat{R} \\ \hat{\Theta} \end{pmatrix}=M_f(z)\begin{pmatrix} \hat{R} \\ \hat{\Theta} \end{pmatrix}, \tag{36}$$

where $\mathbf{1}$ and $\mathbf{0}$ are the two-dimensional unit matrix and the zero matrix. The transformation for a thin astigmatic lens can be expressed as

$$\hat{U}_l^{\dagger}(\mathbf{F})\begin{pmatrix} \hat{R} \\ \hat{\Theta} \end{pmatrix}\hat{U}_l(\mathbf{F})=\begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{F} & \mathbf{1} \end{pmatrix}\begin{pmatrix} \hat{R} \\ \hat{\Theta} \end{pmatrix}=M_l(\mathbf{F})\begin{pmatrix} \hat{R} \\ \hat{\Theta} \end{pmatrix}. \tag{37}$$

The ray matrix of a rotation about the z axis follows from the identity

$$\hat{U}_r^{\dagger}(\alpha)\begin{pmatrix} \hat{R} \\ \hat{\Theta} \end{pmatrix}\hat{U}_r(\alpha)=\begin{pmatrix} \mathbf{R}(\alpha) & 0 \\ 0 & \mathbf{R}(\alpha) \end{pmatrix}\begin{pmatrix} \hat{R} \\ \hat{\Theta} \end{pmatrix}=M_r(\alpha)\begin{pmatrix} \hat{R} \\ \hat{\Theta} \end{pmatrix}. \tag{38}$$

The identities (34) and (37) remain valid for rotating lenses, which makes both the operators $\hat{U}$ and the ray matrices M depend on time. The transformation of a time-dependent ray in the reference plane to another transverse plane z is given by

$$r(z,t)=M(z,t)r(0,t-z/c) \tag{39}$$

in analogy to Eq. (24). A corotating incident ray in the reference plane must give a corotating ray everywhere in the lens guide, and the ray matrices in the rotating frame become independent of time. In complete analogy to Eq. (31), this means that the transformation of a ray in the rotating frame over one period from the reference plane is given by the round-trip ray matrix

$$M_{\text{round}}=M_f(L)M_r(-\Omega L/c)M_2(0)M_f(L)M_r(-\Omega L/c)M_1(0), \tag{40}$$

with $M_1(0)$ and $M_2(0)$ the ray matrices for the lenses 1 and 2 at time 0.

Any ray matrix that describes the transformation of a (sequence of) lossless optical elements obeys the following identity:

$$M^TGM=G, \quad \text{where} \quad G=\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{41}$$

This property generalizes the requirement that the determinant of a ray matrix must be equal to 1 to optical systems that have two independent transverse dimensions. It is easy to show that the ray matrices that we have used obey this identity. The product of matrices that obey this restriction obeys it as well and in mathematical terms the set of 4×4 matrices that obey this identity forms the symplectic group $\mathrm{Sp}(4,\mathbb{R})$. Both the underlying algebra and the physics of such linear phase space transformations has been studied in detail [20].

B. Ladder operators in reference plane

The similarity between Hermite-Gaussian modes of a cavity with spherical mirrors and harmonic-oscillator eigenstates can be traced back to the fact that in the paraxial limit the Heisenberg evolution of the position and direction operators $\hat{R}$ and $\hat{\Theta}$ is linear, so that ladder operators, which are also linear in these operators, preserve their general shape under propagation and optical elements. Though nonorthogonal astigmatism does not have an analog in quantum mechanics, ladder operators can be used to generate a basis of astigmatic modes as well [21]. Recently we have demonstrated that the modes of a cavity with nonaligned astigmatic mirrors are determined by the eigenvectors of the round-trip ray matrix [22]. In the rotating frame, the relation between the propagation operators and the ray matrix for a round trip is basically the same as for a stationary one, and Eqs. (32) and (40) are obviously analogous. This allows us to apply the same technique to obtain explicit expressions for the modes of the rotating cavity. In order to define the ladder operators we shall need the eigenvectors and eigenvalues of the ray matrix Eq. (40). Stability of the rotating cavity requires that the eigenvalues are unitary, and since the matrix M_{round} is real, this implies that its eigenvectors come in two pairs μ_1,μ_1^* and μ_2,μ_2^* that are each other's complex conjugate. The eigenvalue relations are written as

$$M_{\mathrm{round}}\mu_1 = e^{i\chi_1}\mu_1 \quad \text{and} \quad M_{\mathrm{round}}\mu_2 = e^{i\chi_2}\mu_2. \tag{42}$$

