Thermorefractive and thermochemical noise in the beamsplitter of the GEO600 gravitational-wave interferometer

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Braginsky, Gorodetsky, and Vyatchanin have shown that thermorefractive fluctuations are an important source of noise in interferometric gravitational-wave detectors. In particular, the thermorefractive noise in the GEO600 beamsplitter is expected to make a substantial contribution to the interferometer's total noise budget. Here, we present a new computation of the GEO600 thermorefractive noise, which takes into account the beam's elliptical profile and, more importantly, the fact that the laser beam induces a standing electromagnetic wave in the beamsplitter. The use of updated parameters results in the overall reduction of the calculated noise amplitude by a factor of ~ 5 in the low-frequency part of the GEO600 band, compared to the previous estimates. We also find, by contrast with previous calculations, that thermorefractive fluctuations result in white noise between 600 Hz and 39 MHz, at a level of $8.5 \cdot 10^{-24}$ Hz$^{-1/2}$. Finally, we describe a new type of thermal noise, which we call the thermochemical noise. This is caused by a random motion of optically active chemical impurities or structural defects in the direction along a steep intensity gradient of the standing wave. We discuss the potential relevance of the thermochemical noise for GEO600.

I. INTRODUCTION AND MAIN RESULTS FOR THERMOREFRACTIVE NOISE

The optical layout of GEO600, the German-British gravitational-wave detector, differs in an essential way from that of LIGO and VIRGO, the two other working gravitational-wave interferometers. In both LIGO and VIRGO each of the arms consists of a high-quality Fabry-Perot cavity, where most of the light power is concentrated, while the beamsplitter is located outside of both cavities. By contrast, in GEO600 the light, after passing the beamsplitter, is processed only once through each of the doubly folded arms [1]. Therefore, GEO600 is much more sensitive to the noise originating at the beamsplitter than its LIGO/VIRGO counterparts.

The beamsplitter in the GEO600 laser interferometer induces a difference in the optical path lengths between the interferometer's two arms since one of the beams is transmitted through the beamsplitter twice, while the other is being reflected from its surface. This difference in the optical path length can be modified by a change in the refractive index of the beamsplitter due to random fluctuations of temperature in the refracting material [2]. As Braginsky, Gorodetsky, and Vyatchanin have shown [3], the temperature fluctuations result from a random heat flow between different parts of the refractive medium. Since the interrogating beam is not homogeneous but has a Gaussian profile the inhomogeneous temperature fluctuations produce a net (fluctuating) change in the transmitted beam's phase, which is seen as noise in the overall output of the instrument. For a circular Gaussian beam, Braginsky and Vyatchanin [4] have computed the following spectral density of this thermorefractive noise expressed as an equiva-

lent change in the end mirror's position δz:

$$S^{\delta z}(\omega) = \frac{4 k_B \kappa T^2 \beta^2 a}{\pi (C \rho r_0^2 \omega)^2}. \tag{1}$$

Here, k_B is Boltzmann's constant, κ the thermal conductivity, T the temperature, $\beta = \partial n / \partial T$ where n is the refractive index, a is the beamsplitter's thickness, C the specific heat, ρ the density and r_0 is the beam's radius defined in terms of the beam intensity, so that $I = I_0 \exp(-r^2/r_0^2)$. The expression in Eq. (1) has been used to estimate the thermorefractive noise in the GEO600 beamsplitter [5].

However, inside the GEO600 beamsplitter, the beam is not circular: the 45° angle of incidence and the refractive index $n = 1.45$ of fused silica introduces ellipticity of the beam. The ratio of the major and minor axes of the beam's elliptical cross section is given by $\eta = 1.23$, while the minor axis equals the width of the original beam. More importantly, the beam is not homogeneous in the longitudinal direction since a standing wave is formed in the beamsplitter as a result of the beam's reflection from the end mirror. The changes in the material's refractive index [6] close to the nodes of the electric-field intensity have less effect on the wave's phase than changes in the antinodes. Therefore, random thermal fluctuations on the scale of the light wavelength are important, making for a much higher level of noise at frequencies higher than ~ 600 Hz than the Braginsky and Vyatchanin (BV) result would suggest.

In this paper we derive the expression for the thermorefractive noise with these corrections taken into account. We obtain the following result:

TABLE I. Parameter values taken from [7].