From the general property (41) of ray matrices one directly obtains the generalized orthogonality properties

$$\mu_1 G\mu_2 = \mu_1^* G\mu_2 = 0, \tag{43}$$

while the eigenvectors can be normalized in order to obey the identities [7]

$$\mu_1^* G\mu_1 = \mu_2^* G\mu_2 = 2i. \tag{44}$$

We shall now prove that the ladder operators that define the shape of the modes in the reference plane $z=0$ at time 0 are easily expressed in terms of the eigenvectors μ_1 and μ_2 of the ray matrix M_{round}. In analogy to Ref. [22] we introduce two lowering operators

$$\hat{a}_i = \sqrt{\frac{k}{2}}\mu_i G\begin{pmatrix}\hat{R}\\ \hat{\Theta}\end{pmatrix} = \sqrt{\frac{k}{2}}(R_i\hat{\Theta} - \Theta_i\hat{R}), \tag{45}$$

where $i=1,2$. From the generalized orthonormality properties (43) and (44) of the eigenrays μ_i combined with the canonical commutation rules (35) it follows that the ladder operators obey the bosonic commutation rules

$$[\hat{a}_i,\hat{a}_j^\dagger] = \delta_{ij}. \tag{46}$$

Any set of ladder operators that obey these commutation relations defines a complete and orthonormal set of transverse modes according to

$$|v_{nm}\rangle = \frac{1}{\sqrt{n!\,m!}}(\hat{a}_1^\dagger)^n(\hat{a}_2^\dagger)^m|v_{00}\rangle. \tag{47}$$

Apart from an overall phase factor, the fundamental mode (or ground state in the terminology of quantum mechanics) $|v_{00}\rangle$ is determined by the requirement that $\hat{a}_1|v_{00}\rangle=\hat{a}_2|v_{00}\rangle=0$. We conclude that the ladder operators determine the complete set of modes in the rotating frame in the reference plane $z=0$. Explicit expressions will be given below.

C. Ladder operators in arbitrary transverse plane

The eigenrays $\mu_i(0)=\mu_i$ refer to the transformation from the reference plane at $z=0$ to the plane $z=2L$ in the rotating frame. We also need the modes $|v_{mn}(z)\rangle$, and therefore the eigenrays $\mu_i(z)$ in an arbitrary transverse plane z in the lens guide, in the rotating frame. The basic equation for the time-dependent transformation of a ray is given by Eq. (39), so that

$$\mu_i(z) = M(z,0)M_r(-\Omega z/c)\mu_p(0) \tag{48}$$

in analogy to Eq. (30) for the beam profile propagation in the rotating frame. In the special case of propagation over one period, we should take $z=2L$. Then the ray transformation in Eq. (48) is M_{round}, which gives

$$\mu_i(2L) = e^{i\chi_i}\mu_i(0). \tag{49}$$

For notational convenience we separate the four-dimensional eigenvectors in their two-dimensional subvectors as

$$\mu_i(z) = \begin{pmatrix}R_i(z)\\ \Theta_i(z)\end{pmatrix}. \tag{50}$$

Then we compose two 2×2 matrices out of the column vectors $R_i(z)$ and $\Theta_i(z)$, by the definition

$$\mathbf{P}(z) \equiv (R_1(z),R_2(z)) \quad \text{and} \quad \mathbf{T}(z) \equiv (\Theta_1(z),\Theta_2(z)). \tag{51}$$

The relations (43) and (44) can be summarized as

$$\mathbf{P}^T\mathbf{T} - \mathbf{T}^T\mathbf{P} = 0, \quad \mathbf{P}^\dagger\mathbf{T} - \mathbf{T}^\dagger\mathbf{P} = 2i\mathbf{1} \tag{52}$$

in all transverse planes z.