Symbol	(GEO600-) Value
k_B	$1.38 \cdot 10^{-23}$ J/K
κ	1.38 W/mK
T	300 K
β	$8.5 \cdot 10^{-6}$
a	$8.0 \cdot 10^{-2}$ m
η	1.23
k	$8.56 \cdot 10^{6}$ m^{-1}
r_0	$0.71 \cdot 10^{-2}$ m
C	746 J/kg K
ρ	2200 kg/m^3
L	1200 m

$$S^{\delta z}(\omega) = \frac{4k_B\kappa T^2\beta^2 a'}{\pi(C\rho r_0^2\omega)^2}\frac{\eta + \eta^{-1}}{2\eta^2}$$
$$\cdot\left[1 + \frac{2k^2r_0^2\eta}{(\eta + \eta^{-1})(1 + (2kl_{th})^4)}\right]. \tag{2}$$

Here, $a' = a/\cos(i)$ is the optical path length through the beamsplitter, where $i = \arcsin(1/\sqrt{2}n) \sim 29°$ is the angle between the beam axis and the normal to the beamsplitter, $k = 2\pi n/\lambda$ is the wave vector and $l_{th} = \sqrt{\kappa/(C\rho\omega)}$ is the thermal diffusion length. The numerical values relevant for GEO600 are given in Table I. The first term on the right-hand side of Eq. (2) is the contribution due to the transverse elliptic profile of the beam, and it agrees with the result in [4] for $\eta = 1$, while the second term is due to the presence of the standing wave. It is common to express the noise amplitude in terms of the dimensionless metric h for easy comparison to other sources. Figure 1 is a plot of $\sqrt{S^h} = \sqrt{S^{\delta z}}/L$, where L is the interferometers *unfolded* arm length and Fig. 2 is a comparison to the measured noise spectrum and the previous estimate for the beamsplitter thermorefractive noise [8].

At a typical gravitational-wave frequency of 100 Hz the noise amplitude is $\sqrt{S^h} \sim 5.1 \cdot 10^{-23}$ Hz$^{-1/2}$. For low fre-
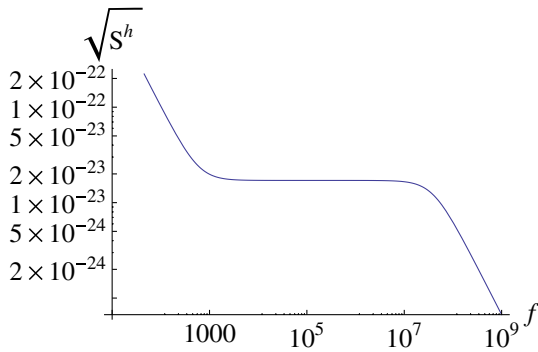

FIG. 1 (color online). Log-log plot of the noise amplitude $\sqrt{S^h(f)}$ in units of Hz$^{-1/2}$ as a function of f in units of Hz with the GEO-600 parameters.
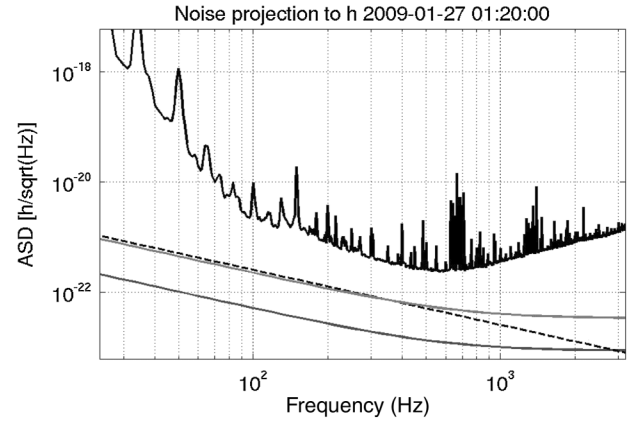

FIG. 2. Plot of measured GEO600 noise (black line), beamsplitter thermorefractive noise from previous calculations [5] (dotted line) and corrected thermorefractive noise using the same parameters (light grey line), and finally of corrected thermorefractive noise using the updated parameters from [7] (dark grey line, [8]).

quencies the effect of the standing wave is negligible, and we have the $1/f$ dependence, consistent with the previous calculations. However, for higher frequencies the standing-wave contribution takes over and the noise spectrum becomes white between 600 Hz and 39 Mhz with an amplitude $\sqrt{S^h} \sim 8.5 \cdot 10^{-24}$ Hz$^{-1/2}$. At the latter frequency the thermal diffusion length becomes comparable to the wavelength of the beam light, and the $1/f$ dependence is recovered but at much higher value than would be predicted by the BV formula.