The dependence of the ladder operators on z in the rotating frame is determined by the requirement that when acting on a rotating solution of the time-dependent paraxial wave equation, they must produce another solution. In view of Eq. (30), this requirement takes the form

$$\hat{a}_i(z) = \hat{U}(z,0)U_r(-\Omega z/c)\hat{a}_i(0)U_r^\dagger(-\Omega z/c)\hat{U}^\dagger(z,0). \tag{53}$$

In the right-hand side of this equation the propagation operators $\hat{U}$ act as an anti-Heisenberg evolution on the operators $\hat{R}$ and $\hat{\Theta}$. In accordance with the general Ehrenfest relation (34), and the relation (41), this gives rise to a product $GM^{-1}=M^TG$ when we substitute the expression (45) for the lowering operator. This leads to the conclusion that the lowering operator obeys the relation

$$\hat{a}_i(z) = \sqrt{\frac{k}{2}}\mu_i(z)G\begin{pmatrix}\hat{R}\\ \hat{\Theta}\end{pmatrix} \tag{54}$$

for all values of z.

VI. STRUCTURE OF THE MODES

We have described a procedure to construct the ladder operators that generate the complete set of transverse modes in all transverse planes of the lens guide. Since the ladder operators are periodic, the generated modes $|v_{nm}\rangle$ in the rotating frame are reproduced after a period $2L$ up to a phase factor. In this section we describe both the spatial and spectral structure of these modes more explicitly.

A. Algebraic expressions of modes

The fundamental mode $|v_{00}(z)\rangle$ in the rotating frame obeys the requirement that the lowering operators $\hat{a}_i(z)$ give zero when acting on it for all transverse planes z in the lens guide. An explicit analytical expression of the normalized mode function as it propagates through the lens guide can be found after a slight generalization of our earlier results for a stationary astigmatic cavity [7,22]. This result showed that the fundamental mode can be expressed in terms of the z-dependent eigenrays, which give rise to the 2×2 matrices $\mathbf{P}(z)$ and $\mathbf{T}(z)$. For rotating cavities, the same result applies in the rotating frame, where the time dependence disappears. In the rotating frame, the beam profile is given by the general Gaussian expression

$$v_{00}(R,z) = \langle R|v_{00}(z)\rangle = \sqrt{\frac{k}{\pi \det \mathbf{P}(z)}}\exp\left(\frac{ik}{2}R\mathbf{T}(z)\mathbf{P}^{-1}(z)R\right). \tag{55}$$

From the properties (52) of the matrices $\mathbf{P}$ and $\mathbf{T}$ it follows that the matrix $\mathbf{TP}^{-1}$ is symmetric. In the intervals between the lenses, the corresponding time-dependent mode $|u_{00}(z,t)\rangle = \hat{U}_r(\Omega t)|v_{00}(z)\rangle$ obeys the time-dependent paraxial wave equation. (10), and the input-output relation for $|u_{00}(z,t)\rangle$ across a lens of type 1 or 2 corresponds to the lens operator $\hat{U}_1(t)$ or $\hat{U}_2(t)$ as in Eq. (17).

The periodicity (49) of the eigenrays μ_i ensures that the matrix $\mathbf{T}(z)\mathbf{P}^{-1}(z)$ is periodic with period $2L$. Moreover, the determinant of $\mathbf{P}$ picks up a phase factor after one period, according to the identity

$$\det \mathbf{P}(2L) = e^{i(\chi_1+\chi_2)}\det \mathbf{P}(0). \tag{56}$$

As a result, the fundamental mode (55) picks up a phase factor $\exp[-i(\chi_1+\chi_2)/2]$ after a period of the lens guide, or over a cavity round trip.

The higher-order modes $|v_{nm}(z)\rangle$ in the rotating frame are obtained from the fundamental mode by using the z-dependent version of Eq. (47)

$$|v_{nm}(z)\rangle = \frac{1}{\sqrt{n\,!\,m!}}[\hat{a}_1^\dagger(z)]^n[\hat{a}_2^\dagger(z)]^m|v_{00}(z)\rangle. \tag{57}$$

The periodicity (49) of the eigenray is reflected in a similar periodicity of the lowering operator, in the form

$$\hat{a}_i(z+2L) = e^{i\chi_i}\hat{a}_i(z), \tag{58}$$

which in turn will give rise to a periodicity of the modes in the rotating frame $|v_{nm}(z)\rangle$. From this equation we find that the raising operator gets an additional phase $\exp(-i\chi_i)$ after a round trip. The phase factor picked up by the mode $|v_{nm}\rangle$ (or by the time-dependent mode $|u_{nm}\rangle$) is therefore specified by the relation

$$|v_{nm}(2L)\rangle = e^{-i\chi_1(n+1/2)-i\chi_2(m+1/2)}|v_{nm}(0)\rangle. \tag{59}$$

The resonance wavelengths of the modes follow from the requirement that the complex electric field