II. METHOD OF CALCULATION

The foundation of this analysis is direct method [9] for calculating thermorefractive noise. The relevant readout variable is the phase of the output beam, translated into a change in the end mirror's position δz. It is a fairly straightforward calculation to show

$$\delta z = \int_V d^3\vec{r} \cdot a' \cdot \beta \cdot \delta T(\vec{r}) \cdot q(\vec{r}), \tag{3}$$

where the integral is over the beamsplitter's volume V, $\delta T(\vec{r})$ is the temperature fluctuation at $\vec{r}$, and $q(\vec{r})$ is the form-factor given, taking the elliptic Gaussian beam and standing wave into account, by

$$q(\vec{r}) = \frac{2}{\pi r_0^2 a'\eta}\exp\left[-\left(\frac{x^2}{r_0^2} + \frac{y^2}{(\eta r_0)^2}\right)\right]\sin^2(kz). \tag{4}$$

We now calculate the noise of this generalized variable via the hypothetical experiment proceeding in three steps:
(1) Periodically inject entropy into the medium with volume density

$$\frac{\delta S(\vec{r}, t)}{dV} = F_0 \cos(\omega t) a' \beta q(\vec{r}). \qquad (5)$$

(2) Calculate the dissipated power W_{diss} as a result of this entropy injection.

To do the latter, we solve the heat equation

$$C\rho \frac{\partial \delta T}{\partial t} - \kappa \nabla^2 \delta T = T \frac{\partial}{\partial t} \frac{\delta S(\vec{r}, t)}{dV}, \qquad (6)$$

while keeping in mind that
   (i) the solution is periodic with frequency ω
  (ii) the wavelength is much smaller than the beamsplitter thickness so we can ignore boundary effects
 (iii) the diffusion of the oscillating temperature in the transverse direction is negligible, giving $\nabla^2 = \partial^2/\partial z^2$, since $r_0 \gg l_{\mathrm{th}}$ for all frequencies of interest.

The dissipated power is then computed by using the standard expression

$$W_{\mathrm{diss}} = \int_V d^3 \vec{r} \frac{\kappa}{T} \langle (\nabla \delta T)^2 \rangle \qquad (7)$$

taken from, e.g. Landau and Lifshitz [10], where $\langle \ldots \rangle$ denotes the time average over one period. Calculate the spectral density of the noise in δz using the fluctuation dissipation theorem [9,11]

$$S^{\delta z}(\omega) = \frac{8 k_B T}{\omega^2} \frac{W_{\mathrm{diss}}}{F_0^2}. \qquad (8)$$

These three steps lead directly to the main result in Eq. (2).

A. Stability of the standing wave

In the discussion above we have assumed that the standing wave is perfectly stationary with respect to the beamsplitter. However, in GEO600 the distance between the power-recycling mirror and the beamsplitter is not interferometrically controlled, and thus the standing wave can move substantially with respect to the beamsplitter [12]. This shift is determined primarily by the motion of the power-recycling mirror, with the rms value of up to a few optical wavelength concentrated in a band around the characteristic suspension frequency of about 1 Hz. We can estimate the effect of this motion as follows:

If a standing wave moves with a uniform velocity v, the form factor q acquires periodic time dependence with the frequency of $2v/\lambda$. Thus, the noise $S_{\mathrm{new}}(f)$ at a given frequency f will become the sum of the original noise $S(f_{1,2})$ evaluated at the two sideband frequencies $f_{1,2} = f \pm 2v/\lambda$. For GEO600, the sidebands are at most a few Hz away from the original frequency, and thus the standing-wave motion is not expected to affect the thermorefractive noise at $f > 10$ Hz

III. THERMOCHEMICAL NOISE

The small scale of the wavelength opens up the possibility for a new type of thermal noise which has not yet been considered. The fused silica used for contemporary beamsplitters contains minute quantities of contaminants such as OH ions, Cl ions, and other defects that have an effect on the refractive index depending on their concentration. As these optically active contaminants diffuse up and down the steep gradient of the standing-wave electric-field intensity, they cause fluctuations of the overall beam's phase shift.

Lets compute this thermochemical noise. Let $P(\vec{r})$ be the fluctuating volume concentration of the optically active impurities. The optical path change due to these impurities is given by

$$\delta z = \int_V d^3 \vec{r} \cdot a' \cdot \alpha \cdot \delta P(\vec{r}) \cdot q(\vec{r}), \qquad (9)$$

where $\alpha = \partial n/\partial P$ and $q(\vec{r})$ is given by Eq. (4).