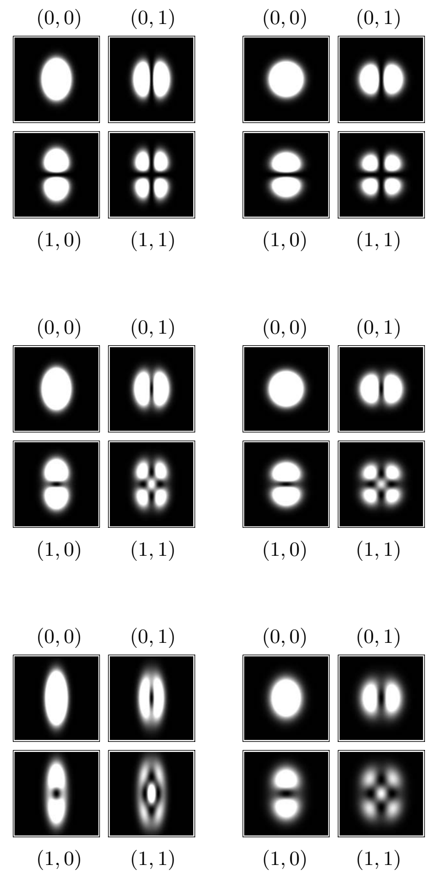

FIG. 2. Intensity patterns of the (0,0), (0,1), (1,0), and (1,1) modes of the cavity between a stationary spherical and a rotating astigmatic mirror at different rotation frequencies. The left plots show the intensity pattern near the spherical mirror while the right plots show the intensity pattern near the astigmatic mirror. The radius of curvature of the spherical mirror is $4L$, where L is the mirror separation. The radius of curvature of the astigmatic mirror in the horizontal direction of the plot is equal to $2L$, while its radius of curvature in the vertical direction is $20L$. From the top to the bottom the rotation frequency is increased from $\Omega=0$ to $\Omega=c\pi/(30L)$ and $\Omega=c\pi/(6L)$.

$$\mathbf{E}_{nm}(R,z,t) = E_0\epsilon u_{nm}(R,z,t)\exp(ikz - i\omega t) \tag{60}$$

is periodic over a round trip. This implies that the wave number k of the transverse modes must obey the identity

$$2kL - \chi_1(n+1/2) - \chi_2(m+1/2) = 2\pi q, \quad q \in \mathbb{Z}, \tag{61}$$

where the integer q is the longitudinal mode index. Note that the round-trip Gouy phases χ_1 and χ_2, and thereby the resonance wavelengths are affected by the rotation. This is obvious since they arise from the eigenvalues of the round-trip ray matrix M_{round}, which according to Eq. (40), contains the angular velocity Ω.

B. Spectral structure

Just as in Eq. (28), the time-dependent mode as viewed from the (nonrotating) laboratory frame follows from the mode $|v_{nm}(z)\rangle$ by a simple rotation, so that

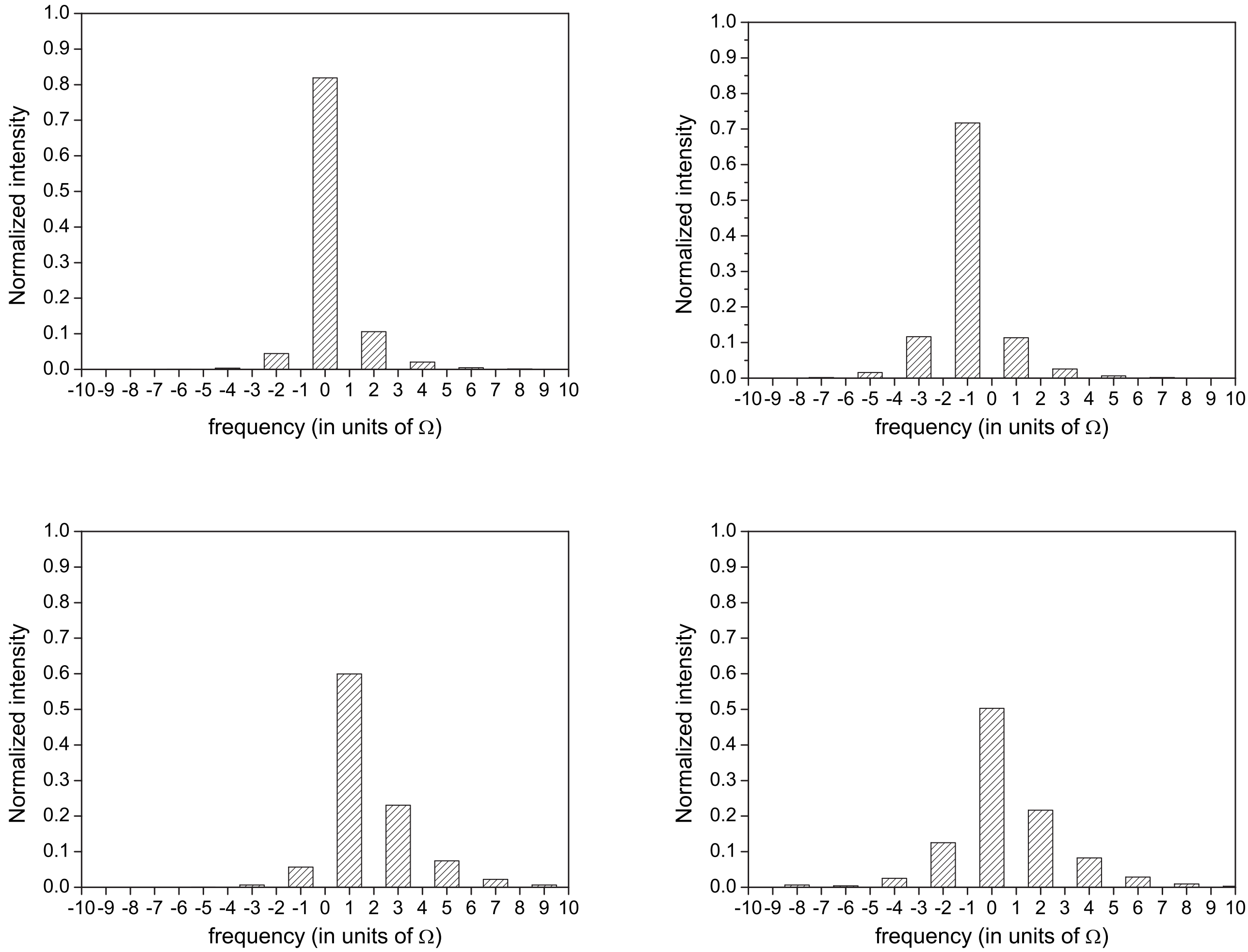

FIG. 3. Spectral structure of the (0,0), (0,1), (1,0), and (1,1) modes of the cavity between a stationary spherical and rotating astigmatic mirror. The radius of curvature of the spherical mirror is equal to $4L$, where L is the mirror separation and the radii of curvature of the astigmatic mirror are equal to $2L$ and $20L$, respectively. The rotation frequency is equal to $\Omega = c\pi/(6L)$.

$$|u_{nm}(z,t)\rangle = \hat{U}_r(\Omega t)|v_{nm}(z)\rangle. \tag{62}$$

The mode function $u_{nm}(R,z,t) = \langle R|u_{nm}(z,t)\rangle$ is a corotating solution of the time-dependent paraxial wave equation (10). Since this mode function depends on time, the electric field (60) is no longer monochromatic. The spectral structure directly follows from the polar expansion

$$u_{nm}(R,z,t) = \sum_l g_{nml}(\rho,z)e^{il(\phi-\Omega t)}. \tag{63}$$

Since the fundamental Gaussian mode (55) is even for inversion of R, the expansion (63) for u_{00} contains only even values of l, so that the fundamental mode only contains sidebands at frequencies $\omega + 2p\Omega$ with integer p and $\omega = ck$. The ladder operators $\hat{a}_i$ are odd for inversion of R, so that the modes $|u_{nm}\rangle$ with even values of $n+m$ only contain the even sidebands $\omega + 2p\Omega$, while the modes with odd values of $n+m$ only contain the odd sidebands $\omega + (2p+1)\Omega$. The separation between neighboring sidebands is always equal to 2Ω, which reflects that the cavity returns to an equivalent orientation after a rotation over an angle of 180°.