We can follow our earlier treatment of thermal noise [13] and calculate the dissipated power in the system under the Hamiltonian

$$H_{\mathrm{int}} = -F_0 \cos(\omega t) \delta z. \qquad (10)$$

This can be easily done by recalling that the formal expression for $P(\vec{r})$ is

$$P(\vec{r}) = \Sigma_i \delta(\vec{r} - \vec{r}_i), \qquad (11)$$

where $\vec{r}_i$ are the positions of individual optical impurities. Then, under the action of the above Hamiltonian, each impurity experiences a force

$$f_i = F_0 \cos(\omega t) a' \alpha \frac{\partial q(\vec{r}_i)}{\partial z_i}. \qquad (12)$$

Under the action of this force, the impurity drifts and energy is dissipated. We assume that the impurities achieve their terminal drift velocity v_i on a time scale much shorter than the frequencies of interest, and use Einstein's relation [14]

$$D f_i = v_i k_B T \qquad (13)$$

to compute the drift velocity [15]. Here, D is the diffusion coëfficient. The dissipated power per particle is $\langle f_i v_i \rangle$; summing over the particles and substituting the result into Eq. (8) yields

$$S^{\delta z}(\omega) = \frac{4 D P \alpha^2 k^2 a'}{\pi r_0^2 \eta \omega^2}. \qquad (14)$$

Taking values of Suprasil 311 SV used in GEO-600 for OH ions [16], $P(OH) = 3.9 \cdot 10^{24}$ m^{-3} and $\alpha = -4.52 \cdot 10^{-31}$ m^3 and estimating a value for D from [17] to be in the order of 10^{-20} m^2/s at room temperature we get

$$\sqrt{S^h(\omega)} = 4.4 \cdot 10^{-26} \text{ Hz}^{-1/2} \qquad (15)$$

at a frequency of 100 Hz, suggesting that a simple version of the thermochemical noise is outside the realm of relevance for the GEO600 interferometer. However, there may be other optically active mobile impurities (e.g., small structural defects or localized 2-state systems) in glass that have not yet been considered. Potential presence of such impurities must be thoroughly investigated in future work.

enabled by the presence of the optical standing wave in the beamsplitter. The naive estimates of this noise, which are based on what is known about the chemical impurities in Suprasil 311, place it beyond the realm of concern. However, one needs to be vigilant about other types of mobile impurities that have not yet been identified, and which could potentially produce a higher thermochemical noise.

IV. DISCUSSION

The calculations presented here demonstrate that the thermorefractive fluctuations result in a noise floor which, while substantially below currently measured noise, may become important for future updates of the GEO600 interferometer. The standing-wave contribution should be taken into account in all future calculations of the thermorefractive noise from transmissive optics.

As a side product, we have identified a new type of thermal noise: the thermochemical noise, which is also

[1] B. Wilke *et al.*, Classical Quantum Gravity **19**, 1377 (2002).
[2] Strictly speaking, the refractive index depends also on the local density of the refractive medium and can change due to thermally fluctuating mechanical motion of the beamsplitter substrate (the elasto-optic effect). However, one can show that this causes a small correction to the overall mechanical thermal noise from the beamsplitter substrate. In this paper we focus on the thermorefractive part of the noise and thus do not calculate explicitly the part of the noise caused by a spontaneous deformation of the beamsplitter.
[3] V. B. Braginsky, M. Gorodetsky, and S. P. Vyatchanin, Phys. Lett. A **271**, 303 (2000).
[4] V. B. Braginsky and S. P. Vyatchanin, Phys. Lett. A **324**, 345 (2004).
[5] Stuart Reid (private communications).
[6] In the beamsplitter's dielectric material the change of the index of refraction is caused by the change in the dielectric constant.
[7] Jerome Degallaix (private communications).
[8] We remark that there is a significant difference between the updated parameters [7] and those used for the calculations done previously [5]. Current β is 8.5 ppm compared to the previous value of 15 ppm, current r_0 is $1/\sqrt{2}$ cm compared to the previous $0.88/\sqrt{2}$ cm, and it is essential to use the length of the unfolded arm $L = 1200$ m in the calculation of S_h as opposed to the previously used $L = 600$ m. All of these corrections, together with the beam ellipticity, reduce the calculated magnitude of the thermorefractive noise.
[9] Y. Levin, Phys. Lett. A **372**, 1941 (2008).
[10] L. D. Landau and E. M. Lifshitz, *Theory of Elasticity* (Pergamon, ,Oxford, 1986), 3rd ed..
[11] H. B. Callen and T. A. Welton, Phys. Rev. **83**, 34 (1951).
[12] This was pointed out by the referee of this paper.
[13] Y. Levin, Phys. Rev. D **57**, 659 (1998).
[14] U. M. B. Marconi *et al.*, Phys. Rep. **461**, 111 (2008).
[15] Another assumption we make is that at frequencies of interest the diffusion scale is much smaller than the light wavelength. This assumption is satisfied with a very high degree of precision.
[16] Ralf Takke (private communications).
[17] K. M. Davis and M. Tomozawa, J. Non-Cryst. Solids **185**, 203 (1995).