C. Cavity field

We have unfolded a cavity with rotating mirrors into a lens guide with rotating lenses and described a method to obtain expressions of the transverse modes that are reproduced after each period of the lens guide. In order to obtain an expression of the electric field inside the cavity the lens-guide modes must be folded back into the cavity

$$\mathbf{E}_{\mathrm{res}}(\mathbf{r},t) = \mathrm{Re}\{-i\epsilon E_0[u(R,z,t)\exp(ikz) - u(R,2L-z,t)\exp(-ikz)]\exp(i\omega t)\} \tag{64}$$

for $0 < z < L$. In the transverse planes near the two mirrors the two terms between the square brackets differ by phase factors $\exp[ikR\mathsf{F}_{1,2}^{-1}(t)R/2]$. For $z \simeq 0$ and $z \simeq L$ the electric field can be expressed as

$$\mathbf{E}_{\mathrm{res}}(\mathbf{r},t) = 2\,\mathrm{Re}\{\epsilon E_0 f_{1,2}(R,t)\sin(kz \pm kR\mathsf{F}_{1,2}^{-1}R/4)\exp(-i\omega t)\}, \tag{65}$$

where the + and − signs apply near mirror 1 and 2, respectively, and $f_{1,2}(R,t)$ is the profile in the imaginary plane "halfway the lenses" in the lens guide picture. A lens guide with nonrotating lenses has inversion symmetry in these planes $u(R,z)=u^{*}(R,-z)$ and, as a result, the profile f is real [7] so that the electric field adopts the phase structure of the lenses (the wave fronts coincide with the mirror surfaces) and the higher-order modes have a Hermite-Gaussian nature. In case of rotating lenses, this is no longer true for a fixed value of time t. The rotation breaks the inversion symmetry and the profile $f(R,t)$ has phase structure as well.

The sine term in Eq. (65) is the generalization of a standing wave to modes with transverse spatial structure and it shows that the electrical field vanishes on the mirror surfaces even though the wave fronts of the time-dependent modes do not fit.

VII. EXAMPLES

In this paper we have presented a method that gives expressions of the modes of uniformly rotating optical cavities. In this section we discuss some explicit examples.

A. Rotating simple astigmatism

The simplest realization of a uniformly rotating cavity consists of a stationary spherical and a rotating astigmatic (or cylindrical) mirror. In the absence of rotation the modes of such a cavity are astigmatic Hermite-Gaussian modes [5]. A cavity of this type has two symmetry planes, which are fixed by the orientation of the astigmatic mirrors and are passing through the optical axis of the cavity. The modes of such a cavity scale differently in the corresponding transverse directions. A typical example of the intensity patterns in the planes near the mirrors of these astigmatic Hermite-Gaussian modes is shown in the upper window of Fig. 2. Notice that the astigmatism of the intensity patterns is most pronounced on the spherical mirror. This is due to the fact that the astigmatism of a mirror is visible in the intensity pattern of the reflected beam only after free propagation over some distance.

If the astigmatic mirror is put into rotation the mode structure significantly changes. This is shown in the other two windows of Fig. 2. As a result of the rotation the cavity no longer has inversion symmetry in the planes through the mirror axes and the optical axis. Instead, it is invariant under inversion in those symmetry planes combined with inversion of the rotation direction. As a result of this symmetry the intensity patterns of the modes are symmetric in the symmetry planes whereas the phase distributions are not. The rotation breaks the inversion symmetry of the corresponding lens guide so that the higher modes are no longer Hermite-Gaussian modes but resemble generalized Gaussian modes with a nature in between Hermite- and Laguerre-Gaussian modes [21,23]. As a result phase singularities (vortices) appear, which are best visible in the center of the (0,1) and (1,0) modes in Fig. 2.

The spectral structure of the rotating modes is illustrated in Fig. 3. These spectra show that the modes are spectrally confined and confirm that they only have odd or even frequency components depending on the parity of the total mode number $n+m$.

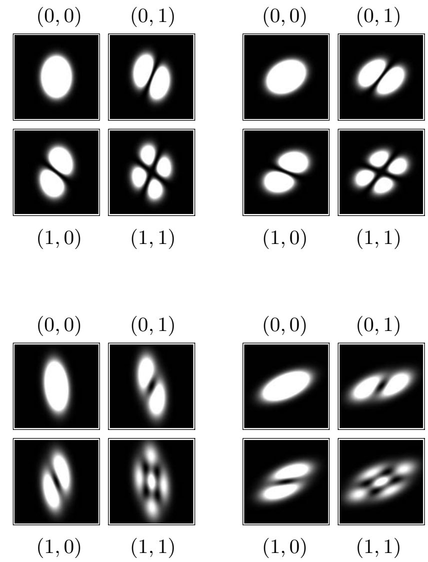

FIG. 4. Modes of an optical cavity between two identical but nonaligned rotating astigmatic mirrors for different rotation frequencies. The mirrors have radii of curvature that are equal to $2L$ and $20L$. The axes of the right mirror coincide with the horizontal and vertical directions of the plots while the axes of the left mirror are rotated over an angle $-\pi/3$. From the top to the bottom the rotation frequency is increased from $\Omega=0$ to $\Omega=c\pi/(6L)$. The latter frequency is chosen such that the angle over which the mirrors are rotated after each round trip is equal but opposite to the angle between the orientations of the two mirrors.

B. Rotating nonorthogonal astigmatism

The mode structure becomes significantly more complex if the cavity has nonorthogonal astigmatism, which is the case if it consists of two nonaligned astigmatic mirrors. Such a cavity does not have symmetry planes through the optical axis. The corresponding lens guide does have inversion symmetry in the imaginary plane "halfway the lenses" so that the higher-order modes are astigmatic Hermite-Gaussian modes. Typical examples of the modes of a stationary astigmatic cavity with nonorthogonal astigmatism are shown in the upper window of Fig. 4.

At first sight one might guess that a physical rotation of the two mirrors effectively modifies their relative orientation so that it can help to reduce the effect of nonorthogonal astigmatism. This is not the case. The effect of a physical rotation of the mirrors is essentially different from the effect of nonorthogonal astigmatism. This is illustrated in the lower window of Fig. 4. The rotation frequency is chosen such that the rotation angle after one round trip is equal but opposite to the angle between the orientation of the two mirrors. Putting the mirrors into physical rotation breaks the inversion symmetry so that the modes are no longer Hermite-Gaussian but generalized Gaussian modes that have nonorthogonal astigmatism. As a result, again, vortices appear.

VIII. CONCLUDING DISCUSSION

We have presented an algebraic method to find explicit expressions of the paraxial modes of an astigmatic optical cavity that is put into a uniform rotation at a constant velocity about its optical axis. The modes in such a time-dependent system are obtained as solutions of the time-dependent paraxial wave equation (10) that rotate along with the mirrors, i.e., are stationary in a corotating frame, and obey the boundary condition that the electric field must vanish on the mirror surfaces at all times. The regime of validity of the time-dependent paraxial wave equation provides an upper limit for the rotation frequency (12).

The method we use to find expressions of the modes that meet the mode criterion involves two pairs of bosonic ladder operators. They generate a complete set of modes in the rotating frame according to Eq. (47). The transformation of the ladder operators from a reference plane in the corotating frame to an arbitrary transverse plane (53) can be expressed in terms of the 4×4 ray matrix that describes the linear transformation of a ray through the same system. As a result the ladder operators that generate the modes can be constructed from the eigenvectors of the ray matrix for a round trip in the corotating frame (40). Similar to the case of a cavity with stationary mirrors geometric stability turns out to be the necessary and sufficient condition for the optical cavity to have modes. The time-dependent expressions of the modes $|u(z,t)\rangle$ in an external observer's frame can be obtained from the corresponding modes in the rotating frame $|v(z)\rangle$ by using Eq. (28).

In the rotating frame, the ray and wave dynamics is modified even though the ray matrices do not depend on time. In Sec. VII we have shown how the mode structure is modified for various rotation frequencies and that they remain spectrally confined as well. In the last part of Sec. VII we have shown some results on the interplay between nonorthogonal astigmatism and rotating mirrors. In both cases the cavity no longer has inversion symmetry so that the higher-order modes are generalized Gaussian modes that have a nature in between Hermite-Gaussian and Laguerre-Gaussian modes. As a result vortices appear.

The mode criterion that we have formulated in this paper cannot be applied to the case of cavities that consist of mirrors rotating at different frequencies. Such systems are significantly more complicated since the rotation cannot be eliminated by a transformation to a corotating frame. In principle, one can define the period of such a system by considering the number of round trips that is needed for both mirrors to return to an equivalent position. Once such a period is defined, the method that we have developed in this paper can be applied to find its modes, provided that the cavity is geometrically stable at all times.

Though the specific setup that we have discussed in this paper might be difficult to experimentally realize, the methods that we have developed here provide a much more general framework to cope with retardation effects in optical setups that have elements with time-dependent settings. The only restriction is that the time-dependent paraxial approximation, which we have formulated in Sec. II of this paper is justified.

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